ANNALES DE L'I. H. P., SECTION A

JOÃO-PAULO DIAS
MARIO FIGUEIRA
JEFFREY RAUCH

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Annales de l'I. H. P., section A, tome 61, nº 1 (1994), p. 75-86
<http://www.numdam.org/item?id=AIHPA_1994__61_1_75_0>

Ann. Inst. Henri Poincaré,

Vol. 61, n° 1, 1994, p. 75-86. *Physique théorique*

Scattering for a one-sided Klein-Gordon equation in quantum gravity

by

João-Paulo DIAS, Mario FIGUEIRA

CMAF, University of Lisdon, Av. Prof. Gama Pinto,
1699 Lisboa Codex, Portugal

and

Jeffrey RAUCH

Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109-1003, U.S.A.

ABSTRACT. – In this paper we consider a Klein-Gordon equation in one space dimension with a class of smooth positive increasing potentials V motivated by a model in quantum gravity. For large times the wave propagate freely to the left. We prove decay in time estimates and asymptotic completeness justifying this phrase.

RÉSUMÉ. – Dans cet article nous considérons une équation de Klein-Gordon dans une dimension d'espace avec une classe de potentiels réguliers positifs et croissants V motivée par un modèle en gravité quantique. Pour des grandes valeurs du temps l'onde se propage librement à gauche. Nous donnons des estimations de décroissance temporelle et prouvons la complétude asymptotique en justifiant cette phrase.

Annales de l'Institut Henri Poincaré - Physique théorique - 0246-0211
Vol. 61/94/01/$ 4.00/© Gauthier-Villars

1. INTRODUCTION AND MAIN RESULTS

This paper is motivated by the results obtained in [2] on the decay of the solutions of the following equation

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \frac{1}{4}(p-1)^2 u + e^{4z} u = 0,$$

where $p \in \mathbb{R}$ is a constant and $(z, y) \in \mathbb{R}^2$.

This equation is obtained, by the transformation $u(z, y) = x^{\frac{p-1}{2}} \psi(x, y)$, $z = \log x$, $x > 0$, from the simplified Wheeler-DeWitt equation with a massless single scalar field y (*cf.* [4])

$$\frac{\partial^2 \psi}{\partial y^2} - x^2 \frac{\partial^2 \psi}{\partial x^2} - px\frac{\partial \psi}{\partial x} + x^4 \psi = 0 \tag{1.1}$$

where $x \in \mathbb{R}_+$ is a scale factor (radius of the Universe), $p \in \mathbb{R}$ is a constant which reflects the factor-ordering ambiguity and $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ is the wave function of the Universe for the minisuperspace model.

As a consequence of the results in this paper we conclude that, for large values of y, the wave function ψ propagates freely near the origin, that is the $x^4 \psi$ term is negligable, and is very small for large values of x.

We study the asymptotic behaviour and scattering properties of the solutions of the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u + V u = 0, \qquad (x, t) \in \mathbb{R}^2 \tag{1.2}$$

where the potential V satisfies

$$\left.\begin{array}{ll} V \in C^3(\mathbb{R}), & \dot{V} \geqq 0, \\ 0 < \dfrac{dV}{dx} \leqq c(1+V), & \left|\dfrac{d^n V}{dx^n}\right| \leqq c_1 \dfrac{dV}{dx}, \\ \multicolumn{2}{c}{n = 2, 3, \quad c, c_1 \text{ positive constants},} \\ V = 0(|x|^{-1-\varepsilon}), & \varepsilon > 0, \quad \text{when } x \to -\infty. \end{array}\right\} \tag{1.3}$$

It is well known that the operator $A_0 : D(A_0) \to L^2(\mathbb{R})$ defined by

$$A_0 u = -\frac{d^2 u}{dx^2} + u + V u \tag{1.4}$$

with domain

$$D(A_0) = \left\{ u \in H^1(\mathbb{R}) | V^{\frac{1}{2}} u \in L^2(\mathbb{R}) \text{ and } -\frac{d^2 u}{dx^2} + V u \in L^2(\mathbb{R}) \right\},$$

is self-adjoint in $L^2(\mathbb{R})$.

Let $H^1_V = \{u \in H^1(\mathbb{R}) | V^{\frac{1}{2}} u \in L^2(\mathbb{R})\}$ with its natural norm and put $\mathcal{H} = H^1_V \times L^2$, $D(A) = D(A_0) \times H^1_V$,

$$A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.5}$$

It is easy to see that the operator A, with domain $D(A)$, is self-adjoint in $\mathcal{H}$.

