ANNALES DE L'I. H. P., SECTION A

GREGORY B. WHITE

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Annales de l'I. H. P., section A, tome 59, nº 3 (1993), p. 269-300
<http://www.numdam.org/item?id=AIHPA_1993__59_3_269_0>

Ann. Inst. Henri Poincaré,

Vol. 59, n° 3, 1993, p. 269-300. *Physique théorique*

Holomorphy of the Dirac resolvent with relatively form bounded potentials

by

Gregory B. WHITE (*)

Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A.

ABSTRACT. — We derive an explicit power series expansion of the Dirac resolvent in a complex variable κ $\left(\text{analogous to } \frac{1}{c}\right)$, valid for potentials which are form bounded with respect to the momentum operator. The zero-order term of this series is the associated Pauli-Schrödinger resolvent (nonrelativistic limit) times a projection. We give estimates for the radius of convergence based on the form boundedness condition. Convergence of the series for $|\kappa|$ small implies that the resolvent is holomorphic at $\kappa=0$ and leads to the standard results on holomorphy of isolated eigenvalues. For Coulomb-like potentials, we give conditions under which the radius of convergence is large enough to encompass the usual perturbed Dirac resolvent ($\kappa=1$). We also derive lower bounds on the associated Pauli-Schrödinger operators from the form boundedness condition. We define our Dirac operator as a form sum, but in order to make this definition for $\kappa \notin \mathbb{R}$, we must first extend the notion of form sum to a class of non-self-adjoint operators including the free Dirac operator. We call this the class of *formable operators* and give conditions under which the form sum of such operators is closed and densely defined.

RÉSUMÉ. — Nous décrivons un développement explicite en série de puissances d'une variable complexe κ $\left(\text{analogue à } \frac{1}{c}\right)$ de la résolvante de

(*) On leave from Occidental College, Los Angeles, CA 90041, U.S.A.

l'opérateur de Dirac. Ce développement est valable pour des potentiels qui sont bornés par l'opérateur de moment au sens des formes. Le terme d'ordre zéro est la résolvante associée de Pauli-Schrödinger (limite non relativiste) multipliée par une projection. Nous donnons une estimation du rayon de convergence basée sur la borne des formes. La convergence de la série pour $|\kappa|$ petit implique que la résolvante est holomorphe à $\kappa=0$ et conduit au résultat standard d'holomorphie des valeurs propres. Pour les potentiels du type de Coulomb nous donnons des conditions pour que le rayon de convergence soit assez grand pour contenir la résolvante de Dirac perturbée habituelle ($\kappa=1$). Nous prouvons aussi des bornes inférieures au sens des formes pour l'opérateur de Pauli-Schrödinger associé. Nous définissons notre opérateur de Dirac comme une somme de formes mais dans le cas $\kappa \notin \mathbb{R}$ nous devons tout d'abord étendre la notion de somme de forme à une classe d'opérateurs non auto-adjoints contenant l'opérateur de Dirac libre. Nous appelons cette classe la classe des *opérateurs formables* et nous donnons des conditions pour que la forme somme de tels opérateurs soit fermée et définie sur un domaine dense.

1. INTRODUCTION

In our study of the Dirac operator we use the abstract setting introduced by Hunziker [H] and Cirincione and Chernoff [CC]. Veselič [V1] was the first to show that the Dirac resolvent converged to the Schrödinger (pseudo)resolvent in the non-relativistic limit ($c \to \infty$) and thereby to show holomorphy of the Dirac resolvent in $\frac{1}{c}$. Hunziker [H], Gesztesy, Grosse, and Thaller [GGT], and Grigore, Nenciu, and Purice [GNP] generalized these results to the case of non-zero magnetic potentials and derived stronger results on the analyticity of eigenvalues. In [GGT] an explicit power series expansion of the Dirac resolvent was given, the zero-order term of which is the Pauli-Schrödinger resolvent times a projection. They also gave a sketch of a method for estimating the radius of convergence of the series. All of these results are valid under the assumption of potentials which are relatively bounded with respect to the momentum operator.

In this paper we generalize the results cited above to the case of potentials which are relatively *form* bounded with respect to the momentum operator. In particular, we give a generalization of the power series expansion of the Dirac resolvent, valid for relatively form bounded potentials. Further, we give explicit estimates of the radius of convergence based on the form boundedness condition and determine conditions under which the usual Dirac resolvent is within the region of convergence.

We complexify the Dirac operator by introducing a complex parameter κ. This has the same effect as allowing the "speed of light" c to become a complex variable and puts us in position to show holomorphy of the resolvent as a function of κ. Using κ instead of $1/c$ as the complex parameter simplifies dimensional analysis of equations and inequalities by allowing us to group units of energy together. At the same time, we get two easy reference points; the usual Dirac operator or resolvent corresponds to $\kappa=1$ and the nonrelativistic limit corresponds to $\kappa=0$.

A major obstacle which we overcome is the definition of the operator itself. With the introduction a complex parameter, the free Dirac operator (with rest mass subtracted) $H^0(\kappa)$ is no longer self-adjoint. In the case of relatively bounded potentials V (*see* e.g. [GGT]), there is no difficulty in defining the operator sum $H(\kappa)=H^0(\kappa)+V$. However, in the case of relatively *form* bounded potentials, we must define the form sum $H(\kappa)=H^0(\kappa)\dot{+}V$, which until now has been defined only for self-adjoint and sectorial operators. Sections 2 through 5 of this paper are devoted to extending this definition to a class of operators which we have called *formable* operators, while laying the foundations for our application to Dirac operators. In Section 2 we define the scale of spaces which we use throughout the paper. In Section 3 we prove some general results about how a self-adjoint operator A may be extended to an operator $A_{\mathscr{D}_J}$ on the scale of spaces and what relationships hold between the operator and its extension. In Section 4, we define the class *formable* operators [designed to include $H^0(\kappa)$] and generalize certain results of Section 3 to this class. In Section 5, we define the form sum as a restriction of the sum of the extended operators. We give a criterion under which the form sum is closed and densely defined and establish conditions under which certain Born series expansion are valid.

In Section 6 we introduce the (complexified) free Dirac operator (rest mass subtracted) $H^0(\kappa)$ and show that it is formable. We introduce a potential V which is form bounded with respect to the momentum operator. Introducing a scale of spaces related to the momentum operator, we define the extended (and perturbed) Dirac operator $(H(\kappa))_{\mathscr{D}_J}$ on the scale of spaces as the sum of the extensions of $H^0(\kappa)$ and V. The (perturbed) Dirac operator $H(\kappa)$ is defined as the form sum $H(\kappa)=H^0(\kappa)\dot{+}V$, a restriction of $(H(\kappa))_{\mathscr{D}_J}$.

We devote Sections 7 and 8 to deriving the power series expansion of the Dirac resolvent. In Section 7, we take the power series expansion of the free Dirac resolvent and extend it to the scale of spaces. We use operator estimates to determine the region of convergence. In Section 8, we obtain an explicit power series expansion (8.7) for the resolvent of $(H(\kappa))_{\mathscr{D}_J}$ via a formal calculation (we substitute the series from Section 7 into the Born series given in Section 5). We show rigorously that the power series is the resolvent of $(H(\kappa))_{\mathscr{D}_J}$ when it converges (Theorem 8.1) and thus we have holomorphy in κ in a z-dependent neighborhood of $\kappa=0$ (Theorem 8.3). When the power series converges, the Dirac operator $H(\kappa)$ is closed and densely defined, and for real κ the Dirac operator is self-adjoint. Restricting this series to $\mathscr{H}$ yields the power series for the resolvent of $H(\kappa)$ and shows holomorphy of $(H(\kappa)-z)^{-1}$. Holomorphy of isolated eigenvalues also follows.

Finally, in Section 9, we develop conditions based on the relative form boundedness condition under which the expansion given in Section 8 converges. For Coulomb-like cases, we give conditions under which the usual perturbed Dirac resolvent ($\kappa=1$) is within the region of convergence. As in [W], our analysis leads to an estimate on the lower bound of the Pauli-Schrödinger operator H_+.

2. BOUNDED FORMS AND THE SCALE OF SPACES

In this section we recall some notions about operators, forms, and Hilbert spaces associated with positive operators and forms. We introduce a scale of spaces associated with a strictly positive operator which will be used both in constructing the form sum and in expanding its resolvent. Much of this introductory material follows Faris [F].

Notation. – Throughout this paper, $\mathscr{H}$ will denote a separable complex Hilbert space and J will denote a strictly positive (self-adjoint) operator on $\mathscr{H}$. We recall that any strictly positive operator J is associated with a Hilbert space $\mathscr{D}_J=D(J^{1/2})\subseteq\mathscr{H}$ with inner product given by

$$(x, y)_{\mathscr{D}_J}=(J^{1/2}x, J^{1/2}y) \tag{2.1}$$

and norm

$$\|x\|_{\mathscr{D}_J}=\|J^{1/2}x\|. \tag{2.2}$$

Conversely, if $\mathscr{D}_J$ is a dense linear subspace of $\mathscr{H}$ and there is an inner product on $\mathscr{D}_J$ in which $\mathscr{D}_J$ is complete, then there is a (possibly different) inner product $(.,.)_{\mathscr{D}_J}$ on $\mathscr{D}_J$ in which $\mathscr{D}_J$ is complete and a strictly positive operator J on $\mathscr{H}$ such that $(x, y)_{\mathscr{D}_J}=(J^{1/2}x, J^{1/2}y)$. Given J, we will often use this associated Hilbert space $\mathscr{D}_J$ without comment.

Definition. — Let h be a sesquilinear form on $\mathscr{H}$. Let $\mathscr{Q}$ be a linear subspace of $\mathscr{H}$ with its own norm $\|\,.\,\|_{\mathscr{Q}}$ such that $\mathscr{Q}\subseteq D(h)$. The *sesquilinear form norm on* $\mathscr{Q}$ is given by

$$\|h\|_{\mathscr{Q}}=\sup\{|h(x, y)|: x, y\in\mathscr{Q} \text{ s. t. } \|x\|_{\mathscr{Q}}=\|y\|_{\mathscr{Q}}=1\}. \tag{2.3}$$

We will often write $h[x]$ for $h(x, x)$. With this notation the *quadratic form norm on* $\mathscr{Q}$ is given by

$$w_{\mathscr{Q}}(h)=\sup\{|h[x]|: x\in\mathscr{Q} \text{ s. t. } \|x\|_{\mathscr{Q}}=1\}. \tag{2.4}$$

A form h is said to be *bounded on* $\mathscr{Q}$ if $\|h\|_{\mathscr{Q}}<\infty$.

