ANNALES DE L'I. H. P., SECTION A

LAURA M. MORATO
STEFANIA UGOLINI

Gaussian solutions to the lagrangian variational problem in stochastic mechanics

Annales de l'I. H. P., section A, tome 60, nº 3 (1994), p. 323-337
<http://www.numdam.org/item?id=AIHPA_1994__60_3_323_0>

Ann. Inst. Henri Poincaré,
Vol. 60, n° 3, 1994, p. 323-337. *Physique théorique*

Gaussian solutions to the Lagrangian variational problem in Stochastic Mechanics

by

Laura M. MORATO
Dip. di Fisica, Universitá di Padova,
35131 Padova, Italy

and

Stefania UGOLINI
Dip. di Fisica, Universitá di Bologna,
I-40126 Bologna, Italy

ABSTRACT. – We present a qualitative study of the time behaviour of Gaussian solutions to the general dynamical equations arising from the Lagrangian variational principle in Stochastic Mechanics.

In particular we give some results on the asymptotic stability of the solutions of the Schrödinger equation with respect to dissipative perturbations.

We also illustrate with the help of numerical calculations the "increasing vorticity phenomenon".

RÉSUMÉ. – Nous présentons une étude qualitative de l'évolution temporelle des solutions gaussiennes pour les équations générales qui résolvent le problème variationnel de la Mécanique Stochastique.

Nous donnons quelques résultats sur la stabilité asymptotique des solutions de l'équation de Schrödinger pour des perturbations dissipatives.

Nous décrivons aussi, au moyen de calculs numériques, le « phénomène de l'accroissement de la rotation ».

1. INTRODUCTION

The aim of this work is to develop a specific example in order to clarify the consistency of the Lagrangian variational principle in Stochastic Mechanics with the usual quantization procedures.

This scheme consists of extremizing the mean classical action, calculated for Markovian diffusions with constant diffusion coefficient, with respect to the path-wise variations which conserve the Brownian term.

Such a method has been introduced in [1] and subsequently re-examined in [2] in the case of Euclidean spaces. The extension to the case of Riemannian manifolds can be found in [3]. For a general discussion and possible extensions see also [4].

The denomination "Lagrangian" is used in order to discriminate this principle from the Eulerian one, where the mean action is considered as functional of the drift fields [5]. At variance with the latter, the Lagrangian principle gives in fact more general equations than the Schrödinger one, the solutions of which form an attracting set. Moreover such equations have the properties of being of dissipative type, non time-reversal invariant and of exibiting an interesting gauge structure [2]. Another peculiar fact is that the corresponding stochastic differential equations have rotational drift fields.

It is worth remarking that the procedure is conceptually very simple and it could also be seen as the most natural generalization of the classical Lagrangian variational principle to the case of diffusive motions in the configuration space. Therefore one could say that, as the Schrödinger equation, also these new equations are *included* in the classical action and, consequently, they appear interesting *per se* in the context of the classical analytical mechanics.

Furthermore they seem to solve a typical conceptual problem connected with the stochastic approach to quantum physics: are the quantum fluctuations dissipative or not? The answer in this setting could be that the dissipative effects are present in the general solutions but they cannot be observed in the usual conservative solutions of the Schrödinger equation which should be seen as a sort of "dynamical equilibrium states".

Unfortunately, for our general equations (a system of parabolic partial differential equations with a singular non linear term) the Cauchy problem turns to be absolutely not trivial.

In this paper we study in detail a specific example, namely that of two-dimensional Gaussian solutions in central symmetry: in this case the system of partial differential equations can be reduced to a non-linear three-dimensional dynamical system.

A first result is that one can prove by standard theorems the existence and the continuation for t going to infinity of such solutions. We then perform a qualitative study of the time-behaviour showing that the solutions of the Schrödinger equation take values in a two-dimensional center manifold.

We also study the asymptotic stability of the orbits lying in the center manifold (corresponding to the solutions of the Schrödinger equation) with respect to dissipative perturbations. In particular we show that such a property holds (at the first order) for a set of orbits lying in a finite region of the center manifold which contains the ground state.

We cannot prove the same result for orbits which do not belong to such a region. This fact is connected with the intriguing phenomenon of the "increasing vorticity". This fact was firstly observed by Guerra [3].

We devolve Section 4 to the description of such a phenomenon through numerical examples.