Denote by e^{-itA} the unitary group in $\mathcal{H}$ generated by $-iA$.

For $\varphi = (u_0, u_1) \in D(A)$, denote by $u(t) = e^{-itA}\varphi$ the corresponding solution of the Cauchy problem $\left(u(0) = u_0, \; \frac{\partial u}{\partial t}(0) = u_1 \right)$ for the equation (1.2).

Let $\mathcal{H}_0 = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $B : D(B) \to \mathcal{H}_0$, $D(B) = H^2 \times H^1$, where B is the self-adjoint operator in $\mathcal{H}_0$ defined by

$$B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} - 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.6}$$

Denote by e^{-itB} the (free) unitary group in $\mathcal{H}_0$ generated by $-iB$. The operator B has constant coefficients, $D(A) \subset D(B)$ and, on $D(A)$,

$$A - B = i \begin{pmatrix} 0 & 0 \\ -V & 0 \end{pmatrix}.$$

The estimates of [2] show that the solutions of (1.2) are small in $x > 0$ for $t \to \infty$. Our main result asserts that the solutions are asymptotically equal to solutions of the free Klein-Gordon equation which travel to the left. For $t \to \infty$, it is free waves travelling to the right (for time increasing) with intervenue. In a sense this is a three space scattering theory.

Write the finite energy free solutions as

$$\sum_{+,-} \int a_\pm(\xi)\, e^{i(x\xi \pm t\langle\xi\rangle \operatorname{sgn}(\xi))}\, d\xi, \qquad \text{where} \quad \langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

The $\pm$ term has group velocity equal to $\mp|\xi|/\langle\xi\rangle$. The waves with plus sign travel to the left and those with a minus to the right. For example, if $a_+ \in C_0^\infty(\mathbb{R} \setminus 0)$ then the plus term has $L^2(\,]0, +\infty[\,)$ norm which tends to zero faster than any power of $1/t$ as t tends to $+\infty$. The same is true for all derivatives (apply the results in app. 1 to XI.3 in [6]).

The energy in $\mathcal{H}_0$ is equal to

$$\sum_{\pm} \int |\langle \xi \rangle a_{\pm}(\xi)|^2 \, d\xi.$$

Thus $\mathcal{H}_0 = \mathcal{H}_+ \oplus_\perp \mathcal{H}_-$ where $\mathcal{H}_\pm$ denotes the space of solutions with $a_\mp \equiv 0$. One has

$$\hat{u}(0, \xi) = (2\pi)^{-\frac{1}{2}} (a_+(\xi) + a_-(\xi)),$$
$$\hat{u}_t(0, \xi) = (2\pi)^{-\frac{1}{2}} \langle \xi \rangle (\operatorname{sgn}(\xi) a_+(\xi) - \operatorname{sgn}(\xi) a_-(\xi))/i.$$

Thus $(u_0, u_1) \in \mathcal{H}_+$ iff $u_1 = -i \langle D \rangle \operatorname{sgn}(D) u_0$. Note the familiar Hilbert transform $\operatorname{sgn}(D)$.

The following result has a proof similar to the proof of theorem 2 in [2].

THEOREM 1. – *Assume* $\varphi = (u_0, u_1) \in D(A^2)$. *Then*

$$\lim_{t\to\infty} \left\| \left(\frac{dV}{dx}\right)^{\frac{1}{2}} \left(|u(\cdot, t)| + \left|\frac{\partial u}{\partial t}(\cdot, t)\right| \right) \right\|_{L^2(\mathbb{R})} = 0.$$

By an adaptation of the proof of the theorem 3 in [2] we will prove the following:

THEOREM 2. – *Assume* $\varphi = (u_0, u_1) \in (\mathcal{D}(\mathbb{R}))^2$. *Then* $u \in C^4(\mathbb{R}^2)$, $u(\cdot, t) \in C_0^4(\mathbb{R})$, $\frac{\partial u}{\partial t}(\cdot, t) \in C_0^3(\mathbb{R})$ *and*

$$\lim_{t\to\infty} \left\| \left(\frac{dV}{dx}\right)^{\frac{1}{2}} \frac{\partial u}{\partial x}(\cdot, t) \right\|_{L^2(\mathbb{R})} = 0.$$

