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G. F. DELL'ANTONIO
R. FIGARI
A. TETA

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Hamiltonians for systems of N particles interacting through point interactions

by

G. F. DELL'ANTONIO

Dipartimento di Matematica,
Universitá di Roma La Sapienza, Italy,
and
Laboratorio Interdisciplinare, S.I.S.S.A., Trieste, Italy.

R. FIGARI

Dipartimento di Scienze Fisiche,
Universitá di Napoli, Italy

and

A. TETA

Dipartimento di Matematica,
Universitá di Roma La Sapienza, Italy

ABSTRACT. – We use renormalization techniques of singular quadratic forms to define and analyze Hamiltonians for N-particle systems interacting through local and translationally invariant zero-range interactions in two dimensions. When the same construction is performed in three dimensions, one obtains quadratic forms which, without symmetry requirements, are unbounded below for $N \geqq 3$ and are in any case unbounded below for any N sufficiently large.

RÉSUMÉ. – En utilisant des techniques de renormalisation de formes quadratiques singulières, on étudie des Hamiltoniens pour systèmes de N particules avec interactions de portée nulle, en dimension deux. Si on utilise la même méthode en dimension trois, on obtient des formes quadratiques qui, pour $N=3$, ont une borne inférieure seulement si l'on impose des conditions de symétrie, et qui, pour N assez grand, n'ont, en tout cas, pas de borne inférieure.

1. INTRODUCTION

We consider a system of N non relativistic spinless particles in R^d interacting through a two-body zero-range force.

To keep the notation to a minimum, we assume that all masses are equal and choose units in which $n=1/2$, $\hbar=1$. It is easily seen that the results we state hold for any choice of the masses. The system is then described informally by the Hamiltonian

$$H \equiv -\Delta - \sum_{i, j=1, i<j}^{N} \mu_{ij}\, \delta(x_i - x_j) \qquad x_i \in R^d \qquad i=1, \ldots, N \quad (1.1)$$

where Δ is the Laplacian in R^{dN} and μ_{ij} are parameters which play the role of coupling constants. One possible way to give a precise meaning to the formal expression (1.1) is to choose for H any of the self-adjoint extensions of $-\Delta^0$ which are translation and rotation invariant (*see* e.g. [1], [2]). Here Δ^0 is the restriction of Δ to C^2_Σ where

$$C^2_\Sigma \equiv \{ f \in C^2(R^{dN}); \quad f=0 \quad \text{in a neighborhood of } \Sigma \equiv \bigcup_{i<j} \sigma_{ij} \}$$

$$\sigma_{ij} \equiv \{ x \in R^{dN};\ x_i = x_j \}, \qquad x \equiv \{ x_1, \ldots, x_N \}.$$

(Notice that σ_{ij} is a hyperplane of codimension d).

Such extensions exist since $-\Delta^0$ commutes with complex conjugation. Each extension is related in a suitable sense to a choice of boundary conditions on the hyperplanes σ_{ij}. We shall call local or translation invariant an extension if locality or translation invariance are properties of the corresponding boundary conditions on Σ. (For the construction of non local extensions *see* e.g. [3]).

Remark that for $d=1$ (1.1) provides a self-adjoint extension for any choice of the parameters μ_{ij}, since the δ-potential is a small form perturbation of $-\Delta$. For $d>3$ the only self-adjoint extension of $-\Delta^0$ is $-\Delta$ (*see* e.g. [4]). We therefore discuss only the cases $d=2, 3$.

A complete characterization of the extensions is in principle possible, using the deficiency subspaces of Δ^0. Spectral properties are not easy to determine in this approach, in particular for $N>3$. In ([1] [2]) a family of extensions of $-\Delta^0$ for $d=3$, $N=3$ were defined and their spectral properties were analyzed. We shall follow a different route and define the local and translation invariant extensions through their quadratic forms.

We want to emphasize that our main concern here is the case $N \geqq 3$. This reveals entirely new difficulties as compared to the case $N=2$ which falls within the theory of one particle Schrödinger equation with zero range potentials, extensively discussed in [5] for space dimension $d=1, 2, 3$. How serious these difficulties are has been already pointed out by Minlos and Faddeev in [2]. We shall come back to this problem in Section 6.

In Section 2 we shall give an overview of the construction of these quadratic forms, emphasizing the qualitative ideas which lie behind our procedure. The basic step, already exploited in the study of interactions in R^3 supported by piecewise smooth curves [6], is to describe the extension by enlarging the form domain of $-\Delta$ through the addition of potentials produced by "charges" ξ_{ij} supported by the hyperplanes σ_{ij}.

For arbitrary N the quadratic form for the case $d=2$ will be defined in Section 3; it will be proved there that it is closed and bounded below.

In Section 4 we make precise, in the sense of Γ-convergence (for the definition and main properties *see* e.g. [7], [8]), the informal limit procedure described in Section 2; this will show that local and translation invariant extensions can be obtained as limit of a sequence of smooth perturbation of $-\Delta$. In fact convergence is proved in a sense somewhat stronger than Γ-convergence and equivalent to strong resolvent convergence.

In Section 5 we analyze the domain and the action of the Hamiltonian defined in Section 3, and we give an explicit expression for the resolvent in terms of the solution of an integral equation for the "charges", a two-dimensional equivalent of the equation introduced for the case $\mathrm{N}=3$, $d=3$ by Ter-Martirosian and Skorniakov [9]. This is an analogue of a well known result in potential theory which permits to reduce a boundary value problem to the solution of an equation for the charges "induced" on the boundary.

Our analysis proves that the Thomas effect (*see* e.g. [10], [11], [12]) is absent in two dimensions for a system of an arbitrary number of particles (our restriction to the case of equal masses is inessential for this result).

(We call Thomas effect the fact that the spectrum of H is unbounded below, reserving the name of Efimov effect to the presence of infinitely many bound states for three-particle systems with regular two-body potentials such that at least two subsystems exhibit a zero-energy resonance.)

In Section 6, following the same procedure as in $d=2$, we construct a quadratic form for the case $d=3$. It turns out that this form is unbounded below; this extends the results in [2].

In Section 7 we prove that the quadratic form constructed in Section 6 is closed and bounded below for $\mathrm{N}=3$ in the subspace of functions which are antisymmetric under exchange of two of the particles. This result was obtained by Minlos and Shermatov ([13], [14]) using different techniques. We also prove boundedness below in the subspace corresponding to angular momentum larger or equal than 1.

Finally, we consider the case of a system of $\mathrm{N}+1$ particles, N of which are identical (scalar) fermions. We show that for N sufficiently large the quadratic form is unbounded below. Therefore the Thomas effect is always present for systems composed of sufficiently many particles; this statement is seen to hold independently of restrictions on angular momentum.

In this work we leave untouched the question of the existence in $d=3$ of extensions which are translation invariant, local and bounded below and whether these extensions can be obtained by our procedure through a further renormalization or through addition of a three-body point interaction.

2. GENERAL REMARKS AND OUTLINE OF THE CONSTRUCTION

In this Section we give a motivation for the construction to be introduced in Section 3 and for the analysis of Γ-convergence in Section 4. We begin with some simple considerations. Denote by $C_0^2(\mathbb{R}^{dN})$ the class of twice differentiable functions on $\mathbb{R}^d$ with compact support. For $u \in C_0^2(\mathbb{R}^{dN})$ define

$$Q^\mu(u) = \int_{\mathbb{R}^{dN}} dx \,|\nabla u(x)|^2 - \sum_{i<j} \mu_{ij} \int_{\mathbb{R}^{dN}} dx \,|u(x)|^2 \,\delta(x_i - x_j). \qquad (2.1)$$

The bilinear form (2.1) can be extended to $L^2(\mathbb{R}^{dN})$ by setting $Q^\mu(u) = +\infty$ when $u \notin C_0^2(\mathbb{R}^{dN})$. If $\mu_{ij} \leqq 0$, $\bar{Q}^\mu \geqq 0$.

The extended functional $\bar{Q}^\mu$ is not lower semi-continuous; denote by $\tilde{Q}^\mu$ the corresponding relaxed functional (the largest lowest semi-continuous functional with values in $[-\infty, +\infty]$ such that $\tilde{Q}^\mu(u) \leqq \bar{Q}^\mu(u)$ $\forall u \in L^2(\mathbb{R}^{dN})$). One can verify that if $\mu_{ij} \leqq 0$ $\forall i<j$, one has $\tilde{Q}^\mu(u) = \int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 \,dx$. In this sense, the Laplacian is the only self-adjoint operator on $L^2(\mathbb{R}^{dN})$ which can be associated directly to (2.1), if $\mu_{ij} \leqq 0$.

If one at least of the μ_{ij} is strictly positive, then

$$\tilde{Q}^\mu(u) = -\infty \;\forall u \in L^2(\mathbb{R}^{dN}).$$

Therefore no self-adjoint operator can be associated "directly" to (2.1).

Still, as mentioned in the introduction, one can give a non trivial meaning to (2.1) by considering it a "symbol" for a self-adjoint extension of $-\Delta^0$. According to the general theory of self-adjoint extension of symmetric operators which commute with complex conjugation, a complete classification can be given in terms of deficiency spaces of $-\Delta^0$ and the unitary maps between them. The parameter space is then a space of functions and the relation with the parameters μ_{ij} occurring in (2.1) is obscure.

In this paper we will study a subclass of extensions, the local and translation invariant ones, corresponding to a choice of a particular local and translation invariant (singular) behaviour at Σ. This subclass of extensions can be described by the energy of charge distributions on Σ. They are obtained through a limit procedure.

In Section 3 we shall show that this form defines uniquely a self-adjoint operator, which coincides with $-\Delta^0$ on $C^2_\Sigma(R^{dN})$, for $d=2$.

We describe now the limit procedure. The first step is to regularize (2.1); denote by $\hat{f}$ the Fourier transform of f. We introduce a cut-off function χ_R, in $\mathbf{p}$-space and "renormalized" parameters β_{ij}; the parameters μ_{ij} will be considered as functions of the β's and of the cut-off. The specific choice of the cut-off function will be irrelevant; we choose χ_R to be the characteristic function of the ball of radius R in R^2. We set

$$\begin{aligned} F^R_\beta(u) = \int_{R^{dN}} d\mathbf{p}\, |\mathbf{p}|^2 |\hat{u}(\mathbf{p})|^2 - \sum_{i<j} \int_{R^{dN}} d\mathbf{p}\, \bar{\hat{u}}(\mathbf{p}) \frac{\mu_{ij}(\beta, R, \mathbf{p})}{4\pi(2\pi)^N} \chi_R((p_i-p_j)/\sqrt{2}) \\ \times \int_{R^d} ds\, \chi_R(s)\, \hat{u}(p_1, \ldots, p_{i-1}, (p_i+p_j)/2 + s/\sqrt{2}, \ldots, p_{j-1}, \\ (p_i+p_j)/2 - s/\sqrt{2}, \ldots, p_N) \quad (2.2) \end{aligned}$$

where $\mathbf{p} \equiv \{p_1, \ldots, p_N\}$, $p_i \in R^d$ and $\beta \equiv \{\beta_{ij}\}$ are new parameters.

It is straightforward to verify that, if $u \in C^2_0(R^{dN})$ and $\lim_{R \to +\infty} \mu_{ij}(\beta, R, \mathbf{p}) = \mu_{ij}$, then $F^R_\beta(u) \to G^\mu(u)$. To obtain a closed form we must therefore choose a different asymptotic behaviour for the functions μ_{ij}; in fact we shall require that $\lim_{R \to +\infty} \mu_{ij}(\beta, R, \mathbf{p}) = 0$. Terefore the forms corresponding to the self-adjoint extensions of $-\Delta^0$ take the value $\|u\|^2_{H^1(R^{2N})}$ on $u \in C^2_0(R^{dN})$.

It will be convenient to rewrite (2.2) introducing a set of volume charges $\rho^{R,u}$ associated to the functions $u \in H^1(R^{dN})$ and defined by

$$\rho^{R,u}(x) \equiv \sum_{i<j} \rho^{R,u}_{ij}(x)$$

$$\hat{\rho}^{R,u}_{ij}(\mathbf{p}) \equiv \chi_R\left(\frac{p_i - p_j}{\sqrt{2}}\right) \xi^{R,u}_{ij}\left(p_1, \ldots, p_{i-1}, \frac{p_i+p_j}{\sqrt{2}}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N\right) \quad (2.3)$$

$$\xi^{R,u}_{ij}(\mathbf{q}) \equiv \frac{\mu_{ij}(\beta, R, \mathbf{q})}{4\pi(2\pi)^N} \int_{R^d} ds\, \chi_R(s)\, \hat{u}\left(q_1, \ldots, q_{i-1}, \frac{q_i+s}{\sqrt{2}}, \ldots, q_{j-1}, \frac{q_i-s}{\sqrt{2}}, q_j, \ldots, q_{N-1}\right) \quad (2.4)$$

where $\mathbf{q}=\{q_1, \ldots, q_{N-1}\}$ and we allow for a small dependence of μ_{ij} on $\mathbf{q}$. For every $\lambda>0$ we introduce "potentials" $G^\lambda \circ \rho^{R,u}$ ($\circ$ stands for convolution) where G^λ is the kernel of $(-\Delta+\lambda)^{-1}$.

Remark that $G^\lambda \circ \rho \in H^2(R^{dN})$ if $\rho \in L^2(R^{dN})$ and moreover the following identity holds

$$\int_{R^{dN}} dx\, \nabla (G^\lambda \circ \rho)(x) . \nabla v(x) + \lambda \int_{R^{dN}} dx\, (G^\lambda \circ \rho)(x)\, v(x) = \int_{R^{dN}} dx\, \bar{\rho}(x)\, v(x) \qquad \forall v \in H^1(R^{dN}). \quad (2.5)$$

Remark 2.1. – Although the choice of the value of the parameter $\lambda>0$ will be inessential for our analysis, one cannot choose $\lambda=0$ since one would then lack control over the behaviour of $G^\lambda \circ \rho^{R,u}(x)$ for large values of $|x|$.

For every choice of $\lambda>0$ one has, using (2.5) and an integration by parts

$$F_\beta^R(u) = \int_{R^{dN}} dx\, [|\nabla(u - G^\lambda \circ \rho^{R,u})|^2 + \lambda |u - G^\lambda \circ \rho^{R,u}|^2 - \lambda |u|^2] + \sum_{i<j} \int_{R^{dN}} dx\, (\bar{\rho}_{ij}^{R,u} u - \bar{\rho}_{ij}^{R,u} G^\lambda \circ \rho_{ij}^{R,u}) - \sum_{ij,\, hk}{}' \int_{R^{dN}} dx\, \bar{\rho}_{ij}^{R,u} G^\lambda \circ \rho_{hk}^{R,u} \quad (2.6)$$

where we have used the symbol $\sum'$ to denote that the sum must be performed on the couples (i, j), $i<j$ and (h, k), $h<k$ with $(i, j) \neq (h, k)$.

For concreteness, we restrict ourselves from now on in this Section to the case $d=2$.