2. THE LAGRANGIAN VARIATIONAL PROBLEM: GENERAL DYNAMICAL EQUATIONS AND THE ENERGY THEOREM

We summarize in this section the basic facts concerning the Lagrangian Variational Problem in Stochastic Mechanics ([1], [2]). Let us consider the classical action functional for the time interval $[t_a, t_b]$:

$$A^{\mathrm{cl}}_{[t_a, t_b]}[q(.)]=\int_{t_a}^{t_b}\left[\frac{1}{2}m\dot{q}(t)^2-\Phi(q(t), t)\right]dt \tag{2.1}$$

where $q:\mathbb{R}\to\mathbb{R}^d$ is the configuration of the system and $\Phi:\mathbb{R}^d\to\mathbb{R}$ is a scalar field. (The case of configurations taking value in a Riemannian manifold and that of electromagnetic interactions is also considered in Stochastic Mechanics [6].)

To fix ideas we can think of one particle of mass m so that $d=3$. Following the original Nelson's scheme in Stochastic Mechanics ([7], [6]). we assume that in the case of a quantum particle there exists a smooth drift field $b:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3$ and a standard 3-dimensional Wiener process W (t) such that $q(t)$ is a strong solution of the stochastic differential equation:

$$dq(t)=b(q(t), t)\,dt+\left(\frac{\hbar}{m}\right)^{1/2}dW(t) \tag{2.2}$$

The Lagrangian Variational problem in Stochastic Mechanics is the most natural generalization of the classical one: one considers as the class of admissible motions the one given by all diffusions for which there exists a smooth drift b so that they are strong solutions of a S.D.E. of type (2.2) for a fixed Wiener process W (t). The action functional is assumed to be the mean classical action.

Denoting by $\{t_i\}_{i=1,\ldots N}$ an equipartition of the interval $[t_a, t_b]$ and by Δ the time difference $(t_a - t_b)/N$, the mean discretized action can be written as:

$$A^N_{[t_a, t_b]}[q(.,.)] = \sum_1^N E\left\{\frac{1}{2}m\frac{(q_{t_{i+1}} - q_{t_i})^2}{\Delta^2} - \Phi(q(t_i), t_i)\right\}$$
$$= \sum_1^N E\left\{\frac{1}{2}m\frac{(q_{t_{i+1}} - q_{t_i})(q_{t_i} - q_{t_{i-1}})}{\Delta^2} - \Phi(q(t_i), t_i)\right\} + \frac{3}{2}\frac{\hbar}{\Delta} + o(\Delta) \quad (2.3)$$

where the last equality follows by estimating $(q_{t_{i+1}} - q_{t_i})$ to the order $(\Delta)^{1/2}$ [6].

By the given definition of the class of admissible motions it follows that we must consider variation processes $\delta q(t)$ such that $q'(t) := q(t) + \delta q(t)$ still belong to that class, *i.e.* there must exist a smooth drift field b' so that:

$$dq'(t) = b'(q(t), t)\,dt + \left(\frac{\hbar}{m}\right)^{1/2} dW(t) \quad (2.4)$$

where $W(t)$ is the same B.M. as in (2.2).

Introducing the smooth vector field $f: \mathbb{R} \to \mathbb{R}^d$ by the equalities:

$$\delta q = \varepsilon h; \qquad b' - b = \varepsilon f + o(\varepsilon) \quad (2.5)$$

ε being a real positive number, one can easily verify that $h(t)$ satisfies, in the limit of ε going to zero, the first order differential equation:

$$\dot{h}(t) = \sum_j \frac{\partial}{\partial x^j} b(q(t), t)\,h_j(t) + f(q(t), t) \quad (2.6)$$

We can observe that since the variation does not involve the divergent terms $\frac{3}{2}\frac{\hbar}{\Delta}$ in (2.3) we can work directly with the regularized action (*see* also [4] for a different method and possible extensions)

$$A_{[t_a, t_b]}(q(.,.)) = \int_{t_a}^{t_b} E\left\{\frac{1}{2}bb_* - \Phi(q(t), t)\right\} dt \quad (2.7)$$

where b_* is the backward drift.

The Lagrangian variational problem then is that of giving conditions on the admissible $q(t)$ so that:

$$\delta A(q(.,.)) = o(\varepsilon) \quad (2.8)$$

This can be done by fixing either initial position and final momentum or, exploiting the backward representation, the final position and the initial momentum. In fact since (2.6) is a first order differential equation is not possible fixing both initial and final positions. Notice also that $h(t)$ is not adapted to the filtration generated by $q(t)$. It is immediate from (2.6) to see that $h(t)$ is a functional of the past of $q(t)$ so that the future increments of the Wiener process are independent of it; but this is no longer true for increments in the past.

This asymmetry in the measurability properties breaks down the time reversal invariance of the equations of motion.