Remarks. — 1) If h is a hermitian form, $h(x, y)=\overline{h(y, x)}$, then

$$\|h\|_{\mathscr{Q}}=w_{\mathscr{Q}}(h). \tag{2.5}$$

In general we have

$$w_{\mathscr{Q}}(h)\leqq\|h\|_{\mathscr{Q}}\leqq 2\, w_{\mathscr{Q}}(h). \tag{2.6}$$

Definition. — Let $\mathscr{Q}$ be an inner product space and denote the space of conjugate linear functionals on $\mathscr{Q}$ by $\mathscr{Q}^*$. For $Y\in\mathscr{Q}^*$ and $x\in\mathscr{Q}$, denote value of Y acting on x by $\langle x, Y\rangle$. Let h_A be a bounded sesquilinear form on $\mathscr{Q}$. We define a map $A_{\mathscr{Q}}: \mathscr{Q}\to\mathscr{Q}^*$ by

$$h_A(x, y)=\langle x, A_{\mathscr{Q}}\, y\rangle. \tag{2.7}$$

Remarks. — 2) *Since* h_A is bounded, $A_{\mathscr{Q}}$ is bounded. In fact, (2.7) establishes a natural bijective norm preserving correspondence between bounded sesquilinear forms on $\mathscr{Q}$ and bounded maps from $\mathscr{Q}$ to $\mathscr{Q}^*$.

Definition. — Let J be a strictly positive operator on $\mathscr{H}$. By considering elements of $\mathscr{H}$ as conjugate linear functionals on $\mathscr{Q}_J$ via the inner product on $\mathscr{H}$, we have $\mathscr{Q}_J\subseteq\mathscr{H}\subseteq\mathscr{Q}_J^*$. Naming the inclusion maps i and j, we call

$$\mathscr{Q}_J \overset{i}{\hookrightarrow} \mathscr{H} \overset{j}{\hookrightarrow} \mathscr{Q}_J^*, \tag{2.8}$$

the *scale of spaces associated with* J. The scale of spaces may be extended infinitely in both direction as

$$\ldots\subseteq\mathscr{Q}_{J^2}\subseteq\mathscr{Q}_J\subseteq\mathscr{H}\subseteq\mathscr{Q}_J^*\subseteq\mathscr{Q}_{J^2}^*\ldots. \tag{2.9}$$

Remarks. — 3) We will refer to operators between spaces in the scale of spaces as *operators on the scale of spaces*. Usually this will refer to operators from $\mathscr{Q}_J$ to $\mathscr{Q}_J^*$ and from $\mathscr{Q}_J^*$ to $\mathscr{Q}_J$. Due to the inclusions, such operators may be considered simply as operators on $\mathscr{Q}_J^*$. We caution the reader that the inner products and norms on $\mathscr{Q}_J$ and $\mathscr{Q}_J^*$ are different (*see* below) and hence a bounded operator from $\mathscr{Q}_J$ to $\mathscr{Q}_J^*$ will not in general be bounded as an operator on $\mathscr{Q}_J^*$.

4) Our notation for the scale of spaces is somewhat different than the usual notation $\ldots\subseteq\mathscr{H}_{+1}\subseteq\mathscr{H}\subseteq\mathscr{H}_{-1}\subseteq\ldots$ (*see* e.g. [RS], [K]). The

difference emphasizes the dependence on J rather than the usual dependence on the operator to be perturbed. In practice, J will be closely related to the operator to be perturbed and the underlying spaces will be same as in the usual scale. However, the norms on these spaces will be given in terms of J.

5) In creating a scale of spaces we suppress the usual identification of $\mathcal{Q}_J$ with $\mathcal{Q}_J^*$ given by the Riesz lemma. However, we may make the isomorphism between these spaces explicit as follows. Define a form h_J by

$$h_J(x, y) = (J^{1/2} x, J^{1/2} y) \qquad \forall x, y \in D(J^{1/2}). \tag{2.10}$$

Clearly h_J is bounded on $\mathcal{Q}_J$, so as in (2.7) we may define $J_{\mathcal{Q}_J} : \mathcal{Q}_J \to \mathcal{Q}_J^*$ by $h_J(x, y) = \langle x, J_{\mathcal{Q}_J} y \rangle$. The natural isomorphism of $\mathcal{Q}_J$ onto $\mathcal{Q}_J^*$ given by the Riesz lemma maps $y \in \mathcal{Q}_J$ to $Y \in \mathcal{Q}_J^*$ by $(x, y)_{\mathcal{Q}_J} = \langle x, Y \rangle$. But $(x, y)_{\mathcal{Q}_J} = (J^{1/2} x, J^{1/2} y) = h_J(x, y) = \langle x, J_{\mathcal{Q}_J} y \rangle$, so $J_{\mathcal{Q}_J}$ is in fact the natural isomorphism. The inner product on $\mathcal{Q}_J^*$ is then

$$(X, Y)_{\mathcal{Q}_J^*} = (J_{\mathcal{Q}_J}^{-1} X, J_{\mathcal{Q}_J}^{-1} Y)_{\mathcal{Q}_J}.$$

For $x, y \in \mathcal{H}$,

$$(x, y)_{\mathcal{Q}_J^*} = (J^{1/2} J_{\mathcal{Q}_J}^{-1} x, J^{1/2} J_{\mathcal{Q}_J}^{-1} y) = (J^{-1/2} x, J^{-1/2} y). \tag{2.11}$$

One may show that $\mathcal{Q}_J^*$ is the completion of $\mathcal{H}$ under this inner product. One may also show that $J_{\mathcal{Q}_J}$ is the operator closure of J in $\mathcal{Q}_J^*$ (*see* Lemma 3.3). In particular, we note that $\mathcal{Q}_J$ is dense in $\mathcal{H}$ and $\mathcal{H}$ is dense in $\mathcal{Q}_J^*$.

Notation. — It will be convenient to define a map $J_{\mathcal{H}}^{1/2} : \mathcal{H} \to \mathcal{Q}_J^*$ by $J_{\mathcal{H}}^{1/2} = J_{\mathcal{Q}_J} J^{-1/2}$. Since $J^{1/2} : \mathcal{Q}_J \to \mathcal{H}$ and $J_{\mathcal{Q}_J} : \mathcal{Q}_J \to \mathcal{Q}_J^*$ are isomorphisms, $J_{\mathcal{H}}^{1/2}$ is also an isomorphism. In fact, one may show that $J_{\mathcal{H}}^{1/2}$ is the closure of $J^{1/2}$ in $\mathcal{Q}_J^*$. We will also write $J_{\mathcal{H}}^{-1/2}$ for $(J_{\mathcal{H}}^{1/2})^{-1}$.

Remarks. — 6) The following relationship for $x \in \mathcal{Q}_J$ and $Y \in \mathcal{Q}_J^*$, will often be useful:

$$\langle x, Y \rangle = h_J(x, J_{\mathcal{Q}_J}^{-1} Y) = (J^{1/2} x, J_{\mathcal{H}}^{-1/2} Y). \tag{2.12}$$

Notation. — We will generally suppress inclusion maps and their inverses. For instance, we will often view multiplication by a constant (usually z) as a map from $\mathcal{Q}_J$ to $\mathcal{Q}_J^*$, but we will simply write z for $z \cdot j \circ i$. For instance, if $A_{\mathcal{Q}_J} : \mathcal{Q}_J \to \mathcal{Q}_J^*$, then $A_{\mathcal{Q}_J} - z$ will make sense as an operator from $\mathcal{Q}_J$ to $\mathcal{Q}_J^*$. Another sense in which we will suppress inclusions is if $Y \in \mathcal{Q}_J^*$ is such that $Y \in j(\mathcal{H})$, then we will consider Y as an element of $\mathcal{H}$ and write (x, Y) for $\langle x, Y \rangle$.

3. EXTENSION OF OPERATORS TO THE SCALE OF SPACES

To construct the form sum we will need to extend certain operators on the Hilbert space $\mathscr{H}$ to operators on the scale of spaces. In this section we show that self-adjoint operators can be extended under certain conditions and we prove some lemmas on the relationship between the operator and its extension. Certain bounded operators such as resolvents have a different type of extension whose properties we explore. We will also have need to restrict operators on the scale of spaces back to $\mathscr{H}$ and we prove a relevant lemma. Finally, we discuss extension and restriction of projections.

We first recall the form associated with a self-adjoint operator.

Definition. — Let A be a self-adjoint operator on $\mathscr{H}$. The *sesquilinear form* h_A *associated with* A is given by

$$h_A(x, y) = (|A|^{1/2} x, U|A|^{1/2} y) \quad \text{for} \quad x, y \in Q(A), \tag{3.1}$$

where $Q(A) = D(|A|^{1/2})$ is called the *form domain* of A and $A = U|A|$ is the polar decomposition.

LEMMA 3.1. — *Let* A *be a self-adjoint operator and* J *a strictly positive operator on* $\mathscr{H}$. *If* $Q(J) \subseteq Q(A)$, *then* h_A *is bounded on* $\mathscr{Q}_J$. *Moreover,* $A_{\mathscr{Q}_J} : \mathscr{Q}_J \to \mathscr{Q}_J^*$ [*defined by* (2.7)] *is bounded.*

Proof:

$$w_{\mathscr{Q}_J}(h_A) = \sup_{\|x\|_{\mathscr{Q}_J}=1} |h_A[x]| = \sup_{\|x\|_{\mathscr{Q}_J}=1} |(|A|^{1/2} x, U|A|^{1/2} x)|$$

$$\leqq \sup_{\|x\|_{\mathscr{Q}_J}=1} \| |A|^{1/2} x\|^2 \leqq \| |A|^{1/2} J^{-1/2}\|^2 < \infty \quad \square \tag{3.2}$$

Remarks. — 1) When $D(A) \subseteq Q(J)$, then for $x, y \in D(A)$, we have $\langle x, A_{\mathscr{Q}_J} y\rangle = (x, Ay)$, so $A_{\mathscr{Q}_J}$ is an extension of A to an operator on the scale of spaces. In fact, if we consider A and $A_{\mathscr{Q}_J}$ as operators on $\mathscr{Q}_J^*$, we may show that $A_{\mathscr{Q}_J} \subseteq \bar{A}^{\mathscr{Q}_J^*}$ where $\bar{A}^{\mathscr{Q}_J^*}$ is the closure of A in $\mathscr{Q}_J^*$. Even when we do not have $D(A) \subseteq Q(J)$, $A_{\mathscr{Q}_J}$ is uniquely associated to A as h_A is uniquely associated with A. We will still speak of $A_{\mathscr{Q}_J}$ as the extension of A to the scale of spaces even though $A_{\mathscr{Q}_J}$ may not strictly speaking be an extension.

The following lemma and its generalization (Lemma 4.3) are useful in relating an operator A, its associated form h_A, and its extension to the scale of spaces $A_{\mathscr{Q}_J}$.

