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MASAHITO OHTA

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Instability of standing waves for the generalized Davey-Stewartson system

by

Masahito OHTA

Department of Mathematical Sciences,
University of Tokyo, Hongo, Tokyo 113, Japan

ABSTRACT. – In this paper we study the instability of standing wave $u_\omega(t, x) = e^{i\omega t} \varphi_\omega(x)$ for the following equation:

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

where $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3 . We prove that if $p \geq 1 + 4/n$, then u_ω is unstable for any $\omega \in (0, \infty)$. This is an extension of the previous result due to Cipolatti [4], who proved that if $p \geq 3$, then u_ω is unstable for any $\omega \in (0, \infty)$. We show a different criterion of instability from that in [4], which can cover the case of $1 + 4/3 \leq p < 3$ for $n = 3$. Furthermore, we prove that if $n = 3$ and $1 < p < 1 + 4/3$, then there exists $\omega_0 = \omega_0(a, p) > 0$ such that u_ω is unstable for any $\omega \in (\omega_0, \infty)$.

RÉSUMÉ. – Dans cet article on étudie l'instabilité de la solution $u_\omega(t, x) = e^{i\omega t} \varphi_\omega(x)$ pour l'équation suivante :

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

où $a > 0$, $1 < p < 1 + 4/(n-2)$ et $n = 2$ ou 3 . On prouve que si $p \geq 1 + 4/n$, u_ω est instable pour tout $\omega \in (0, \infty)$. C'est une extension du résultat précédent par Cipolatti [4], qui prouva que si $p \geq 3$, u_ω est instable pour tout $\omega \in (0, \infty)$. On montre un critère d'instabilité différente de celui de [4], qui peut couvrir le cas de $1 + 4/3 \leq p < 3$ pour $n = 3$. En outre, on prouve que si $n = 3$ et $1 < p < 1 + 4/3$, il existe $\omega_0 = \omega_0(a, p) > 0$ telle que u_ω est instable pour tout $\omega \in (\omega_0, \infty)$.

1. INTRODUCTION AND RESULTS

In the present paper we consider the instability of standing waves for the following nonlinear Schrödinger equation:

$$iu_t + \Delta u + a |u|^{p-1} u + E_1(|u|^2) u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad (1.1)$$

where $a > 0$, $1 < p < 1 + 4/(n-2)$, $n = 2$ or 3 and E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbb{R}^n$.

The equation (1.1) has its origin in fluid mechanics where, for $n = 2$ and $p = 3$, it describes the evolution of weakly nonlinear water waves that travel predominantly in one direction. More precisely, (1.1) is the n -dimensional extension of the generalized Davey-Stewartson system in the elliptic-elliptic case, namely

$$\left. \begin{aligned} iu_t + \lambda u_{xx} + u_{yy} + a |u|^{p-1} u + uv_x &= 0, \\ v_{xx} + \mu v_{yy} &= (|u|^2)_x, \end{aligned} \right\}$$

where $\lambda, \mu > 0$ (see [5]).

By a standing wave, we mean a solution of (1.1) with the form

$$u(t, x) = e^{i\omega t} \varphi_\omega(x),$$

where $\omega > 0$ and φ_ω is a ground state of the following stationary problem:

$$\left. \begin{aligned} -\Delta \psi + \omega \psi - a |\psi|^{p-1} \psi - E_1(|\psi|^2) \psi &= 0, \quad x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \quad \psi &\neq 0. \end{aligned} \right\} \quad (1.2_\omega)$$

DEFINITION 1. – We define the following notations.

$$S_\omega(v) = \frac{1}{2} |\nabla v|_2^2 + \frac{\omega}{2} |v|_2^2 - \frac{a}{p+1} |v|_{p+1}^{p+1} - \frac{1}{4} \int |v|^2 E_1(|v|^2) dx,$$

χ_ω = the set of solutions for (1.2_ω)

$$= \{\psi \in H^1(\mathbb{R}^n) : S'_\omega(\psi) = 0, \quad \psi \neq 0\},$$

$\mathcal{G}_\omega$ = the set of ground states for (1.2_ω)

$$= \{\varphi \in \chi_\omega : S_\omega(\varphi) \leq S_\omega(\psi) \text{ for all } \psi \in \chi_\omega\},$$

$$(\tau_y v)(x) = v(x - y), \quad v \in H^1(\mathbb{R}^n), \quad x, y \in \mathbb{R}^n.$$

Remark 1. – Cipolatti [3] showed that if $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3 , then $\mathcal{G}_\omega$ is not empty for any $\omega \in (0, \infty)$.