Denoting by ρ the time dependent probability density of the trial diffusion and by $v:=(b+b_*)/2$ the current velocity, the solution to the Lagrangian variational problem is given by the dynamical equations:

$$\partial_t \rho + \nabla(\rho v) = 0 \tag{2.9}$$

$$\partial_t v + (v\nabla) v - \frac{\hbar^2}{2m^2} \nabla \left[\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}\right] \mp \frac{\hbar}{2m} (\nabla \ln \rho + \nabla) \times \nabla \times v + \nabla \Phi = 0 \tag{2.10 a, b}$$

It is immediate to notice that in the case $\nabla \times v = 0$ the equations reduce themselves to the Madelung ones, that is to the hydrodynamical version of Schrödinger equation.

The relationship between the general solutions and the subset corresponding to the orthodox ones is provided by the following theorem [2]:

THEOREM 1 (Energy Theorem). – *Let* (ρ, v) *be any solution of* (2.9), (2.10) *and let assume* $\rho = 0$ *at infinity* (1). *Then, defining the mean energy by:*

$$E = \int_{R^d} \left[\frac{1}{2} m \left(\frac{\hbar}{2m} \nabla \ln \rho\right)^2 + \frac{1}{2} m v^2 + \Phi\right] \rho \, dx \tag{2.11}$$

one has:

$$\frac{dE}{dt} = \pm \frac{\hbar}{2} \int_{R^3} (\nabla \times v)^2 \rho \, dx \tag{2.12}$$

We can see that choosing as physical equations (2.9), (2.10 *b*), the set of solutions with $\rho (\nabla \times v)^2 = 0$ is an attracting set.

(1) The case with more general boundary conditions can also be treated [2].

3. GAUSSIAN SOLUTIONS FOR THE TWO-DIMENSIONAL HARMONIC OSCILLATOR

3.1. Existence and continuation for t going to $+\infty$

Denoting by r and θ the polar coordinates in the plane (x, y) we put:

$$\Phi := \Phi(r) = \frac{1}{2} k^2 r^2 \tag{3.1}$$

and look for solutions (ρ, v) of the type:

$$\rho(r, t) = \frac{A}{\pi} \exp(-A r^2) \tag{3.2}$$

$$v(r, t) = a r \hat{r} - \alpha r \hat{\theta} \tag{3.3}$$

where A, a, α are time-dependent scalar parameters with $A > 0$.

Notice that:

$$\nabla \times v = -2\alpha \tag{3.4}$$

so that we often will refer to α as the "vorticity parameter".

The time evolution of these particular solutions to general equations (2.9), (2.10b) for Φ given by (3.1), if they exist, is described by the solutions of the three-dimensional first-order non linear system of O.D.E.:

$$\dot{A} = -2 a A \tag{3.5a}$$

$$\dot{a} = A^2 + \alpha^2 - a^2 - k^2 \tag{3.5b}$$

$$\dot{\alpha} = -2(a + A)\alpha \tag{3.5c}$$

This is immediately derived by introducing (3.1), (3.2) and (3.3) in (2.9), (2.10b) (putting $\hbar = m = 1$). With this position the dimension of A, a and α can be assumed to be that of an energy.

Denoting by ξ the triple (A, a, α) the system (3.5) can be written as:

$$\dot{\xi} = f(\xi) \tag{3.6}$$

where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a $C^{+\infty}$-function. Since this fact guarantees the existence of solutions of (3.5) (at least locally) we can claim that solutions (ρ, v) to (2.9), (2.10b) of type (3.2), (3.3) also must exist at least locally in time.

In terms of (A, a, α) the mean energy, by (2.11), is:

$$E := W(A, a, \alpha) := \frac{1}{2A}(A^2 + a^2 + \alpha^2 + k^2) \tag{3.7}$$

and consequently the energy theorem reads:

$$\frac{dW}{dt} = -2\alpha^2 \tag{3.8}$$

The minimum value of W is k, that is the ground state energy.

The energy theorem allows to prove that all solutions to (3.5) can be continued for t going to $+\infty$. In fact one can easily see that the inside of the spherical surface defined by all representative points ξ_0 s. t. $W(\xi_0)=W_0$ (W_0 being any positive constant), that is the sphere of center $C:=(W_0, 0, 0)$ and radius $R:=\sqrt{W_0^2-k^2}$, because of equality (3.8), is a trapping region for all solutions $\varphi_{\xi_0}(t)$ of (3.5) starting from any point ξ_0 on such a surface.

As is well known this is a sufficient condition for $\varphi_{\xi_0}(t)$ can be continued for t going to infinity [8].