LEMMA 3.2. — *Let* A *be self-adjoint and* J *be strictly positive such that* $\mathscr{Q}_J \subseteq Q(A)$. *Then*

$$\overline{J^{-1/2}|A|^{1/2}}\, U|A|^{1/2} J^{-1/2} = J_{\mathscr{H}}^{-1/2} A_{\mathscr{Q}_J} J^{-1/2}, \tag{3.3}$$

where $\overline{J^{-1/2}|A|^{1/2}}$ *denotes the closure of* $J^{-1/2}|A|^{1/2}$. *Further, for all* $x, y \in \mathcal{Q}_J$,

$$h_A(x, y) = (J^{1/2}x, J_{\mathcal{H}}^{-1/2} A_{\mathcal{Q}_J} y). \tag{3.4}$$

Proof. — By (2.12), for $x, y \in \mathcal{Q}_J$ we have

$$\begin{aligned}\langle x, J_{\mathcal{H}}^{1/2} \overline{J^{-1/2}|A|^{1/2}} U|A|^{1/2} y\rangle &= (J^{1/2}x, \overline{J^{-1/2}|A|^{1/2}} U|A|^{1/2} y)\\ &= (|A|^{1/2}x, U|A|^{1/2}y)\\ &= h_A(x, y) = \langle x, A_{\mathcal{Q}_J} y\rangle,\end{aligned} \tag{3.5}$$

where we have used the fact that the adjoint of $\overline{J^{-1/2}|A|^{1/2}}$ is $|A|^{1/2}J^{-1/2}$. Equation (3.3) now follows immediately. Equation (3.4) also follows from (2.12). □

LEMMA 3.3. — *Let* A *be a self-adjoint operator and* J *a strictly positive operator on* $\mathcal{H}$ *with* D(J) = D(A) *and* Q(J) ⊆ Q(A). *Let* $z \notin \sigma(A)$. *Then* $(A_{\mathcal{Q}_J} - z)^{-1}$ *is defined on all of* $\mathcal{Q}_J^*$, *is bounded as an operator from* $\mathcal{Q}_J^*$ *to* $\mathcal{Q}_J$, $(A_{\mathcal{Q}_J} - z)^{-1}|_{\mathcal{H}} = (A - z)^{-1}$, *and* $A_{\mathcal{Q}_J} = \bar{A}^{\mathcal{Q}_J^*}$.

Proof. — We will first show that $A_{\mathcal{Q}_J} - z$ maps $\mathcal{Q}_J$ onto $\mathcal{Q}_J^*$.

Let $Y \in \mathcal{Q}_J^*$. Since $J^{1/2}(A - z)^{-1}J^{1/2}$ maps $\mathcal{Q}_J$ to $\mathcal{Q}_J$, it has a bounded closure $\overline{J^{1/2}(A - z)^{-1}J^{1/2}}$ in $\mathcal{H}$. Hence, if we set

$$y = J^{-1/2}\overline{J^{1/2}(A - z)^{-1}J^{1/2}} J_{\mathcal{H}}^{-1/2} Y, \tag{3.6}$$

then $y \in \mathcal{Q}_J$. Since $\mathcal{H}$ is dense in $\mathcal{Q}_J^*$, we may choose $\{y_n\} \subseteq \mathcal{H}$ with $y_n \to Y$ in $\mathcal{Q}_J^*$. Then for any $x \in \mathcal{Q}_J$,

$$\begin{aligned}\langle x, (A_{\mathcal{Q}_J} - z)y\rangle &= (|A|^{1/2}x, U|A|^{1/2}y) - z(x, y)\\ &= (J^{1/2}x, (\overline{J^{-1/2}|A|^{1/2}} U|A|^{1/2} - zJ^{-1/2})y)\\ &= \lim_{n\to\infty} (J^{1/2}x, (\overline{J^{-1/2}|A|^{1/2}} U|A|^{1/2} - zJ^{-1/2})\\ &\qquad \times J^{-1/2}\overline{J^{1/2}(A - z)^{-1}J^{1/2}} J^{-1/2} y_n)\\ &= \lim_{n\to\infty} (J^{1/2}x, (J_{\mathcal{H}}^{-1/2}(A_{\mathcal{Q}_J} - z)J^{-1/2})J^{1/2}(A - z)^{-1} y_n)\\ &= \lim_{n\to\infty} (J^{1/2}x, J^{-1/2}y_n) = \langle x, Y\rangle.\end{aligned} \tag{3.7}$$

So $(A_{\mathcal{Q}_J} - z)y = Y$, and $A_{\mathcal{Q}_J} - z$ maps $\mathcal{Q}_J$ onto $\mathcal{Q}_J^*$.

To see that $A_{\mathcal{Q}_J} - z$ is injective, we start by noting that

$$x \to \langle (A - \bar{z})^{-1}x, Y\rangle$$

defines a conjugate linear functional on $\mathcal{H}$ for all $Y \in \mathcal{Q}_J^*$. Hence, we may define $\Phi_z(Y) = y$ where y is the unique element of $\mathcal{H}$ (in fact, y will be in $\mathcal{Q}_J$) such that

$$\langle (A - \bar{z})^{-1}x, Y\rangle = (x, y)_{\mathcal{H}} \qquad \forall x \in \mathcal{H}. \tag{3.8}$$

We must show that $\Phi_z = (A_{\mathcal{Q}_J} - z)^{-1}$.

Let $Y \in \mathcal{Q}_J^*$. Since $A_{\mathcal{Q}_J} - z$ maps $\mathcal{Q}_J$ onto $\mathcal{Q}_J^*$, there exists $y \in \mathcal{Q}_J$ such that $Y = (A_{\mathcal{Q}_J} - z) y$. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned}(x, \Phi_z(Y)) &= \langle (A-\bar{z})^{-1} x, Y \rangle = \langle (A-\bar{z})^{-1} x, (A_{\mathcal{Q}_J} - z) y \rangle \\ &= (|A|^{1/2} (A-\bar{z})^{-1} x, U |A|^{1/2} y) - z ((A-\bar{z})^{-1} x, y) \\ &= (x, y),\end{aligned} \tag{3.9}$$

which shows $\Phi_z = (A_{\mathcal{Q}_J} - z)^{-1}$.

One can easily show that $(A_{\mathcal{Q}_J} - z)^{-1}$ is closed using

$$(x, (A_{\mathcal{Q}_J} - z)^{-1} Y) = \langle (A-\bar{z})^{-1} x, Y \rangle. \tag{3.10}$$

Since $(A_{\mathcal{Q}_J} - z)^{-1}$ is defined on all of the Hilbert space $\mathcal{Q}_J^*$, the closed graph theorem implies that it is bounded. It also follows from (3.10) that $(A_{\mathcal{Q}_J} - z)^{-1} |_{\mathcal{H}} = (A - z)^{-1}$. Finally, since $(A_{\mathcal{Q}_J} - z)^{-1}$ exists and is bounded, $A_{\mathcal{Q}_J}$ is closed. Then $A_{\mathcal{Q}_J} \subseteq \bar{A}^{\mathcal{Q}_J^*}$ implies $A_{\mathcal{Q}_J} = \bar{A}^{\mathcal{Q}_J^*}$. □

We can generalize some of the ideas of the preceding lemma to the case of bounded operators which are not necessarily resolvents.

Definition. – Let R be a bounded operator on $\mathcal{H}$, let J be strictly positive, and assume that $\operatorname{Ran}(R^*) \subseteq \mathcal{Q}_J$. We define the extension of R to $R_{\mathcal{Q}_J^*} : \mathcal{Q}_J^* \to \mathcal{H}$ as follows. For $Y \in \mathcal{Q}_J^*$, let $R_{\mathcal{Q}_J^*} Y$ be the unique element of $\mathcal{H}$ such that

$$\langle R^* x, Y \rangle = (x, R_{\mathcal{Q}_J^*} Y) \qquad \forall x \in \mathcal{H}. \tag{3.11}$$

Remarks. – 2) Clearly, $R_{\mathcal{Q}_J^*} |_{\mathcal{H}} = R$, so $R_{\mathcal{Q}_J^*}$ is an extension of R. In fact, since $\mathcal{H}$ is dense in $\mathcal{Q}_J^*$ and boundedness in $\mathcal{H}$ implies boundedness in $\mathcal{Q}_J^*$, any bounded operator R on $\mathcal{H}$ may be extended by closure in $\mathcal{Q}_J^*$ to a bounded operator $\bar{R}^{\mathcal{Q}_J^*}$ on $\mathcal{Q}_J^*$. One may use (3.11) to show $R_{\mathcal{Q}_J^*} = \bar{R}^{\mathcal{Q}_J^*}$ when the conditions of the definition are met.

3) For a self-adjoint operator A with $D(A) \subseteq \mathcal{Q}_J$, the resolvent $(A - z)^{-1}$ may be extended to $(A - z)^{-1}_{\mathcal{Q}_J^*}$, using the definition. In particular, if A meets the conditions of Lemma 3.3 and $z \notin \sigma(A)$, then we have

$$(A_{\mathcal{Q}_J} - z)^{-1} = (A - z)^{-1}_{\mathcal{Q}_J^*}. \tag{3.12}$$

4) When we take norms of operators on the scale of spaces, it will be important to indicate the initial and final spaces, as the norms in the various spaces differ. When the initial and final spaces are the same, we will denote the norm as $\| \cdot \|$, specifying the space if confusion is likely. When they differ, we will specify both, as in $\| \cdot \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J}$. Using this notation, we note that

$$\| R_{\mathcal{Q}_J^*} \|_{\mathcal{Q}_J^*} = \| J^{-1/2} R_{\mathcal{Q}_J^*} J^{1/2}_{\mathcal{H}} \|_{\mathcal{H}}, \tag{3.13}$$

whereas

$$\| R_{\mathcal{Q}_J^*} \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J} = \| J^{1/2} R_{\mathcal{Q}_J^*} J^{1/2}_{\mathcal{H}} \|_{\mathcal{H}}. \tag{3.14}$$

LEMMA 3.4. — *Let* R *be a bounded operator and let* J *be a strictly positive operator on* $\mathscr{H}$. *Assume that* $\mathrm{Ran}(R)=\mathrm{Ran}(R^*)\subseteq D(J)$. *Then* $\mathrm{Ran}(R_{\mathscr{Q}_J^*})\subseteq \mathscr{Q}_J$ *and*

$$\overline{J^{1/2}RJ^{1/2}}=J^{1/2}R_{\mathscr{Q}_J^*}J_{\mathscr{H}}^{1/2}, \tag{3.15}$$

where the overbar denotes the closure in $\mathscr{H}$. *In particular*, $R_{\mathscr{Q}_J^*}:\mathscr{Q}_J^*\to\mathscr{Q}_J$ *is bounded.*

Proof. — Since $\mathrm{Ran}(R)=\mathrm{Ran}(R^*)\subseteq D(J)$, $J^{1/2}RJ^{1/2}$ and $J^{1/2}R^*J^{1/2}$ map $\mathscr{Q}_J$ to $\mathscr{Q}_J$ and hence each extends to a bounded operator on all of $\mathscr{H}$. Further, one may show that the adjoint of $\overline{J^{1/2}RJ^{1/2}}$ is $\overline{J^{1/2}R^*J^{1/2}}$.