DEFINITION 2. – We shall say that the standing wave $u_\omega(t) = e^{i\omega t} \varphi_\omega$ is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

If $u_0 \in H^1(\mathbb{R}^n)$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $\|u_0 - \varphi_\omega\|_{H^1} < \delta$, then

$$\sup_{0 \leq t < \infty} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|u(t) - e^{i\theta} \tau_y \varphi_\omega\|_{H^1} < \varepsilon.$$

Otherwise, u_ω is said to be *unstable*.

Remark 2. – The unique local existence of H^1 solution for (1.1) was established by Ghidaglia and Saut [6]: If $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3 , then for any $u_0 \in H^1(\mathbb{R}^n)$ there exist $T > 0$ and a unique solution $u(\cdot) \in C([0, T]; H^1(\mathbb{R}^n))$ of (1.1) with $u(0) = u_0$. Furthermore, $u(t)$ satisfies:

$$|u(t)|_2 = |u_0|_2, \quad (1.3)$$

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0), \quad (1.4)$$

for all $t \in [0, T)$, where $\mathcal{E}$ is defined on $H^1(\mathbb{R}^n)$ by

$$\mathcal{E}(v) = \frac{1}{2} |\nabla v|_2^2 - \frac{a}{p+1} |v|_{p+1}^{p+1} - \frac{1}{4} \int |v|^2 E_1(|v|^2) dx.$$

Cipolatti [4] has proved that if $a > 0$, $3 \leq p < 1 + 4/(n-2)$ and $n = 2$ or 3 , then the standing wave $e^{i\omega t} \varphi_\omega$ is unstable for any $\omega \in (0, \infty)$. He has applied a sufficient condition for the instability essentially due to Gonçalves Ribeiro [7], and has constructed the unstable flow by using the Pohozaev multiplier $x \cdot \nabla \varphi_\omega$. For $p \geq 3$, the sufficient condition of instability in [4] is satisfied. On the other hand, under the assumption that the ground state of (1.2_ω) is unique, up to a translation and a phase change, the author [8] showed that if $a > 0$, $1 < p < 1 + 4/n$ and $n = 2$ or 3 , then there exists a sequence (ω_k) such that $\omega_k > 0$, $\omega_k \rightarrow 0$ and $e^{i\omega_k t} \varphi_{\omega_k}$ is stable. When $n = 3$, the result due to Cipolatti [4] is not optimal, that is, we obtain the following results.

THEOREM 1. – If $a > 0$, $1 + 4/n \leq p < 1 + 4/(n-2)$ and $n = 2$ or 3 , then the standing wave $e^{i\omega t} \varphi_\omega$ is unstable for any $\omega \in (0, \infty)$.

THEOREM 2. – If $a > 0$, $n = 3$ and $1 < p < 1 + 4/3$, then there exists $\omega_0 = \omega_0(a, p) > 0$ such that $e^{i\omega t} \varphi_\omega$ is unstable for any $\omega \in (\omega_0, \infty)$.

Remark 3. – As stated above, if $a > 0$, $1 < p < 1 + 4/n$ and $n = 2$ or 3 , then there exist stable standing waves for ω close to 0 (see [8]). Thus, it is natural that the exponent $p = 1 + 4/n$ should appear in Theorems 1 and 2.