3.2. Qualitative study and center manifold

We can observe that $(k, 0, 0)$ is the unique fixed point of (3.5), which of course corresponds to the ground-state.

To do a qualitative study of time-behaviour of the solutions to (3.5) we linearize around such a fixed point. Putting $2k=1$ and

$$\left.\begin{array}{l} x:=A-k \\ y:=a \\ z:=\alpha \end{array}\right\} \tag{3.9}$$

we can write the linearized system in the form:

$$\left.\begin{array}{l} \dot{x}=-y \\ \dot{y}=x \\ \dot{z}=-z \end{array}\right\} \tag{3.10}$$

The eigenvalues are $\lambda_{1,2}=\pm i$, $\lambda_3=-1$. This implies the existence of an invariant two-dimensional center subspace E_c represented by the plane (x, y) and an invariant one-dimensional stable subspace E_s represented by the z-axis. The trajectories lie on cilindrical surfaces as shown in Figure 1.

By the center manifold theorem (*see* for example [9], thm. 3.2.1) we can claim that such a structure is conserved for the original non linear system (3.5). More precisely there exists an invariant two-dimensional center manifold M_c tangent to E_c in $(1, 0, 0)$ and an invariant one-dimensional stable manifold M_s tangent to E_s in the same point. In our case the center manifold is unique and coincides with the plane (A, a) (the "Schrödinger plane").

We can observe that every initial state of finite energy whose representative point belongs to the stable invariant manifold has the peculiarity of relaxing toward the ground state.

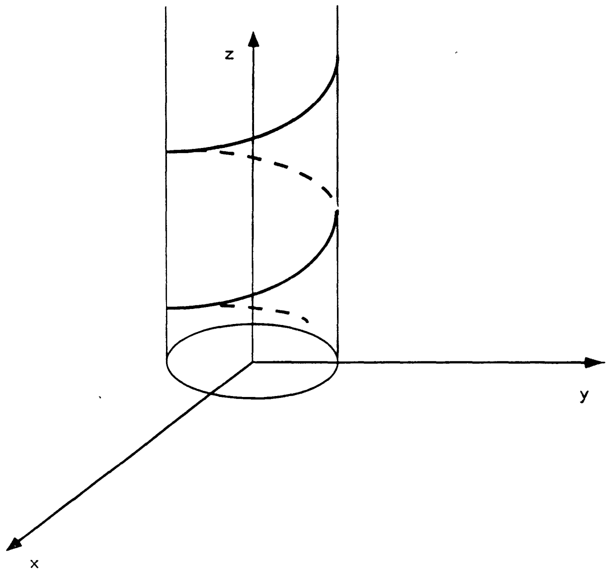

FIG. 1. – A generic trajectory of the linearized system.

3.3. The problem of the asymptotic stability of Schrödinger orbits with respect to dissipative perturbations

Let us consider the system (3.5) reduced to the center manifold, that is to the plane (A, a). To be precise we consider the set of particular solutions $\varphi_{\xi_0^*}(t)$ such that ξ_0^* is of the type $(A_0, a_0, 0)$. Recalling the observations given at the end of Section 3.1 we can see that, since in this case the energy is conserved, the corresponding trajectories are represented by circles. We shall refer to them as the "Schrödinger solutions" and the "Schrödinger orbits", respectively.

Given a Schrödinger solution $\varphi_{\xi_0^*}(t)$ we consider a perturbed solution φ_{ξ_0} by solving (3.5) with initial condition $\xi_0 := (A_0, a_0, \alpha_0)$. We want to investigate the conditions, if they exist, under which the corresponding asymptotic Schrödinger orbit coincides with the unperturbed one.

Clearly this fact is not true in case both ξ_0 and ξ_0^* lie in Schrödinger plane: in fact in this case $\varphi_{\xi_0}(t)$ and $\varphi_{\xi_0^*}(t)$ correspond to two different Schrödinger orbits which conserve respectively the energy $W(\xi_0)$ and $W(\xi_0^*)$.

The interesting case occurs when ξ_0 is of the type (A_0, a_0, α_0) and ξ_0^* is given by $(A_0, a_0, 0)$. In this case we have a purely rotational, and consequently purely dissipative, perturbation of the orbit to which the point ξ_0^* belongs.

We shall see that it is possible to give a simple condition on the initial energy so that the asymptotic orbit coincides with the unperturbed one up to terms of the second order in α_0.

Without loss of generality we can fix the initial conditions in the plane Σ of equation: $a=0$, *i.e.* $\xi_0^*:=(A_0, 0, 0)$ and $\xi_0:=(A_0, 0, \alpha_0)$, and assume $A_0<k$ (see *Fig.* 2).

