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A construction of quasi-modes using coherent states

by

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ABSTRACT. – We present a construction of semi-classical vectors satisfying the Schrödinger equations modulo $\hbar^\infty$. These vectors have nice regular properties, in particular they don't have turning points. An application is given to the calculation of matrix elements. The present work is meant as an introduction to a more general theory which is announced.

RÉSUMÉ. – Nous présentons une construction de vecteurs semi-classiques satisfaisant l'équation de Schrödinger modulo $\hbar^\infty$. Ces vecteurs ont de bonnes propriétés de régularité et en particulier n'ont pas de points tournants. Nous donnons une application au calcul des éléments de matrice. Ce travail est une introduction à une théorie plus générale.

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CONTENTS

1. INTRODUCTION

The theory of semi-classical approximations of eigenstates of self-adjoint operators has a long history in quantum mechanics, since the WKB approximation goes back to its early days ([10], [11]). As it is well known, a defect of the WKB wave function is that it possesses singularities at the so-called "turning points". A formulation of WKB due to Voros [11] avoids this "caustic" problem, by representing locally the eigenfunction in Bargman space in a neighborhood of the corresponding classical trajectory. However this approximation is not defined globally and so doesn't belong to the Hilbert space where the spectral problem is posed.

The goal of this paper is to construct semi-classical approximations of eigenstates having the property that they belong to the Hilbert space where the operator acts, and to its domain as well (in fact they are perfectly smooth functions). Although a more general theory of this construction is available, (*see* [7]), here we want to concentrate on the very simple case of a one dimensional differential operator with a polynomial symbol. In this case everything can be computed explicitly and, we hope, physically, avoiding the microlocal machinery needed in the general case. In the last section we will indicate in what sense the present paper is a special case of a general theory, and elaborate on the geometric background of the construction.

The ideas of this paper can be summarized as follows. Let us consider a pseudo-differential operator $a(x, \hbar D_x)$ on $L^2(\mathbb{R})$, with symbol $a(x, \xi)$ analytic in both x and ξ. We would like to find, for every N, numbers E_N and vectors ψ_N such that

$$\| (a(x, \hbar D_x) - E_N) \psi_N \|_{L^2(\mathbb{R})} \leqq C_N \hbar^{N+1} \tag{1}$$

for some constant C_N and for infinitely-many values of $\hbar$ having zero as a cluster point. E_N and ψ_N are allowed to depend on $\hbar$, and should have a good asymptotic behavior as $\hbar \to 0$. The first idea is to use coherent states to construct ψ_N. Recall that a coherent state – or wave packet – or gaboret – is a vector $\psi_{(x,\xi)}$, indexed by a point (x, ξ) of phase space, (best) localized in phase space around (x, ξ), namely

$$\psi_{(x,\xi)}(y) = \frac{1}{(\pi\hbar)^{1/4}} e^{i(x\xi/2\hbar)} e^{-i(\xi y/\hbar)} e^{-((y-x)^2/2\hbar)}. \tag{2}$$

An easy computation shows that

$$\|(a(x, \hbar D_x) - a(x, \xi))\psi_{(x,\xi)}\| = O(\hbar^{1/2}). \tag{3}$$

However, (3) doesn't give any spectral information: the mean level spacing in one degree of freedom is $O(\hbar)$, and so the spectral prediction of (3), namely that there is an eigenvalue within $\hbar^{1/2}$ of E, is already known to be true, trivially. Moreover, (3) is unsatisfactory because: (i) it possesses a big degeneracy, namely all the $\psi_{(x,\xi)}$ with $a(x, \xi) = E$ are associated to the same eigenvalue, and (ii) $\{\psi_{(x,\xi)}, (x, \xi) \in \mathbb{R}^2 = T^*(\mathbb{R})\}$ is not a basis of $L^2(\mathbb{R})$ and in particular one cannot define an operator which would be diagonal on $\{\psi_{(x,\xi)}\}$.

The idea is to go to next order in $\hbar$ and remove the degeneracy by taking for ψ_N a suitable linear combination of the $\psi_{(x,\xi)}$ with $a(x, \xi) = E$, a combination which would satisfy (1) to order $\hbar^N$.

The main result is that this is indeed possible: to each connected component Γ of the energy surface $\Omega_E = \{(x, \xi), a(x, \xi) = E\}$ and any given N we can associate a vector ψ_Γ, linear combination of $\psi_{(x,\xi)}$ with $(x, \xi) \in \Gamma$, which satisfies (1) to order N for a suitable set of values of $\hbar$. Thus ψ_N will be of the form

$$\psi = \int_0^T s(t, \hbar) e^{if(t)/\hbar} \psi_{(x(t), \xi(t))} \, dt \tag{4}$$

where $(x(t), \xi(t))$ is a parametrization of Γ as a trajectory of the Hamiltonian flow of a, and T is its period. The set $\mathscr{S}$ of values of $\hbar$ for which the estimates (1) holds is determined by a condition of the Bohr-Sommerfeld type. We will in fact discuss *two* Bohr-Sommerfeld conditions: the classical one of the Physics literature, namely

$$\int_\Gamma \xi \, dx = (n + 1/2)\hbar 2\pi, \qquad n \in \mathbb{Z}^+, \tag{5}$$

and a second one which we will call the *geometric BS condition*,

$$\int_\Gamma \xi \, dx = n\hbar 2\pi, \qquad n \in \mathbb{Z}^+. \tag{6}$$

Recall that the original meaning of (5), in the physics literature, is as a rule for finding (approximate) values of the quantum energy levels: one views (5) as a condition on the value of the energy of Γ, with $\hbar$ having its physical value. The values of the energy picked up by (5) predict *exactly* the spectrum of the harmonic oscillator. In general, as is well-known, the prediction is accurate to order $\hbar^2$. For this, the presence of the famous 1/2 is necessary.

In the present semi-classical context, we first fix a regular trajectory Γ arbitrarily, and view both (5) and (6) as defining the set of values of $\hbar$ for which our estimates will hold. When dealing with the geometric BS condition, (6), one has to make a correction of order $\hbar$ to the energy of the trajectory: we will prove that one must take

$$E_1 = E + \frac{\pi}{T}\hbar, \tag{7}$$

where E is the energy of Γ and T is its period. No such correction is necessary if $\hbar$ is determined by the physical BS condition, (5).

About the construction of ψ, one can distinguish two issues: the determination of the amplitude s (which will be a symbol in $1/\hbar$) and of the phase, f, in the ansatz (4). One should notice that in the expression (2) for the coherent states one has tacitly chosen a phase for them: multiplying the right-hand side of (2) by a complex number of modulus one would not change its localization properties. The choice of f [*see* (12)] will be explained in paragraph 5 in terms of geometric quantization. Briefly, the space that labels the coherent states with all possible choices of phases is not the plane, it is the pre-quantum circle bundle $\pi : G \to \mathbb{R}^2$ of the plane. [The choice (2) corresponds to a canonical trivializing global section $\mathbb{R}^2 \to G$.] Our construction should be thought of as taking place up on G, which is a "periodic" version of the Heisenberg group. *The meaning of the integral* (4) *is as a linear combination of the coherent states along a horizontal lift of the trajectory* Γ. The geometric BS condition is that such a horizontal lift be closed. Actually the amplitude s must satisfy certain transport equations that can be solved on sections of the Maslov line bundle of the horizontal lift of Γ, which is to say, on the space of anti-periodic functions on this horizontal lift.

In section 2 we state the main results and prove them in section 3. Section 4 is devoted to a result on the classical limit of matrix elements of an observable and in section 5 we explain the link between our construction and more geometrical objects.

2. THE MAIN RESULTS

We now turn to a precise statement of the main results. Let $a(x, \hbar D_x)$ be a pseudo-differential operator with Weyl symbol $a(x, \xi)$. Although there are several classes of symbols ([9], [8]) for which the results of this section hold, for simplicity we will concentrate here on the case where $a(x, \xi)$ is a real polynomial in (x, ξ). In this case $a(x, \hbar D_x)$ is a differential operator with polynomial coefficients, obtained by the Weyl rule: it is the operator $a(x, \hbar D_x)$ obtained from the polynomial $a(x, \xi)$ by symmetrization ordering in x and $\hbar D_x$. We suppose that $a(x, \hbar D_x)$ is self-adjoint and has discrete spectrum.