Remark 4. – The exponent $p = 1 + 4/n$ is the critical one for the case of the single power nonlinearity:

$$iu_t + \Delta u + |u|^{p-1} u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n.$$

That is, it is well known that if $1 < p < 1 + 4/n$, then all standing waves are stable, and if $1 + 4/n \leq p < 1 + 4/(n-2)$, then all standing waves are unstable (see [1], [2], [10]).

This paper is organized as follows. In Section 2 we first state Theorem 3, which gives a sufficient condition for the instability. Next we prove Theorems 1 and 2 by using Theorem 3. In Section 3 we give the proof of Theorem 3. We should mention that the proof of Theorem 3 is based on the ideas of Shatah and Strauss [9]. In particular, see Section 4 in [9]. In the proof of Theorem 3, we give the unstable direction explicitly by using the scaling $\varphi_\omega^\lambda(x) = \lambda^{n/2} \varphi_\omega(\lambda x)$, $\lambda > 0$, which associates with the pseudo-conformal identity [see (3.4) in Section 3]. In the proof of Theorem 1, we can easily check our sufficient condition $\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0$ by a simple computation. This enables us to remove the restriction of $p \geq 3$ for $n = 3$, which was assumed in the paper [4] by Cipolatti. In the proof of Theorem 2, we compare the norms of the ground states with the case of $a = 0$ in (1.1) by using the variational characterization of the ground states [see Lemma 1 in Section 2]. For the case of $a = 0$, we can use the scaling argument and can estimate the norms of the ground states [see (2.11) and under (2.14)]. The analogous method has been used in [8] to show the existence of the stable standing waves.

In what follows, we omit the integral variables with respect to the spatial variable x , and we omit the integral region when it is the whole space $\mathbb{R}^n$. We denote the norms of $L^q(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ by $|\cdot|_q$ and $\|\cdot\|_{H^1}$, respectively.

2. PROOF OF THEOREMS 1 AND 2

In this section we prove Theorems 1 and 2 by using the following Theorem 3, which will be proved in the next section.

THEOREM 3. – *Suppose that $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3 . If $\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, then the standing wave $e^{i\omega t} \varphi_\omega$ is unstable, where $v^\lambda(x) = \lambda^{n/2} v(\lambda x)$, $\lambda > 0$.*

Proof of Theorem 1. – A simple computation shows

$$\begin{aligned} \mathcal{E}(\varphi_\omega^\lambda) &= \frac{\lambda^2}{2} |\nabla \varphi_\omega|^2 - \frac{a}{p+1} \lambda^{\frac{n}{2}(p-1)} |\varphi_\omega|_{p+1}^{p+1} \\ &\quad - \frac{1}{4} \lambda^n \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \partial_\lambda \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} &= |\nabla \varphi_\omega|_2^2 - \frac{n(p-1)}{2(p+1)} a |\varphi_\omega|_{p+1}^{p+1} \\ &\quad - \frac{n}{4} \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} &= |\nabla \varphi_\omega|_2^2 - \frac{n(p-1)}{2(p+1)} \left(\frac{n}{2} (p-1) - 1 \right) a |\varphi_\omega|_{p+1}^{p+1} \\ &\quad - \frac{n}{4} (n-1) \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx. \end{aligned} \quad (2.3)$$

Here, we have used the fact that $E_1(v(\lambda \cdot))(x) = E_1(v)(\lambda x)$, $\lambda > 0$ (see [3], Lemma 2.1). Since $|\varphi_\omega^\lambda|_2 = |\varphi_\omega|_2$ and $\varphi_\omega \in \chi_\omega$, we have

$$\partial_\lambda \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} = \partial_\lambda S_\omega(\varphi_\omega^\lambda)|_{\lambda=1} = \langle S'_\omega(\varphi_\omega), \partial_\lambda \varphi_\omega^\lambda|_{\lambda=1} \rangle = 0. \quad (2.4)$$

Substituting (2.2) and (2.4) into (2.3), we have

$$\begin{aligned} \partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} &= \left(\frac{n}{2} \right)^2 \frac{p-1}{p+1} \left(1 + \frac{4}{n} - p \right) a |\varphi_\omega|_{p+1}^{p+1} \\ &\quad + \frac{n}{4} (2-n) \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx. \end{aligned} \quad (2.5)$$