In analogy with the Poincaré map method, let Γ denote the periodic trajectory $\varphi_{\xi_0^*}(t)$ and consider the plane Σ as Poincaré section for the flow map $\varphi_\xi(t)$ generated by (3.6). In fact Σ is transverse to the flow and there is a single intersection of Γ with Σ (if $0<A<k$).

Beeing ξ_0^* the intersection point of the periodic orbit Γ with the plane Σ (with $A_0<k$), starting from the point ξ_0 which is in the neighborhood $U\subseteqq\Sigma$ of ξ_0^* we can consider the map $P:U\to\Sigma$ defined by $P(\xi_0):=\varphi_{\xi_0}(\tau)$, where τ is the time for the trajectory with initial point ξ_0 to return to Σ for the first time.

We denote by $\{A_i\}_{i=0,1,\ldots}$ and $\{\alpha_i\}_{i=0,1,\ldots}$ respectively the sequences of the first and third coordinates of points $P(\xi_0)$, $P^2(\xi_0)$, ... that is of the intersections of the trajectory $\varphi_{\xi_0}(t)$ with the plane Σ [with $A(t)<k$]. Easy considerations of the geometrical and analytical nature imply that $A_i\leqq k$, $\forall i$.

Let also denote by $\{W_i\}_{i=0,1,\ldots}$ and $\{\tau_i\}_{i=0,1,\ldots}$, $\tau_0=0$, the sequences of the corresponding energies, according to (3.7), and crossing times.

We can prove the following proposition:

PROPOSITION 1. – *Let $\varphi_{\xi_0}(t)$ and $\varphi_{\xi_0^*}(t)$ be the solutions to* (3.5) *starting from the initial points ξ_0 and ξ_0^* respectively. Let $\{A_i\}_{i=0,1,\ldots}$ be the corresponding sequence of crossing coordinates as defined above. Then if ξ_0 satisfies the condition:*

$$W(\xi_0)<k\sqrt{2} \tag{3.11}$$

we have:

$$\lim_{i\to\infty}|A_i-A_0|<O(\alpha_0^2) \tag{3.12}$$

Proof. – Let denote $\varphi_{\xi_0}(t)$ by $(A(t), a(t), \alpha(t))$ and $\varphi_{\xi_0^*}(t)$ by $(A^*(t), a^*(t), 0)$. We have:

$$\begin{aligned} \dot{A}&=-2aA, & A(0)&=A_0\\ \dot{a}&=A^2+\alpha^2-a^2-k^2, & a(0)&=0\\ \dot{\alpha}&=-2(a+A)\alpha, & \alpha(0)&=\alpha_0 \end{aligned}$$

We notice that the corresponding decreasing sequence of energies satisfies the relation:

$$0<k\leqq W_i<W_0,\qquad i=1, 2, \ldots \tag{3.13}$$

Thus the sequence $\{W_i\}_{i=0,1,\ldots}$ is bounded. Let W_∞ be its limit. We also have (recalling section 3.1 and that $A_i<k$):

$$A_i = W_i - \sqrt{W_i^2 - (k^2 + \alpha_i^2)} \tag{3.14}$$

so that, by continuity of A_i as function of W_i, and α_i we have:

$$A_\infty := \lim_{i\to\infty} A_i = W_\infty - \sqrt{W_\infty^2 - k^2} \tag{3.15}$$

We consider separately the two cases $A_\infty < A_0$ and $A_\infty > A_0$.

(i) *Case* $A_\infty < A_0$. Denoting by W_* the (conserved) energy of the unperturbed orbit, that is:

$$W_* := \frac{1}{2A_0}(A_0^2 + k^2) \tag{3.16}$$

we have:

$$W_* < W_\infty < W_0 \tag{3.17}$$

Substituting in (3.17) we get:

$$0 < (A_0 - A_\infty)\left(\frac{k^2}{A_0 A_\infty} - 1\right) < \frac{\alpha_0^2}{A_0} \tag{3.18}$$

But since $A_\infty < A_0 < k$ we also have:

$$(A_0 - A_\infty)\left(\frac{k^2}{A_0^2} - 1\right) < (A_0 - A_\infty)\left(\frac{k^2}{A_0 A_\infty} - 1\right) < \frac{\alpha_0^2}{A_0} \tag{3.19}$$