The two sums in (2.6) are more conveniently expressed in Fourier transform, taking the form

$$\sum_{i<j} \int_{R^{2N-2}} d\mathbf{p}\, |\xi_{ij}^{R,u}(\mathbf{q})|^2 \left(\frac{4\pi (2\pi)^N}{\mu_{ij}(\beta, R, \mathbf{q})} - \pi \log \frac{R^2 + |\mathbf{q}|^2 + \lambda}{|\mathbf{q}|^2 + \lambda} \right) - \sum_{ij,\, hk}{}' \int_{R^{2N}} d\mathbf{p}\, \chi_R((p_i - p_j)/\sqrt{2})\, \chi_R((p_h - p_k)/\sqrt{2}) \times \frac{\xi_{ij}^{R,u}(p_1, \ldots, p_{i-1}, (p_i+p_j)/\sqrt{2}, \ldots, p_N)\, \xi_{hk}^{R,u}(p_1, \ldots, p_{h-1}, (p_h+p_k)/\sqrt{2}, \ldots, p_N)}{|\mathbf{p}|^2 + \lambda}. \quad (2.7)$$

To avoid divergences in the limit $R \to \infty$ we choose

$$\mu_{ij}(\beta, R, \mathbf{q}) \equiv \frac{4\pi (2\pi)^N}{\beta_{ij} + \pi \log (R^2 + |\mathbf{q}|^2)}. \quad (2.8)$$

We shall discuss later the relation between the parameters β_{ij} and the boundary conditions at σ_{ij}.

Remark 2.2. – For the informal manipulations of this Section, we could have chosen, instead of (2.8), the $\mathbf{q}$-independent relation

$$\mu_{ij}(\beta, \mathrm{R}) = \frac{4\pi(2\pi)^{\mathrm{N}}}{\beta_{ij} + \pi \log \mathrm{R}^2}. \tag{2.9}$$

This would give the same weak limit, but the proof of Γ-convergence we shall give in Section 4 does not hold if the μ_{ij} are chosen as in (2.9).

If one chooses μ_{ij} as in (2.8), the approximating form can be written as

$$\mathrm{F}_\beta^{\mathrm{R}}(u) = \mathscr{F}^{\mathrm{R},\lambda}(u) + \Phi_\beta^{\mathrm{R},\lambda}(\xi^{\mathrm{R},u}) \tag{2.10}$$

where

$$\mathscr{F}^{\mathrm{R},\lambda}(u) = \int_{\mathbf{R}^{2\mathrm{N}}} dx\,[|\nabla(u - \mathrm{G}^\lambda \circ \rho^{\mathrm{R},u})|^2 + \lambda\,|u - \mathrm{G}^\lambda \circ \rho^{\mathrm{R},u}|^2 - \lambda\,|u|^2] \tag{2.11}$$

$$\begin{aligned}\Phi_\beta^{\mathrm{R},\lambda}(\xi^{\mathrm{R},u}) = {} & \sum_{i<j} \int_{\mathbf{R}^{2\mathrm{N}-2}} d\mathbf{q}\,|\hat{\xi}_{ij}^{\mathrm{R},u}(\mathbf{q})|^2 \\ & \times \left\{ \beta_{ij} + \pi \log\left[(|\mathbf{q}|^2 + \lambda)\left(1 - \frac{\lambda}{\mathrm{R}^2 + |\mathbf{q}|^2 + \lambda}\right)\right]\right\} \\ & - \sum_{ij,\,hk}' \int_{\mathbf{R}^{2\mathrm{N}}} d\mathbf{p}\,\chi_{\mathrm{R}}((p_i - p_j)/\sqrt{2})\,\chi_{\mathrm{R}}((p_h - p_k)/\sqrt{2}) \\ & \times \frac{\hat{\xi}_{ij}^{\mathrm{R},u}(p_1, \ldots, p_{i-1}, (p_i + p_j)/\sqrt{2}, \ldots, p_{\mathrm{N}})\,\hat{\xi}_{hk}^{\mathrm{R},u}(p_1, \ldots, p_{h-1}, (p_h + p_k)/\sqrt{2}, \ldots, p_{\mathrm{N}})}{|\mathbf{p}|^2 + \lambda}.\end{aligned} \tag{2.12}$$

It is now reasonable to agree (a formal proof will be given in Section 4) that in the limit $\mathrm{R} \to \infty$ one obtains the form

$$\mathrm{F}_\beta(u) = \mathscr{F}^\lambda(u) + \Phi_\beta^{\lambda,1}(\xi^u) + \Phi^{\lambda,2}(\xi^u) \tag{2.13}$$

$$\mathscr{F}^\lambda(u) = \int_{\mathbf{R}^{2\mathrm{N}}} dx\,[|\nabla(u - \mathrm{G}^\lambda \circ \xi^u)|^2 + \lambda\,|u - \mathrm{G}^\lambda \circ \xi^u|^2 - \lambda\,|u|^2] \tag{2.14}$$

$$\Phi_\beta^{\lambda,1}(\xi^u) = \sum_{i<j} \int_{\mathbf{R}^{2\mathrm{N}-2}} d\mathbf{q}\,|\hat{\xi}_{ij}^u(\mathbf{q})|^2\,[\beta_{ij} + \pi \log(|\mathbf{q}|^2 + \lambda)] \tag{2.15}$$

$$\begin{aligned}\Phi^{\lambda,2}(\xi^u) = {} & -\sum_{ij,\,hk}' \int_{\mathbf{R}^{2\mathrm{N}}} d\mathbf{p} \\ & \times \frac{\hat{\xi}_{ij}^u(p_1, \ldots, p_{i-1}, (p_i + p_j)/\sqrt{2}, \ldots, p_{\mathrm{N}})\,\hat{\xi}_{hk}^u(p_1, \ldots, p_{h-1}, (p_h + p_k)/\sqrt{2}, \ldots, p_{\mathrm{N}})}{|\mathbf{p}|^2 + \lambda}\end{aligned} \tag{2.16}$$

where $\xi^u \equiv \{\xi_{ij}^u,\ i<j\}$ is a collection of "charges" supported by σ_{ij}, obtained as suitable limits of the volume charges $\rho_{ij}^{\mathrm{R},u}$. Correspondingly the $\mathrm{G}^\lambda \circ \xi_{ij}^u$ are now the potentials in $\mathbf{R}^{2\mathrm{N}}$ due to the charges ξ_{ij}^u.

Notice that the bilinear form F_β does not depend on the parameter λ and that the "charge renormalization" (2.8) is independent of λ.

On the other hand, each term on the r.h.s. of (2.13) does depend on λ; we shall exploit the arbitrariness in the choice of λ to provide a lower bound on F_β and therefore on the spectrum of the corresponding Hamiltonian H_β.

3. QUADRATIC FORMS AND SELFADJOINT EXTENSIONS FOR $d=2$

In this Section we prove that the form F_β defined in (2.13) is closed and bounded below. In Section 5 we shall describe the domain of the corresponding operator, and give a representation for its resolvent. We shall also prove that the operator is characterized by boundary conditions which are local and translation invariant.

The method we use is an adaptation of the one employed in [6] to study perturbations of the Laplacian in R^3 which are supported by piecewise smooth curves. Notice that there, as well as here, the support of the perturbation has codimension two.

We begin with some preliminary results about the form Φ_β^λ.

Let $\mathscr{H} = \bigoplus_{i<j} L^2(\sigma_{ij}, dy_{ij})$ where dy_{ij} is Lebesgue measure on the hyperplanes σ_{ij}. A generic element of $\mathscr{H}$ will be denoted by $\xi \equiv \{\xi_{ij}, i<j\}$. We denote by $\|\xi\|_{\mathscr{H}}$ the norm of ξ in $\mathscr{H}$.

LEMMA 3.1. – *For any* $\lambda > \exp(\beta^- + \pi^2 N(N-1)/2)$, $\beta^- \equiv \min_{i,j}\{0, \beta_{ij}\}$, *the quadratic form* $\Phi_\beta^\lambda \equiv \Phi_\beta^{\lambda,1} + \Phi^{\lambda,2}$ *on* $\mathscr{H}$, *with domain* $D(\Phi_\beta^\lambda) \equiv \{\xi \in \mathscr{H}, \Phi_\beta^{\lambda,1}(\xi) < \infty\}$ *is closed and coercive. In particular,* $D(\Phi_\beta^\lambda)$ *is a Hilbert space with norm given by* $\Phi_\beta^{\lambda,1}$.

Proof. – Fix λ as indicated. Then $\Phi_\beta^{\lambda,1}$ is obviously closed and coercive on $D(\Phi_\beta^\lambda)$.

In order to study $\Phi^{\lambda,2}$ it is convenient to treat separately the following two cases in the sum over the indices.

(*a*) $\{i, j\} \cap \{h, k\} = \{\varnothing\}$

(*b*) one of the indices i, j takes the same value as one of the indices h, k.

Case (*a*). – In view of the symmetry of the kernel $(|\mathbf{p}|^2 + \lambda)^{-1}$ under interchange of indices, it is sufficient to consider the case $i=1, j=2, h=3$,

$k=4$. We must then estimate

$$\int_{\mathbf{R}^{2N}} d\mathbf{p} \frac{|\hat{\xi}_{12}((p_1+p_2)/\sqrt{2}, p_3, p_4, \ldots, p_N)|\,|\hat{\xi}_{34}(p_1, p_2, (p_3+p_4)/\sqrt{2}, \ldots, p_N)|}{|\mathbf{p}|^2+\lambda} \equiv \Psi_1(\xi). \quad (3.1)$$

We use the following inequality, valid for $f, g \in L^2(\mathbf{R}^2)$ and any $c>0$

$$\begin{aligned}\int_{\mathbf{R}^2} dx \int_{\mathbf{R}^2} dx' \frac{|f(x)|\,|g(x')|}{x^2+x'^2+c} &\leqq \frac{1}{4}\int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \int_0^{\infty} dt \int_0^{\infty} dt' \frac{|f_0(\sqrt{t}, \varphi)|\,|g_0(\sqrt{t'}, \varphi')|}{t+t'} \\ &\leqq \frac{\pi}{4}\int_0^{2\pi} d\varphi \left(\int_0^{\infty} dt\,|f_0(\sqrt{t}, \varphi)|^2\right)^{1/2} \int_0^{2\pi} d\varphi' \left(\int_0^{\infty} dt'\,|g_0(\sqrt{t'}, \varphi')|^2\right)^{1/2} \\ &\leqq \frac{\pi^2}{2}\|f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)} \quad (3.2)\end{aligned}$$

where $f_0(\sqrt{t}, \varphi)=f(x^1, x^2)$ of $x^1=\sqrt{t}\cos\varphi$, $x^2=\sqrt{t}\sin\varphi$.

In deriving (3.2) we have made use of the fact that $(t+t')^{-1}$ is the integral kernel of a bounded operator T on $L^2(\mathbf{R}^+)$ with norm π. This known result can be easily obtained by noticing that the isometry $U: L^2(\mathbf{R}^+) \to L^2(\mathbf{R})$ defined by

$$(Uf)(z) \equiv e^{-z/2} f(e^z) \quad (3.3)$$

is such that UTU^{-1} acts as convolution by $(e^{y/2}+e^{-y/2})$ and its norm is easily computed by Fourier transform.

From estimate (3.2) and a repeated use of Schwartz's inequality one concludes

$$\Psi_1(\xi) \leqq \frac{\pi^2}{2}\|\hat{\xi}_{12}\|_{L^2(\mathbf{R}^{2N-2})}\|\hat{\xi}_{34}\|_{L^2(\mathbf{R}^{2N-2})}. \quad (3.4)$$

Case (*b*). – Without loss of generality, we take $i=h=1$, $j=2$, $k=3$. We introduce the new variables

$$q_1=\frac{\sqrt{2}}{3}p_1+\frac{\sqrt{2}}{3}p_2-\frac{4}{3\sqrt{2}}p_3,$$

$$q_2=\frac{\sqrt{2}}{3}p_1+\frac{\sqrt{2}}{3}p_3-\frac{4}{3\sqrt{2}}p_2, \qquad q_3=\frac{p_1+p_2+p_3}{\sqrt{3}}$$

and define

$$\eta_{1,m}(q_{m-1}, q_3, \ldots, q_{N-1}) \equiv \hat{\xi}_{1,m}\left(\frac{q_{m-1}}{2}+\sqrt{\frac{2}{3}}q_3, \frac{-q_{m-1}}{\sqrt{2}}+\frac{q_3}{\sqrt{3}}, \ldots, q_{N-1}\right), \qquad m=2, 3.$$

Then

$$\Psi_2(\xi) \equiv \int_{\mathbb{R}^{2N}} d\mathbf{p}$$

$$\times \frac{|\hat{\xi}_{12}((p_1+p_2)/\sqrt{2}, p_3, \ldots, p_N)| \, |\hat{\xi}_{13}((p_1+p_3)/\sqrt{2}, p_2, \ldots, p_N)|}{|\mathbf{p}|^2+\lambda}$$

$$= \frac{\sqrt{3}}{2} \int dp_4 \ldots \int dp_N \int dq_1 \, dq_2 \, dq_3$$

$$\times \frac{|\eta_{12}(q_1, q_3, q_4, \ldots, p_N)| \, |\eta_{13}(q_2, q_3, p_4, \ldots, p_N)|}{q_1^2+q_2^2+q_1 \cdot q_2+q_3^2+\sum_{m \geq 4} p_m^2+\lambda}. \quad (3.5)$$

We now use the inequality

$$q_1^2+q_2^2+q_1 \cdot q_2 \geqq \frac{1}{2}(q_1^2+q_2^2)$$

and proceed as in case (*a*) to conclude

$$\Psi_2(\xi) \leqq \pi^2 \|\hat{\xi}_{12}\|_{L^2(\mathbb{R}^{2N-2})} \|\hat{\xi}_{23}\|_{L^2(\mathbb{R}^{2N-2})}. \quad (3.6)$$

From (3.4), (3.6) we conclude that $\Phi^{\lambda, 2}$ is a bounded quadratic form on $\mathscr{H}$. It is also easy to verify that if $\xi \in D(\Phi_\beta^\lambda)$ then

$$\Phi_\beta^\lambda(\xi) \geqq \left[\log \lambda - \beta^- - \pi^2 \frac{N(N-1)}{2}\right] \|\xi\|_{\mathscr{H}}^2.$$

This concludes the proof of Lemma 3.1. □

We shall also need the following estimates for the potential $G^\lambda \circ \xi$ produced by the charges ξ.

LEMMA 3.2. — *For every* $\lambda > 0$ *one has*

(*a*) $$\|G^\lambda \circ \xi\| < c(\lambda, N) \|\xi\|_{\mathscr{H}}, \qquad \lim_{\lambda \to \infty} c(\lambda, N) = 0 \quad (3.7)$$

(*b*) $$G^\lambda \circ \xi \notin H^1(\mathbb{R}^{2N}) \quad \textit{if} \quad \xi \in D(\Phi_\beta^\lambda), \qquad \xi \neq 0. \quad (3.8)$$

Here $\|.\|$ indicates the norm in $L^2(\mathbb{R}^{2N})$.