From the definition of E_1 and the Parseval identity, we have

$$\begin{aligned} \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx &= \int |\varphi_\omega|^2 \mathcal{F}^{-1} \sigma_1(\xi) \mathcal{F}(|\varphi_\omega|^2) dx \\ &= \int \sigma_1(\xi) |\mathcal{F}(|\varphi_\omega|^2)|^2 d\xi > 0, \end{aligned}$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$. Therefore, from (2.5), we have

$$\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0, \quad (2.6)$$

if $n = 2$ and $p > 1 + 4/n$ or $n = 3$ and $p \geq 1 + 4/n$. The proof follows from (2.6) and Theorem 3, except the case $n = 2$ and $p = 1 + 4/n$. For this special case, see Proposition 3.16 in [4]. $\square$

Before giving the proof of Theorem 2, we make some preparations.

DEFINITION 3. – For any $v \in H^1(\mathbb{R}^n)$, we define that

$$\begin{aligned} K_\omega(v) &= S_\omega(v) - \frac{1}{n} |\nabla v|_2^2 \\ &= \left(\frac{1}{2} - \frac{1}{n}\right) |\nabla v|_2^2 + \frac{\omega}{2} |v|_2^2 \\ &\quad - \frac{a}{p+1} |v|_{p+1}^{p+1} - \frac{1}{4} \int |v|^2 E_1(|v|^2) dx, \\ P(v) &= |\nabla v|_2^2 - \frac{n(p-1)}{2(p+1)} a |v|_{p+1}^{p+1} - \frac{n}{4} \int |v|^2 E_1(|v|^2) dx. \end{aligned}$$

Remark 5. – The functional K_ω is called the Pohozaev functional. Since $K_\omega(v) = \frac{1}{n} \partial_\lambda S_\omega(v(\cdot/\lambda))|_{\lambda=1}$ for $v \in H^1(\mathbb{R}^n)$, we have $K_\omega(\psi) = 0$ for $\psi \in \chi_\omega$. Moreover, from (2.2), we have $P(v) = \partial_\lambda \mathcal{E}(v^\lambda)|_{\lambda=1}$ for $v \in H^1(\mathbb{R}^n)$. The functional P is closely related to the pseudo-conformal conservation law [see, e.g., (3.4) in Section 3].

LEMMA 1. – Assume that $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3 .

(1) If $v \in H^1(\mathbb{R}^n)$ satisfies $K_\omega(v) < 0$, then there exists $\tilde{v} \in H^1(\mathbb{R}^n)$ such that $K_\omega(\tilde{v}) = 0$, $\tilde{v} \neq 0$ and $|\nabla \tilde{v}|_2 < |\nabla v|_2$,

(2) $K_\omega(\varphi_\omega) = 0$ and

$$|\nabla \varphi_\omega|_2^2 = \inf \{ |\nabla v|_2^2 : v \in H^1(\mathbb{R}^n), v \neq 0, K_\omega(v) = 0 \},$$

(3) $S_\omega(\varphi_\omega) = \inf \{ S_\omega(v) : v \in H^1(\mathbb{R}^n), |\nabla v|_2 = |\nabla \varphi_\omega|_2 \}$.

Proof. – (1) Let $K_\omega(v) < 0$. For any $\eta > 0$, we have

$$\begin{aligned} K_\omega(\eta v) &= \left(\frac{1}{2} - \frac{1}{n}\right) \eta^2 |\nabla v|_2^2 + \frac{\omega}{2} \eta^2 |v|_2^2 - \frac{a}{p+1} \eta^{p+1} |v|_{p+1}^{p+1} \\ &\quad - \frac{1}{4} \eta^4 \int |v|^2 E_1(|v|^2) dx. \end{aligned}$$

Thus, we have $K_\omega(\eta v) > 0$ for η small enough. Therefore, there exists $\eta_0 \in (0, 1)$ such that $K_\omega(\eta_0 v) = 0$. Taking $\tilde{v} = \eta_0 v$, we have $|\nabla \tilde{v}|_2 = \eta_0 |\nabla v|_2 < |\nabla v|_2$. This implies (1).