We denote by φ_t the Hamilton flow associated to $a(x, \xi)$:

$$\phi_t(x_0, \xi_0) = (x(t), \xi(t)), \tag{8}$$

with

$$\left.\begin{aligned} \dot{x}(t) &= \frac{\partial a}{\partial \xi}, & \dot{\xi}(t) &= -\frac{\partial a}{\partial x} \\ x(0) &= x_0, & \xi(0) &= \xi_0. \end{aligned}\right\} \tag{9}$$

Since we are in one dimensional the flow is integrable and each regular trajectory is periodic and fills up a connected component of the surface energy

$$\Omega_E = \{(x, \xi), a(x, \xi) = E\}. \tag{10}$$

Let Γ such a trajectory with period T, and let

$$\alpha = 1/2(\xi\, dx - x\, d\xi) \tag{11}$$

be a potential 1-form on $\mathbb{R}^2 = T^*(\mathbb{R})$, so that $-d\alpha$ is the symplectic form on $T^+(\mathbb{R})$. To each (x, ξ) we associate the vector of $L^2(\mathbb{R})$ defined by equation (2). To Γ and to a smooth function $s(t)$ of a real variable, anti-periodic of period T [that is $s(t+T) = -s(t)$] we associate the vector

$$\psi_{\Gamma, s} = \frac{1}{(2\pi\hbar)^{1/4}\sqrt{T}} \int_0^T s(t)\, e^{i\int_0^t ((x\dot{\xi} - \dot{x}\xi)/2\hbar)\, d\tau}\, \psi_{x(t), \xi(t)}\, dt \tag{12}$$

where $(x(t), \xi(t))$ is a parametrization of Γ as a trajectory of the Hamiltonian flow of a. This means that we choose once and for all an origin in Γ (Changing the origin simply multiplies our quasimode by a constant). The choice of this particular linear combination of coherent states has several motivations in terms of geometrical considerations exposed in [7]. Another motivation comes from the propagation of coherent states as shown in Hagedorn [3] and Litteljohn [6]: $\psi_{\Gamma, s}$ is the mean of the semiclassical evolution of a coherent state pined on Γ, conveniently weighted.

Remarks. – (1) The fact that $s(t)$ has to be T anti-periodic is going to become clear later on, and is related to the metaplectic representation [$s(t)$ depends on $\hbar$ as well]. (2) The phase factor inside the integral is crucial to the construction. Its geometry will be examined in paragraph 5.

THEOREM 2.1. — *With the previous assumptions, let* Γ *be a regular trajectory of energy* E, *period* T, *"action"* $A=\frac{1}{2}\int_{\Gamma}(\xi\,dx-x\,d\xi)$ *and flow* $(x(t),\ \xi(t))$, $t\in[0,\ T]$. *Then there exist a sequence of numbers* (c_k) *and a sequence of smooth* T *antiperiodic functions*, $(\alpha_k(t))$, *such that, if we let*

$$E_N(\hbar)=E+\sum_{n=1}^{N} c_k \hbar^k$$

and

$$s_N(t)=\sum_{k=0}^{N}\alpha_k(t)\,\hbar^k,$$

for each $N\in\mathbb{N}$, $N\geqq 1$, *there exists* $C_N>0$ *such that for all* $\hbar$ *small enough and of the form*

$$\hbar=\frac{A}{2\pi(n+1/2)},\qquad n\in\mathbb{N}^* \tag{13}$$

one has

$$\|\psi_{\Gamma,\,s_N}\|=1+O(\hbar) \tag{14}$$

and

$$\|(a(x,\ \hbar D_x)-E_N(\hbar))\,\psi_{\Gamma,\,s_N}\|\leqq C_N\,\hbar^{N+1}. \tag{15}$$

Moreover

$$\left.\begin{aligned} &c_1=0\quad \textit{so}\ E_1=E\\ &\textit{and}\\ &\alpha_0(t)=s_0(t)=\sqrt{\frac{1}{\sqrt{2}}(\dot{x}(t)-i\,\dot{\xi}(t))}\end{aligned}\right\} \tag{16}$$

[*in* (16) *the standard branch of the square root is implied*].

An easy (and standard) consequence of Theorem 1 is:

COROLLARY 2.2. — *With the same hypothesis, there exists eigenvalues* $\lambda(\hbar)$ *of* $a(x,\ \hbar D_x)$ *satisfying, for each* N *and all* $\hbar$ *of the form* (13)

$$|\lambda(\hbar)-E_N(\hbar)|\leqq C_N\,\hbar^{N+1}.$$

Moreover, if there exists a constant γ *such that*

$$\sigma(a(x,\ \hbar D_x))\cap[E-\gamma\hbar,\ E+\gamma\hbar]$$

has only simple eigenvalues, where $\sigma(a(x,\ \hbar D_x))$ *is the spectrum of* $a(x,\ \hbar D_x)$, *then, if* ψ_λ *is the eigenvector of eigenvalue* λ, *we have*

$$\forall N,\quad \exists D_N,\quad \|\psi_{\Gamma,\,s_N}-\psi_\lambda\|\leqq D_N\,\hbar^{N+1}. \tag{17}$$

Remarks. — 1. The example of the harmonic oscillator shows that (13) is in fact necessary; it is related to the Bohr-Sommerfeld quantization condition which says that the semi-classical eigenvalues are the energies E satisfying

$$A=\int_{\Gamma \text{ of energy } E} \xi\, dx=(n+1/2)\hbar.$$

2. The condition of non degeneracy in the Corollary is necessary and corresponds to having only one connected component Γ on Ω_E. Otherwise, as well known, the eigenvector may be a linear combination of ψ_{Γ_i}, with $\Gamma_i \subset \Omega_E$.

3. A simple computation shows that, if $a(x, \hbar D_x)=+\frac{\hbar^2 D_x^2+x^2}{2}$ (the harmonic oscillator), the corresponding approximation ψ_{Γ, s_1} is in fact exact.

Working with the geometric Bohr-Sommerfeld condition, (6), the result is the following:

THEOREM 2.3. — *In the statement of the previous theorem we can replace "anti-periodic" by "periodic", and* (13) *by*

$$\hbar=\frac{A}{2\pi n}, \qquad n\in\mathbb{N}^* \tag{18}$$

and the conclusions (14) *and* (15) *still hold. The formulae* (16) *should be replaced with*

$$\left.\begin{aligned} & c_1=\frac{\pi}{T} \quad \textit{so} \quad E_1=E+\hbar\frac{\pi}{T}, \\ \textit{and} \qquad & \\ & \alpha_0(t)=s_0(t)=e^{\pi i t/T}\sqrt{\frac{1}{\sqrt{2}}(\dot{x}(t)-i\dot{\xi}(t))} \end{aligned}\right\} \tag{19}$$

3. PROOF OF THEOREM 2.1

3.1 Bargman space and preliminaries

Since the proof is going to take place in the Bargman space, we first briefly recall some basic facts about it (*see* [1]). Bargman space, B, arises when one introduces a complex polarization on $T^*\mathbb{R}=\mathbb{R}^2$; our normalization is

$$z=\frac{1}{\sqrt{2}}(x+i\xi). \tag{20}$$

B is defined as:

$$\mathrm{B}=\left\{ g(z,\bar{z})=e^{-(z\bar{z}/2\hbar)}f(z),\ \text{with } f \text{ entire analytic and} \right.$$
$$\left. \int_{\mathbb{C}} |g(z,\bar{z})|^2\,dz\,d\bar{z}<+\infty \right\}.$$

B is a Hilbert space with reproducing kernel, which means that there exists a family of vectors $\rho_z \in \mathrm{B}$, indexed by $z \in \mathbb{C}$, such that

$$\forall g \in \mathrm{B},\quad g(z,\bar{z})=\int \overline{\rho_z(z')}\,g(z',\bar{z}')\,dz'\,d\bar{z}'. \tag{21}$$