Proof. — (*a*) Using the explicit expression for the Fourier transform of $G^\lambda \circ \xi$ one can verify that (3.7) is implies by the inequality

$$\int_{\mathbb{R}^{2N}} d\mathbf{p} \frac{|\hat{\xi}_{12}((p_1+p_2)/\sqrt{2}, p_3, \ldots, p_N)|^2}{(|\mathbf{p}|^2+\lambda)^2} \leqq \frac{\pi}{\lambda} \|\xi_{12}\|_{L^2(\mathbb{R}^{2N-2})}^2 \quad (3.9)$$

and this follows easily performing explicitly the integration with respect to the variable $p_1 - p_2/\sqrt{2}$.

(*b*) To verify (3.8), choose $\xi \in D(\Phi_\beta^\lambda)$, $\xi \neq 0$, $R > 0$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} d\mathbf{p} \prod_{l=1}^{N} \chi_R(p_l) |\mathbf{p}|^2 \Big| \sum_{i<j} (G^\lambda \circ \xi_{ij})(\mathbf{p}) \Big|^2 \\
&= \sum_{i<j} \int_{\mathbb{R}^{2N}} d\mathbf{p} \prod_{l=1}^{N} \chi_R(p_l) \\
&\quad \times \frac{|\hat{\xi}_{ij}(p_1, \ldots, p_{i-1}, (p_i+p_j)/\sqrt{2}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N)|^2}{|\mathbf{p}|^2+\lambda} \\
&\quad -\lambda \sum_{i<j} \int_{\mathbb{R}^{2N}} d\mathbf{p} \prod_{l=1}^{N} \chi_R(p_l) \\
&\quad \times \frac{|\hat{\xi}_{ij}(p_1, \ldots, p_{i-1}, (p_i+p_j)/\sqrt{2}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N)|^2}{(|\mathbf{p}|^2+\lambda)^2} \\
&\quad +\lambda \sum_{ij,\, hk}' \int_{\mathbb{R}^{2N}} d\mathbf{p} \prod_{l=1}^{N} \chi_R(p_l) \frac{|\mathbf{p}|^2}{|\mathbf{p}|^2+\lambda} \\
&\quad \times \overline{\hat{\xi}}_{ij}(p_1, \ldots, p_{i-1}, (p_i+p_j)/\sqrt{2}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N) \\
&\quad \times \hat{\xi}_{hk}(p_1, \ldots, p_{h-1}, (p_h+p_k)/\sqrt{2}, \ldots, p_{k-1}, p_{k+1}, \ldots, p_N). \qquad (3.10)
\end{aligned}$$

By (3.9), Lemma 3.1 and dominated convergence, the second and third term in (3.10) are bounded above by $c\|\xi\|_{\mathscr{H}}^2$ for some $c>0$. We show now that the first term in (3.10), which is strictly positive (unless $\xi \equiv 0$), diverges to $+\infty$ when $R \to \infty$. By dominated convergence, this will imply *b*).

Since every term in the sum is non-negative, and by assumption $\xi \not\equiv 0$, it is enough to evaluate the term $i=1, j=2$ under the assumption $\xi_{12} \not\equiv 0$. One has

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} d\mathbf{p} \prod_{l=1}^{N} \chi_R(p_l) \frac{|\hat{\xi}_{12}((p_1+p_2)/\sqrt{2}, p_3, \ldots, p_N)|^2}{|\mathbf{p}|^2+\lambda} \\
&\geqq \pi \log R^2 \int_{\mathbb{R}^{2N-4}} dp_3 \ldots dp_N \prod_{l=3}^{N} \chi_R(p_l) \int_{\mathbb{R}^2} ds\, \chi_R(s) |\hat{\xi}_{12}(s, p_3, \ldots, p_N)|^2 \\
&\quad -\pi \int_{\mathbb{R}^{2N-4}} dp_3 \ldots dp_N \prod_{l=3}^{N} \chi_R(p_l) \\
&\quad \times \int_{\mathbb{R}^2} ds\, \chi_R(s) |\hat{\xi}_{12}(s, p_3, \ldots, p_N)|^2 \log(s^2+p_3^2+\ldots+\lambda). \qquad (3.11)
\end{aligned}$$

The last term in (3.11) is in absolute value smaller than $c\, \Phi_\beta^{\lambda,1}(\xi)$. On the contrary the first term in the r.h.s. of (3.11) is larger than $c(\xi) \log R^2$, where $c(\xi) > 0$ if $\xi \not\equiv 0$. Therefore the left hand side in (3.10) diverges as $c \log R$ when $R \to \infty$, and this proves (*b*). □

We study now the properties of the form F_β defined by (2.13)-(2.16) on the domain $D(F_\beta)$

$$D(F_\beta) \equiv \{ u \in L^2(R^3),\ \exists\, \xi^u \in D(\Phi_\beta^\lambda)\ \text{s.t.}\ u - G^\lambda \circ \xi^u \in H^1(R^{2N}) \}. \quad (3.12)$$

Notice that the charge ξ^u associated to $u \in D(F_\beta)$ is uniquely determined in view of Lemma 3.2, part (*b*) and independent of λ since $G^\lambda \xi - G^{\lambda'} \xi \in H^1(R^{2N})$ if $\xi \in D(\Phi_\beta^\lambda)$. Moreover F_β and $D(F_\beta)$ do not depend on the value chosen for the parameter λ. This can be checked directly, but it is also a consequence of the fact, proved in Section 4, that F_β is the Γ-limit of approximating forms which are explicitly independent of λ.

The charges ξ^u can be computed as pointwise and $L^2(R^{2N-2})$ limits. One has, for almost every choice of $q_1, \ldots, q_{N-1}$ in any bounded subset of $R^{2(N-1)}$

$$\xi_{12}^u(q_1, \ldots, q_{N-1}) = \lim_{R \to \infty} \frac{1}{2\pi \log R} \times \int_{R^2} ds\, \chi_R(s)\, \hat{u}((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, q_2, \ldots, q_{N-1}) \quad (3.13)$$

and similar expressions for ξ_{ij}^u, $i<j$, $i \neq 1$, $j \neq 2$.

To prove (3.13), remark that if $u \in D(F_\beta)$ one has

$$\begin{aligned}
&\frac{1}{2\pi \log R} \int_{R^2} ds\, \chi_R(s)\, \hat{u}((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, \ldots, q_{N-1}) \\
&= \frac{1}{2\pi \log R} \int_{R^2} ds\, \chi_R(s)\, \hat{w}((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, \ldots, q_{N-1}) \\
&\quad + \frac{1}{2\pi \log R} \sum_{\substack{i<j \\ (i,j) \neq (1,2)}} \int_{R^2} ds\, \chi_R(s) \\
&\quad \times \frac{\xi_{ij}^u((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, \ldots, (q_i+q_j)/\sqrt{2}, \ldots, q_{N-1})}{s^2 + |\mathbf{q}|^2 + \lambda} \\
&\quad + \frac{1}{2\pi \log R} \pi \log \frac{R^2 + |\mathbf{q}|^2 + \lambda}{|\mathbf{q}|^2 + \lambda} \xi_{12}^u(q_1, \ldots, q_{N-1}) \quad (3.14)
\end{aligned}$$

where $w \equiv u - G^\lambda \circ \xi^u \in H^1(R^{2N})$. Using Schwartz's inequality the first term on the r.h.s. of (3.14) is bounded pointwise in $R^{2(N-1)}$ by

$$\frac{1}{2\sqrt{\pi} \log R} \left(\log \frac{R^2 + |\mathbf{q}|^2 + \lambda}{|\mathbf{q}|^2 + \lambda} \right)^{1/2} \times \left(\int_{R^2} ds\, |\hat{w}((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, \ldots, q_{N-1})|^2 (s^2 + |\mathbf{q}|^2 + \lambda) \right)^{1/2} \quad (3.15)$$

which converges to zero a.e. in $\mathbb{R}^{2(N-1)}$ and also in $L^2(\mathbb{R}^{2(N-1)})$ since $w \in H^1(\mathbb{R}^{2N})$.

Reasoning as in the proof of Lemma 3.1, also the second term is shown to converge to zero a.e. and in $L^2(\mathbb{R}^{2(N-1)})$ when $R \to \infty$.

The third term converges to ξ_{12}^u pointwise and in $L^2(\mathbb{R}^{2(N-1)})$ by construction.

We prove now that the quadratic form F_β defines a self-adjoint operator which represents a perturbation of $-\Delta$ supported by the set $\Sigma \equiv \bigcup_{i<j} \sigma_{ij}$.

THEOREM 3.3. – *The quadratic form* F_β *defined by* (2.13)-(2.16) *on the domain* (3.12) *is bounded below and closed. Moreover, if* $u \in H^1(\mathbb{R}^{2N})$, *then*

$$F_\beta(u) = \int_{\mathbb{R}^{2N}} dx \, |\nabla u|^2.$$

Proof. – Choose $\lambda > \exp(\beta^- + \pi^2 N(N-1)/2)$. By Lemma 3.1 one concludes that a lower bound for $F_\beta(u)$ is $-\exp(\beta^- + \pi^2 N(N-1)/2) \, \|u\|^2$ since the first two terms in (2.14) are non-negative and $\Phi_\beta^\lambda(\xi^u) \geqq 0$, $\forall u$, under the assumptions made on λ.

To prove that F_β is closed, we proceed as in [6]; it is of course sufficient to prove that $F_\beta^\lambda(u) \equiv F_\beta(u) + \lambda \int dx \, |u|^2$ is closed.

Let $\{u_n\}$, $u_n \in D(F_\beta)$ be such that

$$\lim_{n \to \infty} \|u_n - u\| = 0, \qquad \lim_{n, m \to \infty} F_\beta^\lambda(u_n - u_m) = 0. \tag{3.16}$$

Then $\{u_n - G^\lambda \circ \xi^{u_n}\}$ is a Cauchy sequence in $H^1(\mathbb{R}^{2N})$; denote by w its limit.

Since $\lim_{n \to \infty} \Phi_\beta^\lambda(\xi^{u_n} - \xi^{u_m}) = 0$, $\{\xi^{u_n}\}$ is a Cauchy sequence in $\mathscr{H}$; denote by ξ its limit.

By Lemma 3.2, $\lim_{n \to \infty} \|G^\lambda \circ \xi^{u_n} - G^\lambda \circ \xi\| = 0$.

We conclude that $u_n - (w + G^\lambda \circ \xi)$ converges to zero in $L^2(\mathbb{R}^{2N})$, and therefore

$$u = w + G^\lambda \circ \xi \in D(F_\beta)$$

and moreover

$$\lim_{n \to \infty} F_\beta^\lambda(u - u_n) = 0.$$

In order to conclude the proof of the theorem notice that from uniqueness of ξ^u for $u \in D(F_\beta)$ it follows that $\xi^u \equiv 0$ if $u \in H^1(\mathbb{R}^{2N})$. □

4. THE FORM F_β AS A Γ-LIMIT OF APPROXIMATING FORMS

In this Section we give some details of the proof of Γ-convergence to F_β of the approximating forms F_β^R introduced in Section 2.

For a given quadratic form T in a Hilbert space $\mathscr{H}$ with domain D(T) we introduce the extension $\bar{T}$ to the entire space by

$$\bar{T}(u) \equiv \begin{cases} T(u) & u \in D(T) \\ +\infty & u \notin D(T) \end{cases}.$$

It is easy to verify that T is closed if and only if $\bar{T}$ is lower semicontinuous (l.s.c.).

We recall next the definition of Γ-convergence for quadratic forms. We refer to ([7], [8]) for an extensive treatment.

We denote by $\|\cdot\|_{\mathscr{H}}$ the Hilbert norm of $\mathscr{H}$.

DEFINITION 4.1. — $\bar{T}$ is the Γ-limit of $\bar{T}_n$ if and only if

(i) $\forall u \in D(T), \quad \exists \{u_n\},\ u_n \in D(T_n), \quad \lim_{n\to\infty} \|u_n - u\|_{\mathscr{H}} = 0$

such that

$$T(u) = \lim_{n\to\infty} T_n(u_n) \tag{4.1}$$

(ii) $\forall u \in \mathscr{H}, \quad \forall \{u_n\},\ u_n \in \mathscr{H}, \quad \text{if} \quad \lim_{n\to\infty} \|u_n - u\|_{\mathscr{H}} = 0$

then

$$\bar{T}(u) \leqq \liminf_{n\to\infty} \bar{T}_n(u_n). \tag{4.2}$$

We shall recall some properties of Γ-limits which will be useful in the sequel. Proofs can be found in ([7], [8]).

LEMMA 4.2. — (i) *The Γ-limit is unique when it exists.*

(ii) *If* T *is* l.c.s. *and bounded below, then in definition* 4.1 *one can substitute* D(T) *with* D(A), *where* A *is the s.a. operator associated to* T.

(iii) *If* $\Gamma - \lim \bar{T}_n = \bar{T}$, *and* S *is a bounded quadratic form, then* $\Gamma - \lim(\bar{T}_n + S) = \bar{T} + S$.

(iv) *Let* $\{T_n\}$, T *be quadratic forms uniformly bounded below and l.s.c. If* $\{A_n\}$, A *are the corresponding operators, the following statements are equivalent*

(*a*) $\Gamma - \lim \bar{T}_n = T$, $\bar{T}(u) \leq \liminf \bar{T}_n(u_n)\ \forall \{u_n\}$ *which converge to u*

(*b*) $\lim_{n\to\infty} A_n = A$ *in the strong resolvent sense*

We verify now Γ-convergence of F_β^R to F_β when $R \to \infty$. For a sequence $R_m \to \infty$ we write F_β^m for $F_\beta^{R_m}$. We denote with H_β^m and H_β the selfadjoint operators corresponding respectively to the forms F_β^m and F_β.

Choose $\lambda > \exp(\beta^- + \pi^2 N(N-1)/2$ and let

$$F_\beta^{m,\lambda}(u) \equiv F_\beta^m(u) + \lambda \| u \|^2. \tag{4.3}$$

THEOREM 4.3. – (i) *For every* $u \in D(H_\beta)$ *one can find a sequence* $\{u_m\}$, $u_m \in H^1(R^{2N})$ *such that*

$$\lim_{m \to \infty} \|u_m - u\| = 0 \tag{4.4}$$

$$\lim_{m \to \infty} F_\beta^{m,\lambda}(u_m) = F_\beta^\lambda(u). \tag{4.5}$$

(ii) *For every* $u \in L^2(R^{2N})$ *and for any sequence* $\{u_m\}$ *in* $L^2(R^{2N})$ *weakly convergent to u, one has*

$$\bar{F}_\beta^\lambda(u) \leq \liminf_{m \to \infty} \bar{F}_\beta^{m,\lambda}(u^m). \tag{4.6}$$

Remark 4.4. – As a consequence of Theorem 4.3 and of Lemma 4.2, we conclude that the selfadjoint operators H_β^R associated to F_β^R converge in the strong resolvent sense to the s.a. operator H_β.

Proof. – We prove first (i).