(2) See Lemma 2.1 in [8].

(3) Consider any $v \in H^1(\mathbb{R}^n)$ such that $|\nabla v|_2 = |\nabla \varphi_\omega|_2$.

Then, it follows from (1) and (2) that $K_\omega(v) \geq 0$. Thus, we have

$$S_\omega(v) = \frac{1}{n} |\nabla v|_2^2 + K_\omega(v) \geq \frac{1}{n} |\nabla \varphi_\omega|_2^2 = S_\omega(\varphi_\omega),$$

which implies (3). $\square$

Remark 6. – We note that even if there are many distinct ground states, we have

$$|\nabla \varphi_\omega^{(1)}|_2^2 = n S_\omega(\varphi_\omega^{(1)}) = n S_\omega(\varphi_\omega^{(2)}) = |\nabla \varphi_\omega^{(2)}|_2^2, \quad \varphi_\omega^{(1)}, \varphi_\omega^{(2)} \in \mathcal{G}_\omega.$$

Proof of Theorem 2. – Let $a > 0$, $n = 3$ and $1 < p < 1 + 4/3$.

From (2.5), we have $\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, provided that

$$\frac{|\varphi_\omega|_{p+1}^{p+1}}{\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx} < \frac{p+1}{3(p-1)(1+4/3-p)a}. \quad (2.7)$$

If we have

$$\lim_{\omega \rightarrow \infty} \frac{|\varphi_\omega|_{p+1}^{p+1}}{\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx} = 0, \quad (2.8)$$

then there exists $\omega_0 = \omega_0(a, p) > 0$ such that (2.7) holds for any $\omega \in (\omega_0, \infty)$. Thus, the proof follows from Theorem 3. Therefore, it is enough to show (2.8). Let $\tilde{\varphi}_\omega$ be a ground state of

$$-\Delta \psi + \omega \psi - E_1(|\psi|^2) \psi = 0, \quad x \in \mathbb{R}^n, \quad (2.9)$$

and let

$$\tilde{K}_\omega(v) = \frac{1}{6} |\nabla v|_2^2 + \frac{\omega}{2} |v|_2^2 - \frac{1}{4} \int |v|^2 E_1(|v|^2) dx.$$

From Lemma 1 (1) and (2), we have

$$K_\omega(\tilde{\varphi}_\omega) < \tilde{K}_\omega(\tilde{\varphi}_\omega) = 0 \quad \text{and} \quad |\nabla \varphi_\omega|_2^2 < |\nabla \tilde{\varphi}_\omega|_2^2. \quad (2.10)$$

We set $\tilde{\varphi}(x) = (1/\sqrt{\omega}) \tilde{\varphi}_\omega(x/\sqrt{\omega})$. Then, $\tilde{\varphi}$ is a ground state of (2.9) with $\omega = 1$. If we put $d_1 = |\nabla \tilde{\varphi}|_2^2$, from Remark 6, we have $d_1 = |\nabla \varphi|_2^2$ for all ground states φ of (2.9) with $\omega = 1$. By the change of variables, we also have $|\nabla \tilde{\varphi}_\omega|_2^2 = d_1 \sqrt{\omega}$. Therefore, from (2.10), we obtain

$$|\nabla \varphi_\omega|_2^2 \leq d_1 \sqrt{\omega} \quad \text{for any } \omega \in (0, \infty). \quad (2.11)$$

From the definition of P and (2.4), we have $P(\varphi_\omega) = 0$, which together with (2.11) implies

$$|\varphi_\omega|_{p+1}^{p+1} + \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx \leq C_1 \sqrt{\omega}, \quad \omega \in (0, \infty), \quad (2.12)$$

for some $C_1 = C_1(a, p) > 0$.