Explicitly

$$\rho_z(z')=\frac{1}{\pi\hbar}e^{z'\bar{z}/\hbar}e^{-(|z|^2/2\hbar)}e^{-(|z'|^2/2\hbar)}. \tag{22}$$

There is a very natural relationship between the ρ_z and the coherent states $\psi_{x,\xi}$ defined earlier, given by:

LEMMA 3.1. – *The map* $\mathrm{U}:\mathrm{L}^2(\mathbb{R})\to\mathrm{B}$ *defined by the integral kernel*

$$\mathrm{A}(z,y)=\frac{1}{\sqrt{\hbar}(\pi\hbar)^{3/4}}e^{-((z^2+y^2+2\sqrt{2}\,zy+z\bar{z})/2\hbar)},$$

namely

$$(\mathrm{U}\psi)(z)\equiv\int \mathrm{A}(z,y)\,\psi(y)\,dy, \tag{23}$$

is a unitary map. Moreover,

$$\mathrm{U}(\psi_{x,\xi})=\sqrt{\pi\hbar}\,\rho_z \tag{24}$$

where z and (x,ξ) are related by (20).

For the proof of this see the original paper by Bargman [1]. Another well known "basic" fact about coherent states and Weyl symbols is:

LEMMA 3.2. – *Let $a(x,\hbar\mathrm{D}_x)$ be as in the hypothesis of Theorem* 1, *namely given by a polynomial Weyl symbol. Then*

$$(\psi_{x_2,\xi_2},\,a(x,\hbar\mathrm{D}_x)\,\psi_{x_1,\xi_1})$$
$$=\pi\hbar\left[e^{-(\hbar/2)\,\partial_{\bar{z}_1}\partial_{z_2}}a\left(\frac{z_2+\bar{z}_1}{\sqrt{2}},\frac{z_2-\bar{z}_1}{\sqrt{2i}}\right)\right]\rho_{z_1}(z_2,\bar{z}_2) \tag{25}$$

Proof. – First note that:

$$\mathrm{U}\, a(x, \hbar \mathrm{D}_x)\, \mathrm{U}^{-1} \rho_{z_1}(z_2, \bar{z}_2) = e^{-(|z_1|^2/2\hbar)} e^{-(|z_2|^2/2\hbar)} b(\bar{z}_1, z_2),$$

b being analytic in $\bar{z}_1$ and z_2. We first compute the diagonal term of b, namely:

$$\pi \hbar\, b(\bar{z}_1, z_1) = e^{|z_1|^2/\hbar} (\psi_{x_1, \xi_1}, a(x, \hbar \mathrm{D}_x) \psi_{x_1 \xi_1}).$$

A direct computation gives [2],

$$b(\bar{z}_1, z_1) = e^{-(\hbar/2)\partial_{z_1} \partial_{\bar{z}_1}} a\left(\frac{z+\bar{z}_1}{\sqrt{2}}, \frac{z-\bar{z}_1}{\sqrt{2}\, i} \right).$$

We obtain $b(\bar{z}_1, z_2)$ by analytic continuation in z_1. □

Remarks. – (i) The operator $e^{-(\hbar/2)\partial_{\bar{z}_1} \partial_{z_2}}$ makes sense when applied to a polynomial.

(ii) The function

$$h(z, \bar{z}) = a\left(\frac{z+\bar{z}}{\sqrt{2}}, \frac{z-\bar{z}}{\sqrt{2}\, i} \right) \tag{26}$$

is nothing but the Hamiltonian $a(x, \xi)$ written in complex coordinates. Since a was assumed to be a polynomial, h has a unique extension to a function $h(z_1, \bar{z}_2)$ of two complex variables, holomorphic in the first and anti-holomorphic in the second. This is the function to which the operator $e^{-(\hbar/2)\partial_{\bar{z}_1} \partial_{z_2}}$ is being applied.

To prove our Theorem we need to estimate

$$\| a(x, \hbar \mathrm{D}_x) - \mathrm{E}_\mathrm{N}) \psi_{\Gamma, s_\mathrm{N}} \|_{\mathrm{L}^2(\mathbb{R})} = \| \mathrm{U}(a(x, \hbar \mathrm{D}_x) - \mathrm{E}_\mathrm{N}) \mathrm{U}^{-1} \mathrm{U} \psi_{\Gamma, s_\mathrm{N}} \|_\mathrm{B}. \tag{27}$$

By the previous formulae

$$\mathrm{U} \psi_{\Gamma, s_\mathrm{N}} = \frac{1}{\mathrm{T}} \int_0^\mathrm{T} s_\mathrm{N}(t)\, e^{i \int_0^t ((x \dot{\xi} - \dot{x} \xi)/2\hbar) d\tau} \rho_{z(t)}\, dt$$

which, since $z(t) = (x(t) + i \xi(t))/\sqrt{2}$, equals

$$= \frac{1}{\mathrm{T}} \int_0^\mathrm{T} s_\mathrm{N}(t)\, e^{-\int_0^t ((z\dot{\bar{z}} - \dot{z}\bar{z})/2\hbar) d\tau} \rho_{z(t)}\, dt \tag{28}$$

We will obtain the estimate of (27) from a pointwise estimate of

$$|\{ \mathrm{U}(a(x, \hbar \mathrm{D}_x) - \mathrm{E}_\mathrm{N}) \mathrm{U}^{-1} (\mathrm{U} \psi_{\Gamma, s_\mathrm{N}}) \}(z_1)|.$$

Let us define

$$\mathrm{I}(z_1) := \mathrm{U}(a(x, \hbar \mathrm{D}_x) - \mathrm{E}_\mathrm{N}) \mathrm{U}^{-1} \mathrm{U} \psi_{\Gamma, s_\mathrm{N}}(z_1). \tag{29}$$

Then one computes:

$$I(z_1)=\left(\frac{\pi\hbar}{2}\right)^{1/4}\frac{1}{\sqrt{T}}\int_0^T s_N(t)\,e^{-\int_0^t((z\dot{\bar z}-\dot z\bar z)/2\hbar)\,d\tau}$$

$$\times U(a(x,\hbar D_x)-E_N)U^{-1}\rho_{z(t)}(z_1)\,dt=\left(\frac{\pi\hbar}{2}\right)^{1/4}\frac{1}{\sqrt{T}}$$

$$\times\int_0^T\left(e^{-(\hbar/2)\,\partial_{\bar z(t)}\partial_{z_1}}\left(a\left(\frac{z_1+\bar z(t)}{\sqrt{2}},\frac{z_1-\bar z(t)}{\sqrt{2}i}\right)-E_N\right)\right)s_N(t)\,e^{-\int_0^t((z\dot{\bar z}-\dot z\bar z)/2\hbar)\,d\tau}$$

$$\times\rho_{z(t)}(z_1)\,dt$$

which can finally be written as

$$I(z_1)=\frac{e^{-(|z_1|^2/2\hbar)}}{2^{1/4}(\pi\hbar^{2/4})\sqrt{T}}\int_0^T(A(z_1,\bar z(t))-E_N)\,s_N(t)$$

$$\times\left(e^{-\int_0^t((z\dot{\bar z}-\dot z\bar z)/2\hbar)\,dt}\right)e^{z_1\bar z(t)/\hbar}\,e^{-(|z(t)|^2/2\hbar)}\,dt \tag{30}$$

where

$$A(z_1,\bar z(t))=e^{-(\hbar/2)\,\partial_{\bar z(t)}\partial_{z_1}}\,a\left(\frac{z_1+\bar z(t)}{\sqrt{2}},\frac{z_1-\bar z(t)}{\sqrt{2}i}\right)$$

$$=h(z_1,\bar z(t))-\frac{\hbar}{2}\frac{\partial^2 h}{\partial z_1\,\partial\bar z}(z_1,\bar z(t))+\ldots$$