For fixed $u \in D(H_\beta)$ define

$$u_m \equiv w + G^\lambda \circ g_m^u, \qquad w \equiv u - G^\lambda \circ \xi^u \tag{4.7}$$

$$g_m^u(\mathbf{p}) = \sum_{i<j} \chi_{R_m}\left(\frac{p_i - p_j}{\sqrt{2}}\right) \times \xi_{ij}^{R,u}\left(p_1, \ldots, p_{i-1}, \frac{p_i + p_j}{\sqrt{2}}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N\right). \tag{4.8}$$

By construction $u_m \in H^1(R^{2N})$, $\forall m$, and one can easily verify (4.4). To verify (4.5) it will be convenient to rewrite $F_\beta^{m,\lambda})u_m)$ as

$$F_\beta^{m,\lambda}(u_m) = \mathscr{F}^{m,\lambda}(u) + \Phi_\beta^{m,\lambda}(\xi^m) \tag{4.9}$$

[*see* Eq. (2.10) in Section 2]. As in (2.15), (2.16) we will denote by $\Phi_\beta^{m,\lambda,1}$ and $\Phi^{m,\lambda,2}$ respectively the diagonal and off-diagonal part of $\Phi_\beta^{m,\lambda}$.

By explicit computation one obtains, e.g. for $i=1, j=2$

$$\begin{aligned}\xi_{12}^m(\mathbf{q}) = \frac{1}{\beta_{12} + \pi \log(R_m^2 + |\mathbf{q}|^2)} & \times \left[\int_{R^2} ds\, \chi_{R_m}(s)\, \hat{w}((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, q_2, \ldots, q_{N-1}) \right. \\ & + \sum_{\substack{h<k \\ (h,k)\neq(1,2)}} \int_{R^2} ds\, \chi_{R_m}(s)\, \chi_{R_m}\left(\frac{q_{h-1} - q_{k-1}}{\sqrt{2}}\right) \\ & \times \frac{\xi_{hk}^u((q_1+s)/\sqrt{2}, (q_1-s)/\sqrt{2}, \ldots, (q_{h-1}+q_{k-1})/\sqrt{2}, \ldots, q_{k-2}, q_k, \ldots, q_{N-1})}{s^2 + |\mathbf{q}|^2 + \lambda} \\ & \left. + \pi \log \frac{R_m^2 + |\mathbf{q}|^2 + \lambda}{|\mathbf{q}|^2 + \lambda}\, \xi_{12}^u(\mathbf{q}) \right]. \end{aligned} \tag{4.10}$$

For the sake of simplicity in (4.10) we did not write explicitly the two terms $h=1$ and $h=2$. It is easily checked that the estimate which follows applies to those terms as well. From (4.10), using Schwartz's inequality, one obtains, for any α, $0 \leqq \alpha \leqq 2$

$$\int_{\mathbf{R}^{2N-2}} d\mathbf{q} \log^\alpha (|\mathbf{q}|^2+\lambda) |\xi^m(\mathbf{q})-\xi^u(\mathbf{q})|^2 \leq \frac{c}{\log^{2-\alpha} \mathbf{R}_m} \left[\int_{\mathbf{R}^{2N}} d\mathbf{p} (|\mathbf{p}|^4+1) |\hat{w}|^2(\mathbf{p}) + \sum_{i<j} \int_{\mathbf{R}^{2N-2}} d\mathbf{q} \log^2 (|\mathbf{q}|^2+\lambda) |\xi^u_{ij}(\mathbf{q})|^2 \right] \quad (4.11)$$

and an identical estimate for any $(i, j) \neq (1, 2)$.

We shall prove in Section 5 that if $u \in \mathrm{D}(\mathrm{H}_\beta)$ then u can be written in an unique way as $u=w+\mathrm{G}^\lambda \circ \xi^u$ with $w \in \mathrm{H}^2(\mathbf{R}^{2N})$ and $\log(|\mathbf{q}|^2+\lambda)\,\xi^u_{hk}(\mathbf{q}) \in \mathrm{L}^2(\mathbf{R}^{2N-2})\,\forall h, k, h<k$. Estimate (4.11) indicates then that for any $u \in \mathrm{D}(\mathrm{H}_\beta)$ the approximation procedure introduced above is such that the ξ^m converge to the ξ^u in the $\mathrm{L}^2(\mathbf{R}^{2N-2})$ norm ($\alpha=0$) and in the norm induced by the form $\Phi^{\lambda, 1}_\beta$ ($\alpha=1$).

Notice that for R_m sufficiently large

$$\frac{1}{2}\log(|\mathbf{q}|^2+\lambda) \leqq \log\left[(|\mathbf{q}|^2+\lambda)\left(1-\frac{\lambda}{\mathrm{R}^2+|\mathbf{q}|^2+\lambda}\right)\right] \leqq \log(|\mathbf{q}|^2+\lambda) \quad (4.12)$$

so that the norms induced by $\Phi^{m, \lambda}_\beta$ and by Φ^λ_β are equivalent. From (4.11) and Lemma 3.1

$$|\Phi^{m, \lambda}_\beta(\xi^m)-\Phi^\lambda_\beta(\xi^u)| \leqq c\,|\Phi^{\lambda, 1}_\beta(\xi^m)-\Phi^{\lambda, 1}_\beta(\xi^u)| \leqq c(\log \mathrm{R}_m)^{-1} \quad (4.13)$$

and moreover

$$\|\mathrm{G}^\lambda \circ (g^m-\rho^m)\|_{\mathrm{H}^1(\mathbf{R}^{2N})} \leqq \int_{\mathbf{R}^{2N}} d\mathbf{p} \frac{|\hat{g}^m(\mathbf{p})-\hat{\rho}^m(\mathbf{p})|^2}{|\mathbf{p}|^2+\lambda} \leqq \pi \mathrm{N} \sum_{i<j} \int_{\mathbf{R}^{2N-2}} d\mathbf{p} \log\left(\frac{\mathrm{R}_m^2+|\mathbf{q}|^2+\lambda}{|\mathbf{q}|^2+\lambda}\right) |\xi^u_{ij}(\mathbf{q})-\xi^m_{ij}(\mathbf{q})|^2 \quad (4.14)$$

so that, using estimate (4.11), with $\alpha=0$

$$\lim_{m \to \infty} \|\mathrm{G}^\lambda \circ (g^m-\rho^m)\|_{\mathrm{H}^1(\mathbf{R}^{2N})}=0.$$

Since by construction $u^m-\mathrm{G}^\lambda \circ \rho^m=w+\mathrm{G}^\lambda \circ (g^m-\rho^m)$, the proof of (i) is complete.

We prove now (ii).

Suppose that $\{u_m\}$, $u_m \in \mathrm{L}^2(\mathbf{R}^{2N})$, is such that u_m converges to u weakly in $\mathrm{L}^2(\mathbf{R}^{2N})$.

If $\liminf_{m \to \infty} F^{m, \lambda}(u_m) = +\infty$, assertion (ii) is trivally satisfied. We can therefore assume that there exists a positive constant c and a subsequence (still denoted by $\{u_m\}$) for which

$$F^{m, \lambda}(u_m) < c. \tag{4.15}$$

We conclude that for m sufficiently large

$$\| u_m - G^\lambda \circ \rho^m \|_{H^1(R^{2N})} < c \tag{4.16}$$

$$\Phi_\beta^{m, \lambda, 1}(\xi^m) < c. \tag{4.17}$$

Denote by w and ξ respectively the weak limit of $\{u_m - G^\lambda \circ \rho^m\}$ in $H^1(R^{2N})$ and the weak limit of $\{\xi^m\}$ in $D(\Phi_\beta^{\lambda, 1})$. One easily verifies that $G^\lambda \circ \rho^m$ tends to $G^\lambda \circ \xi$ weakly in $L^2(R^{2N})$. We conclude then that $w + G^\lambda \circ \xi$ is the weak$-L^2(R^{2N})$ limit of u^m and coincide with u by our assumptions.

By lower semicontinuity of the Hilbert norms with respect to the weak convergence, one has then

$$\| u - G^\lambda \circ \xi \|_{H^1(R^{2N})} \leq \liminf_{m \to \infty} \| u_m - G^\lambda \circ \rho^m \|_{H^1(R^{2N})} \tag{4.18}$$

$$\Phi_\beta^{\lambda, 1}(\xi) \leq \liminf_{m \to \infty} \Phi_\beta^{m, \lambda, 1}(\xi^m). \tag{4.19}$$

Taking into account the continuity of $\Phi^{\lambda, 2}$, inequalities (4.18), (4.19) conclude the proof of part (ii) of theorem 4.3. $\square$

5. THE HAMILTONIAN FOR $d=2$

In this Section we study the selfadjoint operator H_β defined by the quadratic form F_β introduced in Section 2 and analyzed in Section 3.

Consider the selfadjoint positive operator Γ_β^λ on $\mathscr{H}$ associated to the form Φ_β^λ, with $\lambda > \exp(\beta^- + \pi^2 N(N-1)/2)$. It is easy to verify that

$$D(\Gamma_\beta^\lambda) = \{ \xi \in \mathscr{H} \text{ s.t. } \forall\, i, j \quad \hat{\xi}_{ij}(\mathbf{q}) \log(|\mathbf{q}|^2 + \lambda) \in L^2(R^{2N-2}) \} \tag{5.1}$$

and that, if $\xi \in D(\Gamma_\beta^\lambda)$

$$(\Gamma_\beta^\lambda \xi)_{ij}(\mathbf{q}) = [\beta_{ij} + \log(|\mathbf{q}|^2 + \lambda)]\, \hat{\xi}_{ij}(\mathbf{q}) - \sum_{\substack{h<k \\ (h,k) \neq (i,j)}} \int_{R^2} ds$$

$$\times \frac{\xi_{hk}(\ldots, q_{i-1}, (q_i + s)/\sqrt{2}, \ldots, q_{j-1}, (q_i - s)/\sqrt{2}, \ldots, (q_{h-1} - q_{k-1})/\sqrt{2}, \ldots, q_{N-1})}{s^2 + |\mathbf{q}|^2 + \lambda} \tag{5.2}$$

(the dependence on the q_i's in (5.2) is in fact suitable only for the case

$h>j$; the terms relative to different type of pairs (i, j), (h, k) can be easily written down and were omitted in order to simplify the notation).

We claim that the Hamiltonian H_β, $\beta \equiv \{\beta_{ij}\}$ describing a system of N particles of mass 1/2 in R^2 interacting through two-body zero-range forces, is characterized by

$$D(H_\beta) = \{ u \in D(F_\beta) \text{ s.t. } \xi^u \in D(\Gamma_\beta^\lambda),\ u - G^\lambda \circ \xi^u \in H^2(R^{2N}) \\ (u - G^\lambda \circ \xi^u)|_{\sigma_{ij}} = (\Gamma_\beta^\lambda \xi^u)_{ij} \} \tag{5.3}$$

for some $\lambda > 0$, and therefore for all $\lambda > 0$. Notice that $u - G^\lambda \circ \xi^u \in H^2(R^{2N})$ has an L^2-trace on each σ_{ij}, and the last equality in (5.3) is in the L^2-sense. Moreover if $u \in D(H_\beta)$, for all $\lambda > 0$ one has

$$(H_\beta + \lambda) u = (-\Delta + \lambda)(u - G^\lambda \circ \xi^u) \tag{5.4}$$

Remark 5.1. – It follows from (5.3), (5.4) that if u_0 is an eigenvector of H_β relative to a negative eigenvalue $-\lambda_0$, then $u_0 = G^{\lambda_0} \circ \xi^{u_0}$ (notice that $-\Delta + \lambda$ is invertible for every positive λ) and moreover ξ^{u_0} satisfies $\Gamma_\beta^{\lambda_0} \xi^{u_0} = 0$. Conversely if there exist λ_0 and ξ such that $\Gamma_\beta^{\lambda_0} \xi = 0$ then by (5.4) $u_0 \equiv G^{\lambda_0} \circ \xi$ is an eigenvector relative to the eigenvalue $-\lambda_0$ and, being $G^{\lambda_0} \circ \xi \notin H^1(R^{2N})$, $\xi = \xi^{u_0}$ by uniqueness. Therefore the negative eigenvalues of H_β are those λ's for which the equation $\Gamma_\beta^{\lambda_0} \xi = 0$ has a solution.

The proof of (5.3), (5.4) can be obtained following the procedure described in [6]. For the convenience of the reader we recall it briefly here.

If $u \in D(H_\beta)$, then by definition $\exists g \in L^2(R^{2N})$ s.t. $\forall v \in D(F_\beta)$

$$F_\beta(u, v) = (g, v). \tag{5.5}$$

Choose $v \in H^1(R^{2N})$. In this case $\xi^v \equiv 0$ and (5.5) becomes

$$\int_{R^{2N}} dx\, (\nabla v, \nabla(u - G^\lambda \circ \xi^u)) + \lambda \int_{R^{2N}} dx\, (v, (u - G^\lambda \circ \xi^u)) \\ = \int_{R^{2N}} dx\, v(g + \lambda u) \tag{5.6}$$

which implies

$$u - G^\lambda \circ \xi^u \in H^2(R^{2N}). \tag{5.7}$$

Using this information and integrating by parts in the first integral in (5.6) one obtains (5.4).

Choose now v so that $v = G^\lambda \circ \xi^v$, $\xi \in D(\Phi_\beta^\lambda)$. Then (5.5) gives

$$\Phi_\beta^\lambda(\xi^u, \xi^v) = (\lambda u + g, G^\lambda \circ \xi^v) = ((-\Delta + \lambda)(u - G^\lambda \circ \xi^u), G^\lambda \circ \xi^v) \\ = \sum_{i<j} \int_{\sigma_{ij}} dy_{ij} (u - G^\lambda \circ \xi^u)(y_{ij})\, \xi_{ij}^v(y_{ij}) \leq c \| u - G^\lambda \circ \xi^u \|_{H^2(R^{2N})} \| \xi^v \|_{\mathscr{H}} \tag{5.8}$$

where a standard estimate on traces of functions in Sobolev spaces was used. From (5.7), (5.8) we conclude that $D(H_\beta)$ is contained in the set of functions appearing on the right hand side of (5.3).

Conversely, following the same steps, it is not difficult to verify the other inclusion.

Let us remark here that the condition

$$(u - G^\lambda \circ \xi^u)\big|_{\sigma_{ij}} = (\Gamma^\lambda_\beta \xi^u)_{ij} \qquad \forall i<j \tag{5.9}$$

is equivalent to the following boundary condition, satisfied by $u \in D(H_\beta)$ almost everywhere on σ_{ij}

$$\lim_{|x_i - x_j| \to 0} \Bigg[u(x_1, \ldots, x_N) - \frac{1}{2\pi} \xi^u_{ij}(x_1, \ldots, (x_i + x_j)/\sqrt{2}, \ldots, x_N) \log \frac{1}{|x_i - x_j|} \Bigg] = \beta_{ij} \xi^u_{ij}(x_1, \ldots, (x_i + x_j)/\sqrt{2}, \ldots, x_N). \tag{5.10}$$

The equivalence is easily checked if one takes into account that

$$\Bigg[\lim_{|x_i - x_j| \to 0} \Bigg(G^\lambda \circ \xi^u_{ij}(x_1, \ldots, x_N) - \frac{1}{2\pi} \xi^u_{ij}\left(x_1, \ldots, \frac{x_i + x_j}{\sqrt{2}}, \ldots, x_N\right) \log \frac{1}{|x_i - x_j|} \Bigg) \Bigg]^{\wedge}(\mathbf{q})$$
$$= \lim_{|x_i - x_j| \to 0} \hat{\xi}^u_{ij}(\mathbf{q}) \left[K_0\left(\sqrt{|\mathbf{q}|^2 + \lambda} \frac{|x_i - x_j|}{\sqrt{2}} \right) - \frac{1}{2\pi} \log \frac{1}{|x_i - x_j|} \right] = -\hat{\xi}^u_{ij}(\mathbf{q}) \log(|\mathbf{q}|^2 + \lambda) \tag{5.11}$$

where $K_0(.)$ is the zero-th order Hankel function with imaginary argument [15].