Let $Y\in\mathscr{Q}_J^*$. Since $\overline{J^{1/2}RJ^{1/2}}$ is bounded, if we set

$$y=J^{-1/2}\overline{J^{1/2}RJ^{1/2}}J_{\mathscr{H}}^{-1/2}Y, \tag{3.16}$$

then $y\in\mathscr{Q}_J$, and we will have $\mathrm{Ran}(R_{\mathscr{Q}_J^*})\subseteq\mathscr{Q}_J$ if we can show

$$J^{-1/2}\overline{J^{1/2}RJ^{1/2}}J_{\mathscr{H}}^{-1/2}=R_{\mathscr{Q}_J^*}. \tag{3.17}$$

But (3.17) is a consequence of the fact that for all $x\in\mathscr{H}$,

$$\begin{aligned}(x, J^{-1/2}\overline{J^{1/2}RJ^{1/2}}J_{\mathscr{H}}^{-1/2}Y)&=(\overline{J^{1/2}R^*J^{1/2}}J^{-1/2}x, J_{\mathscr{H}}^{-1/2}Y)\\&=(J^{1/2}R^*x, J_{\mathscr{H}}^{-1/2}Y)\\&=\langle R^*x, Y\rangle=(x, R_{\mathscr{Q}_J^*}Y),\end{aligned} \tag{3.18}$$

where we have used equation (2.12). Now (3.15) follows immediately.

Boundedness of $R_{\mathscr{Q}_J^*}:\mathscr{Q}_J^*\to\mathscr{Q}_J$ follows from (3.15), (3.14) and the fact that $\overline{J^{1/2}RJ^{1/2}}$ is bounded on $\mathscr{H}$. □

LEMMA 3.5. — *Let* $\mathscr{A}$ *be a closed, densely defined operator and let* J *be a strictly positive operator on* $\mathscr{H}$ *such that* $D(\mathscr{A})\subseteq\mathscr{Q}_J$. *Let* $z\notin\sigma(\mathscr{A})$ *and assume that there is a bounded operator* $R:\mathscr{Q}_J^*\to\mathscr{H}$ *such that* $R|_{\mathscr{H}}=(\mathscr{A}-z)^{-1}$. *Then* $(\mathscr{A}-z)_{\mathscr{Q}_J^*}^{-1}=R$.

Proof. — Let $Y\in\mathscr{Q}_J^*$. Since $\mathscr{H}$ is dense in $\mathscr{Q}_J^*$, we may choose $y_n\in\mathscr{H}$ with $y_n\to Y$ in $\mathscr{Q}_J^*$. Then for all $x\in\mathscr{H}$, we have

$$\begin{aligned}(x,(\mathscr{A}-z)_{\mathscr{Q}_J^*}^{-1}Y)&=\langle(\mathscr{A}^*-\bar z)^{-1}x, Y\rangle=\lim_{n\to\infty}((\mathscr{A}^*-\bar z)^{-1}x, y_n)\\&=\lim_{n\to\infty}(x,(\mathscr{A}-z)^{-1}y_n)=\lim_{n\to\infty}(x, Ry_n)\\&=(x, RY),\end{aligned} \tag{3.19}$$

where we have used the continuity of both R and the inner product. □

Let $P_\pm$ be orthogonal projections on $\mathscr{H}$ which commute with a strictly positive operator J such that $P_++P_-=I$. Then $P_+|_{\mathscr{Q}_J}$ and $P_-|_{\mathscr{Q}_J}$ are orthogonal projections on $\mathscr{Q}_J$ and $P_+|_{\mathscr{Q}_J}+P_-|_{\mathscr{Q}_J}=I$. As with $\mathscr{H}$, we may write $\mathscr{Q}_J=\mathscr{Q}_{J_+}\oplus\mathscr{Q}_{J_-}$, where $\mathscr{Q}_{J_\pm}=P_\pm\mathscr{Q}_J$. Similarly, we may extend $P_\pm$ to mappings on $\mathscr{Q}_J^*$ by

$$\langle x, P_\pm Y\rangle\equiv\langle P_\pm x, Y\rangle \tag{3.20}$$

for $x \in \mathcal{Q}_J$ and $Y \in \mathcal{Q}_J^*$. Again using the fact that $P_\pm$ commutes with J, one can show that $P_\pm$ are orthogonal projections on $\mathcal{Q}_J^*$ and $\mathcal{Q}_J^* = \mathcal{Q}_{J+}^* \oplus \mathcal{Q}_{J-}^*$ where $\mathcal{Q}_{J\pm}^* = P_\pm \mathcal{Q}_J^*$. Generally, for projections on $\mathcal{Q}_J$ or $\mathcal{Q}_J^*$ derived in this manner from projections on $\mathcal{H}$, we will simply write P_+ or P_- without referencing the domain, unless it is not clear from context.

4. FORMABLE OPERATORS

For our application to Dirac operators we wish to define the form sum of certain non-self-adjoint operators. To do so we will need to be able to extend these operators on $\mathcal{H}$ to operators on the scale of spaces. In this section we define an appropriate class of operators and generalize the lemmas of Section 3 to this class.

Given a linear operator $\mathcal{A}$ on $\mathcal{H}$, we may define a form $s_{\mathcal{A}}$ by

$$s_{\mathcal{A}}(x, y) = (x, \mathcal{A} y) \qquad \forall x, y \in D(s_{\mathcal{A}}) = D(\mathcal{A}) \tag{4.1}$$

In particular, for a self-adjoint operator A, if we set $J = |A| + 1$, then $\mathcal{Q}_J = Q(J) = Q(A)$, and it is easy to see that h_A is the unique extension of s_A to $\mathcal{Q}_J$ which is bounded on $\mathcal{Q}_J$. This unique extension property may be generalized to a broader class of operators.

Definition. – Let C be an operator on $\mathcal{H}$ and let $\mathcal{Q}$ be a linear subspace of $\mathcal{H}$. We will say that C *preserves* $\mathcal{Q}$ if C is an injective map of $\mathcal{Q}$ onto itself.

Definition. – Let $\mathcal{A}$ be a closed, densely defined operator on $\mathcal{H}$. We will say that $\mathcal{A}$ is *formable as* CAC if there exists a self-adjoint operator A and a bounded invertible operator C such that $\mathcal{A} = CAC$ and C and C^* preserve Q(A).

THEOREM 4.1. – *Let $\mathcal{A}$ be formable as* CAC. *Let* $J = |A| + 1$. *Define the form $h_{\mathcal{A}}$ by*

$$h_{\mathcal{A}}(x, y) = h_A(C^* x, C y) \qquad \forall x, y \in D(h_{\mathcal{A}}) = Q(A). \tag{4.2}$$

Then $h_{\mathcal{A}}$ is bounded on $\mathcal{Q}_J$ and is the unique extension of $s_{\mathcal{A}}$ to $\mathcal{Q}_J$.

Proof. – We have $D(A) \subseteq \mathcal{Q}_J$, $D(\mathcal{A}) = C^{-1} D(A)$ and C^{-1} preserves $\mathcal{Q}_J = Q(A)$, so $D(\mathcal{A}) \subseteq \mathcal{Q}_J$. From this and the definition, it is clear that $h_{\mathcal{A}}$ is an extension of $s_{\mathcal{A}}$.

Since C preserves $\mathcal{Q}_J$, C is defined on all of $\mathcal{Q}_J$ as an operator from $\mathcal{Q}_J$ to $\mathcal{Q}_J$. C is closed as an operator on $\mathcal{Q}_J$ since it is bounded on $\mathcal{H}$. Hence, by the closed graph theorem, C is bounded on $\mathcal{Q}_J$. Similarly, C^* is bounded on $\mathcal{Q}_J$. So C and C^* are bounded invertible operators on $\mathcal{Q}_J$.

Since $D(A) = D(J)$ is $\| \cdot \|_{\mathcal{Q}_J}$-dense in $\mathcal{Q}_J$, $D(\mathcal{A}) = C^{-1} D(A)$ is $\| \cdot \|_{\mathcal{Q}_J}$-dense in $\mathcal{Q}_J$. Uniqueness of the extension will follow if $h_{\mathcal{A}}$ is bounded on $\mathcal{Q}_J$. We have just seen that $\|C\|_{\mathcal{Q}_J}$ and $\|C^*\|_{\mathcal{Q}_J}$ exist. We also know $\|h_A\|_{\mathcal{Q}_J}$ exists by Lemma 3.1. Hence, boundedness follows from

$$\begin{aligned} w_{\mathcal{Q}_J}(h_{\mathscr{A}}) &= \sup_{\|x\|_{\mathcal{Q}_J}=1} |h_{\mathscr{A}}[x]| = \sup_{\|x\|_{\mathcal{Q}_J}=1} |(|A|^{1/2} C^* x, U|A|^{1/2} C x)| \\ &= \sup_{\|x\|_{\mathcal{Q}_J}=1} \|Cx\|_{\mathcal{Q}_J} \|C^* x\|_{\mathcal{Q}_J} \\ &\qquad \times \left| \left(|A|^{1/2} \frac{C^* x}{\|C^* x\|_{\mathcal{Q}_J}}, U|A|^{1/2} \frac{Cx}{\|Cx\|_{\mathcal{Q}_J}} \right) \right| \\ &\leqq \|C^*\|_{\mathcal{Q}_J} \|C\|_{\mathcal{Q}_J} \|h_A\|_{\mathcal{Q}_J} \quad \square \end{aligned} \tag{4.3}$$

Remarks. — 1) Given that $\mathscr{A}$ is formable as CAC for some A, the operator C is not completely determined, even if we restrict ourselves to unitary C (e. g. replace C by $-C$). However, the preceding theorem shows that the form $h_{\mathscr{A}}$ is independent of the particular choice of C.

2) In general, given a formable operator $\mathscr{A}$, it is not clear whether the choice of the self-adjoint operator A which satisfies the definition is unique. The question of whether D(A) and Q(A) are unique is also left open. As these questions are open, the form $h_{\mathscr{A}}$ may depend on A. In our applications A will be specified.

We omit the proofs of the following lemmas, as they follow the proofs of Lemmas 3.1 through 3.3 (with the insertion of C, C*, $\mathscr{A}$, and $\mathscr{A}^*$ where appropriate).