So that:

$$(A_0 - A_\infty) < \frac{A_0 \alpha_0^2}{k^2 - A_0^2} \tag{3.20}$$

(ii) *Case* $A_0 < A_\infty \leqq k$. By the Energy Theorem we have, from (3.8):

$$W_0 - W_i = 2\int_0^{\tau_i} \alpha_0^2 \left[\exp\left(-4\int_0^t (A(s) + a(s))\,ds\right)\right] dt \tag{3.21}$$

Suppose now that for every $t \geqq 0$ the following condition is satisfied:

$$(A(t) + a(t)) > 0, \qquad \forall t > 0 \tag{3.22}$$

and, in addition,

$$\Omega := \inf_{s\geqq 0} (A(s) + a(s)) > 0 \tag{3.23}$$

Then it is immediate to see that, in such a case,

$$W_0 - W_i < \frac{\alpha_0^2}{2\Omega}(1 - \exp(-4\Omega\tau_i)), \qquad i = 1, 2, \ldots \tag{3.24}$$

and therefore:

$$W_0 - W_\infty < \frac{\alpha_0^2}{2\Omega} \tag{3.25}$$

Substituting the explicit expressions of W_0 and W_∞ in (3.25) we find:

$$0<(A_\infty-A_0)\left(\frac{k^2}{A_0 A_\infty}-1\right)<\alpha_0^2\left(\frac{1}{\Omega}-\frac{1}{A_0}\right) \tag{3.26}$$

which implies:

$$(A_\infty-A_0)\left(\frac{k}{A_0}-1\right)<\alpha_0^2\left(\frac{1}{\Omega}-\frac{1}{A_0}\right) \tag{3.27}$$

Let now examine condition (3.22). As observed above, $\varphi_{\xi_0}(t)$ must remain confined in the trapping region given by the interior of the sphere centered in $(W_0, 0, 0)$ and with radius $R_0=\sqrt{W_0^2-k^2}$, W_0 denoting the initial energy. Thus $A(t)$ and $a(t)$ must remain confined in the disk given by the intersection between such a sphere and the plane (A, a). A sufficient condition in order (3.22) is verified is then that the line of equation $(A+a)=0$ does not cross such a disk. An elementary calculation shows then that the condition is:

$$W_0<k\sqrt{2} \tag{3.28}$$

The corresponding region in the Schrödinger plane is qualitative represented in Figure 2.

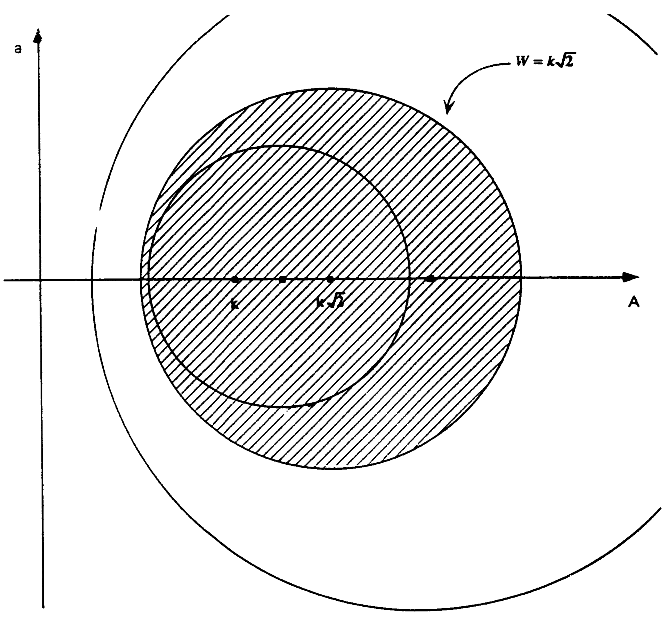

FIG. 2. – Representation of Schrödinger orbits.
The dashed region corresponds to condition (3.28).

Finally we can directly prove that:

$$\Omega \geqq \Omega_0 \tag{3.29}$$

where Ω_0 is the minimum of the function $f: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$; $(\mathrm{A}, a) \to \mathrm{A} + a$, $\mathrm{A} > 0$, on the intersection between the spherical surface corresponding to W_0 and the plane (A, a), that is the circle of equation:

$$(\mathrm{A} - \mathrm{W}_0)^2 + a^2 = \mathrm{R}_0^2, \qquad \mathrm{R}_0 = \sqrt{\mathrm{W}_0^2 - k^2} \tag{3.30}$$

Let Ω_t be the minimum of the function f on the surface corresponding to $\mathrm{W}_t := \mathrm{W}(\xi(t))$. An easy calculation, applying the Lagrange multipliers method, gives:

$$\Omega_t = \mathrm{W}_t - \mathrm{R}_t \sqrt{2}, \qquad \mathrm{R}_t = \sqrt{\mathrm{W}_t^2 - k^2} \tag{3.31}$$