Equation (5.10) expresses a linear relation between the first two terms in an asymptotic expansion of the function u near $P \in \sigma_{ij}$; moreover the relation is independent of P. Therefore H_β is a local and translation invariant extension of $-\Delta^0$.

Notice that the relation (5.10) is a direct generalization the boundary condition defining the two-body zero-range interaction in two dimensions (*see* e.g. [5]).

We end this Section giving an explicit description of the resolvent of H_β. For every $\lambda > \exp(\beta^- + \pi^2 N(N-1)/2)$ and $f \in L^2(\mathbb{R}^{2N})$, one has

$$(H_\beta + \lambda)^{-1} f = G^\lambda \circ f + G^\lambda \circ \eta \tag{5.12}$$

where η is the unique solution of the integral equation

$$(\Gamma^\lambda_\beta \eta)_{ij} = G^\lambda \circ f\big|_{\sigma_{ij}} \tag{5.13}$$

Equation (5.13) generalizes the equation of Ter-Martirosian and Skorniakov [9] which was further analyzed in [1], [2] for the case of three particles in R^3. In fact, (5.12), (5.13) continues to hold, by analytic continuation of both sides, in the complex λ-plane cut along the negative real axis, with the exception of those positive values of λ for which $\ker \Gamma_\beta^\lambda \neq \{0\}$. We recover here the fact already noted in Remark 4.1 that $-\lambda_0$ is in the negative part of the spectrum of H_β precisely if $\ker \Gamma_\beta^\lambda \neq \{0\}$.

We shall not discuss further here the properties of the spectrum of H_β, and we add only the remark that the eigenfunctions have an exponential decay in the directions away from Σ, with a rate which increases with λ_0. The decay along Σ is linked to the decay of the solutions of $\Gamma_\beta^{\lambda_0} \xi = 0$ and will in general not be exponential.

6. QUADRATIC FORMS FOR $d=3$

In this Section we begin the discussion of the case of dimension three. The informal part of the discussion follows the pattern of Section 2.

The first sum in (2.7) has now the form

$$\sum_{i<j} \int_{R^{3(N-1)}} d\mathbf{q} \, |\xi_{ij}^{R,u}(\mathbf{q})|^2 \times \left(\frac{(2\pi)^{3N/2} (4\pi)^{3/2}}{\mu_{ij}(\beta, R)} - 4\pi R + 4\pi \sqrt{\mathbf{q}^2+\lambda} \arctan \frac{R}{\sqrt{\mathbf{q}^2+\lambda}} \right) \tag{6.1}$$

while the second sum in (2.7) is unchanged, apart from the fact that the integration is now over R^{3N}.

Correspondingly one chooses

$$\mu_{ij}(\beta, R) = (2\pi)^{3N/2} (4\pi)^{3/2} (\beta_{ij} + 4\pi R)^{-1} \tag{6.2}$$

The approximating forms can than be written as in (2.10)-(2.12), with the obvious changes, and the expected limit form is

$$F_\beta(u) = \mathscr{F}^\lambda(u) + \Phi_\beta^{\lambda, 1}(\xi^u) + \Phi^{\lambda, 2}(\xi^u) \tag{6.3}$$

where

$$\mathscr{F}^\lambda(u) = \int_{R^{3N}} dx \left[|\nabla(u - G^\lambda \circ \xi^u)|^2 + \lambda |u - G^\lambda \circ \xi^u|^2 - \lambda |u|^2 \right] \tag{6.4}$$

$$\Phi_\beta^{\lambda, 1}(\xi^u) = \sum_{i<j} \int_{R^{3(N-1)}} d\mathbf{q} \, |\xi_{ij}^u(\mathbf{q})|^2 \left[\beta_{ij} + 2\pi^2 \sqrt{\mathbf{q}^2+\lambda} \right] \tag{6.5}$$

$$\Phi^{\lambda, 2}(\xi^u) = -\sum_{ij, hk}{}' \int_{R^{3N}} d\mathbf{p} \times \frac{\hat{\xi}_{ij}^u(p_1, \ldots, p_{i-1}, (p_i+p_j)/\sqrt{2}, \ldots, p_N) \hat{\xi}_{hk}^u(p_1, \ldots, p_{h-1}, (p_h+p_k)/\sqrt{2}, \ldots, p_N)}{|\mathbf{p}|^2+\lambda} \tag{6.6}$$

with domain

$$\left.\begin{array}{l} \mathrm{D}(\mathrm{F}_\beta)=\{u\in \mathrm{L}^2(\mathrm{R}^{3\mathrm{N}}),\ u=w+\mathrm{G}^\lambda\circ\xi^u,\ w\in \mathrm{H}^1(\mathrm{R}^{3\mathrm{N}}),\\ \qquad \xi^u\equiv\{\xi^u_{ij}\},\ \xi^u_{ij}\in \mathrm{H}^{1/2}(\mathrm{R}^{3(\mathrm{N}-1)})\}\end{array}\right\} \tag{6.7}$$

Notice that $\mathscr{F}$ and $\Phi^{\lambda,1}_\beta$ are, by construction, well defined on $\mathrm{D}(\mathrm{F}_\beta)$. It is immediately checked that also $\Phi^{\lambda,2}$ is well defined on the same domain. We give details of the proof for the case $\{ij\}\cap\{h,k\}\neq\varnothing$; the other case is treated similarly. Choosing for definitness $i=1$, $j=2$, $h=1$, $k=3$ and introducing the same variables as in the case (*b*) of Lemma 3.1 one easily obtains

$$\int_{\mathrm{R}^{3\mathrm{N}}} d\mathbf{p}\frac{\bar{\hat{\xi}}_{12}((p_1+p_2)/\sqrt{2},p_3,\ldots,p_\mathrm{N})\hat{\xi}_{13}((p_1+p_3)/\sqrt{2},p_2,\ldots,p_\mathrm{N})}{|\mathbf{p}|^2+\lambda}$$
$$\leqq \mathrm{C}\int dp_4\ldots dp_\mathrm{N}\int dq_3\int dq_1\,dq_2$$
$$\times\frac{|\zeta_{12}(q_1,q_3,p_4,\ldots,p_\mathrm{N})\zeta_{13}(q_2,q_3,p_4,\ldots,p_\mathrm{N})|}{\sqrt{|q_1|}(q_1^2+q_2^2)\sqrt{|q_2|}}$$

where

$$\zeta_{12}(q_1,q_3,p_4,\ldots,p_\mathrm{N})=\left[\frac{3}{4}q_1^2+q_3^2+\lambda\right]^{1/4}$$
$$\times\hat{\xi}_{12}\left(\frac{q_1}{2}+\sqrt{\frac{2}{3}}q_3,\ -\frac{q_1}{\sqrt{2}}+\frac{q_3}{\sqrt{3}},p_4,\ldots,p_\mathrm{N}\right). \tag{6.8}$$

The function ζ_{13} is defined in an analogous way and we have used the fact that $(q_1^2+q_2^2)/2\leqq q_1^2+q_2^2+q_1\cdot q_2\leqq(q_1^2+q_2^2)\,3/2$.

Performing the integration in the angular variables of q_1 and q_2, using the notation $\bar{\zeta}(|q|,\ldots)\equiv\left(\int d\Omega|\zeta|^2(|q|,\Omega,\ldots)\right)^{1/2}$, one has then

$$\int dp_4\ldots dp_\mathrm{N}\int dq_3\int dq_1\,dq_2$$
$$\times\frac{|\zeta_{12}(q_1,q_3,p_4,\ldots,p_\mathrm{N})|\,|\zeta_{13}(q_2,q_3,p_4,\ldots,p_\mathrm{N})|}{\sqrt{|q_1|}(q_1^2+q_2^2)\sqrt{|q_2|}}$$
$$\leqq\left[\sup_{|q_1|}\int_0^\infty dq_2\frac{|q_1|^{1/2}|q_2|^{1/2}}{q_1^2+q_2^2}\right].$$
$$\times\int dp_4\ldots dp_\mathrm{N}\int dq_3\left\{\int_0^\infty d|q_1|\,|q_1|^2\bar{\zeta}_{12}^2(|q_1|,\ldots)\right\}^{1/2}$$
$$\times\left\{\int_0^\infty d|q_2|\,|q_2|^2\bar{\zeta}_{13}^2(|q_2|,\ldots)\right\}^{1/2}$$
$$\leqq \mathrm{C}\,\Phi^{\lambda,1}_\beta(\xi). \tag{6.9}$$

As a result of these estimates, we have also shown that F_β is closed in the norm defined by $\mathscr{F}^\lambda + \Phi_\beta^{\lambda, 1}$ (*i.e.* the H^1-norm for w and the $H^{1/2}$-norm for ξ).

Also the result described in Lemma 3.2 holds. We shall reformulate it in a somewhat sharper form, which will be useful later.

LEMMA 6.1.

$$\text{(i)} \qquad \| G^\lambda \circ \xi_{ij} \| = \| \xi_{ij} \|_{-1/2, \lambda} \tag{6.10}$$

$$\text{(ii)} \qquad G^\lambda \circ \xi \notin H^1(R^{3N}) \quad \textit{if} \quad \xi \in D(\Phi_\beta^{\lambda, 1}), \qquad \xi \neq 0 \tag{6.11}$$

where

$$\| \xi_{ij} \|^2_{-1/2, \lambda} \equiv \pi^2 \int_{R^{3(N-1)}} (|\mathbf{q}|^2 + \lambda)^{-1/2} | \xi_{ij}(\mathbf{q}) |^2 d\mathbf{q}. \tag{6.12}$$

Notice that the norm defined by (6.12) is equivalent to the norm of $H^{-1/2}(R^{3(N-1)})$. In fact it is immediately checked that one can find two positive constants $c_-(\lambda)$, $c_+(\lambda)$ such that

$$c_-(\lambda) \| \xi \|_{H^{-1/2}(R^{3(N-1)})} \leqq \| \xi \|_{-1/2, \lambda} \leqq c_+(\lambda) \| \xi \|_{H^{-1/2}(R^{3(N-1)})}. \tag{6.13}$$

Proof. – (i) This is an explicit computation, using the expression for the Fourier transform of $G^\lambda \circ \xi_{ij}$.

(ii) The proof is a lenghty but straightforward computation, along the lines described in detail in Lemma 3.2 for the case $d=2$. We shall not repeat it here. □

A result equivalent to Lemma 3.1 does *not* hold in dimension three. The form F_β is neither closed nor bounded below in general (*i.e.* if no further constraint is imposed on its domain).

It can be used however, as we shall presently see, to construct an operator which commutes with complex conjugation and has dense domain in $L^2(R^{3N})$. This operator admits therefore self-adjoint extensions, all of which are unbounded below.

For $N=3$, a family of such self-adjoint extensions has been constructed in [1].

The form F_β commutes, at least formally, with the translation group, the rotation group $O(3)$ and the group Π of permutations of particle indices (recall that we have chosen all masses to be equal).

We shall only consider extensions which commute with the translation group. In view of this fact, one can take into account the tensor product decomposition of $L^2(R^9)$ obtained using center-of-mass and relative coordinates and introduce a different form F'_β, which coincides with F_β on a common domain (dense in $L^2(R^3)$) and is obtained by decomposing u in a way different from (6.7). To prove the results described in this Section we find it more convenient to work with F_β. In the next section, to prove boundedness below of the restriction of the form on suitable subspaces, it will be more convenient to make use of F'_β. We shall give then more details about the relation between the two representations.

As for the other two groups, the formal invariance of F_β allows to consider separately its restriction to subspaces corresponding to irreducible representations of $O(3)$ or of Π (or of some of their subgroups); it can be expected that some of these restrictions be bounded below. In particular, we shall prove in the next Section that, for $N=3$, this is the case if one considers the subspace of functions which are antisymmetric under exchange of two of the indices, and also if one considers the subspace of functions which have zero mean under the operation of rotation of the relative coordinates, leaving the baricentric coordinates invariant.

We remark that the existence of a self-adjoint extension which is bounded below was established for the first of these two cases in ([13], [14]).

We prove now that F_β is unbounded below if no further constraint is introduced in $D(F_\beta)$, as defined in (6.7).

We prove first that this is the case for $N=3$.

LEMMA 6.2. — *There exists a sequence* $\{u_m\}$, $u_m \in F_\beta^{(3)}$, $\|u_m\|^2 \leqq 1$ *such that*

$$\lim_{m \to \infty} F_\beta^{(3)}(u_m) = -\infty \tag{6.14}$$

(*here* $F_\beta^{(N)}$ *denotes the bilinear form* F_β *for a system of* N *particles*).

Proof. — To prove that F_β is unbounded below, it is sufficient to restrict attention to a subset of functions in $D(F_\beta)$. We choose functions $u(x_1, x_2, x_3)$ which are symmetric under permutation of any two of the particle indices, and have moreover the form

$$u(x_1, x_2, x_3) = (G^\lambda \circ \xi)(x_1, x_2, x_3) \tag{6.15}$$

where, as usual $\xi \equiv \{\xi_{ij}, i<j, i, j=1, 2, 3\}$, ξ_{ij} is supported on the hyperplane $\sigma_{ij} = \{x \in R^9 \mid x_i = x_j\}$, and $\xi_{ij} \in D(\Phi_\beta^{\lambda, 1})$, $\forall i<j$. Notice that all the (putative) negative bound states of any operator associated to $F_\beta^{(3)}$ have the form (6.15).

It is easy to see that if (6.15) defines a symmetric function, then the corresponding charges are all equal: $\xi_{12}(q_1, q_2) = \xi_{13}(q_1, q_2) = \xi_{23}(q_2, q_1)$.

It is convenient to introduce the variables defined in Lemma 3.1, case (*b*), and the new charge $\eta(p, q)$, defined by

$$\eta(p, q) \equiv \sqrt{\frac{3\sqrt{3}}{2}}\,\xi_{12}\left(\frac{p}{2} + \sqrt{\frac{2}{3}}q, \; -\frac{p}{\sqrt{2}} + \frac{q}{\sqrt{3}}\right) \tag{6.16}$$

We choose η's which have the product form

$$\eta(p, q) = f(p)\, g(q) \tag{6.17}$$

where g has compact support and $\|g\|_{L^2(R^3)} = 1$.