This integral is the object we must study. The proof of our Theorem consists in estimating $I(z_1)$ asymptotically by the (complex) stationary phase method [4], and showing that for a precise choice of a_N and E_N, $I(z_1)$ can be made of order $O(\hbar^{N+1})$. Let's begin by finding the critical points of the phase appearing in (30), which is

$$\phi(t,z_1)=-\int_0^t\frac{(z\dot{\bar z}-\dot z\bar z)}{2}\,d\tau+z_1\bar z(t)-\frac{z(t)\bar z(t)}{2}-\frac{z_1\bar z_1}{2} \tag{31}$$

where z_1 is regarded as a parameter. We get:

$$\frac{\partial\phi}{\partial t}=(z_1-z(t))\,\dot{\bar z}(t). \tag{32}$$

Since $\dot{\bar z}=\dfrac{\dot x-i\dot\xi}{\sqrt{2}}$ and Γ is a regular trajectory (and so $\dot x$ and $\dot\xi$ do not vanish simultaneously), $\dot{\bar z}\neq 0$. Hence the phase is stationary precisely for t such that $z_1=z(t)$. Accordingly, we will break the analysis in the two cases: z_1 on Γ or not. Stationary phase for $z_1\in\Gamma$ will dictate what the α_k

and c_k should be. The main difficulty of the proof is in dealing with the case $z_1 \notin \Gamma$: for each such z_1 $\mathrm{I}(z_1)$ is rapidly decreasing but *not* uniformly as z_1 approaches Γ. We will deal with that case in paragraph 3.4; we now look at the case $z_1 \in \Gamma$.

3.2. Estimates on Γ.

Let us first take $z_1 \in \Gamma$; then there is a critical point at $t = t_1$ such that $z(t_1) = z_1$. Moreover

$$\frac{\partial^2}{\partial t^2} \phi(t_1, z_1) = -|\dot{z}(t_1)|^2 < 0 \qquad \text{and} \qquad \phi(t_1, z_1) = -\int_0^{t_1} \frac{z\dot{\bar{z}} - \dot{z}\bar{z}}{2\hbar} d\tau$$

is purely imaginary.

LEMMA 3.3. — *There exist coefficients $\beta_k(t_1, z_1)$ such that* (30) *is equal to*

$$-\frac{e^{-(|z_1|^2/2\hbar)}}{\sqrt{\mathrm{T}}} \frac{(2\pi\hbar)^{1/4}}{|\dot{z}(t_1)|} e^{-\int_0^{t_1} (z\dot{\bar{z}} - \dot{z}\bar{z}/2\hbar)\, d\tau} \left[\sum_{k=1}^{\mathrm{N}} \hbar^k \beta_k(t_1, z_1)\right] \tag{33}$$

modulo $\hbar^{\mathrm{N}+1}$, uniformly on $z_1 \in \Gamma$. Moreover,

$$\beta_1(t_1, z_1) = \frac{\partial \alpha_0}{\partial t}(t_1) - \left(\frac{\ddot{\bar{z}}(t_1)}{2\dot{\bar{z}}(t_1)} + ic_1\right) \alpha_0(t_1). \tag{34}$$

Proof. — The proof is a direct application of [4] Theorem 7.7.5. The same theorem gives explicit formulas for the coefficients β_k. For example, the coefficient of $\hbar^0$ is

$$\left(a\left(\frac{z_1 + \bar{z}_1}{\sqrt{2}}, \frac{z_1 - \bar{z}_1}{\sqrt{2}\, i}\right) - \mathrm{E}\right) \alpha_0$$

equal to zero automatically in the present case when $z_1 \in \Gamma$. Hence the sum in (33) indeed starts with $k = 1$. A more involved calculation, which we will omit, proves (34). □

More generally, computing the coefficients given by the stationary phase method and re-ordering according to increasing powers of $\hbar$, one finds that the general form of $\beta_k(t_1, z_1)$ is

$$\beta_k(t_1, z_1) = \frac{\partial \alpha_{k-1}}{\partial t} - \frac{\ddot{\bar{z}}}{2\dot{\bar{z}}} \alpha_{k-1} - c_k \alpha_0(t_1) + g_k(t_1), \tag{35}$$

where $g_k(t_1)$ is a smooth function of α_{j-1}, c_j with $1 \leqq j \leqq k-1$. We will use this fact in a moment.

Now the α_j and c_j in our main Theorems are obtained by solving the equations

$$\beta_k(t_1, z_1) = 0, \qquad k = 1, 2, \ldots \tag{36}$$

and imposing the appropriate periodicity condition on the α_j. Solving $\beta_1(t, z_1)=0$ gives, by (34),

$$\alpha_0(t_1)=\sqrt{\dot{\bar{z}}}\,e^{itc_1} \tag{37}$$

up to a multiplicative constant.

Remarks. – (i) If we choose $c_1=0$, α_0 is T-antiperiodic,

(ii) If we choose $c_1=\frac{\pi}{T}$, then α_0 is T-periodic.

For general k, equations (35) and (36) and the method of variation of parameters give that

$$\alpha_{k-1}(t)=\alpha_0(t)\left[tc_k-\int_0^t \frac{g_k(s)}{\alpha_0(s)}ds\right]. \tag{38}$$

If we choose

$$c_k=\frac{1}{T}\int_0^T \frac{g_k(s)}{\alpha_0(s)}ds \tag{39}$$

then, arguing by induction, (38) is a periodic (resp. anti-periodic) function $\Leftarrow$ of t if α_0 is periodic (resp. anti-periodic).

In conclusion, in this subsection we have proved the existence of the α_k, c_k with the desired periodicity properties, such that $I(z_1)$ is $O(\hbar^N)$ for all N, uniformly on $z_1 \in \Gamma$.

3.3 The norm estimate

Next we prove the norm estimate (14). By definition,

$$\|\psi_{\Gamma, s_N}\|^2=\frac{1}{(2\pi\hbar)^{1/2}}\frac{1}{T}$$
$$\times\int_0^T dt\int_0^T dt'\, e^{i\int_{t'}^t ((x\dot{\xi}-\dot{x}\xi)/2\hbar)\,d\tau}\,\overline{s_N(t)}\,s_N(t')\langle \psi_{x(t), \xi(t)}, \psi_{x(t'), \xi(t')}\rangle, \tag{40}$$

which, by unitarity of U and (28), can be written as

$$\|\psi_{\Gamma, s_N}\|^2=\left(\frac{\pi\hbar}{2}\right)^{1/2}\frac{1}{T}\int_0^T dt\int_0^T dt'\,\overline{s_N(t)}\,s_N(t')\,e^{-\int_{t'}^t ((z\dot{\bar{z}}-\bar{z}\dot{z})/2\hbar)\,d\tau}\,\rho_{z(t')}(z(t)).$$

Introducing the expression (22) for the reproducing kernel, we finally obtain

$$\|\psi_{\Gamma, s_N}\|^2=\frac{1}{(2\pi\hbar)^{1/2}}\frac{1}{T}$$
$$\times\int_0^T dt\int_0^T dt'\,\overline{s_N(t)}\,s_N(t')\,e^{-\int_{t'}^t ((z\dot{\bar{z}}-\bar{z}\dot{z})/2\hbar)\,d\tau}$$
$$\times\, e^{z(t)\bar{z}(t')/\hbar}\,e^{-(|z(t')|^2/2\hbar)}\,e^{-(|z(t)|^2/2\hbar)}. \tag{41}$$

We now apply the method of stationary phase to this integral, for small $\hbar$. The critical points of the phase are given by the equation $t=t'$. A straightforward calculation shows that this is a non-degenerate critical manifold, and that the stationary phase method gives

$$\| \psi_{\Gamma, s_N} \|^2 = \frac{1}{T} \int_0^T \frac{|s_N(t)|^2}{|\dot{z}(t)|} dt + O(\hbar).$$

Since

$$s_N(t) = \sqrt{\dot{\bar{z}}(t)} + O(\hbar),$$

we obtain the desired result (14). This calculation explains the perhaps surprising normalizations appearing in the definition of ψ_{Γ, s_N}.