LEMMA 4.2. — *Let $\mathscr{A}$ be formable as* CAC *and let* J *be strictly positive with $\mathcal{Q}_J \subseteqq Q(A)$. Then $h_{\mathscr{A}}$ is bounded on $\mathcal{Q}_J$. Moreover, $\mathscr{A}_{\mathcal{Q}_J}: \mathcal{Q}_J \to \mathcal{Q}_J^*$ is bounded.*

LEMMA 4.3. — *Let $\mathscr{A}$ be formable as* CAC *and let* J *be strictly positive with $\mathcal{Q}_J \subseteqq Q(A)$. Then*

$$\overline{J^{-1/2} C |A|^{1/2}} U |A|^{1/2} C J^{-1/2} = J_{\mathscr{H}}^{-1/2} \mathscr{A}_{\mathcal{Q}_J} J^{-1/2}, \tag{4.4}$$

where $\overline{J^{-1/2} C |A|^{1/2}}$ denotes the closure of $J^{-1/2} C |A|^{1/2}$. Further, for all $x, y \in \mathcal{Q}_J$,

$$h_{\mathscr{A}}(x, y) = (J^{1/2} x, J_{\mathscr{H}}^{-1/2} \mathscr{A}_{\mathcal{Q}_J} y). \tag{4.5}$$

LEMMA 4.4. — *Let $\mathscr{A}$ be formable as* CAC *and let* J *be strictly positive with* $D(J) = D(A)$ *and $\mathcal{Q}_J \subseteqq Q(A)$. Assume that* C *and* C* *preserve* D(A). *Let $z \notin \sigma(\mathscr{A})$. Then $(\mathscr{A}_{\mathcal{Q}_J} - z)^{-1}: \mathcal{Q}_J^* \to \mathcal{Q}_J$ is a bounded operator defined on all of $\mathcal{Q}_J^*$, $(\mathscr{A}_{\mathcal{Q}_J} - z)^{-1}|_{\mathscr{H}} = (\mathscr{A} - z)^{-1}$, $(\mathscr{A}_{\mathcal{Q}_J} - z)^{-1} = \overline{(\mathscr{A} - z)_{\mathcal{Q}_J^*}^{-1}}$, and $\mathscr{A}_{\mathcal{Q}_J} = \bar{\mathscr{A}}^{\mathcal{Q}_J^*}$.*

5. THE FORM SUM

Having extended formable operators to the scale of spaces, we may now define the form sum of such operators as a restriction of the sum of the extensions. We show that if the sum of the extended operators is closed as an operator on $\mathcal{Q}_J^*$ and its resolvent is bounded as an operator from $\mathcal{Q}_J^*$ to $\mathcal{Q}_J$, then the form sum is also closed and densely defined on $\mathcal{H}$. We call this fact the "basic criterion" for closedness of the restriction and in Section 8 we will apply this criterion to the Dirac operator. We also prove two corollaries in this section, one appropriate to Dirac operators and the other appropriate to the associated Pauli-Schrödinger operators, giving conditions under which Born series expansions are valid and the basic criterion holds. The ideas of this section are closely related to and generalize Faris [F], Theorem 5.2.

Definition. — Let $\mathcal{A}$ be formable as $C_A A C_A$ and let $\mathcal{V}$ be formable as $C_V V C_V$. Let J be a strictly positive operator and assume that $\mathcal{Q}_J \subseteq Q(A)$ and $\mathcal{Q}_J \subseteq Q(V)$. Let $\mathcal{B}_{\mathcal{Q}_J} = \mathcal{A}_{\mathcal{Q}_J} + \mathcal{V}_{\mathcal{Q}_J}$. Let

$$D(\mathcal{B}) = \{ x \in \mathcal{Q}_J : \mathcal{B}_{\mathcal{Q}_J} x \in \mathcal{H} \}, \tag{5.1}$$

and define the *form sum* of $\mathcal{A}$ and $\mathcal{V}$ to be

$$\mathcal{B} \equiv \mathcal{A} \mathring{+} \mathcal{V} \equiv \mathcal{B}_{\mathcal{Q}_J} |_{D(\mathcal{B})}. \tag{5.2}$$

Remarks. — 1) By Lemma 4.2, $\mathcal{A}_{\mathcal{Q}_J}$ and $\mathcal{V}_{\mathcal{Q}_J}$ are bounded, so $\mathcal{B}_{\mathcal{Q}_J}$ is defined on all of $\mathcal{Q}_J$ as an operator sum. For $x, y \in \mathcal{Q}_J$,

$$\langle x, \mathcal{B}_{\mathcal{Q}_J} y \rangle = \langle x, \mathcal{A}_{\mathcal{Q}_J} y \rangle + \langle x, \mathcal{V}_{\mathcal{Q}_J} y \rangle = h_{\mathcal{A}}(x, y) + h_{\mathcal{V}}(x, y), \tag{5.3}$$

so when $x \in \mathcal{Q}_J$ and $y \in D(\mathcal{B})$, we have

$$(x, \mathcal{B} y) = h_{\mathcal{A}}(x, y) + h_{\mathcal{V}}(x, y). \tag{5.4}$$

The following theorem may be considered as a "basic criterion for closedness of the restriction" in analogy to "the basic criterion for self-adjointness".

THEOREM 5.1 (Basic criterion). — *Let* J *be a strictly positive operator. Let $\mathcal{B}_{\mathcal{Q}_J}$ be a bounded map of $\mathcal{Q}_J$ into $\mathcal{Q}_J^*$ and let $\mathcal{B}$ be defined by* (5.1) *and* (5.2). *Assume that $\mathcal{B}_{\mathcal{Q}_J}$ is closed as an operator on $\mathcal{Q}_J^*$ and for some $z \notin \sigma(\mathcal{B}_{\mathcal{Q}_J})$*

$$\| (\mathcal{B}_{\mathcal{Q}_J} - z)^{-1} \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J} < \infty. \tag{5.5}$$

Then $\mathcal{B}$ is closed and densely defined on $\mathcal{H}$. Moreover, under these conditions, $z \notin \sigma(\mathcal{B})$, $(\mathcal{B} - z)^{-1} = (\mathcal{B}_{\mathcal{Q}_J} - z)^{-1} |_{\mathcal{H}}$, and $(\mathcal{B} - z)_{\mathcal{Q}_J^*}^{-1} = (\mathcal{B}_{\mathcal{Q}_J} - z)^{-1}$.

Proof. — To show that $\mathcal{B}$ is closed, it suffices to show that for some $z_0 \in \mathbb{C}$, $\mathcal{B} - z_0$ is an injective map of $D(\mathcal{B})$ onto $\mathcal{H}$ and $(\mathcal{B} - z_0)^{-1} : \mathcal{H} \to \mathcal{H}$ is bounded. Now $\mathcal{B} - z$ is injective and onto since $\mathcal{B}_{\mathcal{Q}_J} - z$ is an injective map of $\mathscr{Q}_J$ onto $\mathscr{Q}_J^*$. By definition $(\mathscr{B}-z)^{-1}=(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}|_{\mathscr{H}}$ and $(\mathscr{B}-z)^{-1}$ is bounded since

$$\begin{aligned}\|(\mathscr{B}-z)^{-1}\| &\leqq \|J^{1/2}\|\,\|J^{1/2}(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}J_{\mathscr{H}}^{1/2}\|\,\|J^{-1/2}\| \\ &= \|J^{-1/2}\|^2\,\|(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}\|_{\mathscr{Q}_J^*\to\mathscr{Q}_J}<\infty. \end{aligned} \tag{5.6}$$

Hence, $z\notin\sigma(\mathscr{B})$ and we may conclude that $\mathscr{B}=\mathscr{A}\mathring{+}\mathscr{V}$ is closed.

We now show that $D(\mathscr{B})$ is dense in $\mathscr{H}$. Let $y\in\mathscr{Q}_J$ and let $Y=(\mathscr{B}_{\mathscr{Q}_J}-z)y\in\mathscr{Q}_J^*$. Since $\mathscr{H}$ is dense in $\mathscr{Q}_J^*$, there exists $\{X_n\}\subseteqq\mathscr{H}$ such that $X_n\to Y$ in $\mathscr{Q}_J^*$. Let $x_n=(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}X_n$. Since $(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}$ is bounded, continuity implies $x_n\to y$ in $\mathscr{Q}_J$. But $\{x_n\}\subseteqq D(\mathscr{B})$, so $D(\mathscr{B})$ is $\|\,.\,\|_{\mathscr{Q}_J}$-dense in $\mathscr{Q}_J$. Since $\|\,.\,\|_{\mathscr{Q}_J}$-density implies $\|\,.\,\|_{\mathscr{H}}$-density and $\mathscr{Q}_J$ is dense in $\mathscr{H}$, $D(\mathscr{B})$ is dense in $\mathscr{H}$.

Finally, we note that by Lemma 3.5, $(\mathscr{B}-z)_{\mathscr{Q}_J^*}^{-1}=(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}$. □

Remarks. – 2) If A and V are self-adjoint with $\mathscr{Q}_J\subseteqq Q(A)$ and $\mathscr{Q}_J\subseteqq Q(V)$, then using (5.3), $A\mathring{+}V$ is symmetric. Thus, if the basic criterion for closedness of the restriction holds for some z, $\bar{z}\notin\mathbb{R}$, then $A\mathring{+}V$ is self-adjoint by the basic criterion for self-adjointness.

The following two corollaries are presented with the Dirac and Pauli-Schrödinger operators, respectively, in mind.

COROLLARY 5.2 *Let* $\mathscr{A}$ *be formable as* $C_A A C_A$ *and let* $\mathscr{V}$ *be formable as* $C_V V C_V$. *Let* J *be a strictly positive operator and assume that* $\mathscr{Q}_J\subseteqq Q(A)$ *and* $\mathscr{Q}_J\subseteqq Q(V)$. *Define* $\mathscr{B}_{\mathscr{Q}_J}=\mathscr{A}_{\mathscr{Q}_J}+\mathscr{V}_{\mathscr{Q}_J}$. *Define* $D(\mathscr{B})$ *and* $\mathscr{B}$ *as in* (5.1) *and* (5.2). *Assume that* $D(J)=D(A)$, C_A *and* C_A^* *preserve* $D(A)$, *and that there is a* $z\notin\sigma(\mathscr{A})$ *such that*

$$\|(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}\mathscr{V}_{\mathscr{Q}_J}\|_{\mathscr{Q}_J}<1. \tag{5.7}$$

Then the results of Theorem 5.1 *follow. Moreover, the Born series*

$$(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}=(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}-(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}\mathscr{V}_{\mathscr{Q}_J}(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}+\dots \tag{5.8}$$

and its restriction to $\mathscr{H}$

$$(\mathscr{B}-z)^{-1}=(\mathscr{A}-z)^{-1}-(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}\mathscr{V}_{\mathscr{Q}_J}(\mathscr{A}-z)^{-1}+\dots \tag{5.9}$$

are valid.