Also, from $K_\omega(\varphi_\omega) = 0$ and (2.12), we have

$$\begin{aligned} \frac{\omega}{2} |\varphi_\omega|_2^2 &\leq \frac{a}{p+1} |\varphi_\omega|_{p+1}^{p+1} + \frac{1}{4} \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx \leq C_2 \sqrt{\omega}, \\ \omega &\in (0, \infty), \end{aligned}$$

for some $C_2 = C_2(a, p) > 0$.

Thus, we have

$$|\varphi_\omega|_2^2 \leq 2 C_2 \omega^{-1/2}, \quad \omega \in (0, \infty). \quad (2.13)$$

Next, we shall estimate $|\nabla \varphi_\omega|_2^2$ from below. Let $\hat{\varphi}_\omega$ be a ground state of

$$-\Delta \psi + \omega \psi - 2 E_1(|\psi|^2) \psi = 0, \quad x \in \mathbb{R}^n,$$

and let

$$\hat{K}_\omega(v) = \frac{1}{6} |\nabla v|_2^2 + \frac{\omega}{2} |v|_2^2 - \frac{1}{2} \int |v|^2 E_1(|v|^2) dx.$$

If we have

$$\frac{2a}{p+1} |\varphi_\omega|_{p+1}^{p+1} < \frac{1}{6} |\nabla \varphi_\omega|_2^2 + \frac{\omega}{2} |\varphi_\omega|_2^2, \quad \omega > \omega_1, \quad (2.14)$$

for some $\omega_1 > 0$, then we obtain $\hat{K}_\omega(\varphi_\omega) < 2 K_\omega(\varphi_\omega)$, for $\omega > \omega_1$, and in the same way as (2.11) we can also show that $d_2 \sqrt{\omega} \leq |\nabla \varphi_\omega|_2^2$ for $\omega > \omega_1$, where $d_2 = |\nabla \hat{\varphi}_1|_2^2$. In fact, from the Gagliardo and Nirenberg inequality, we have

$$\begin{aligned} \int |\varphi_\omega|^{p+1} dx &\leq \int |\varphi_\omega|^{2+4/3} dx + \int |\varphi_\omega|^2 dx \\ &\leq C \left(\int |\varphi_\omega|^2 dx \right)^{2/3} \int |\nabla \varphi_\omega|^2 dx + \int |\varphi_\omega|^2 dx, \end{aligned} \quad (2.15)$$

for some $C > 0$. It follows from (2.13) and (2.15) that there exists $\omega_1 = \omega_1(a, p) > 0$ such that (2.14) holds for any $\omega \in (\omega_1, \infty)$.

Therefore, we have

$$d_2 \sqrt{\omega} \leq |\nabla \varphi_\omega|_2^2, \quad \omega \in (\omega_1, \infty). \quad (2.16)$$

Furthermore, from (2.11), (2.13) and (2.15), we have

$$|\varphi_\omega|_{p+1}^{p+1} \leq C_3 (\omega^{-1/3} \cdot \omega^{1/2} + \omega^{-1/2}) \leq C_4 \omega^{1/6}, \quad \omega \in (\omega_2, \infty), \quad (2.17)$$

where positive constants ω_2 , C_3 and C_4 depend only on a and p .

Also, from (2.16), (2.17) and $P(\varphi_\omega) = 0$, we have

$$\begin{aligned} \frac{3}{4} \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx &= |\nabla \varphi_\omega|_2^2 - \frac{3(p-1)}{2(p+1)} a |\varphi_\omega|_{p+1}^{p+1} \\ &\geq C_5 \omega^{1/2} - C_6 \omega^{1/6} \geq C_7 \omega^{1/2}, \quad \omega \in (\omega_3, \infty), \end{aligned} \quad (2.18)$$

where positive constants ω_3 , C_5 , C_6 and C_7 depend only on a and p .