3.4. Estimates near Γ.

To obtain the norm estimate of Theorem 2.1, we need to estimate (30) with $z_1 \notin \Gamma$. We begin by noticing that the analysis can be restricted to a neighborhood Ω of Γ: away from Γ, $I(z_1)$ can be easily bounded (in modulus) by

$$\phi(z_1) = C \int_0^T e^{-(|z_1 - z(t)|^2/2\hbar)} dt. \tag{42}$$

On can derive from this that

$$\int_{\mathbb{C} \setminus \Omega} |\phi(z_1)|^2 dz_1 d\bar{z}_1 \leqq C'' e^{-(D/\hbar)}. \tag{43}$$

Thus we are left to estimate $I(z_1)$ in Ω. We will work first under the standard Bohr-Sommerfeld condition, and with $c_1 = 0$; we will point out at the end of this section how to change the argument to prove Theorem 2.3.

As already noticed, for $z_1 \in \Omega \setminus \Gamma$ fixed, (30) decreases rapidly with $\hbar$ but not uniformly in z_1, as z_1 approaches Γ. We will however establish the following key estimate:

THEOREM 3.4. – *As $\hbar \to 0$ along the* BS *values and with $z_1 \in \Omega$, one has an asymptotic expansion of* $I(z_1)$ *of the form*

$$I(z_1) \sim \frac{(2\pi\hbar)^{1/4}}{\sqrt{T}} \frac{e^{-|z_1|^2/2\hbar}}{Q(z_1)} e^{W(z_1)/\hbar} \left(\sum_{k=0}^{\infty} \hbar^k \tilde{\beta}_k(z_1) \right) \tag{44}$$

with $e^{W(z_1)/\hbar}$ and all the $\beta_k(z_1)$ analytic functions of $z_1 \in \Omega$, and Q *a smooth function of z_1 independent of $\hbar$ (equal to $|\dot{z}(t_1)|$ if $z_1 \in \Gamma$). The expansion is uniform on $z_1 \in \Omega$. Moreover*:

a) If $z_1 = z(t_1) \in \Gamma$,

$$e^{W(z_1)/\hbar} = e^{\int_0^{t_1} \bar{z}\dot{z}\, d\tau/\hbar}$$

and

$$\tilde{\beta}_k(z_1)=\beta_k(t_1, z_1).$$

b) $$\operatorname{Re}\left(W(z_1)-\frac{|z_1|^2}{2}\right)\leqq 0, \qquad \forall z_1\in\Omega.$$

c) $$\operatorname{Re}\left(W(z_1)-\frac{|z_1|^2}{2}\right)=0 \qquad \text{if} \quad z_1\in\Gamma.$$

As a consequence, if we pick the α_k, c_k so that $\beta_k(t_1, z_1)=0$ for $z_1\in\Gamma$, then $\tilde{\beta}_k$ must vanish on Ω, by analyticity. By integration over Ω this, together with (43), gives the desired norm estimates.

The idea behind the proof of Theorem 3.4 consists of deforming the contour of integration of (30) in order to "cath" a given z_1 as a critical point of the phase. This can be done since:

PROPOSITION 3.5. – *Under the Bohr-Sommerfeld condition,* (5), *the integral in the expression* (30) *for* $I(z_1)$, *is the integral over* Γ *of an analytic function defined in a neighborhood* Ω *of* Γ.

The analytic extension of the integrand is also an oscillatory function. After proving Proposition 3.5, we will define suitable deformations of the contour of integration and apply the method of complex stationary phase to obtain the estimates. This way of using analyticity in a neighborhood of Γ to obtain the estimate is inspired by methods used by Voros [11].

Let us begin the proof of Proposition 3.5. Re-write the integral (30) in the form

$$I(z_1)=\frac{1}{(2\pi\hbar)^{1/4}}\,\frac{1}{\sqrt{T}}\,I'(z_1), \tag{45}$$

where $I'(z_1)=\displaystyle\int B(z_1, t)\, e^{\phi(t, z_1)/\hbar}\,dt$ with

$$B(z_1, t)=\left[e^{[-(\hbar/2)\,\partial_{z_1}\partial_{z(t)}]}\left(a\left(\frac{z_1+\bar{z}(t)}{\sqrt{2}}, \frac{z_1-\bar{z}(t)}{\sqrt{2}\,i}\right)-E_N\right)\right]s_N(t)$$

and

$$\phi(t, z_1)=-\int_0^t\frac{z\dot{\bar{z}}-\dot{z}\bar{z}}{2}\,d\tau+z_1\bar{z}(t)-\frac{|z|^2}{2}-\frac{|z_1|^2}{2}$$

$$=-\int_0^T z\dot{\bar{z}}\,d\tau+z_1\bar{z}(t)-\frac{|z_1|^2}{2}. \tag{46}$$

(45) can then be rewritten as

$$I'(z_1)=\int_\Gamma\frac{B(z_1, t)}{\dot{\bar{z}}}\,e^{\phi(t, z_1)/\hbar}\,d\bar{z}$$

where, implicitly, $t=t(\bar{z}_1)$ is the inverse of the function $\bar{Z}(t)$. In order to obtain the analytic continuation of the integrand, we define a multi-valued function $S(\bar{z})$ as the solution of the equation

$$h(S'(\bar{z}), \bar{z})=E \tag{47}$$

whose derivative restricted to Γ is

$$S'|_{\Gamma}=z, \tag{48}$$

where

$$h(z, \bar{z})=a\left(\frac{z+\bar{z}}{\sqrt{2}}, \frac{z-\bar{z}}{\sqrt{2}i}\right)$$

and E is the value of a on Γ.

The function h is nothing but the symbol of $a(x, \hbar D_x)$ on the complex coordinates $z=\dfrac{x+i\xi}{\sqrt{2}}$ and $\bar{z}=\dfrac{x-i\xi}{\sqrt{2}}$, while (47) is the Hamilton-Jacobi equation defining a canonical transformation that would change h into a function of $\bar{z}$ alone.

LEMMA 3.6. – *If* Γ *is a regular trajectory, the problem* (47, 48) *has a unique* (*multi--valued*) *solution analytic in an annular neighborhood* Ω *of* Γ. *The derivative* $G=S'$ *is single-valued on* Ω *and*

$$\int_{\Gamma} G(\bar{z})\, d(\bar{z})=i\mathrm{A}, \tag{49}$$

where $\mathrm{A}=\dfrac{1}{2}\displaystyle\int_{\Gamma}\xi\, ds-x d\xi$ *is the action of* Γ.

Proof. – We first show that the equation

$$h(G(\bar{z}), \bar{z})=E$$

has a unique single-valued analytic solution in an anular neighborhood of Γ and satisfying

$$G(\bar{z})=z \quad \text{on } \Gamma. \tag{50}$$

$G(\bar{z})=z$ is obviously a solution on Γ by definition of $\Gamma=\{z, h(z, \bar{z})=E\}$. Local existence and analyticity in a neighborhood of Γ is given by the implicit function theorem, since

$$\frac{\partial}{\partial z}h(z, \bar{z})=\dot{\bar{z}}\neq 0 \qquad \text{(given that } \Gamma \text{ is regular).}$$