Proof. – For $z\notin\sigma(\mathscr{A})$, we know by Lemma 4.4 that $(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}$ is defined on all of $\mathscr{Q}_J^*$. Then by condition (5.7), the expansion

$$(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}=(1+(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}\mathscr{V}_{\mathscr{Q}_J})^{-1}(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1} \tag{5.10}$$

is valid. From (5.10) it is clear that $\mathscr{B}_{\mathscr{Q}_J}-z$ is an injective map of $\mathscr{Q}_J$ onto $\mathscr{Q}_J^*$ and using Lemma 4.4 and (5.7) we see that $\|(\mathscr{B}_{\mathscr{Q}_J}-z)^{-1}\|_{\mathscr{Q}_J^*\to\mathscr{Q}_J}<\infty$. Hence, $\mathscr{B}_{\mathscr{Q}_J}$ is closed in $\mathscr{Q}_J^*$ and the conditions of Theorem 5.1 are satisfied.

Expanding $(1+(\mathscr{A}_{\mathscr{Q}_J}-z)^{-1}\mathscr{V}_{\mathscr{Q}_J})^{-1}$ in a geometric series in (5.10) gives (5.8). Restricting to $\mathscr{H}$ and using Lemma 4.4 gives (5.9). □

COROLLARY 5.3. — *Let* A *and* V *be self-adjoint and let* J *be strictly positive. Assume that* $D(J^2)=D(A)$, $D(J)\subseteq Q(A)$, *and* $Q(J)\subseteq Q(V)$. *Define* $\mathcal{B}_{\mathcal{Q}_{J^2}}=A_{\mathcal{Q}_{J^2}}+V_{\mathcal{Q}_J}$ *(with domain* $\mathcal{Q}_{J^2}$*). Let*

$$D(B)=\{x\in\mathcal{Q}_{J^2}:B_{\mathcal{Q}_{J^2}}x\in\mathcal{H}\} \tag{5.11}$$

and define the form sum

$$B\equiv A\overset{\circ}{+}V\equiv B_{\mathcal{Q}_{J^2}}|_{D(B)}. \tag{5.12}$$

Let $z\notin\sigma(A)$ *and assume*

$$\|(A_{\mathcal{Q}_{J^2}}-z)^{-1}V_{\mathcal{Q}_J}\|_{\mathcal{Q}_J}<1. \tag{5.13}$$

Then B *is closed and densely defined on* $\mathcal{H}$. *Moreover, under these conditions,* $z\notin\sigma(B)$, $(B-z)^{-1}=(B_{\mathcal{Q}_{J^2}}-z)^{-1}|_{\mathcal{H}}$, *and* $(B-z)^{-1}_{\mathcal{Q}^*_{J^2}}=(B_{\mathcal{Q}_{J^2}}-z)^{-1}$. *For* $x\in\mathcal{Q}_{J^2}$ *and* $y\in D(B)$, *we have*

$$(x, By)=h_A(x, y)+h_V(x, y). \tag{5.14}$$

Also under these conditions, the Born series

$$(B_{\mathcal{Q}_{J^2}}-z)^{-1}=(A_{\mathcal{Q}_{J^2}}-z)^{-1}-(A_{\mathcal{Q}_{J^2}}-z)^{-1}V_{\mathcal{Q}_J}(A_{\mathcal{Q}_{J^2}}-z)^{-1}+\dots \tag{5.15}$$

and its restrictions

$$(B_{\mathcal{Q}_{J^2}}-z)^{-1}|_{\mathcal{Q}^*_J}=(A-z)^{-1}_{\mathcal{Q}^*_J}-(A-z)^{-1}_{\mathcal{Q}^*_J}V_{\mathcal{Q}_J}(A-z)^{-1}_{\mathcal{Q}^*_J}+\dots \tag{5.16}$$

and

$$(B-z)^{-1}=(A-z)^{-1}-(A-z)^{-1}_{\mathcal{Q}^*_J}V_{\mathcal{Q}_J}(A-z)^{-1}+\dots \tag{5.17}$$

are valid.

Proof. — The results up through (5.14) follow as they do in Corollary 5.2 and we have

$$(B_{\mathcal{Q}_{J^2}}-z)^{-1}=(1+(A_{\mathcal{Q}_{J^2}}-z)^{-1}V_{\mathcal{Q}_J})^{-1}(A_{\mathcal{Q}_{J^2}}-z)^{-1}. \tag{5.18}$$

Using (5.11), (5.18) expands into the Born series (5.15). Now $(A_{\mathcal{Q}_{J^2}}-z)^{-1}=(A-z)^{-1}_{\mathcal{Q}^*_{J^2}}$ by (3.12), and $(A-z)^{-1}_{\mathcal{Q}^*_{J^2}}|_{\mathcal{Q}^*_J}=(A-z)^{-1}_{\mathcal{Q}^*_J}$ as can be seen from the definition (3.11). Then restricting both sides of (5.15) to $\mathcal{Q}^*_J$ and noting that $\mathrm{Ran}(V_{\mathcal{Q}_J})\subseteq\mathcal{Q}^*_J$ yields (5.16). Similarly, restricting (5.15) to $\mathcal{H}$ yields (5.17). □

Remarks. — 3) By Remark 2), if the conditions of Corollary 5.3 hold for $z=z_0$ and for $z=\bar{z}_0$ where $\mathrm{Im}\, z_0\neq 0$, then B is self-adjoint.

6. THE ABSTRACT DIRAC OPERATOR

The abstract setting for the Dirac operator is due to Hunziker [H] and Cirincione and Chernoff [CC] and is general enough to be applicable to Dirac operators on curved spaces. Our notation also reflects that of

Gesztesy, Grosse, and Thaller [GGT] and Grigore, Nenciu, and Purice [GNP].

Let $\mathscr{D}$ and β be self-adjoint operators on $\mathscr{H}$ with the properties

$$\beta^2 = I \qquad \text{and} \qquad \mathscr{D} = -\beta \mathscr{D} \beta. \tag{6.1}$$

Then β is unitary and has two spectral projections P_+ and P_- corresponding to the eigenvalues $+1$ and -1. We may write $\mathscr{H}$ as the orthogonal direct sum $\mathscr{H}^+ \oplus \mathscr{H}^-$, where $\mathscr{H}^\pm$ is the range of $P_\pm$.

In general, $\mathscr{D}$ will be unbounded and we assume it is defined and self-adjoint on some dense domain $D(\mathscr{D})$. By (6.1), β preserves $D(\mathscr{D})$. With respect to the decomposition of $\mathscr{H}$, we have

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \qquad \text{and} \qquad \mathscr{D} = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix}, \tag{6.2}$$

where we have used (6.1) to deduce the form of $\mathscr{D}$. Note that the operator $D : \mathscr{H}^+ \to \mathscr{H}^-$ is closed and densely defined with adjoint $D^* : \mathscr{H}^- \to \mathscr{H}^+$. It will be convenient to define operators $D_\pm : \mathscr{H}^\pm \to \mathscr{H}^\pm$ by

$$D_+ = (D^* D)^{1/2} \qquad \text{and} \qquad D_- = (DD^*)^{1/2}. \tag{6.3}$$

For simplicity we will assume $\sigma(D_+) = \sigma(D_-)$ and we note that

$$|\mathscr{D}| = \begin{bmatrix} D_+ & 0 \\ 0 & D_- \end{bmatrix}. \tag{6.4}$$

For $\kappa \in \mathbb{C} \setminus \{0\}$, we define our *free Dirac operator* (with rest mass subtracted) as

$$H^0(\kappa) = \frac{c}{\kappa} \mathscr{D} - \frac{2mc^2}{\kappa^2} P_-. \tag{6.5}$$

The operator $\mathscr{D}$ may be thought of as the momentum operator $\sigma . \mathbf{p}$, possibly including a magnetic potential (*see* Remark 3 below). The real constants m and c represent the "rest mass" and the "velocity of light", respectively. The variable κ is a dimensionless complex parameter which we will let go to zero to achieve the nonrelativistic limit of the resolvent. Using κ as the expansion variable rather than c^{-1} allows us to group units of energy together and makes for easy dimensional analysis of equations. By allowing c to retain its usual value, $H^0(\kappa)$ becomes the usual free Dirac operator (with rest mass subtracted) when $\kappa = 1$. For $\kappa \neq 0$, $D(H^0(\kappa)) = D(\mathscr{D})$ and $H^0(\kappa)$ is self-adjoint when κ is real.

Writing $H^0(\kappa)$ in matrix form, we have

$$H^0(\kappa) = \begin{bmatrix} 0 & \frac{c}{\kappa} D^* \\ \frac{c}{\kappa} D & \frac{2mc^2}{\kappa^2} \end{bmatrix}. \tag{6.6}$$

If we let

$$C(\kappa)=P_{+}+\frac{1}{\kappa}P_{-}=\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\kappa} \end{bmatrix}. \tag{6.7}$$

then $C(\kappa)$ is bounded invertible, $H^0(\kappa)=C(\kappa)H^0(1)C(\kappa)$, and $C(\kappa)$ and $C(\kappa)^*$ preserve $D(\mathscr{D})$ and $Q(\mathscr{D})$. It is clear that $D(H^0(\kappa))=D(\mathscr{D})=D(|\mathscr{D}|)$. To see that $Q(H^0(1))=D(|\mathscr{D}|^{1/2})$, one may use a spectral respresentation for $c\mathscr{D}+mc^2\beta$ and the fact that $H^0(1)=c\mathscr{D}+mc^2\beta-mc^2$, to show that

$$D(|H^0(1)|^{1/2})=D(|c\mathscr{D}+mc^2\beta|^{1/2}).$$

Since $|c\mathscr{D}+mc^2\beta|=(c^2\mathscr{D}^2+m^2c^4)^{1/2}$, using a spectral representation for $\mathscr{D}$ one may show that $D(|c\mathscr{D}+mc^2\beta|^{1/2})=D(|\mathscr{D}|^{1/2})$. Since $C(\kappa)$ preserves $Q(H^0(1))=D(|\mathscr{D}|^{1/2})$, $H^0(\kappa)$ is formable as $C(\kappa)H^0(1)C(\kappa)$.

Before adding the potential term to the operator, we recall the following definition ([RS], [K]).