From (2.17) and (2.18), we have

$$\frac{|\varphi_\omega|_{p+1}^{p+1}}{\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2) dx} \leq C_8 \omega^{-1/3}, \quad \omega \in (\omega_3, \infty),$$

for some $C_8 = C_8(a, p) > 0$, which implies (2.8). $\square$

3. PROOF OF THEOREM 3

In this section we first prepare three lemmas and next prove Theorem 3. Throughout this section, we assume that $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3 . Moreover, since we fix the parameter ω , we drop the subscript ω . Thus, we write φ for φ_ω , S for S_ω , and so on.

LEMMA 2. – For any $\varepsilon > 0$, there exists $\delta > 0$ and a mapping

$$\lambda : N_\varepsilon(\varphi) \rightarrow (1 - \delta, 1 + \delta)$$

$$\text{such that } |\nabla v^\lambda|_2 = |\nabla \varphi|_2 \quad \text{for } v \in N_\varepsilon(\varphi),$$

$$\text{where } N_\varepsilon(\varphi) = \{v \in H^1(\mathbb{R}^n) : \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|v - e^{i\theta} \tau_y \varphi\|_{H^1} < \varepsilon\}.$$

Proof. – Since $|\nabla v^\lambda|_2 = \lambda |\nabla v|_2$ for any $\lambda > 0$ and $v \in H^1(\mathbb{R}^n)$, it suffices to take $\lambda(v) = |\nabla \varphi|_2 / |\nabla v|_2$. $\square$

LEMMA 3. — *If $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$, then there exist $\varepsilon_1, \delta_1 > 0$ with the following property: for any $v \in N_{\varepsilon_1}(\varphi)$ satisfying $|v|_2 = |\varphi|_2$, there exists $\lambda(v) \in (1 - \delta_1, 1 + \delta_1)$ such that $\mathcal{E}(\varphi) \leq \mathcal{E}(v) + (\lambda(v) - 1)P(v)$, where P is defined as in Definition 3.*

Proof. — From the assumption $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$ and the continuity of $\partial_\lambda^2 \mathcal{E}(v^\lambda)$ in λ and v , there exist $\varepsilon_1, \delta_1 > 0$ such that $\partial_\lambda^2 \mathcal{E}(v^\lambda)|_{\lambda=1} < 0$ for any $\lambda \in (1 - \delta_1, 1 + \delta_1)$ and $v \in N_{\varepsilon_1}(\varphi)$. Since $\partial_\lambda \mathcal{E}(v^\lambda)|_{\lambda=1} = P(v)$, the Taylor expansion at $\lambda = 1$ gives

$$\mathcal{E}(v^\lambda) \leq \mathcal{E}(v) + (\lambda - 1)P(v), \quad \lambda \in (1 - \delta_1, 1 + \delta_1), \quad v \in N_{\varepsilon_1}(\varphi). \quad (3.1)$$

From Lemma 2, if we take ε_1 small enough, for any $v \in N_{\varepsilon_1}(\varphi)$, there exists $\lambda(v) \in (1 - \delta_1, 1 + \delta_1)$ such that $|\nabla v^{\lambda(v)}|_2 = |\nabla \varphi|_2$. Furthermore, from Lemma 1 (3), If $|v|_2 = |\varphi|_2$, we have

$$\mathcal{E}(v^{\lambda(v)}) = S(v^{\lambda(v)}) - \frac{\omega}{2} |v^{\lambda(v)}|_2^2 \geq S(\varphi) - \frac{\omega}{2} |\varphi|_2^2 = \mathcal{E}(\varphi). \quad (3.2)$$

Therefore, from (3.1) and (3.2), we have

$$\mathcal{E}(\varphi) \leq \mathcal{E}(v) + (\lambda(v) - 1)P(v)$$

for any $v \in N_{\varepsilon_1}(\varphi)$ satisfying $|v|_2 = |\varphi|_2$. $\square$

DEFINITION 4. — Put

$$\mathcal{A} = \{v \in N_{\varepsilon_1}(\varphi) : \mathcal{E}(v) < \mathcal{E}(\varphi), |v|_2 = |\varphi|_2, P(v) < 0\},$$

and for any $u_0 \in N_{\varepsilon_1}(\varphi)$, we define the exit time from $N_{\varepsilon_1}(\varphi)$ as follows:

$$T(u_0) = \sup \{T > 0 : u(t) \in N_{\varepsilon_1}(\varphi), 0 \leq t \leq T\},$$

where $u(t)$ is the solution of (1.1) with $u(0) = u_0$.