Then $F_\beta^{(3)}(u)$ becomes

$$F_\beta^{(3)}(u) = -\lambda \int_{R^3} dx\,|u|^2 + \int_{R^3} dq\,|g(q)|^2 \int_{R^3} dp \left[\bar\beta + 2\pi^2 \sqrt{\frac{3}{4}p^2 + q^2 + \lambda}\right] |f(p)|^2 - 2\int_{R^3} dq\,|g(q)|^2 \int_{R^6} dp_1\,dp_2 \frac{\bar f(p_1) f(p_2)}{p_1^2 + p_2^2 + p_1 . p_2 + q^2 + \lambda} \quad (6.18)$$

with $\bar\beta = \frac{1}{3}\sum_{i<j} \beta_{ij}$.

For q in a compact set one has

$$\sqrt{\frac{3}{4}p^2 + q^2 + \lambda} - \sqrt{\frac{3}{4}p^2 + \lambda} \leq \frac{C}{\sqrt{3/4\,p^2 + q^2 + \lambda}} \quad (6.19)$$

Moreover the kernel

$$\frac{1}{p_1^2 + p_2^2 + p_1 . p_2 + q^2 + \lambda} - \frac{1}{p_1^2 + p_2^2 + p_1 . p_2 + \lambda} \quad (6.20)$$

defines a bounded integral operator in $H^{-1/2}(R^6)$.

Taking into account (6.10) and choosing f real and rotationally invariant

$$f(p) = h(|p|), \qquad \operatorname{Im} h = 0 \quad (6.21)$$

from (6.18) we obtain

$$F_\beta^{(3)}(u) \leq (-\lambda + C) \int_{R^3} dx\,|u|^2 + 4\pi\bar\beta \int_0^\infty d\rho\,\rho^2 |h(\rho)|^2 + \Psi^\lambda(h) \quad (6.22)$$

where

$$\Psi^\lambda(h) \equiv 8\pi^3 \int_0^\infty d\rho\,\rho^2 \sqrt{\frac{3}{4}\rho^2 + \lambda}\,h^2(\rho) - 16\pi^2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2\,\rho_1\rho_2 \log\frac{\rho_1^2 + \rho_2^2 + \rho_1\rho_2 + \lambda}{\rho_1^2 + \rho_2^2 - \rho_1\rho_2 + \lambda} h(\rho_1) h(\rho_2) \quad (6.23)$$

In order to diagonalize the operator defined by $\Psi^\lambda(h)$ in $L^2(R^3)$ we introduce the following change of variables

$$\sinh x = \frac{\sqrt{3}}{2} \frac{\rho}{\sqrt{\lambda}} \qquad \forall x \geqq 0 \quad (6.24)$$

and

$$\varphi(x) \equiv \sqrt{\frac{32\pi^3}{3\sqrt{3}}}\,\lambda \sinh x \cosh x\, h\left(\frac{2\sqrt{\lambda}}{\sqrt{3}} \sinh|x|\right) \qquad x \in R. \quad (6.25)$$

Using elementary properties of hyperbolic functions and taking Fourier transform we have

$$\Psi^\lambda(h)=\bar{\Psi}^\lambda(\varphi)=\int_{\mathrm{R}} dx\,\varphi^2(x)$$
$$-\frac{4}{\sqrt{3}\,\pi}\int_{\mathrm{R}} dx\int_{\mathrm{R}} dx'\,\varphi(x)\,\varphi(x')\log\frac{2\cosh(x-x')+1}{2\cosh(x-x')-1}$$
$$=\int_{\mathrm{R}} dk\,(1-\mathrm{T}(k))\,\hat{\varphi}^2(k) \quad (6.26)$$

where

$$\mathrm{T}(k)=\frac{8}{\sqrt{3}}\frac{\sinh \pi/6\,k}{k\cosh \pi/2\,k} \quad (6.27)$$

is $\sqrt{2\pi}$ times the Fourier transform of

$$(4/\sqrt{3}\,\pi)\log[(2\cosh x+1)/(2\cosh x-1)]$$

(*see* for example [16]). Notice that T is a positive, even and decreasing function of $k\in\mathrm{R}$, with $\max_{k\in\mathrm{R}}\mathrm{T}(k)=\mathrm{T}(0)=\frac{4\pi}{3\sqrt{3}}>1.$

Consider a function $\hat{\varphi}_0(k)$ with compact support, $\int_{\mathrm{R}} dk\,\hat{\varphi}_0^2(k)=1$ such that

$$\frac{2\pi}{3\sqrt{3}}<\mathrm{T}(k)<\frac{4\pi}{3\sqrt{3}} \qquad \forall\, k\in\operatorname{supp}\hat{\varphi}_0 \quad (6.28)$$

One has

$$\bar{\Psi}(\varphi_0)\leqq\left(1-\frac{2\pi}{3\sqrt{3}}\right)\int_{\mathrm{R}} dk\,\hat{\varphi}_0^2(k)=\left(1-\frac{2\pi}{3\sqrt{3}}\right)\int_{\mathrm{R}} dx\,\varphi_0^2(x) \quad (6.29)$$

Notice that if $\varphi_0(x)$ satisfies (6.28), so does $\varphi_0(x+b)$, $\forall\, b\in\mathrm{R}$. Choose a sequence

$$\varphi_m(x)\equiv\frac{1}{\gamma_m}\varphi_0(x-m) \qquad m\in\mathrm{N}, \qquad \gamma_m>0 \quad (6.30)$$

and denote by u_m the corresponding sequence in $\mathrm{D}(\mathrm{F}_\beta^{(3)})$ defined via (6.25), (6.21), (6.17), (6.16), (6.15).

Using Lemma 6.1, (6.24), (6.25) and (6.29), from (6.22) we have

$$\mathrm{F}_\beta^{(3)}(u_m)\leqq(-\mathrm{C}+\mathrm{C}'/\lambda)\frac{1}{\gamma_m^2}\int_{\mathrm{R}} dy\frac{\varphi_0^2(y)}{\cosh^2(y+m)}+\frac{\mathrm{C}'}{\sqrt{\lambda}}\frac{1}{\gamma_m^2}$$
$$\int_{\mathrm{R}} dy\frac{\varphi_0^2(y)}{\cosh(y+m)}-\left(\frac{2\pi}{3\sqrt{3}}-1\right)\frac{1}{\gamma_m^2} \quad (6.31)$$

By dominated convergence the two integrals in (6.31) converge to zero for m large and one can find a sequence γ_m converging to zero so that $F_\beta^{(3)}(u_m)$ diverges to $-\infty$ for m tending to infinity, concluding the proof of the Lemma. □

We consider next the case of N particles. One has

LEMMA 6.3. – *$F_\beta^{(N)}$ is unbounded below if no symmetry restriction is placed on* $D(F_\beta^{(N)})$.

Denote by $\sigma_-(N) \in [-\infty, 0)$ the largest lower bound for $F_\beta^{(N)}$

$$\sigma_-(N) \equiv \inf_{u \in D(F_\beta^{(N)}),\ \| u \| = 1} F_\beta^{(N)}(u) \tag{6.32}$$

Lemma 6.3 is a consequence of Lemma 6.2 and of the following Lemma 6.4, which expresses the rather natural fact that the lower bound for $F_\beta^{(N)}$ is a non-increasing function of N

LEMMA 6.4. – $\sigma_-(N) \leq \sigma_-(M)$ *if* $N \leq M$.

Proof. – In the proof of the Lemma we denote by $G_\lambda^{(M)}$ the kernel of $(-\Delta+\lambda)^{-1}$ in R^{3M}.

Notice first that, if $u_1 \in D(F_\beta^{(M)})$ and $u_2 \in H^1(R^{(N-M)})$, then $u \equiv u_1 . u_2$ defined by

$$u(x_1 \dots x_N) \equiv u_1(x_1 \dots x_M)\, u_2(x_{m+1} \dots x_N) \tag{6.33}$$

belongs to $D(F_\beta^{(N)})$.

To prove this recall that for every $\lambda > 0$ one can decompose (uniquely) u_1 as

$$u_1 = w + G_\lambda^{(M)} \circ \xi \tag{6.34}$$

By construction $w u_2 \in H^1(R^{3N}) \subset D(F_\beta^{(N)})$.

We prove now that

$$(G_\lambda^{(M)} \circ \xi) . u_2 - G_\lambda^{(N)} \circ (\xi u_2) \in H^1(R^{3N}) \tag{6.35}$$

Using Fourier transformations, (6.35) is equivalent to $a_{ij} < \infty$ $\forall\, i, j = 1 \dots M$, $i<j$ where e.g.

$$\begin{aligned} a_{12} &= \int dp_1 \dots dp_N (p_1^2 + \dots + p_N^2 + \lambda) \left[\frac{1}{p_1^2 + \dots + p_M^2 + \lambda} - \frac{1}{p_1^2 + \dots + p_N^2 + \lambda} \right]^2 \\ &\quad \times | \hat{\xi}((p_1+p_2)/\sqrt{2}, \dots, p_M)\, \hat{u}_2(p_{M+1}, \dots, p_N) | \\ &= \int dp_1 \dots dp_N \frac{(p_{M+1}^2 + \dots + p_N^2)^2 |\hat{\xi}|^2 |\hat{u}_2|^2}{(p_1^2 + \dots + p_N^2 + \lambda)(p_1^2 + \dots + p_M^2 + \lambda)^2} \\ &\leq \int dp_{M+1} \dots dp_N (p_{M+1}^2 + \dots + p_N^2) | \hat{u}_2(p_{M+1}, \dots, p_N) |^2 \\ &\quad \times \int dp_1 \dots dp_M \frac{| \hat{\xi}((p_1+p_2)/\sqrt{2}, \dots, p_M) |^2}{(p_1^2 + \dots + p_M^2 + \lambda)^2} \\ &\leq C \| u_2 \|^2_{H^1(R^{3(N-M)})} \| \xi \|^2_{L^2(R^{3(M-1)})} \end{aligned} \tag{6.36}$$

Define now $u_2^b(x_{M+1} \ldots x_N)$ by

$$u_2^b(x_{M+1}, \ldots, x_N) \equiv u_2(x_{M+1}+b, \ldots, x_N+b) \tag{6.37}$$

Using the continuity properties implied by (6.36) and Lebesgue's dominated convergence theorem, it is now easy to prove, for almost every b

$$\lim_{\gamma \to \infty} F_\beta^{(N)}(u_1 . u_2^{\gamma b}) = F_\beta^{(M)}(u_1) = \| u_2 \|^2_{H^1(R^{3(N-M)})} \tag{6.38}$$

Therefore

$$\sigma_-(N) \leqq \inf_{u_1, u_2, \| u_1 \| = \| u_2 \| = 1} (F_\beta^{(M)}(u_1) + \| u_2 \|^2_{H^1(R^{3(N-M)})}) = \sigma_-(M) \tag{6.39}$$

concluding the proof of the Lemma. □

Remark 6.5. — One can prove the stronger inequality

$$\sigma_-(N) \leqq \sigma_-(M) + \sigma_-(N-M) \qquad 1 \leqq M \leqq N \tag{6.40}$$

but the proof is not straightforward because one cannot use product functions. The difficulty lies in the fact that if $u_1 \in D(F_\beta^{(M)})$ and $u_2 \in D(F_\beta^{(M-N)})$, then in general $u_1 . u_2 \notin D(F_\beta^{(N)})$, if M and $M-N$ are larger or equal to 2. The proof must then proceed through approximations and a limiting procedure.

7. SYMMETRY CONSTRAINTS AND BOUNDEDNESS BELOW

We start with the case $N=3$ and we shall prove that, if one introduces suitable symmetry constraints on the domain of $F_\beta^{(3)}$, then this form is bounded below, and defines a unique self-adjoint operator, bounded below. To simplify the notation we shall drop the dependence on the number of particles.

It is convenient to consider a slightly different quadratic form F'_β, which coincides with F_β on the common domain $D(F_\beta) \cap D(F'_\beta)$ which is dense in $L^2(R^9)$. Both F_β and F'_β are continuous in the topology induced by $\mathscr{F}(u) + \Phi_\beta^{\lambda, 1}(\xi^u)$ [*see* (6.3)].

Therefore if one of them is bounded below, so is the other one, and they define the same self-adjoint operator.

The new form F'_β is obtained through an informal construction which parallels the one described in Section 2 but places more emphasis on the separation between the center-of-mass and relative coordinates.

Consider the decomposition

$$L^2(R^9) = L^2(R^3) \otimes L^2(R^6) \tag{7.1}$$

corresponding to this choice of coordinates.

One can construct approximate bilinear forms $F'_{\beta, R}$, using volume charge distributions as in Section 2, but making now use of the Green's function $\tilde{G}^\lambda$ of the restriction of the Laplacian to the second factor in (7.1).

In particulier, the restriction of $F'_{\beta, R}$ to the first factor in (7.1) coincides for every R with the bilinear form associated with the free Laplacian in R^3.

Proceedings as in Section 2 we obtain the following result for the quadratic form in the limit $R \to \infty$

$$F'_\beta(vu) = \| v \|^2_{L^2(R^3)} \tilde{F}_\beta(u) + \| v \|^2_{H^1(R^3)} \| u \|^2_{L^2(R^6)} \tag{7.2}$$

$$D(F'_\beta) = \{ vu \in L^2(R^3) \otimes L^2(R^6) \,|\, v \in H^1(R^3),\ u \in D(\tilde{F}_\beta) \} \tag{7.3}$$

$$D(\tilde{F}_\beta) = \left\{ \begin{array}{l} u \in L^2(R^6) \,|\, \exists\, \xi^u \equiv (\xi^u_{12}, \xi^u_{13}, \xi^u_{23}),\ \xi^u_{ij} \in H^{1/2}(R^3)\ \forall\, i<j, \\ u - \tilde{G}^\lambda \circ \xi^u \in H^1(R^6) \end{array} \right\} \tag{7.4}$$

$$\tilde{G}^\lambda \circ \xi^u(k_1, k_2) = \sum_{i<j} \tilde{G}^\lambda \circ \xi^u_{ij}(k_1, k_2) = \frac{\xi^u_{12}(k_1) + \xi^u_{13}(k_2) + \xi^u_{23}(k_1+k_2)}{k_1^2 + k_2^2 + k_1 . k_2 + \lambda} \tag{7.5}$$

$$\tilde{F}_\beta(u) = \int_{R^6} dk_1\, dk_2\, (k_1^2 + k_2^2 + k_1 . k_2 + \lambda) \,| \hat{u} - \tilde{G}^\lambda \circ \xi^u |^2 - \lambda \int_{R^6} dx\, | u |^2 + \Phi^{\lambda, 1}_\beta(\xi^u) + \Phi^{\lambda, 2}(\xi^u) \tag{7.6}$$

$$\Phi^{\lambda, 1}_\beta(\xi^u) = \sum_{i<j} \int_{R^3} dq\, | \xi^u_{ij}(q) |^2 \left(\beta_{ij} + 2\pi^2 \sqrt{\frac{3}{4} q^2 + \lambda} \right) \tag{7.7}$$

$$\Phi^{\lambda, 2}(\xi^u) = - \sum_{ij,\, hk}{}' \int_{R^6} dq_1\, dq_2 \frac{\bar{\xi}^u_{ij}(q_1)\, \xi^u_{hk}(q_2)}{q_1^2 + q_2^2 + q_1 . q_2 + \lambda}. \tag{7.8}$$

To compare the domains of F_β and F'_β notice that $u \in D(F_\beta)$ if there exists $w \in H^1(R^9)$ such that

$$(\hat{u} - \hat{w})(k_1, k_2, k_3) = \frac{\hat{\xi}_{12}((k_1+k_2)/\sqrt{2}, k_3) + \hat{\xi}_{13}((k_1+k_3)/\sqrt{2}, k_2) + \hat{\xi}_{23}(k_1, (k_2+k_3)/\sqrt{2})}{k_1^2 + k_2^2 + k_3^2 + \lambda} \tag{7.9}$$

where $\xi_{ij} \in H^{1/2}(R^6)\ \forall\, i<j$, while $u \in D(F'_\beta)$ if there exist $w \in H^1(R^9)$ and $v \in H^1(R^3)$ such that

$$(\hat{u} - \hat{w})(k_1, k_2, k_3) = v(k_1 + k_2 + k_3) \frac{\eta_{12}(k_1 - k_2) + \eta_{13}(k_3 - k_1) + \eta_{23}(k_3 - k_2)}{k_1^2 + k_2^2 + k_3^2 - k_1 . k_2 - k_1 . k_3 - k_2 . k_3 + \lambda} \tag{7.10}$$

where $\eta_{ij} \in H^{1/2}(R^3)\ \forall\, i<j$. Using these explicit formulae it is easy to verify that $D(F_\beta) \not\subset D(F'_\beta)$, $D(F'_\beta) \not\subset D(F_\beta)$ and that $D(F_\beta) \cap D(F'_\beta)$ is dense in $L^2(R^9)$.