In addition,

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}} \frac{dV}{dx} \left(|u|^2 + \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 \right) dx\, dt < +\infty \tag{1.7}$$

COROLLARY. – (i) *If* $\varphi \in (\mathcal{D}(\mathbb{R}))^2$ *and, for* $a \in \mathbb{R}$, $H_a = H^1(]a, +\infty[) \times L^2(]a, +\infty[)$, *then* $\lim_{t\to\infty} \|e^{-itA}\varphi\|_{H_a} = 0$. (ii) *If* $P_{\text{ac}} = P_{\text{ac}}(A)$ *denote the orthogonal projection of* $\mathcal{H}$ *onto the subspace of absolute continuity for* A, *then* $P_{\text{ac}} = \text{Id}$.

The first part of the corollary is an immediate consequence of theorems 1 and 2, since $\frac{dV}{dx} \geqq a_1 > 0$ in $[a, +\infty[$ [by (1.3)]. The second part will be provided as a consequence of (1.7).

Similar results can be proved for the corresponding Schrödinger equation $i\frac{\partial v}{\partial t}+\frac{\partial^2 v}{\partial x^2}-V\,v=0$ (*cf.* [3] for the special case $V(x)=e^{4x}$).
Choose

$$\left.\begin{array}{l} \chi\in C^\infty(\mathbb{R}), \qquad 0\leqq\chi\leqq 1, \\ \begin{cases}\chi=1 & \text{in }]-\infty,\, b], \\ \chi=0 & \text{in } [b_1,\ +\infty[, \qquad b_1>b\end{cases}\end{array}\right\} \tag{1.8}$$

It is easy to see that

$$\begin{aligned}\mathcal{H}_\mp &= \left\{\varphi\in\mathcal{H}_0 \,|\, \|(1-\chi)\,e^{-itB}\,\varphi\,\|_{\mathcal{H}_0} \xrightarrow[t\to\mp\infty]{} 0\right\} \\ &= \left\{\varphi\in\mathcal{H}_0|\varphi=\lim_{t\to\mp\infty} e^{itB}\,\chi\,e^{-itB}\,\varphi \text{ in } \mathcal{H}_0\right\}.\end{aligned}$$

Our main results are

THEOREM 3 (Existence of the wave operators). – *For each $\varphi\in\mathcal{H}_\mp$ there exists a unique φ_- (resp. φ_+) in $\mathcal{H}$ such that*

$$\lim_{t\to\mp\infty}\|\chi\,e^{-iBt}\,\varphi-e^{-iAt}\,\varphi_\mp\|_{\mathcal{H}}=0 \tag{1.9}$$

We have $\varphi_\mp = M^\mp\,\varphi = \lim_{t\to\mp\infty} e^{iAt}\chi\,e^{-iBt}\,\varphi$ in $\mathcal{H}$ and the maps $M^\mp:\mathcal{H}_\mp\to\mathcal{H}$ are bounded.

THEOREM 4 (Completeness of the wave operators). – *For each $\psi\in\mathcal{H}$ there exists a unique ψ_- (resp. ψ_+) in $\mathcal{H}_\mp$ such that*

$$\lim_{t\to\mp\infty}\|e^{-iAt}\,\psi-\chi\,e^{-iBt}\,\psi_\mp\|_{\mathcal{H}_0}=0 \tag{1.10}$$

We have $\psi_\mp=\lim_{t\to\mp\infty} e^{iBt}\,e^{-iAt}\,\psi$ in $\mathcal{H}_0$, and the maps $\psi\to\psi_\mp$ are bounded inverses of the maps $M^\mp$ defined in theorem 3. In addition, if $V(x)=e^{\lambda x}$, $\lambda>0$, then the map $S=(M^+)^{-1}M^-:\mathcal{H}_-\to\mathcal{H}_+$ is an isometry.

For the proof of theorems 3, 4, we use the asymetric Birman-Kato theory, more precisely the theorem in [5], and (for theorem 4) the corollary of theorems 1 and 2. Note that the asymmetry of the problem forces us to use the Birman-Kato theory two times, once for existence and, in a different way, for completeness. Analogous results can be proved for the Schrödinger equation

$$i\frac{\partial v}{\partial t}+\frac{\partial^2 v}{\partial x^2}-V\,v=0$$

using decay estimates extending those of [3] proved for the special case $V(x) = e^{4x}$.