Definition. — Let A and V be self-adjoint operators. We say that V is *form bounded with respect to* A (or V is A-*form bounded*) if

(i) $Q(A)\supseteq Q(V)$ and

(ii) there exist constants $a>0$ and $b\geq 0$ such that

$$|h_V[x]|\leq a|h_A[x]|+b(x,x) \qquad \forall x\in Q(A) \tag{6.8}$$

If (6.8) holds for a particular choice of a and b, we will say that V is A-form bounded *with constants a, b*. The infimum of all such a's is called the relative bound. If the relative bound is zero, we say that V is *infinitessimally* A-form bounded.

To define the full Dirac operator, we introduce a self-adjoint operator V, representing the electrostatic potential, such that $V=\beta V\beta$ and V is $c|\mathscr{D}|$-form bounded. That is, V has the matrix representation

$$\begin{bmatrix} V_{+} & 0 \\ 0 & V_{-} \end{bmatrix}, \tag{6.9}$$

where $V_{\pm}$ are self-adjoint operators on $\mathscr{H}^{\pm}$, and V satisfies

$$|h_V[x]|\leq a h_{c|\mathscr{D}|}[x]+b(x,x) \qquad \forall x\in Q(\mathscr{D}) \tag{6.10}$$

for some $a>0$ and $b>0$. (Note: The case $b=0$ can be dealt with as a limit.)

Remarks. — 1) Since V is diagonal relative to the decomposition of $\mathscr{H}$, it is more natural (and weaker) to use $|\mathscr{D}|$ rather than $\mathscr{D}$ in the form boundedness condition. In fact, if V is $c\mathscr{D}$-form bounded, one may use the fact that $\mathscr{D}$ is off-diagonal to show that V is a bounded operator on

$\mathscr{H}(\|V\| \leqq b)$. By using (6.4), (6.9), and (6.10), we see that $V_\pm$ is $cD_\pm$-form bounded. When V is positive, this is equivalent to saying that $V_\pm^{1/2}$ is $(cD_\pm)^{1/2}$-bounded.

2) Condition (6.10) also follows from the assumption that V is $\mathscr{D}$-bounded (as opposed to *form* bounded, *see* e. g. [RS], Thm X.18). Thus, our results will generalize results where relative boundedness has been assumed (e. g. [GGT], [GNP]).

Let $k=b/a$, where a and b are as in (6.10). For the remainder of this paper we will let

$$J = c|\mathscr{D}| + k \tag{6.11}$$

and we will define $J_+ = cD_+ + k$ and $J_- = cD_- + k$, so

$$J = \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix}. \tag{6.12}$$

Then V is J-form bounded with constants $a_1 = a$, $b_1 = 0$. Since J is positive and $\mathscr{Q}_J = Q(H^0(1)) \subseteqq Q(V)$, we may define the *extended Dirac operator* by

$$(H(\kappa))_{\mathscr{Q}_J} = (H^0(\kappa))_{\mathscr{Q}_J} + V_{\mathscr{Q}_J}. \tag{6.13}$$

As in equations (5.1) and (5.2), we may then define the *abstract (perturbed) Dirac operator* as the form sum

$$H(\kappa) = H^0(\kappa) \dotplus V. \tag{6.14}$$

In Sections 8 and 9 we will find conditions on a, b, κ, and z under which $H(\kappa)$ is closed and densely defined.

Since V is $c|\mathscr{D}|$-form bounded, V is infinitessimally $(1/2m)\mathscr{D}^2$-form bounded and the form sum $(1/2m)\mathscr{D}^2 \dotplus V$ is self-adjoint with domain contained in $D(\mathscr{D})$ (*cf.* Cor. 5.3 or [F], Thm 5.2). Thus the Pauli-Schrödinger operators

$$H_\pm = \frac{1}{2m} D_\pm^2 \dotplus \pm V_\pm \tag{6.15}$$

are self-adjoint with $D(H_\pm) \subseteqq D(D_\pm) \subseteqq \mathscr{H}^\pm$. H_+ and H_- are identified as the nonrelativistic Hamiltonian for the electron and the positron, respectively. We also define

$$H_\pm^0 = \frac{1}{2m} D_\pm^2. \tag{6.16}$$

We will have occasion to use operators such as $\begin{bmatrix} (H_+ - z)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, which by abuse of notations we will often write as $(H_+ - z)^{-1} P_+$. We note that the discussion at the end of Section 3 on orthogonal projections applies to P_+ and P_-.

Remarks. — 3) As in related papers ([CC], [GGT], [GNP], [W]), magnetic fields can easily be incorporated. Let $\mathscr{D}_0$ and $\mathbf{A}$ be self-adjoint operators on $\mathscr{H}$ which anticommute with β, such that $\mathbf{A}$ is $c\mathscr{D}_0$-form bounded with relative bound $a_{\mathbf{A}}<1$. Here $\mathscr{D}_0$ may be thought of as the momentum operator and $\mathbf{A}$ as the vector potential. If we define $\mathscr{D}$ by the form sum

$$\mathscr{D}=\mathscr{D}_0 \dotplus \frac{1}{c}\mathbf{A},$$

then $\mathscr{D}$ is self-adjoint. It is easy to show that $\mathscr{D}$ anticommutes with β using the definition of the form sum.

7. EXPANSION OF THE FREE DIRAC RESOLVENT

Our first step in expanding the resolvent of the perturbed Dirac operator is to expand the resolvent of the free Dirac operator. Under the assumption of relatively bounded potentials, it is known that the Dirac resolvent is holomorphic in κ $\left(\text{or } \frac{1}{c}, \textit{ see } [\text{V1}], [\text{H}], [\text{GGT}]\right)$ in a z-dependent neighborhood of $\kappa=0$ for $\operatorname{Im} z\neq 0$. An explicit power series expansion was given in [GGT]. We begin with this expansion for the free Dirac resolvent ($V=0$) and extend this expansion to the scale of spaces. It turns out that both expansions converge in the same region. In the next section, our strategy for the form bounded case will be to use the extended free expansion and the Born series to derive the extended perturbed expansion.

The following theorem is a consequence of equation (2.34) of [GGT] and the estimates which follow it.

THEOREM 7.1. — *Let* $H^0(\kappa)$ *be defined as in equation* (6.5) *and let* $z\notin\sigma(H^0_+)$. *Let*

$$R^0(z,\kappa)=\begin{Bmatrix} (H^0(\kappa)-z)^{-1} & \textit{if } |\kappa|>0 \\ (H^0_+-z)^{-1}P_+ & \textit{if } |\kappa|=0 \end{Bmatrix} \tag{7.1}$$

Then $R^0(z,\kappa)$ *is norm holomorphic in* κ *in a* z*-dependent neighborhood of* $\kappa=0$. *Explicitly, for* $z\notin\sigma(H^0_+)$, *if* $|\kappa|$ *is small enough, then*

$$R^0(z,\kappa)=\sum_{n=0}^{\infty}\kappa^n K_n(z) \tag{7.2}$$

where

$$K_0(z)=(H^0_+-z)^{-1}P_+ \tag{7.3}$$

and for $n \geqq 1$

$$K_{2n-1}(z) = \frac{z^{2n-2}}{(2mc^2)^n} \begin{pmatrix} 0 & c\,D^*(H^0_- - z)^{-n} \\ c\,D\,(H^0_+ - z)^{-n} & 0 \end{pmatrix}, \tag{7.4}$$

$$K_{2n}(z) = \frac{z^{2n-1}}{(2mc^2)^n} \begin{pmatrix} H^0_+ (H^0_+ - z)^{-n-1} & 0 \\ 0 & (H^0_- - z)^{-n} \end{pmatrix}. \tag{7.5}$$

In particular, (7.2) *is valid if*

$$(i) \quad |\kappa| < \left(\frac{2mc^2}{|z|}\right)^{1/2} \quad \textit{and} \quad \operatorname{Re} z < 0$$

or

$$(ii) \quad |\kappa| < \frac{(2mc^2)^{1/2}\,|\operatorname{Im} z|^{1/2}}{|z|}.$$

Remarks. — 1) The region of convergence in the z-plane given by conditions (i) and (ii) of Theorem 7.1 consists of the union of the open half disk $|z| < \dfrac{2mc^2}{|\kappa|^2}$, $\operatorname{Re} z < 0$ and the two open disks $\left| z \pm \dfrac{mc^2}{|\kappa|^2} i \right| < \dfrac{mc^2}{|\kappa|^2}$. For z not on the positive real axis, we can always choose $|\kappa|$ small enough that one of these conditions holds.

Since $H^0(\kappa)$ is formable, it may be extended to $(H^0(\kappa))_{\mathscr{Q}_J} : \mathscr{Q}_J \to \mathscr{Q}_J^*$. We may also extend $(H^0(\kappa) - z)^{-1}$ and the K_n to the scale of spaces and it is natural to ask whether there is a suitable extension of Theorem 7.1 to the scale of spaces. In order to answer this question, we will require estimates on $\| (K_n(z))_{\mathscr{Q}_J^*} \|_{\mathscr{Q}_J^* \to \mathscr{Q}_J}$. These estimates will be used again in discussing the convergence of the power series expansion of the perturbed resolvent.

To write our estimates of $\| (K_n(z))_{\mathscr{Q}_J^*} \|_{\mathscr{Q}_J^* \to \mathscr{Q}_J}$ more compactly, we define quantities $e_1(z, k)$, $e_2(z)$, and $e_3(z)$ and estimate them as follows:

$$\| J_\pm (H^0_\pm - z)^{-1} \|_{\mathscr{H}^\pm} \leqq \sup_{x \in \sigma(D_\pm)} \frac{cx + k}{|(1/2m)x^2 - z|}$$

$$\equiv e_1(z, k) \leqq \left\{ \begin{array}{ll} \dfrac{1}{2|z|}\left(k + \sqrt{k^2 + 2mc^2|z|}\right) & \text{if } z \in \mathbb{R}^- \\ \left(\dfrac{mc^2}{|\operatorname{Im} z|}\right)^{1/2} + \dfrac{k + \sqrt{2mc^2 |\operatorname{Re} z|}}{|\operatorname{Im} z|} & \text{if } z \in \mathbb{C} \setminus \mathbb{R} \end{array} \right\} \tag{7.6}$$

$$\| H^0_\pm (H^0_\pm - z)^{-1} \|_{\mathscr{H}^\pm} \leqq \sup_{x \in \sigma(D_\pm)} \frac{(1/2m)x^2}{|(1/2m)x^2 - z|}$$

$$\equiv e_2(z) \leqq \left\{ \begin{array}{ll} 1 & \text{if } z \in \mathbb{R}^- \\ 1 + \left|\dfrac{\operatorname{Re} z}{\operatorname{Im} z}\right| & \text{if } z \in \mathbb{C} \setminus \mathbb{R} \end{array} \right\} \tag{7.7}$$

$$\|(H^0_\pm - z)^{-1}\|_{\mathscr{H}^\pm} \leqq \sup_{x \in \sigma(D_\pm)} \frac{1}{|(1/2m)x^2 - z|}$$
$$\equiv e_3(z) \leqq \begin{cases} 1/|z| & \text{if } z \in \mathbb{R}^- \\ 1/|\operatorname{Im} z| & \text{if } z \in \mathbb{C}\setminus\mathbb{R} \end{cases} \quad (7.8)$$

We note that for $z \in \mathbb{R}^-$, we have equality in the above if $\sigma(D_\pm) = \mathbb{R}^+ \cup \{0\}$. The estimates for $z \in \mathbb{C}\setminus\mathbb{R}$ have not been optimized. We also note that if z is on the positive real axis, but $z \notin \sigma(H^0_\pm)$, then e_1, e_2, and e_3 are finite.