We shall consider the restriction of $\tilde{F}_\beta$ to the subspace of functions which are antisymmetric under interchange of the indices 2 and 3, and also the restriction to the subspace of functions which have zero mean under the rotation group (notice that the permutation group acts trivially on the first factor in (7.1)).

Recently ([13], [14]) Minlos and Shermatov have shown boundedness below for the Schrödinger operator corresponding to the first case. They construct the Hamiltonian applying the Krein's theory of s.a. extensions of symmetric operators. Then, using the Mellin transform, they diagonalize the Faddeev equation for the bound states and, by direct inspection, they show that the spectrum is bounded below.

In the Proposition below we shall find the same result using minimax methods.

PROPOSITION 7.1. – *Let* $F_{\beta,a}$ *the restriction of* $\tilde{F}_\beta$ *to the subspace of functions which are antisymmetric under interchange of the indices* 2 *and* 3. *One can find* $\lambda_0>0$ *such that*

$$F_{\beta,a}(u) \geqq -\lambda_0 \|u\|^2_{L^2(\mathbb{R}^6)} \qquad \forall u \in D(F_{\beta,a}). \tag{7.11}$$

Proof. – We already remarked that there is no loss of generality in assuming that the functions considered are symmetric under interchange of the indices 1 and 2. Every such $u\in D(F_{\beta,a})$ can be uniquely decomposed as

$$u = w^\lambda + \tilde{G}^\lambda \circ \xi \tag{7.12}$$

where

$$w^\lambda \in H^1(\mathbb{R}^6), \qquad \xi \equiv (\sqrt{2}\,\eta, -\sqrt{2}\,\eta, 0) \qquad \text{with} \quad \eta \in H^{1/2}(\mathbb{R}^3) \tag{7.13}$$

and correspondingly $F_{\beta,a}$ takes the form

$$F_{\beta,a}(u) = \int_{\mathbb{R}^6} dk_1\, dk_2\, (k_1^2+k_2^2+k_1.k_2+\lambda)\,|\hat{u}-\tilde{G}^\lambda \circ \xi|^2 - \lambda \int_{\mathbb{R}^6} dx\,|u|^2 + \Phi^{\lambda,1}_{\beta,a}(\eta) + \Phi^{\lambda,2}_a(\eta) \tag{7.14}$$

$$\Phi^{\lambda,1}_{\beta,a}(\eta) = \int_{\mathbb{R}^3} dq\,|\eta(q)|^2 \left(\beta + 2\pi^2 \sqrt{\frac{3}{4}q^2+\lambda}\right) \tag{7.15}$$

$$\Phi^{\lambda,2}_a(\eta) = \int_{\mathbb{R}^6} dq_1\, dq_2 \frac{\bar{\eta}(q_1)\,\eta(q_2)}{q_1^2+q_2^2+q_1\cdot q_2+\lambda}. \tag{7.16}$$

Take a positive number ε, with $0<\varepsilon<1$ if $\beta<0$ and $\varepsilon=1$ if $\beta\geqq 0$. Then

$$\begin{aligned} F_{\beta,a}(u) &\geqq -\lambda \int_{\mathbb{R}^6} dx\,|u|^2 + \beta \|\eta\|^2_{L^2(\mathbb{R}^3)} + (1-\varepsilon)\,\Phi^{\lambda,1}_{0,a}(\eta) + \varepsilon \Phi^{\lambda,1}_{0,a}(\eta) + \Phi^{\lambda,2}_a(\eta) \\ &\geqq -\lambda \int_{\mathbb{R}^6} dx\,|u|^2 + (\beta + 2\pi^2(1-\varepsilon)\sqrt{\lambda})\,\|\eta\|^2_{L^2(\mathbb{R}^3)} + \varepsilon \Phi^{\lambda,1}_{0,a}(\eta) + \Phi^{\lambda,2}_a(\eta) \end{aligned}$$

$$\geqq -\lambda \int_{\mathbb{R}^6} dx\,|u|^2 + \varepsilon\,[\Phi_{0,a}^{\lambda,1}(\eta) - \Phi_{0,a}^{0,1}(\eta)] + [\Phi_a^{\lambda,2}(\eta) - \Phi_a^{0,2}(\eta)] + \varepsilon\Phi_{0,a}^{0,1}(\eta) + \Phi_a^{0,2}(\eta) \qquad (7.17)$$

where $\sqrt{\lambda} > |\beta|/2\pi^2(1-\varepsilon)$ if $\beta < 0$ and $\lambda > 0$ if $\beta \geqq 0$.

As in (6.19), (6.20) one can see that

$$|\Phi_{0,a}^{\lambda,1}(\eta) - \Phi_{0,a}^{0,1}(\eta)| \leqq c\lambda \|\eta\|_{-1/2,\lambda}^2 \qquad (7.18)$$

$$|\Phi_a^{\lambda,2}(\eta) - \Phi_a^{0,2}(\eta)| \leqq c\lambda \|\eta\|_{-1/2,\lambda}^2. \qquad (7.19)$$

Then, using Lemma 6.1, equations (7.18) and (7.19) imply

$$F_{\beta,a}(u) \geqq -c\lambda \int_{\mathbb{R}^6} dx\,|u|^2 + \varepsilon\Phi_{0,a}^{0,1}(\eta) + \Phi_a^{0,2}(\eta). \qquad (7.20)$$

The Proposition is proved if we can show that for some ε, $0 < \varepsilon < 1$, $\varepsilon\varphi_{0,a}^{0,1}(\eta) + \varphi_a^{0,2}(\eta)$ is non negative for any $\eta \in H^{1/2}(\mathbb{R}^3)$.

Introducing spherical coordinates (ρ, ω), $\rho > 0$, $\omega \equiv (\theta, \varphi) \in S^2$ and defining

$$h(\rho, \omega) = \eta(q), \qquad q \in \mathbb{R}^3 \qquad (7.21)$$

we have

$$\varepsilon\Phi_{0,a}^{0,1}(\eta) + \Phi_a^{0,2}(\eta) = \pi^2\sqrt{3\varepsilon}\int_{S^2} d\omega \int_0^\infty d\rho\,\rho^3\,|h(\rho,\omega)|^2 + \int_{S^2\times S^2} d\omega_1\,d\omega_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 \frac{\rho_1^2\rho_2^2\,\bar{h}(\rho_1,\omega_1)\,h(\rho_2,\omega_2)}{\rho_1^2 + \rho_2^2 + \rho_1\rho_2\cos(\omega_1,\omega_2)} \qquad (7.22)$$

where $\cos(\omega_1, \omega_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2)$. Now with the change of coordinates

$$\rho = e^x, \qquad x \in \mathbb{R} \qquad (7.23)$$

$$\varphi(x, \omega) = \pi\sqrt{\varepsilon\sqrt{3}\,e^{2x}}\,h(e^x, \omega) \qquad (7.24)$$

the last integral in (7.22) is reduced to a convolution and it is then diagonalized taking Fourier transform

$$\begin{aligned}\varepsilon\Phi_{0,a}^{0,1}(\eta) + \Phi_a^{0,2}(\eta) &= \int_{S^2} d\omega \int_{\mathbb{R}} dx\,|\varphi(x,\omega)|^2 \\ &\quad + \frac{1}{2\pi^2\sqrt{3\varepsilon}} \int_{S^2\times S^2} d\omega\,d\omega' \int_{\mathbb{R}^2} dx\,dx' \frac{\bar{\varphi}(x,\omega)\,\varphi(x',\omega')}{\cosh(x-x') + 1/2\cos(\omega,\omega')} \\ &= \int_{S^2} d\omega \int_{\mathbb{R}} dk\,|\hat{\varphi}(k,\omega)|^2 \\ &\quad + \int_{S^2\times S^2} d\omega\,d\omega' \int_{\mathbb{R}} dk\,\bar{\hat{\varphi}}(k,\omega)\,\hat{\varphi}(k,\omega')\,S(k,\gamma) \qquad (7.25)\end{aligned}$$

where

$$\gamma = \cos^{-1}\left(\frac{1}{2}\cos(\omega, \omega')\right), \qquad \frac{\pi}{3} \leqq \gamma \leqq \frac{2}{3}\pi, \tag{7.26}$$

and

$$S(k, \gamma) = \frac{1}{\pi\sqrt{3}\,\varepsilon \sin\gamma} \frac{\sinh \gamma k}{\sinh \pi k} \tag{7.27}$$

is $\sqrt{2\pi}$ times the Fourier transform of $[2\pi^2\sqrt{3}\,\varepsilon(\cosh x + \cos\gamma)]^{-1}$ (*see e.g.* [16]).

It is easily checked that $S(k, \gamma)$ is positive for $k \geqq 0$, $\pi/3 \leqq \gamma \leqq 2\pi/3$, and strictly increasing as a function of γ for any $k \geqq 0$. Therefore

$$\min_{\gamma} S(k, \gamma) \equiv S^-(k) = \frac{2}{3\pi\varepsilon} \frac{\sinh \pi k/3}{\sinh \pi k} \tag{7.28}$$

$$\max_{\gamma} S(k, \gamma) \equiv S^+(k) = \frac{2}{3\pi\varepsilon} \frac{\sinh 2\pi k/3}{\sinh \pi k} \tag{7.29}$$

Notice that $S^+(k) - S^-(k)$ is even, positive and decreasing in k. So that it attains its maximum value at $k=0$. Using this fact and the Schwartz inequality we have from (7.25)

$$\begin{aligned}
&\int_{\mathbb{R}} dk \int_{S^2} d\omega \bar{\hat{\varphi}}(k, \omega) \int_{S^2} d\omega' \hat{\varphi}(k, \omega') S(k, \gamma) \\
&= \frac{1}{2}\int_{\mathbb{R}} dk\,(S^+(k) + S^-(k)) \left| \int_{S^2} d\omega\, \hat{\varphi}(k, \omega) \right|^2 \\
&\quad + \int_{\mathbb{R}} dk \int_{S^2} d\omega\, \bar{\hat{\varphi}}(k, \omega) \int_{S^2} d\omega'\, \hat{\varphi}(k, \omega') \\
&\quad \times \left[S(k, \gamma) - \frac{S^+(k) + S^-(k)}{2} \right] \\
&\geqq -\frac{1}{2}\int_{\mathbb{R}} dk\,(S^+(k) - S^-(k)) \int_{S^2} d\omega\, |\hat{\varphi}(k, \omega)| \int_{S^2} d\omega'\, |\hat{\varphi}(k, \omega')| \\
&\geqq -\frac{4}{9}\varepsilon \int_{\mathbb{R}} dk \int_{S^2} |\hat{\varphi}(k, \omega)|^2.
\end{aligned} \tag{7.30}$$

In view of (7.30) and (7.25) we get

$$\varepsilon \Phi_{0;\,a}^{0;\,1}(\eta) + \Phi_a^{0,\,2}(\eta) \geqq 0 \qquad \text{for} \quad \frac{4}{9} < \varepsilon < 1 \tag{7.31}$$

and this concludes the proof of the Proposition. □

Along the same line of reasoning as in the above proof one can also show that $\Phi_{\beta,\,a}^{\lambda;\,1} + \Phi_a^{\lambda,\,2}$ is closed and bounded below on $H^{1/2}(\mathbb{R}^3)$.

Then, proceeding exactly as in the two dimensional case, it is easy to verify that $F_{\beta,\,a}$ is closed on $D(F_{\beta,\,a})$ and that it is the Γ-limit of approximating forms which are smooth perturbations of the form of the Laplacian.

The s.a. operator $H_{\beta, a}$ on $L^2(R^6)$ defined by $F_{\beta, a}$ is bounded below and describes (in the center of mass reference frame) the dynamics of two identical scalar fermions interacting with a (different) particle through a two-body, local and translation invariant zero-range interaction. Action, domain and resolvent of $H_{\beta, a}$ can be easily characterized following the same steps as for the case $d=2$.

Remark 7.1. – Let $\sigma_-(\beta)$ be the infimum of the spectrum of $H_{\beta, a}$. It is easy to see that

$$\sigma_-(\beta)=\min(0, -\lambda_0(\beta))$$

where $\lambda_0(\beta)$ is defined by

$$\lambda_0(\beta)\equiv \inf_{\lambda>0} \inf\left\{\zeta\in L^2(R^3) \,\Big|\, \frac{\beta}{2\pi^2}\int_{R^3}\frac{|\zeta(q)|^2}{\sqrt{3/4\,q^2+\lambda}}dq+(\zeta, T_\lambda\zeta)\geqq 0\right\}$$

and

$$(\zeta, T_\lambda\zeta)\equiv\int_{R^3} dq\,|\zeta(q)|^2+\frac{1}{2\pi^2}$$
$$\int_{R^6} dq_1\,dq_2\frac{\bar{\zeta}(q_1)\zeta(q_2)}{((3/4)\,q_1^2+\lambda)^{1/4}\,(q_1^2+q_2^2+q_1\,.\,q_2+\lambda)\,((3/4)\,q_2^2+\lambda)^{1/4}}.$$

For the system we are examining in this Section one has $T_\lambda\geqq 0$.

It follows that $\lambda_0(\beta)=0$ if $\beta\geqq 0$ and therefore $\sigma_-(\beta)=0$ if $\beta\geqq 0$ (*i.e.* if $\beta\geqq 0$ there are no negative energy bound states for the system).