The two first authors were supported by the JNICT project PMCT/C/CEN/18/90 and by the Gulbenkian Foundation/Deptm. of Mathematics of the University of Lisbon project. The third author was supported by the NSF grant DMS 8601783 and by the ONR grant NO 01492J1245.

2. PROOF OF THEOREM 2

Assume $\varphi = (u_0, u_1) = \left(u(0), \frac{\partial u}{\partial t}(0)\right) \in (\mathcal{D}(\mathbb{R}))^2 \subset D(A^4)$. Since $V \in C^3$ it is easy to see that the corresponding solution of the Cauchy problem u belongs to $C^4(\mathbb{R}^2)$ and that for each t, $u(\cdot, t) \in C_0^4(\mathbb{R})$, $\frac{\partial u}{\partial t}(\cdot, t) \in C_0^3(\mathbb{R})$. We have, for $v = \frac{\partial u}{\partial x}$,

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v + V v + \frac{dV}{dx} u = 0 \tag{2.1}$$

Multiply the equation (2.1) by $\frac{\partial \bar{v}}{\partial t}$, then integrate in $x \in \mathbb{R}$ and take the real part. This yields, with

$$E(v(t)) = \frac{1}{2} \int_{\mathbb{R}} \left[\left|\frac{\partial v}{\partial t}\right|^2 + \left|\frac{\partial v}{\partial x}\right|^2 + |v|^2 + V|v|^2\right](x, t)\, dx,$$

$$\frac{\partial}{\partial t} E(v(t)) + \operatorname{Re} \int_{\mathbb{R}} \frac{dV}{dx} u \frac{\partial \bar{v}}{\partial t}\, dx = 0,$$

$$\frac{\partial}{\partial t}\left[E(v(t)) - \frac{1}{2}\int_{\mathbb{R}} \frac{d^2 V}{dx^2} |u|^2\, dx\right] = \operatorname{Re} \int_{\mathbb{R}} \frac{dV}{dx} \frac{\partial u}{\partial t} \frac{\partial \bar{u}}{\partial x}\, dx.$$

Thus, for $t \geqq 0$, $\delta > 0$,

$$\begin{aligned} E(v(t)) \leqq & E(v(0)) + c_2 E(u_0) + \delta \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} |v|^2\, dx\, ds \\ & + c(\delta) \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} \left|\frac{\partial u}{\partial t}\right|^2 dx\, ds \end{aligned} \tag{2.2}$$

Now, multiply the equation (2.1) by $\frac{\partial \bar{v}}{\partial x}$, integrate in $x \in \mathbb{R}$ and take the real part to find

$$\operatorname{Re} \frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial x} \, dx - \frac{1}{2} \int_{\mathbb{R}} \frac{dV}{dx} \left| \frac{\partial u}{\partial x} \right|^2 + \operatorname{Re} \int_{\mathbb{R}} \frac{dV}{dx} u \frac{\partial^2 \bar{u}}{\partial x^2} \, dx = 0$$

and

$$\operatorname{Re} \int_{\mathbb{R}} \frac{dV}{dx} u \frac{\partial^2 \bar{u}}{\partial x^2} \, dx = - \int_{\mathbb{R}} \frac{dV}{dx} \left| \frac{\partial u}{\partial x} \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}} \frac{d^3 V}{dx^3} |u|^2 \, dx.$$

Hence,

$$\operatorname{Re} \frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial x} \, dx = \frac{3}{2} \int_{\mathbb{R}} \frac{dV}{dx} |v|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \frac{d^3 V}{dx^3} |u|^2 \, dx$$

and so, by (1.3),

$$\int_0^t \int_{\mathbb{R}} \frac{dV}{dx} |v|^2 \, dx \, ds \leqq c_3 \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} |u|^2 \, dx + c_4 \, E(v(0)) + c_4 \, E(v(t)) \tag{2.3}$$

Now, (2.2) and (2.3) yield, with a suitable $\delta > 0$,

$$E(v(t)) \leqq c_5 + c_6 \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} \left[|u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] dx \, ds \tag{2.4}$$