We observe that:

$$\Omega_0 < \Omega_t \leqq \mathrm{A}(t) + a(t) \tag{3.32}$$

where the explicit expression of Ω_0 is of course:

$$\Omega_0 = \mathrm{W}_0 - \sqrt{\mathrm{W}_0^2 - k^2}\sqrt{2} \tag{3.33}$$

Thus Ω_0 depends only on the initial energy and is strictly positive if $\mathrm{W}_0 < k\sqrt{2}$. So this condition guarantees that both (3.22) and (3.23) are satisfied. Consequently we can write:

$$(\mathrm{A}_\infty - \mathrm{A}_0)\left(\frac{k}{\mathrm{A}_0} - 1\right) < \alpha_0^2\left(\frac{1}{\Omega_0} - \frac{1}{\mathrm{A}_0}\right) \tag{3.34}$$

so that:

$$\mathrm{A}_\infty - \mathrm{A}_0 < \alpha_0^2 \frac{\mathrm{A}_0 - \Omega_0}{\Omega_0 (k - \mathrm{A}_0)} \tag{3.35}$$

It is immediate to recognize, recalling the definition of W_0, that Ω_0 goes to a positive constant when α_0 goes to zero. Thus the proposition is proved.

We can conclude that the Schrödinger orbits are "at the first order" asymptotically stable with respect to purely dissipative perturbations.

4. NUMERICAL EXAMPLES AND THE INCREASING VORTICITY PHENOMENON

We have seen in Section 3 that a sufficient condition for the first order asymptotic stability of a Schrödinger orbit $(\mathrm{A}^*(t), a^*(t))_{t \in \mathbb{R}^+}$ with respect to purely rotational dissipative perturbations, is given by the condition that the associated energy is less than $k\sqrt{2}$.

The reason for we cannot prove an analogous result for orbits which do not belong to such a region is connected with the "increasing vorticity phenomenon", which actually is immediately expected by equation (3.5).

We observe in fact that the further degree of freedom given by the vorticity α implies that there exists a "separation plane" given by the equation:

$$h(\xi)=0 \qquad \textit{with} \quad h(\xi):=h(\mathrm{A}, a, \alpha):=(a+\mathrm{A}) \tag{4.1}$$

such that the yorticity decreases when the system respresentation point ξ remains in the region where $h>0$, it increases when $h<0$ and it remains constant when $h=0$.

This phenomenon was firsly observed by F. Guerra for a more complicated example (*see* [3]).

We have numerically integrated system (3.5) by the coupled Adams and Runge-Kutta methods. In particular Figure 3 illustrates the time-evolution for an initial state $\xi_0=(5.0, 2.0, 10.0)$ with energy $\mathrm{W}_0=13.3$

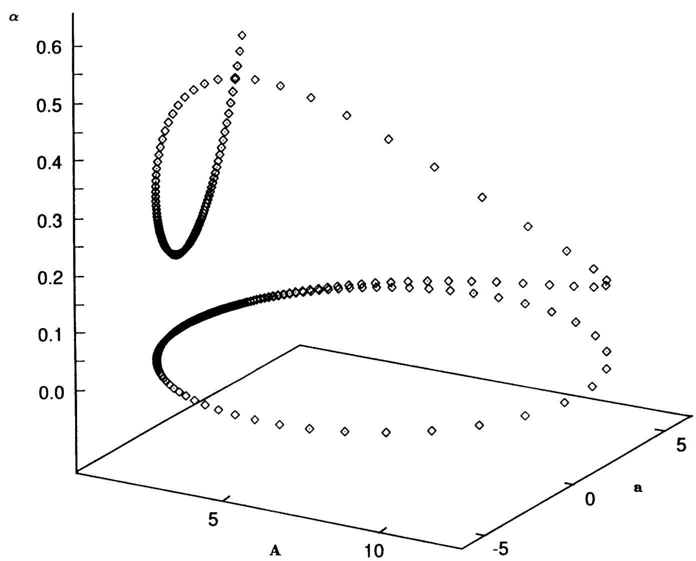

FIG. 3. – A typical time-evolution of a generic solution of system (3.5).

and external frequency $k=2.0$ (in units $\hbar=m=1$). The relaxing time is 3.2 (within 10^{-10}) and in Figure 4 the corresponding evolution of the vorticity α and of the energy $\mathrm{W}(t)$ is plotted.