If $\beta<0$, an easy scaling argument shows that $\lambda_0(\beta)$ is given by

$$\lambda_0(\beta)=\left(\frac{|\beta|}{2\pi^2 c}\right)^2, \qquad c\equiv \inf_{\zeta\in L^2(R^3)}\frac{(\zeta, T_1\zeta)}{\displaystyle\int_{R^3}|\zeta(q)|^2((3/4)\,q^2+1)^{-1/2}\,dq}.$$

Notice that $c\leqq 1$, as one can see taking test functions with compact support concentrating at the origin. We remark that $c=1$ if one takes functions which are product of the two-body bound state (*seé e.g.* [5]) and a free particle.

It would be interesting to obtain a sharper estimate on c. Indeed, if $c<1$, then the three-particle system has a bound state with energy smaller than $(-|\beta|/2\pi^2)^2$.

We shall now prove boundedness below also for the restriction of F'_β to functions which have zero mean for the group of rotations of the internal variables.

Consider the following action of the rotation group

$$x_i-x_j\to R(x_i-x_j), \qquad x_1+x_2+x_3\to x_1+x_2+x_3, \qquad (7.32)$$

with $R \in O(3)$. Its dual action on functions, in the decomposition (7.1), is trivial on the first factor and is the usual action of the rotation group on the second factor. In particular, if u has the form $u = \tilde{G}^{\lambda} \circ \xi$, $\xi \in L^2(R^3)$, $O(3)$ acts by

$$\xi(p) \to \xi^{R}(p) \equiv \xi(Rp). \tag{7.33}$$

Therefore, on the charges ξ, the restriction we are considering can be described as the requirement that "there are no s-wave charges".

If one considers the entire space $L^2(R^9)$ this restriction on the angular momentum takes a somewhat more complicated form and will not be described here.

As we have seen in the course of the proof of Proposition 7.1, the proof of boundedness below only involves properties of the term $\Phi_{\beta}^{\lambda, 1}(\xi) + \Phi^{\lambda, 2}(\xi)$ in (7.6) and therefore we will take explicitely into account only the condition

$$\int_{R \in O(3)} \xi(Rp)\, d\mu(R) = 0 \tag{7.34}$$

where μ is the Haar measure on $O(3)$.

We prove

PROPOSITION 7.2. — *Let $F_{\beta, 0}$ be the restriction of $\tilde{F}_{\beta}$ to the subspace of functions which are such that* (7.34) *is satisfied. One can find $\lambda_1 > 0$ such that*

$$F_{\beta, 0}(u) = -\lambda_1 \| u \|^2_{L^2(R^6)}, \qquad \forall u \in D(F_{\beta, 0}). \tag{7.35}$$

Proof. — We proceed along the line of the proof of Proposition 7.1. Take ε such that $0 < \varepsilon < 1$ if $\min_{i<j} \beta_{ij} < 0$ and $\varepsilon = 1$ if $\beta_{ij} \geqq 0\ \forall i<j$. Then

$$\begin{aligned} F_{\beta, 0}(u) \geqq -\lambda \int_{R^6} dx\, |u|^2 + \sum_{i<j} [\beta_{ij} + (1-\varepsilon)\, 2\pi^2 \sqrt{\lambda}]\, \| \xi_{ij} \|^2_{L^2(R^3)} \\ + \varepsilon [\Phi_{0, 0}^{\lambda, 1}(\xi) - \Phi_{0, 0}^{0, 1}(\xi)] + [\Phi_{0}^{\lambda, 2}(\xi) - \Phi_{0}^{0, 2}(\xi)] + \varepsilon \Phi_{0, 0}^{0, 1}(\xi) + \Phi_{0}^{0, 2}(\xi) \end{aligned} \tag{7.36}$$

where $\Phi_{\beta, 0}^{\lambda, 1}$ and $\Phi_{0}^{\lambda, 2}$ are the restrictions of $\Phi_{\beta}^{\lambda, 1}$ and $\Phi^{\lambda, 2}$, defined in (7.7), (7.8), to the subspace of functions satisfying (7.34).

Now we take $\sqrt{\lambda} > -\min_{i<j} \beta_{ij} [2\pi^2 (1-\varepsilon)]^{-1}$ if $\min_{i<j} \beta_{ij} < 0$ and $\lambda > 0$ if $\beta_{ij} \geqq 0\ \forall i<j$ and we estimate the third and the forth term in r.h.s. of (7.36) exactly as in (7.18) and (7.19). Then again the proof is reduced to show that for some positive ε, $0 < \varepsilon < 1$, $\varepsilon \Phi_{0, 0}^{0, 1}(\xi) + \Phi_{0}^{0, 2}(\xi)$ is non negative for $\xi_{ij} \in H^{1/2}(R^3)$ satisfying (7.34) $\forall i<j$.

Proceedings as in the previous case [*see* (7.21), ..., (7.30)] and using the fact that the charges have zero mean over S^2 we have

$$\begin{aligned}\varepsilon\Phi^{0,\,1}_{0,\,0}(\xi)+\Phi^{0,\,2}_{0}(\xi)&=\sum_{i<j}\int_{S^2}d\omega\int_{R}dk\,|\hat{\varphi}_{ij}(k,\omega)|^2\\&\quad-2\sum_{ij,\,hk}'\int_{S^2\times S^2}d\omega\,d\omega'\int_{R}dk\,\bar{\hat{\varphi}}_{ij}(k,\omega)\,\hat{\varphi}_{hk}(k,\omega')\,S(k,\gamma)\\&\geqq\sum_{i<j}\int_{S^2}d\omega\int_{R}dk\,|\hat{\varphi}_{ij}(k,\omega)|^2\\&\quad-\frac{2}{9\pi\varepsilon}\sum_{ij,\,hk}'\int_{R}dk\int_{S^2}d\omega\,|\hat{\varphi}_{ij}(k,\omega)|\int_{S^2}d\omega'\,|\hat{\varphi}_{hk}(k,\omega')|\\&\geqq\left(1-\frac{8}{9\varepsilon}\right)\sum_{i<j}\int_{R}dk\int_{S^2}d\omega\,|\hat{\varphi}_{ij}(k,\omega)|^2\end{aligned}\tag{7.37}$$

where

$$\xi_{ij}(q)=h_{ij}(\rho,\omega),\qquad q\in R^3,\qquad \rho>0,\qquad \omega\in S^2\tag{7.38}$$

$$\rho=e^x,\qquad x\in R\tag{7.39}$$

$$\varphi_{ij}(x,\omega)=\pi\sqrt{\varepsilon\sqrt{3}\,e^{2x}}\,h_{ij}(e^x,\omega)\tag{7.40}$$

and $S(k,\gamma)$ has been defined in (7.27). Taking ε satisfying $8/9<\varepsilon<1$ completes the proof. □

As in the previous case we can conclude that $F_{\beta,0}$ is also closed on $D(F_{\beta,0})$ and then the corresponding Hamiltonian $H_{\beta,0}$ is s.a. and bounded below.

Finally we consider the system of $N+1$ particles, formed by N identical (scalar) fermions and a different particle and we show that the corresponding quadratic form $F^{N,\,1}_{\beta}$ is unbounded below for N sufficiently large.

Taking into account the symmetry constraint it is easily seen that the interaction is described by a single charge $\xi\in H^{1/2}(R^{3N})$. The form $F^{N,\,1}_{\beta}$ is then defined as in (6.3), (6.4) with $\Phi^{\lambda,\,1}_{\beta}$ and $\Phi^{\lambda,\,2}_{\beta}$ replaced by

$$\Phi^{\lambda,\,1}_{\beta,\,N}(\xi)=N\int_{R^{3N}}d\mathbf{q}\,|\hat{\xi}(\mathbf{q})|^2\left(\beta+2\pi^2\sqrt{|\mathbf{q}|^2+\lambda}\right)\tag{7.41}$$

$$\Phi^{\lambda,\,2}_{N}(\xi)=N(N-1)\int_{R^{3(N+1)}}d\mathbf{p}\times\frac{\bar{\hat{\xi}}((p_1+p_{N+1})/\sqrt{2},p_2,\ldots,p_N)\,\hat{\xi}((p_2+p_{N+1})/\sqrt{2},p_1,\ldots,p_N)}{|\mathbf{p}|^2+\lambda}.\tag{7.42}$$

We choose a function $u = G^{\lambda} \circ \xi$ and, introducing the variables defined in Lemma 3.1 case (*b*), we define as before

$$\eta(t, s, p_3, \ldots, p_N) = \sqrt{N\frac{\sqrt{3}}{2}}\,\xi\left(\frac{t}{2} + \sqrt{\frac{2}{3}}s,\ -\frac{t}{\sqrt{2}} + \frac{s}{\sqrt{3}}, p_3, \ldots, p_N\right). \quad (7.43)$$

In terms of the new charge η we obtain

$$\begin{aligned}\Phi^{\lambda,1}_{\beta,N}(\xi) + \Phi^{\lambda,2}_{N}(\xi) &= \bar{\Phi}^{\lambda,1}_{\beta,N}(\eta) + \bar{\Phi}^{\lambda,2}_{N}(\eta)\\ &= \int_{R^{3N}} dq_3 \ldots dq_N\, ds\, dt\, |\eta(t, s, q_3, \ldots, q_N)|^2\\ &\quad\times\left(\beta + 2\pi^2\sqrt{\frac{3}{4}t^2 + s^2 + q_3^2 + \ldots + q_N^2 + \lambda}\right)\\ &\quad + (N-1)\int_{R^{3(N+1)}} dp_3 \ldots dp_N\, ds\, dt_1\, dt_2\\ &\quad\times\frac{\bar{\eta}(t_1, s, p_3, \ldots, p_N)\,\eta(t_2, s, p_3, \ldots, p_N)}{t_1^2 + t_2^2 + t_1 . t_2 + s^2 + p_3^2 + \ldots + p_N^2 + \lambda}. \quad (7.44)\end{aligned}$$

All the arguments of the charge η, with the exception of the first one, have the role of parameters and can be rescaled through the change of variables $t \to z$

$$z = \frac{t}{\sqrt{s^2 + q_3^2 + \ldots + q_N^2 + \lambda}} \quad (7.45)$$

and the redefinition of the charge

$$\zeta(z, s, q_3, \ldots, q_N) = (s^2 + q_3^2 + \ldots + q_N^2 + \lambda)\,\eta\left(\sqrt{s^2 + q_3^2 + \ldots + q_N^2 + \lambda}\, z, s, q_3, \ldots, q_N\right). \quad (7.46)$$

Finally we choose ζ in the form of a product

$$\zeta^{a,g}(x_1, \ldots, x_N) = a(\rho_1)\cos\theta_1 . g(x_2, \ldots, x_N) \quad (7.47)$$

where $(\rho_1, \theta_1, \varphi_1)$ are the spherical coordinates of $x_1 \in R^3$, a is a positive, smooth function defined in $[0, +\infty)$ and $\|g\|_{L^2(R^{3(N-1)})} = 1$. Then

$$\begin{aligned}\bar{\Phi}^{\lambda,1}_{\beta,N}(\eta) + \bar{\Phi}^{\lambda,2}_{N}(\eta) &= \tilde{\Phi}^{\lambda,1}_{\beta,N}(\zeta^{a,g}) + \tilde{\Phi}^{\lambda,2}_{N}(\zeta^{a,g})\\ &= \frac{4\pi}{3}\beta\int_{R^{3(N-1)}} dx_2 \ldots dx_N \frac{|g(x_2, \ldots, x_N)|^2}{\sqrt{x_2^2 + \ldots + x_N^2 + \lambda}}\int_0^\infty d\rho_1\, \rho_1^2\, a^2(\rho_1)\\ &\quad + \frac{8\pi^3}{3}\int_0^\infty d\rho_1\, \rho_1^2\sqrt{\frac{3}{4}\rho_1^2 + 1}\, a^2(\rho_1)\\ &\quad + (N-1)\int_0^\infty d\rho_1\int_0^\infty d\rho_2\, \rho_1^2\, a(\rho_1)\, \rho_2^2\, a(\rho_2)\\ &\quad\times\int_{S^2\times S^2} d\omega_1\, d\omega_2 \frac{\cos\theta_1\cos\theta_2}{\rho_1^2 + \rho_2^2 + \rho_1\rho_2\cos(\omega_1, \omega_2) + 1}. \quad (7.48)\end{aligned}$$

It is easily checked that, for N sufficiently large, the sum of the last two terms in (7.48) is in fact negative. Following the line of Lemma 6.2, it is now straightforward to show that the form $F_\beta^{N,1}$ is unbounded below for N sufficiently large.

Remark 7.2. – Obviously the quadratic form for a system of N identical fermions plus M identical fermions of a different type is also unbounded below for min (N, M) large.

Remark 7.3. – The last result shows that a system on N particles in R^3 interacting through local and translation invariant point interactions is unstable for N large, even if the maximal (non trivial) symmetry constraint on the form domain is required.

8. CONCLUSIONS

We conclude with some remarks and indication of open problems.

It would be desirable to prove that one can construct the point interaction described here as limit of sequence of smooth two-body potentials, and to study the possible role of three-body potentials in providing limit forms which are bounded below. Alternatively, one could study the influence on the spectrum due to an additional term in the energy form for the charges, supported by $\sigma_{ij} \cap \sigma_{ik} j \neq k$.

Another interesting problem would be to establish a firm link between the Thomas effect and the Efimov effect [17], which we take to be the following: if V is a two-body potential which correspond to a positive spectrum and a zero-energy resonance, then for $N \geqq 3$ a system of N particles pairwise interacting through V has infinitely many negative bound states accumulating at zero. Note that for $d=2$ there is no Efimov effect for $N=3$ [18] and we have proved in Section 3 that the Thomas effect is absent for $d=2$ and any N (in this context it would be interesting to prove that the Efimov effect is absent for $d=2$ also for $N \geqq 4$). If $d=3$, the Efimov effect is present and it can be expected [19] that one has $\lim E_{n+1}/E_n = c$, where $\{E_n\}$ is the sequence of the eigenvalues and the constant c depends only on the mass ratios of the particles. It would seem that the Thomas effect is in some sense a dilation of the Efimov effect, and its spectrum could provide the "universal" asymptotic behaviour of the eigenvalues in the Efimov effect.

Finally one should also extend to the case of space dimensions $d=2$ the very interesting result of Dimock [20] on the relation between the scattering matrix for a point interaction and the scattering matrix for a relativistic scalar field theory with quartic interaction.

In space dimension $d=1$ Dimock proved that the two-particle channel of the scattering matrix of the relativistic field converges in the non-relativistic limit to the scattering matrix of the Schrödinger operator with a point interaction. In space dimensions $d=2$ our results are significant for matrix elements other than in two-particle channel.
The interest in this problem for $d=2$ lies in the fact that both the point interaction and the scalar field theory require a "renormalization" which in both cases corresponds to the subtraction of a "self-energy", as indicated informally at the beginning of the paper.
We remark that it would also be interesting and non trivial to extend, for space dimension $d=1$, the result of Dimock to cover the case on N-particle channels of the scattering matrix.

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