On the other hand, it is easy to see that the proof of proposition 3.1 in [2] yields

$$\int_0^{+\infty} \int_{\mathbb{R}} \frac{dV}{dx} \left(|u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial t^2} \right|^2 \right) dx \, ds \leqq 4 \left(E(u_0) + E(u_1) + E\left(\frac{\partial^2 u}{\partial t^2}(0) \right) \right) \tag{2.5}$$

Hence, using (2.3), (2.4) and (2.5) we obtain

$$\int_0^{\infty} \int_{\mathbb{R}} \frac{dV}{dx} |v|^2 \, dx \, ds < +\infty \tag{2.6}$$

Taking the t derivative in equation (2.1), replacing v by $\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$, and imitating the proof using (2.5), yields

$$\int_0^\infty \int_{\mathbb{R}} \frac{dV}{dx} \left|\frac{\partial v}{\partial t}\right|^2 dx\, ds < +\infty \tag{2.7}$$

The two inequalities (2.6), (2.7) with a classical argument employed for example in the proof of the theorem 2 in [2], imply

$$\lim_{t \to +\infty} \left\| \left(\frac{dV}{dx}\right)^{\frac{1}{2}} \frac{\partial u}{\partial x}(\cdot,\, t) \right\|_{L^2(\mathbb{R})} = 0.$$

Time reversibility completes the proof of theorem 2. ■

Proof of part (ii) of the corollary. – For $\varphi \in (\mathcal{D}(\mathbb{R}))^2$ a dense subset of $\mathcal{H}$, we have

$$(\varphi,\, e^{-itA}\,\varphi)_{\mathcal{H}} = \int_{\mathbb{R}} e^{-it\lambda}\, d(E_\lambda\,\varphi,\, \varphi).$$

The second part of theorem 2 implies that the l.h.s. belongs to $L^2(\mathbb{R})$. Therefore, the Fourier transform of the bounded measure $d(E_\lambda\,\varphi,\, \varphi)$ belongs to $L^2(\mathbb{R})$. Thus the measure is absolutely continuous so $P_{\text{ac}}(\varphi) = \varphi$. ■

3. PROOF OF THEOREMS 3 AND 4

Proof of theorem 3. – Let $\mathcal{H}_1 = H^2(\mathbb{R}) \times H^1(\mathbb{R})$, $\mathcal{H}_2 = \mathcal{H} = H_V^1 \times L^2$, $H_1 = B_{|H^3 \times H^2}$, $H_2 = A$ and $J : \mathcal{H}_1 \to \mathcal{H}_2$ the linear continuous operator defined by $J = \chi\, H_1^{-1}$, where χ is the function defined in (1.8). We have, for $\varphi \in H^3 \times H^2$,

$$\begin{aligned} e^{iH_2 t}\,\chi\, e^{-iH_1 t}\,\varphi &= e^{iH_2 t}\,\chi\, H_1^{-1}\, e^{-iH_1 t}\, H_1\,\varphi \\ &= e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi, \end{aligned}$$

with $\psi = H_1\,\varphi \in \mathcal{H}_1$.

Furthermore, $J D(H_1) \subset D(H_2)$ and

$$H_2\, J - J H_1 = i \begin{pmatrix} 0 & 0 \\ \dfrac{d^2\chi}{dx^2} + 2\,\dfrac{d\chi}{dx}\,\dfrac{d}{dx} - V\chi & 0 \end{pmatrix} H_1^{-1}$$

[extended to $\mathcal{H}_1$, as an element of $\mathcal{B}(\mathcal{H}_1,\, \mathcal{H}_2)$].

On the other hand, we have

$$H_1^{-1} = \begin{pmatrix} 0 & i\left(-\frac{d^2}{dx^2}+1\right)^{-1} \\ -i & 0 \end{pmatrix}.$$

Thus, if $\psi = (u, v) \in \mathcal{H}_1$, we get

$$(H_2 J - J H_1)\,\psi = i\left[\begin{pmatrix} 0 \\ \left(\frac{d^2\chi}{dx^2} - V\chi\right)\left(\frac{d^2}{dx^2}+1\right)^{-1}(iv) \end{pmatrix} + \begin{pmatrix} 0 \\ 2\frac{d\chi}{dx}\left(-\frac{d^2}{dx^2}+1\right)^{-1}\left(i\frac{dv}{dx}\right) \end{pmatrix}\right].$$