Being equal to z on Γ, the solution is necessarily single-valued. To show (49), use (50) plus the identity

$$z\, d\bar{z}=\frac{1}{2}(d|z|^2+i(\xi\, dx-x d\xi)). \quad \square$$

Proof of Proposition 3.5. – Since

$$\left(\frac{d}{dt}\phi(t, z_1)\right) = -z\dot{\bar{z}} + z_1\dot{\bar{z}} = -S'(\bar{z})\dot{\bar{z}} + z_1\dot{\bar{z}},$$

a primitive of $G(\bar{z})$, $S(\bar{z})$, can be chosen so that

$$\phi(t, z_1) = -S(\bar{z}(t)) + z_1\bar{z}(t) - \frac{|z_1|^2}{2}. \tag{51}$$

Moreover, since $\alpha_k(t) = \alpha_0(t)\tilde{\alpha}_k(t)$,

$$s_N(t) = \sqrt{\dot{\bar{z}}}\,\tilde{a}_N(t)$$

where $\tilde{a}_N(t)$ is T-periodic. Writting

$$B(z_1, t) = \sqrt{\dot{\bar{z}}}\,g(z_1, \dot{\bar{z}}(t)), \tag{52}$$

with

$$g(z_1, \dot{\bar{z}}(t)) = \left[e^{[-(\hbar/2)\,\partial_{z_1}\,\partial_{\bar{z}(t)}]}\left(a\left(\frac{z_1+\bar{z}(t)}{\sqrt{2}}, \frac{z_1-\bar{z}(t)}{\sqrt{2}\,i}\right) - E_N\right)\right]\tilde{a}_N(t), \tag{53}$$

and noting that, by the equations of motion

$$\dot{\bar{z}}(t) = \frac{\partial h}{\partial z}(z(t), \bar{z}(t)) = \frac{\partial h}{\partial z}(S'(\bar{z}(t)), \bar{z}(t)),$$

we see that we can write I′ as:

$$I' = \int_\Gamma g(z_1, \bar{z})\frac{e^{-(S(\bar{z})+z_1\bar{z}-(z_1\bar{z}_1/2))/\hbar}}{\sqrt{\partial_z h(S'(\bar{z}), \bar{z})}}\,d\bar{z}. \tag{54}$$

From this expression we see that Proposition 3.5 will follow from the following:

LEMMA 3.7. – *If* Γ *satisfies the Bohr-Sommerfeld condition* (13),

$$\frac{e^{-(S(\bar{z})+z_1\bar{z}-(z_1\bar{z}_1/2))/\hbar}}{\sqrt{\partial_z h(S'(\bar{z}), \bar{z})}}$$

is a single-valued analytic function of $\bar{z}$ *globally for* z *in a neighborhood of* Γ.

Proof. – The function $S(\bar{z})$ is analytic in $\bar{z}$, so it is enough to see that

$$\frac{e^{-(S(\bar{z})/\hbar)}}{\sqrt{\partial_z h(S'(\bar{z}), \bar{z})}} \tag{55}$$

is a single-valued function. We see from (49) that, under the BS condition,

$$\oint S'(\bar{z})\,d\bar{z} = 2\pi i\left(n+\frac{1}{2}\right)\hbar, \tag{56}$$

where $\oint$ means integration over any contour homologous to Γ. It follows that, after z winds around once in the annulus Ω, $e^{-(S(\bar{z})/\hbar)}$ changes sign, and once can see that the square root in (55) changes sign also, since on Γ

$$\partial_z h(S'(\bar{z}), \bar{z}) = \dot{\bar{z}}$$

and the velocity vector $\dot{z}$ winds around the origin once as t ranges from zero to T. □

This finishes the proof of Proposition 3.5. By the Cauchy formula we obtain:

COROLLARY 3.8. – *Under the Bohr-Sommerfeld condition, the integral* I′ *given by* (54) *doesn't depend on the contour inside the neighborhood defined by Lemma* 1.

Having proved Proposition 3.5, we now define a suitable family of deformations of Γ yielding contours for which I, defined as an integral on this contour, will have a given z_1 as a critical point. Such deformations will be given by a family of ODEs parametrized by z_1. We first notice that the function S′ has an inverse S'^{-1} on $\Omega : S'^{-1}$ is analytic and single valued on Ω since it satisfies

$$h(z, S'^{-1}(z)) = E$$

on Ω, and

$$S'^{-1}(z) = \bar{z}$$

on Γ. Let us now consider the equation

$$\dot{z}_1(t) = h_{\bar{z}}(z_1, S'^{-1}(z)), \qquad z_1(0) = z_1, \tag{57}$$

where

$$h_{\bar{z}}(z_1, \bar{z}'_1) = \frac{\partial}{\partial \bar{z}'_1} h(z_1, z'_1).$$

If $z_1 \in \Gamma$, $z_1(t)$ is a trajectory of the original Hamiltonian flow, since $S'^{-1}(z_1) = \bar{z}_1$ on Γ. Consider now the curve:

$$\bar{\zeta}(t) = S'^{-1}(z_1(t)) \tag{58}$$

LEMMA 3.9.

$$\dot{\bar{\zeta}}(t) = h_z(S'(\bar{\zeta}), \bar{\zeta}) \qquad \textit{and} \qquad \bar{z}(0) = S'^{-1}(z_1)$$

where $h_z(z, \bar{z}') = \dfrac{\partial h}{\partial z}(z, \bar{z}')$. *Moreover*, $\bar{\zeta}(t)$ *and* $z_1(t)$ *are both periodic flows with the same period* T *as* Γ.

Proof. – The first part follows easily from the chain rule. Since $h_z(S'(\bar{\zeta}), \bar{\zeta})$ and $h_{\bar{z}}(z_1, S'^{-1}(z_1))$ are analytic functions in $\bar{\zeta}$ and z_1, they induce analytic flows. In particular if $\phi(\tau, \bar{\zeta}(0)) = \bar{\zeta}(t)$ by (58), then $\phi(T, \cdot)$

is analytic in $\bar{\zeta}(0)$. Moreover, $\phi(\mathrm{T}, \bar{\zeta})=\bar{\zeta}$ if $\bar{\zeta} \in \Gamma$. By analyticity, $\phi(\mathrm{T}, \bar{\zeta})=\bar{\zeta}$ on Ω. The same argument is valid for $z_1(t)$. □

Proof of Theorem 3.4. – Let us call Γ_1 the curve defined by (58) for $0 \leqq t \leqq \mathrm{T}$. Then

$$\mathrm{I}(z_1)=\int_{\Gamma_1} \dots \tag{59}$$

The fact that $\mathrm{I}(z_1)$ has a critical point at $t=t_1$ is by the definition of $z_1(t)$. Conditions (*b*) and (*c*) follow from simple calculations. The important fact is (44), which we obtain by applying the method of stationary phase to the integral (59). We need to show: (i) analyticity of the coefficients $\tilde{\beta}$, and (ii) uniformity on $z_1 \in \Omega$. Both of these follow from the method of stationary phase itself, and the fact that the flows $z_1(t)$ and $\zeta(t)$ depend analytically on the initial condition z_1. The coefficients $\tilde{\beta}$ are the result of applying differential operators with analytic coefficients to

$$g(z_1, \bar{\zeta}(t)) \sqrt{h_z(\mathrm{S}'(\bar{\zeta}(t)), \bar{\zeta}(t))},$$

and so are analytic themselves. Uniformity on Ω follows because the constants appearing in the method of stationary phase are bounded in the C^∞ topology of the phase. Condition (*a*) follows from the fact that when $z_1 \in \Gamma$ (59) reduces to (30). □

Proof of Theorem 2.3. – Theorem 2.3 is proved in exactly the same fashion, with the following minor changes. The geometric BS condition translates into the fact that the exponential

$$e^{-(\mathrm{S}(\bar{z})+z_1 \bar{z}-(z_1 \bar{z}_1/2))/\hbar}$$

is now a single valued analytic function of $\bar{z} \in \Omega$. As noticed in remark (ii) following (37) and in the remarks following (38), choosing $c_1=\dfrac{\pi}{\mathrm{T}}$ makes the amplitude T periodic, and hence its analytic extension single-valued. So, again, $\mathrm{I}(z_1)$ is the contour integral of a single-valued analytic function on Ω, and the rest of the proof is identical. □

4. CLASSICAL LIMIT OF MATRIX ELEMENTS

In this section we are concerned by the following problem. Consider a pseudo-differential operator of the form described before $a(x, \hbar \mathrm{D}_x)$. Its symbol $a(x, \xi)$ constitutes a one dimensional classical Hamiltonian; therefore, in a neighborhood of a regular trajectory, Ω it possesses a system of action-angle variables: there exists a canonical transformation C

$$\begin{aligned} \mathrm{C}:\ \Omega &\to \mathbb{R}^+ \times \mathrm{S}^1 \\ (x, \xi) &\mapsto (\mathrm{A}, \varphi) \end{aligned}$$

such that $a(\mathrm{C}^{-1}(\mathrm{A}, \varphi)) = \alpha(\mathrm{A})$, a function independent of φ. The results of previous sections show that to each value of the action A of the form

$$\mathrm{A} = (h + 1/2)\hbar$$

is associated an appropriate eigenvector localized precisely on the trajectory of energy $\alpha(\mathrm{A})$.