The time evolution of energy dissipation was also studied in detail for different values of ξ_0 and k. We have observed that, given fixed initial values $a_0 \neq 0$ and $\mathrm{A}_0>0$, for small α_0 and high k there is a great energy dissipation, so that the asymptotic orbit is close to ground state $(k, 0, 0)$. In the opposite case there is almost no dissipation.

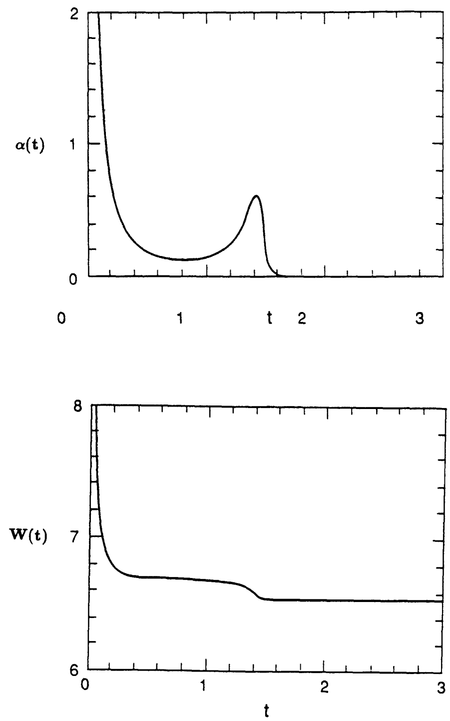

FIG. 4. – Time-evolution of the vorticity parameter α (t) end energy W (t) of the same solution of (3.5) represented in Figure 3.

5. CONCLUSIONS AND OUTLOOK

We have proved the existence and the continuability for t going to infinity of two-dimensional Gaussian solutions of the general equations (2.9), (2.10*b*).

The detailed study of the particular case under consideration has shown a good consistency of the new structure with the orthodox quantization: in fact we have identified the Schrödinger plane with a center manifold and proved that there exists a finite region, containing the ground-state representative point, such that any Schrödinger orbit lying in it is (at the first order) asymptotically stable with respect to purely rotational (*i.e.* dissipative) perturbations.

We have also described the somehow unexpected phenomenon of the increasing vorticity. It would be interesting to investigate whether such a phenomenon can occur in regions of the space of configurations where the probability density decreases, so that the vorticity would eventually concentrate in the zeros of the asymptotic density.

Since to a normalized smooth solution (ρ, v) of the system (2.9), (2.10 *b*) corresponds a diffusion process it is natural considering the consistency problem also from the probabilistic point of view of the convergence of diffusions.

This aspect will be illustrated in a forthcoming paper [10].

ACKNOWLEDGEMENTS

We are particulary grateful to Francesco Guerra for having illustrated to us his work on this subject and for valuable discussions. We wish also to thank Marco Brunella, Eric Carlen and Andrea Posilicano for their nice help.

REFERENCES

[1] L. M. MORATO, Path-Wise Stochastic Calculus of Variations with the Classical Action and Quantum Systems, *Phys. Rev. D.*, Vol. **31**, 1985, pp. 1982-1987.
[2] M. I. LOFFREDO and L. M. MORATO, Lagrangian Variational Principle in Stochastic Mechanics: Gauge Structure, *J. Math. Phys.*, Vol. **30,** 1989, pp. 354-360, M. I. LOFFREDO (private note).
[3] F. GUERRA, Private Note; E. ALDROVANDI, D. DOHRN and F. GUERRA, The Lagrangian Approach to Stochastic Variational Principles on Curved Manifolds, *Acta Applicandae Math.*, Vol. **26,** 1992, pp. 219-236.
[4] F. GUERRA, Stochastic Variational Principles and Quantum Mechanics, *Ann. Inst. Henri Poincaré*, Vol. **49,** 1988, pp. 315-324.
[5] F. GUERRA and L. M. MORATO, Quantization of Dynamical Systems and Stochastic Control Theory, *Phys. Rev. D*, Vol. **27,** pp. 1774-1786.
[6] E. NELSON, *Quantum Fluctuations*, Princeton Univ. Press, 1985.
[7] E. NELSON, *Dynamical Theories of Brownian Motion*, Princeton Univ. Press, 1967.
[8] V. I. ARNOLD, *Equazioni Differenziali Ordinarie*, Edizioni Mir, 1979.
[9] J. GUCKENHEIMER and P. HOLMES, Non linear Oscillations, Dynamical Systems, and Bifurcations of Vector Fiels, *Applied Mathematical Science*, Vol. **42,** Springer-Verlag.
[10] L. M. MORATO and S. UGOLINI, in preparation.

(*Manuscript received November* 30, 1992.)