Theorem 3.2 in [1] implies that the operator $H_2 J - J H_1$ is a trace class operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. This and the theorem in [5] imply that the limits

$$\lim_{t\to\pm\infty} e^{iAt}\,\chi\, e^{-iBt}\varphi = \lim_{t\to\pm\infty} e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi$$

exist in $\mathcal{H}_2 = \mathcal{H}$. By density we can take $\varphi \in \mathcal{H}_0 = H^1 \times L^2$, and it follows immediatelly that, if $\varphi_\pm = \lim_{t\to\pm\infty} e^{iAt}\,\chi\, e^{-iBt}\,\varphi$, then $\|\varphi_\pm\|_{\mathcal{H}} \leqq c\|\varphi\|_{\mathcal{H}_0}$.

To prove unicity, suppose that $\varphi \in \mathcal{H}_+$ and $0 = \varphi_+ = \lim_{t\to+\infty} e^{iAt}\,\chi\, e^{-iBt}\,\varphi$ in $\mathcal{H}$. Then

$$\|\chi\; e^{-itB}\,\varphi\,\|_{\mathcal{H}_0} \leqq \|\chi\; e^{-itB}\,\varphi\,\|_{\mathcal{H}} \xrightarrow[t\to+\infty]{} 0.$$

Moreover,

$$\|(1-\chi)\; e^{-itB}\,\varphi\,\|_{\mathcal{H}_0} \xrightarrow[t\to+\infty]{} 0.$$

Hence,

$$\|\varphi\|_{\mathcal{H}_0} = \|\, e^{-itB}\,\varphi\,\|_{\mathcal{H}_0} \xrightarrow[t\to+\infty]{} 0. \quad \blacksquare$$

PROOF OF THEOREM 4. – Let $\mathcal{H}_1 = D(A^2)$, $\mathcal{H}_2 = \mathcal{H}_0 = H^1 \times L^2$, $H_1 = A_{|D(A^3)}$, $H_2 = B$ and $J : \mathcal{H}_1 \to \mathcal{H}_2$ defined by $J = \chi\, H_1^{-1}$, where

$$H_1^{-1} = \begin{pmatrix} 0 & i\left(-\frac{d^2}{dx^2}+1+V\right)^{-1} \\ -i & 0 \end{pmatrix}.$$

Since the projection (in $\mathcal{H}_1$) onto the subspace of absolute continuity for H_1 is $P_{\mathrm{ac}|\mathcal{H}_1} = \mathrm{Id}$, we have, for $\varphi \in D(A^3)$,

$$e^{iH_2 t}\chi\, e^{-iH_1 t}\varphi = e^{iH_2 t}\chi\, H_1^{-1}\, e^{-iH_1 t} H_1\,\varphi = e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi,$$

with $\psi = H_1\,\varphi \in \mathcal{H}_1$.

Furthermore, for $\psi_1 = (u,\, v) \in \mathcal{H}_1$ we obtain (by extending $H_2\,J - JH_1$ to $\mathcal{H}_1$):

$$(H_2\,J - JH_1)\,\psi_1 = i\left[\begin{pmatrix} 0 \\ \left(\dfrac{d^2\chi}{dx^2} - V\chi\right)\left(-\dfrac{d^2}{dx^2} + 1 + V\right)^{-1}(iv)\end{pmatrix} + \begin{pmatrix} 0 \\ 2\,\dfrac{d\chi}{dx}\,\dfrac{d}{dx}\left(\left(-\dfrac{d^2}{dx^2} + 1 + V\right)^{-1}(iv)\right)\end{pmatrix}\right].$$

To estimate the last term write

$$\begin{aligned}
&\frac{d\chi}{dx}\,\frac{d}{dx}\left(\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}(v)\right)\\
&\quad = \frac{d\chi}{dx}\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}\frac{dv}{dx}\\
&\quad - \frac{d\chi}{dx}\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}\left(\frac{dV}{dx}\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1} v\right),
\end{aligned}$$

where we have used the fact that $v \in D\left(-\dfrac{d^2}{dx^2} + 1 + V\right)$ implies that $\dfrac{dV}{dx}\left(-\dfrac{d^2}{dx^2} + 1 + V\right)^{-1} v \in L^2$ and the map from $D\left(-\dfrac{d^2}{dx^2} + 1 + V\right)$ to L^2 so defined is continuous (the proof is exactly as for the case $V(x) = e^{4x}$ which is presented in section 3 of [3]). We conclude, by the theorem 3.2 in [1] that the operator $H_2\,J - JH_1$ is a trace class operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. Hence, by the theorem in [5] we conclude that the limits