Let us now consider another operator $b(x, \hbar \mathrm{D}_x)$. We would like to compute semi-classically the following matrix elements:

$$\mathrm{C}_{n, m} = (\psi_n, b(x, \hbar \mathrm{D}_x)\psi_m) \tag{60}$$

where ψ_n is the (normalized) eigenvector of $a(x, \hbar \mathrm{D}_x)$ of quantum number n, concentrated on the trajectory of action $\mathrm{A} = (n + 1/2)\hbar$.

THEOREM 4.1. – *Let $a(x, \hbar \mathrm{D}_x)$ and $b(x, \hbar \mathrm{D}_x)$ be as above and let (A, φ) be the action-angle variables of $a(x, \xi)$. Let*

$$\alpha(\mathrm{A}, \varphi) = \alpha(\mathrm{A}) \qquad \textit{and} \qquad \beta(\mathrm{A}, \varphi) = b(\mathrm{C}^{-1}(\mathrm{A}, \varphi))$$

the symbols of $a(x, \hbar \mathrm{D}_x)$ and $b(x, \hbar \mathrm{D}_x)$ expressed in the variables (A, φ). Let

$$\beta(\mathrm{A}, \varphi) = \frac{1}{\sqrt{2\pi}} \sum_k \beta_k(\mathrm{A}) e^{ik\varphi} \tag{61}$$

where

$$\beta_k = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{S}_1} \beta(\mathrm{A}, \varphi) e^{ik\varphi} d\varphi.$$

Then, under the hypothesis of Corollary 2.2,

$$|(\psi_n, b(x, \hbar \mathrm{D}_x)\psi_m) - \beta_{m-n}(\mathrm{A})| = O(\hbar) \tag{62}$$

as $n, m \to \infty$ with $|m - n|$ bounded and $\hbar = \dfrac{\mathrm{A}}{h + 1/2}$.

This theorem says that, in the classical limit, the matrix elements $\mathrm{C}_{n, m}$ tend to the Fourier coefficients on the symbol of $b(x, \hbar \mathrm{D}_x)$ expressed on the action-angle variables of the symbol of $a(x, \hbar \mathrm{D}_x)$.

The following is a small variation of Theorem 2:

THEOREM 4.2. – $\mathrm{C}_{n, m} = (\psi_n, b(x, \hbar \mathrm{D}_x)\psi_m)$ *has an asymptotic expansion of the form*:

$$\sum_{l=0}^{\infty} \beta_{n-m}^{l} \hbar^l$$

as (n, m) *tend to infinity as in the previous theorem all* β^l_{n-m} *computable in terms of* $\beta^0_{n,m}=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{S_1}\beta(A, \varphi)\,e^{-i(n-m)\varphi}\,d\varphi$.

We will sketch the proof. Since we are under the hypothesis of Corollary 2.2, the eigenvectors ψ_n can be approximated by a vector of the form $\psi_{\Gamma, s}$ given by (12) with s given by (16), namely

$$\alpha_0(t)=\sqrt{\frac{\dot{x}(t)-i\dot{\xi}(t)}{\sqrt{2}}}$$

with $(x(t), \xi(t))$ is the flow on the trajectory Γ satisfying $\displaystyle\int_\Gamma x\,d\xi=(n+1/2)\hbar$. The same is valid for ψ_m with $\alpha'_0(t)=\sqrt{\dfrac{\dot{x}'(t)-i\dot{\xi}'(t)}{\sqrt{2}}}$, $(x'(t), \xi'(t))$ being the flow on Γ' of action $(m+1/2)\hbar$. Then $C_{n,m}$ is expressed, using Lemma 3.2 as an integral of the form:

$$\begin{aligned}C_{n,m}=\int_{z(t)\in\Gamma}\int_{z'(t)\in\Gamma'} & a(t)\,\bar{a}(t')\,b(t, t')\,e^{z(t)\,\bar{z}'(t')/\hbar}\,e^{-(|z(t)|^2/2\hbar)}\\ & \times\ldots e^{-(|z'(t')|^2/2\hbar)}\,e^{-\int_0^t(z\dot{\bar{z}}-\dot{z}\bar{z}/2\hbar)\,d\tau-\int_{t'}^0(z'\dot{\bar{z}}'-\dot{z}'\bar{z}'/2\hbar)\,d\tau'}\,dt\,dt' \qquad (63)\end{aligned}$$

where

$$b(t, t')=e^{-\hbar\,\partial_{z(t)}\partial_{\bar{z}'(t')}}\,b\left(\frac{z(t)+\bar{z}'(t')}{2}, \frac{z(t)-\bar{z}'(t')}{2i}\right).$$

By the same method used to prove Theorem 2.1, the left hand side of (63) can be replaced by an integral of the form

$$C_{n,m}=\int_\Gamma\int_{\Gamma'} e^{z\bar{z}/\hbar}\,e^{(-\phi(z)/\hbar)-(\overline{\phi'(z')}/\hbar)}\,b(z, \bar{z}')\,\frac{dz\,d\bar{z}'}{\sqrt{\dot{z}\dot{\bar{z}}'}} \qquad (64)$$

when $z\in\Gamma$, $z'\in\Gamma'$ and ϕ and ϕ' satisfy the Hamilton-Jacobi equations associated to energies $\alpha((n+1/2)\hbar)$ for ϕ and $\alpha((m+1/2)\hbar)$ for ϕ', namely

$$h\left(\frac{d\phi}{dz}(z), z\right)=\alpha((n+1/2)\hbar)$$

and

$$h\left(\frac{d\phi'(z')}{dz'}, z'\right)=\alpha((m+1/2)\hbar). \qquad (65)$$

Since for $\hbar$ small enough Γ and Γ' are close, we can replace in (64) the contour Γ' by Γ. Moreover we get from (65) that

$$\left(\frac{d\phi}{dz}-\frac{d\phi'}{dz}\right)\frac{\partial a}{\partial\bar{z}}=(n-m)\,\hbar\,\frac{\partial\alpha}{\partial A}+O(\hbar^2).$$

But $\frac{\partial h}{\partial \bar{z}} = \dot{z}$ and $\frac{\partial \alpha}{\partial A} = \omega(A) = \frac{1}{T}$, where T is the period of Γ. This means that

$$\left(\frac{d\phi}{dz} - \frac{d\phi'}{dz}\right)\dot{z} = (n-m)\hbar\frac{1}{T} + O(\hbar^2).$$

Then:

$$C_{n,m} = \int_{z(t)\in\Gamma}\int_{\bar{z}(t')\in\Gamma} e^{z(t)\bar{z}(t')/\hbar}\, e^{-(-\phi(z(t))/\hbar) - (\overline{\phi(z(t'))}/\hbar)}\, b(t, t')\ldots dt\, dt' + O(\hbar^2).$$

The stationary phase method gives now, in the integration over dt', a critical point at t' defined by

$$(z - \overline{\phi(z(t'))}) = 0$$

i.e. $t' = t$. A little computation gives

$$C_{n,m} = \frac{1}{T}\int_0^T e^{i(n-m)\omega(A)t}\, b(t, t)\, dt + 0(\hbar).$$

To finish the proof notice that $\frac{t}{T} = \varphi$, and that the first term of $b(t, t)$ is

$$b\left(\frac{z(t)+\bar{z}(t)}{\sqrt{2}}, \frac{z(t)-\bar{z}(t)}{i\sqrt{2}}\right)$$

Theorem 4.2 can be proved by the method of stationary phase, computing explicitly the terms in the expansion.