LEMMA 4. — *If $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$, then for any $u_0 \in \mathcal{A}$ there exists $\delta_0 = \delta_0(u_0) > 0$ such that $P(u(t)) \leq -\delta_0$ for $0 \leq t < T(u_0)$.*

Proof. — Take $u_0 \in \mathcal{A}$ and put $\delta_2 = \mathcal{E}(\varphi) - \mathcal{E}(u_0) > 0$. From Lemma 3 and the conservation laws (1.3) and (1.4), we have

$$\delta_2 \leq (\lambda(u(t)) - 1)P(u(t)), \quad 0 \leq t < T(u_0). \quad (3.3)$$

Thus, we have $P(u(t)) \neq 0$ for $0 \leq t < T(u_0)$. Since the mapping $t \mapsto P(u(t))$ is continuous and $P(u_0) < 0$, we have $P(u(t)) < 0$ for $0 \leq t < T(u_0)$. Therefore, from Lemma 3 and (3.3), we have

$$-P(u(t)) \geq \frac{\delta_2}{1 - \lambda(u(t))} \geq \frac{\delta_2}{\delta_1}, \quad 0 \leq t < T(u_0).$$

Hence, putting $\delta_0 = \delta_2/\delta_1$, we have

$$P(u(t)) \leq -\delta_0 \quad \text{for } 0 \leq t < T(u_0). \quad \square$$

Proof of Theorem 3. – Since $\partial_\lambda \mathcal{E}(\varphi^\lambda)|_{\lambda=1} = 0$, $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$ and $P(\varphi^\lambda) = \lambda \partial_\lambda \mathcal{E}(\varphi^\lambda)$, we have $\mathcal{E}(\varphi^\lambda) < \mathcal{E}(\varphi)$ and $P(\varphi^\lambda) < 0$ for $\lambda > 1$ sufficiently close to 1. Furthermore, since $|\varphi^\lambda|_2 = |\varphi|_2$ and $\lim_{\lambda \rightarrow 1} \|\varphi^\lambda - \varphi\|_{H^1} = 0$, we have $\varphi^\lambda \in \mathcal{A}$ for $\lambda > 1$ sufficiently close to 1.

Since it follows from Theorem 2.4 in [3] that $\int |x|^2 |\varphi^\lambda(x)|^2 dx < \infty$, we have

$$\frac{d^2}{dt^2} \int |x|^2 |u_\lambda(t, x)|^2 dx = 8P(u_\lambda(t)), \quad 0 \leq t < T(\varphi^\lambda), \quad (3.4)$$

where $u_\lambda(t)$ is the solution of (1.1) with $u_\lambda(0) = \varphi^\lambda$. From Lemma 4, there exists $\delta_\lambda > 0$ such that

$$P(u_\lambda(t)) \leq -\delta_\lambda, \quad 0 \leq t < T(\varphi^\lambda). \quad (3.5)$$

Hence, from (3.4) and (3.5), we can conclude that $T(\varphi^\lambda) < \infty$.

Since $\lim_{\lambda \rightarrow 1} \|\varphi^\lambda - \varphi\|_{H^1} = 0$, the proof is completed. $\square$

Remark 7. – Cipolatti also showed in [3] the existence of ground states of (1.2_ω) in the case when $a < 0$, $1 < p \leq 3$ and $n = 2$ or 3 , and proved in [4] that all standing waves are unstable in that case. We note that our proof is applicable to that case.

Remark 8. – It is an open problem whether the above u_λ blows up in finite time or not. For the case of local nonlinearity, see Berestycki and Cazenave [1].

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