$$\lim_{t\to\pm\infty} e^{iBt}\chi\, e^{-iAt}\varphi = \lim_{t\to\pm\infty} e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi$$

exist in $\mathcal{H}_2 = \mathcal{H}_0$. By density, the same limits exist for

$$\varphi \in \mathcal{H} = H_V^1 \times L^2.$$

For such φ, let

$$\varphi_+ = \lim_{t \to +\infty} e^{iBt} \chi e^{-iAt} \varphi \quad \text{in} \quad \mathcal{H}_0.$$

The next step is to prove that

$$\varphi_+ = \lim_{t \to +\infty} e^{iBt} e^{-iAt} \varphi \quad \text{in} \quad \mathcal{H}_0.$$

This implies that

$$\| e^{-iAt} \varphi - \chi e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \underset{t \to +\infty}{\longrightarrow} 0$$

and so $\varphi_+ \in \mathcal{H}_+$ and $\|\varphi_+\|_{\mathcal{H}_0} = \lim_{t \to +\infty} \|e^{iBt} e^{-iAt} \varphi\|_{\mathcal{H}_0} \leqq \|\varphi\|_{\mathcal{H}}$.

Choose $\varphi_n \in (\mathcal{D}(\mathbb{R}))^2$, $n = 1, 2, \cdots$, such that $\varphi_n \underset{n \to \infty}{\longrightarrow} \varphi$ in $\mathcal{H}$. We have

$$\begin{aligned} \| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} &\leqq \|\chi e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \\ &\quad + \|(1-\chi) e^{-iAt} \varphi \|_{\mathcal{H}_0} \\ &\leqq \|\chi e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \\ &\quad + \|(1-\chi) e^{-iAt} \varphi_n \|_{\mathcal{H}_0} + c_2 \|\varphi - \varphi_n\|_{\mathcal{H}}. \end{aligned}$$

Since, for each n, and by the corollary of theorems 1 and 2, part (i), $\|(1-\chi) e^{-iAt} \varphi_n \|_{\mathcal{H}_0} \underset{t \to +\infty}{\longrightarrow} 0$, we conclude, by standard arguments, that $\| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \underset{t \to +\infty}{\longrightarrow} 0$.

Now, we derive

$$\begin{aligned} \| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} &\leqq \|\chi (e^{-iAt} \varphi - e^{-iBt} \varphi_+) \|_{\mathcal{H}_0} \\ &\quad + \|(1-\chi) e^{-iAt} \varphi \|_{\mathcal{H}_0} \\ &\leqq c_3 \|e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \\ &\quad + \|(1-\chi) e^{-iAt} \varphi_n \|_{\mathcal{H}_0} + c_2 \|\varphi - \varphi_n\|_{\mathcal{H}}, \end{aligned}$$

are therefore, by similar arguments,

$$\| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \underset{t \to +\infty}{\longrightarrow} 0.$$

Finally, let $S = (M^+)^{-1} M^- : \mathcal{H}_- \to \mathcal{H}_+$ which is bijective and bicontinuous. Assume $V(x) = e^{\lambda x}$, $\lambda > 0$. Let $\psi = M^- \varphi_-$, $\varphi_+ = (M^+)^{-1} \psi$. Since $D(A^2)$ is dense in $\mathcal{H}$ we can assume $\psi \in D(A^2)$. Hence, by theorem 1 and since $\dfrac{dV}{dx} = \lambda V$, we get

$$\|\psi\|_{\mathcal{H}} = \lim_{t \to \infty} e^{-itA} \psi\|_{\mathcal{H}} = \lim_{t \to \infty} \|e^{iBt} \psi\|_{\mathcal{H}_0}.$$

Moreover, we have $\varphi_\pm = \lim_{t \to +\infty} e^{itB} e^{-itA} \psi$ in $\mathcal{H}_0$ and so

$$\|\varphi_\pm\|_{\mathcal{H}_0} = \lim_{t \to \pm\infty} \| e^{-itA} \psi\|_{\mathcal{H}_0} = \|\psi\|_{\mathcal{H}}. \quad \blacksquare$$

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(Manuscript received July 12, 1993;
revised version received October 10, 1993.)