Remark. — In the case of the harmonic oscillator, the formula given by Theorem 4.2 is well known in physics [5].

5. GEOMETRIC INTERPRETATION AND GENERALIZATIONS

In this section we show how our construction is related to the theory of Hermite distribution as pointed out in the end of the introduction. This will give a geometrical interpretation of the BS condition, and explain the relationship of our construction to the Heisenberg group.

Let's begin, in analogy with the previous considerations, with a closed simple curv, $\gamma_0 \subset \mathbb{R}^2$. Denote its action by A:

$$A = \frac{1}{2}\int_{\gamma_0} p\,dq - q\,dp.$$

Let

$$P = \mathbb{R}^2 \times S^1,$$

(where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$) be the trivial principal circle bundle endowed with the connection form

$$\alpha_A = d\theta + \frac{\pi}{A}(pdq - qdp). \tag{66}$$

Then P is a pre-quantum circle bundle of $\mathbb{R}^2$ with the symplectic form

$$\Omega_A = -\operatorname{curv}(\alpha_A) = \frac{2\pi}{A} dq\, dp.$$

The choice of the connection form is so that the curve γ_0 has a closed horizontal lift, $\gamma \subsetneq P$.

The space P is a "reduced" version of the Heisenberg group,

$$\mathbb{H} = \mathbb{R}^2 \times \mathbb{R}.$$

More specifically, consider on $\mathbb{H}$ the group law

$$\Leftarrow \quad (p, q, t)\cdot(p', q', t') = (p+p', q+q', t+t'+(pq'-qp')/2).$$

$\mathbb{H} \to \mathbb{R}^2$ is the natural pre-quantum $\mathbb{R}$ bundle of $(\mathbb{R}^2, dqdp)$ if we put on $\mathbb{H}$ the connection form

$$\alpha = dt + \frac{1}{2}(pdq - qdp).$$

LEMMA 5.1. – *P is the quotient of* $\mathbb{H}$ *by the subgroup of the center*

$$Z_A = \{(0, 0, kA);\ k \in \mathbb{Z}\},$$

and the connection α_A *is the quotient of* α.

Proof. – The quotient $\mathbb{H}/Z_A$ only takes place in the $\mathbb{R}$ component; since the variable θ in (66) is 2π periodic, it is locally related to the variable t by

$$\theta = \frac{2\pi}{A} t. \tag{67}$$

It is clear that the connection form, α_A, of the quotient connection is a multiple of α. By the condition $\alpha_A(\partial_\theta) = 1$, and (67), we get

$$\frac{2\pi}{A}\left(dt + \frac{1}{2}(pdq - qdp)\right) = \alpha_A. \quad \square$$

We now recall a few well-known facts about the geometry and harmonic analysis of the Heisenberg group (a good reference for this material is the book by G. Folland [12]). The Heisenberg group is the boundary of a strictly pseudoconvex domain (the Siegel upper half plane). Let us denote by $\mathscr{H}$ the Bergman space of square-integrable functions on $\mathbb{H}$ which are non-tangential boundary values of holomorphic functions on the Siegel upper half plane. ($\mathbb{H}$ is an unimodular group with invariant measure *dqdpdt*.) The group $\mathbb{H}$ is represented on $\mathcal{H}$, the representation being induced by the left action of $\mathbb{H}$ on the Siegel upper half plane. This representation is reducible: it is in fact a direct integral of the Heisenberg representations of $\mathbb{H}$:

$$\mathcal{H} = \int_{\hbar=0}^{\infty} \mathcal{H}_\hbar \, d\hbar,$$

where $\mathcal{H}_\hbar$ is the only irreducible representation of $\mathbb{H}_\hbar$ where $-i\partial/\partial t$ is represented by the operator "multiplication by $\hbar$". Notice that the representations $\mathcal{H}_\hbar$ with $\hbar = k 2\pi/A$, $k = 1, 2, \ldots$ are precisely those that pass to the quotient $P = \mathbb{H}/Z_A$, and the quotient representations are *faithful*.

More precisely, P itself is the boundary of a strictly pseudoconvex domain, $\mathcal{D}$, and its Bergman space is

$$\mathcal{H}_P = \bigoplus_{k=0}^{\infty} \mathcal{H}_{k\, 2\pi/A}.$$

The domain $\mathcal{D}$ has two realizations: (i) as the quotient of the Siegel upper-half plane by Z_A, and (ii) as

$$\mathcal{D} = \{(z, w) \in \mathbb{C} \times \mathbb{C};\ |z| \leqq e^{-|w|^2/2}\}.$$

The latter description of $\mathcal{D}$ is as the unit disk bundle of the dual of the holomorphic hermitian line bundle over $\mathbb{R}^2 = \mathbb{C}$ with curvature *dqdp*.

Our analysis is based on the micro-local structure of the Szegö projector:

$$\Pi:\quad L^2(P) \to \mathcal{H}_P.$$

We will see that a fundamental geometric object associated with Π is the following symplectic submanifold of $T^* P$:

$$\Sigma = \{(x, r(\alpha_A)_x;\ x \in P,\ r > 0\}.$$

THEOREM 5.2. – *The wave-front set of the Schwartz kernel of* Π *is equal to*

$$\Sigma \overset{\triangle}{\times} \Sigma = \{(\sigma, \sigma);\ \sigma \in \Sigma\}.$$

More precisely, Π *is a Fourier integral operator of Hermite type associated with this isotropic submanifold of* $T^* P \times T^* P$, *in the sense of Boutet de Monvel and Guillemin* [13].

In [13], Boutet de Monvel and Guillemin construct a symbol calculus of Fourier integral operators of Hermite type. Although we won't go here into this theory, we will indicate however how our results can be recast into a symbolic calculation of Hermite distributions. Consider the *Borel sum* of the states ψ_γ which we have defined:

$$\Theta := \sum_{k=1}^{\infty} e^{ik\theta} \psi_{\gamma, k}.$$

First we claim the following:

THEOREM 5.3. – *Θ is a distribution of the form $\Pi(u)$, where u is a distribution on* P *conormal to a horizontal lift γ of γ_0. The wave-front set of Θ is contained in*

$$C_\gamma := \{ (x, r(\alpha_A)_x; \; x \in \gamma, \; r > 0 \}.$$

In fact, Θ is a Hermite distribution associated with the coisotropic submanifold C_γ.

Remark. – Let u be a distribution on P conormal to γ. The fact that γ is horizontal means that the connection form α_A is conormal to it, and Theorem 5.2 implies that the projection $\Pi(u)$ has wave-front set in C_γ.

Now for every value of $\hbar$ of the form $\hbar = k\,2\pi/A$, the Schrödinger operator $a(x, \hbar D_x)$ can be realized in $\mathscr{H}_\hbar$. Denote the resulting operator A_k. One can then form the "Borel sum" of these, *i. e.* the operator

$$B = \bigoplus_{k=1}^{\infty} A_k$$

acting on $\mathscr{H}$. Our second claim is that B *is a zeroth order Toeplitz operator on* P, that is of the form

$$B = \Pi Q \Pi,$$

where Q is a zeroth order pseudodifferential operator on P which can be chosen to commute with Π and with D_θ. It is not hard to see that in constructing our quasi-mode we are solving the equation

$$[B - E(D_\theta)](\Theta) \in C^\infty(P), \tag{68}$$

where E is an unknown real classical symbol of order zero. This equation can be solved symbolically, by an iterative procedure. At every stage one has to solve a transport equation along γ; the corrections to the energy (which are the terms in the asymptotic expansion of E) are the zeroth Fourier coefficients of the right-hand side of the equation; they must be subtracted to ensure global solvability of the transport equation. In this guise our method clearly generalizes to many other settings. Details will appear in [7].

We would like to conclude by two generalizations of the preceding construction. In the case of a multidimensional analytic integrable hamiltonian the preceding proof will apply and give a construction of quasi modes associated to invariant tori. Using the more general theory of Hermite distributions [2], a similar construction is possible for stable periodic trajectories of (non-integrable) multidimensional hamiltonians. Both cases will be presented in [7].

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