For $z \notin \sigma(H^0_\pm)$, $\operatorname{Ran}(K_n(z)) \subseteqq D(J) \subseteqq \mathscr{Q}_J$ for $n = 0, 1, 2, \ldots$, and we may define $(K_n(z))_{\mathscr{Q}_J^*}$ via (3.11). Using (3.14) and Lemma 3.4, we have

$$\|(K_n(z))_{\mathscr{Q}_J^*}\|_{\mathscr{Q}_J^* \to \mathscr{Q}_J} = \|J^{1/2}(K_n(z))_{\mathscr{Q}_J^*} J^{1/2}_{\mathscr{H}}\| = \|\overline{J^{1/2} K_n(z) J^{1/2}}\|. \quad (7.9)$$

We will be able to estimate this last norm for each n by using the estimates above.

Turning first to $K_0(z)$, we have

$$J^{1/2} K_0(z) J^{1/2} = J^{1/2}_+ (H^0_+ - z)^{-1} J^{1/2}_+ P_+ \subseteqq J_+ (H^0_+ - z)^{-1} P_+ \quad (7.10)$$

So,

$$\|(K_0(z))_{\mathscr{Q}_J^*}\|_{\mathscr{Q}_J^* \to \mathscr{Q}_J} = \|\overline{J^{1/2} K_0(z) J^{1/2}}\|$$
$$= \|J_+ (H^0_+ - z)^{-1}\|_{\mathscr{H}^+} \leqq e_1(z, k) \quad (7.11)$$

From (7.5), for $n \geqq 1$ we have

$$J^{1/2} K_{2n}(z) J^{1/2} = \frac{z^{2n-1}}{(2mc^2)^n} \begin{pmatrix} J^{1/2}_+ H^0_+ (H^0_+ - z)^{-n-1} J^{1/2}_+ & 0 \\ 0 & J^{1/2}_- (H^0_- - z)^{-n} J^{1/2}_- \end{pmatrix}$$
$$\subseteqq \frac{z^{2n-1}}{(2mc^2)^n} \begin{pmatrix} J_+ (H^0_+ - z)^{-1} H^0_+ (H^0_+ - z)^{-n} & 0 \\ 0 & J_- (H^0_- - z)^{-n} \end{pmatrix}. \quad (7.12)$$

From (7.12) and (7.9) we see that

$$\|(K_{2n}(z))_{\mathscr{Q}_J^*}\|_{\mathscr{Q}_J^* \to \mathscr{Q}_J} \leqq \frac{|z|^{2n-1}}{(2mc^2)^n} e_1(z, k) \max(e_2(z), 1)(e_3(z))^{n-1}. \quad (7.13)$$

To estimate $\|\overline{J^{1/2} K_{2n-1}(z) J^{1/2}}\|$, we use $\|A\| = \|A^* A\|^{1/2}$. We have

$$(J^{1/2} K_{2n-1}(z) J^{1/2})^* (J^{1/2} K_{2n-1}(z) J^{1/2})$$
$$\subseteqq \left(\frac{|z|^{2n-2}}{(2mc^2)^n}\right)^2 \begin{pmatrix} c D_+ J_+ (H^0_+ - \bar{z})^{-n} c D_+ J_+ (H^0_+ - z)^{-n} & 0 \\ 0 & c D_- J_- (H^0_- - \bar{z})^{-n} c D_- J_- (H^0_- - z)^{-n} \end{pmatrix} \quad (7.14)$$

where we have used $D_- D = D D_+$ and $D_+ D^* = D^* D_-$ (which follow from the fact that $|\mathscr{D}|$ commutes with $\mathscr{D}$) as well as

$$(H^0_- - z)^{-1} D \subseteqq D (H^0_+ - z)^{-1}$$

and $(H^0_+ - z)^{-1} D^* \subseteq D^* (H^0_- - z)^{-1}$ (*see* [De]). It follows from (7.14) and (7.9) that

$$\| (K_{2n-1}(z))_{\mathcal{Q}_J^*} \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J} \leqq \frac{|z|^{2n-2}}{(2mc^2)^n} (2mc^2 e_2(z) + k e_1(z, 0)) (e_3(z))^{n-1}. \quad (7.15)$$

For convenience, we have made estimates which are uniform in $n \geqq 1$. We note, however, that these estimates can be improved for $n > 1$.

From (7.13) and (7.15), we see that in general for $n \geqq 1$, we have

$$\| (K_n(z))_{\mathcal{Q}_J^*} \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J} \leqq M(z, k) \left(\frac{|z|}{(2mc^2)^{1/2}} (e_3(z))^{1/2} \right)^n, \quad (7.16)$$

where

$$M(z) = \max \left(\frac{2mc^2 e_2(z) + k e_1(z, 0)}{|z| (2mc^2)^{1/2} (e_3(z))^{1/2}}, \frac{e_1(z, k) \max(e_2(z), 1)}{|z| e_3(z)} \right). \quad (7.17)$$

We are now in a position to extend Theorem 7.1:

THEOREM 7.2. — *Let* $H^0(\kappa)$ *be defined as in equation* (6.5) *and let* $z \notin \sigma(H^0_+)$. *Let*

$$R^0_{\mathcal{Q}_J^*}(z, \kappa) = \begin{cases} ((H^0(\kappa))_{\mathcal{Q}_J} - z)^{-1} & \text{if } |\kappa| > 0 \\ (H^0_+ - z)^{-1}_{\mathcal{Q}_{J+}^*} P_+ & \text{if } |\kappa| = 0 \end{cases} \quad (7.18)$$

Then $R^0_{\mathcal{Q}_J^*}(z, \kappa)$ *is norm holomorphic in* κ *in a z-dependent neighborhood of* $\kappa = 0$. *Explicitly, for* $z \notin \sigma(H^0_+)$, *if* $|\kappa|$ *is small enough, then*

$$R^0_{\mathcal{Q}_J^*}(z, \kappa) = \sum_{n=0}^{\infty} \kappa^n (K_n(z))_{\mathcal{Q}_J^*}. \quad (7.19)$$

In particular, if conditions (i) *or* (ii) *of Theorem* 7.1 *hold, then* (7.19) *is valid.*

Proof. — B the operator estimates (7.16) we have

$$\sum_{n=1}^{\infty} |\kappa|^n \| (K_n(z))_{\mathcal{Q}_J^*} \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J} \leqq M(z) \sum_{n=1}^{\infty} |\kappa|^n \left(\frac{|z|}{(2mc^2)^{1/2}} (e_3(z))^{1/2} \right)^n. \quad (7.20)$$

So $\sum_{n=0}^{\infty} |\kappa|^n \| (K_n(z))_{\mathcal{Q}_J^*} \|_{\mathcal{Q}_J^* \to \mathcal{Q}_J}$ converges in norm for $z \notin \sigma(H^0_+)$ for $|\kappa|$ small enough. Let $x \in \mathcal{H}$ and $Y \in \mathcal{Q}_J^*$. Then, using Theorem 7.1,

$$\left(x, \left(\sum_{n=1}^{\infty} \kappa^n (K_n(z))_{\mathcal{Q}_J^*} \right) Y \right)$$

$$= \sum_{n=1}^{\infty} (x, \kappa^n (K_n(z))_{\mathcal{Q}_J^*} Y) = \sum_{n=1}^{\infty} \langle \bar{\kappa}^n K_n(\bar{z}) x, Y \rangle$$

$$= \langle (H^0(\bar{\kappa}) - \bar{z})^{-1} x, Y \rangle = (x, (H^0(\kappa) - z)^{-1}_{\mathcal{Q}_J^*} Y) \quad (7.21)$$

is valid for $|\kappa|>0$ small enough. Since $\mathcal{H}$ is dense in $\mathcal{Q}_J^*$, (7.21) shows weak, and hence norm, convergence of $\sum_{n=0}^{\infty} \kappa^n (K_n(z))_{\mathcal{Q}_J^*}$ to $(H^0(\kappa)-z)^{-1}_{\mathcal{Q}_J^*}$. Again from Theorem 7.1, it follows that for $|\kappa|>0$ small enough, $z \notin \sigma(H^0(\kappa))$ and so by Lemma 4.4, $((H^0(\kappa))_{\mathcal{Q}_J}-z)^{-1}=(H^0(\kappa)-z)^{-1}_{\mathcal{Q}_J^*}$. In particular, all of the above holds when conditions (*i*) or (*ii*) hold. Our conclusions now follow immediately. □

8. EXPANSION OF THE PERTURBED DIRAC RESOLVENT

We begin this section with a formal calculation of the power series expansion for the resolvent of the extended Dirac operator. We show that this power series is indeed the resolvent of the extended Dirac operator whenever it converges. When the series converges for some $\kappa \neq 0$, the Dirac operator (6.14) is closed and densely defined and results on the holomorphy of the Dirac resolvent follow. Finally we discuss analyticity of isolated eigenvalues.

While one may specify conditions under which each of the steps in the following expansion of the resolvent are valid, the result of the calculation is valid under broader conditions. Hence, we prefer to do the calculation formally and justify it afterwards. However, we give (justifiable) heuristics for each step.

As in equation (5.8) of Corollary 5.2, we express the extended Dirac resolvent $((H^0(\kappa))_{\mathcal{Q}_J}+V_{\mathcal{Q}_J}-z)^{-1}$ as

$$((H^0(\kappa))_{\mathcal{Q}_J}-z)^{-1}-((H^0(\kappa))_{\mathcal{Q}_J}-z)^{-1}V_{\mathcal{Q}_J}((H^0(\kappa))_{\mathcal{Q}_J}-z)^{-1}+\dots \qquad (8.1)$$

and then we substitute the expansion (7.19) to get

$$\left(\sum_{n=0}^{\infty} \kappa^n (K_n(z))_{\mathcal{Q}_J^*}\right)-\left(\sum_{n=0}^{\infty} \kappa^n (K_n(z))_{\mathcal{Q}_J^*}\right)V_{\mathcal{Q}_J}\left(\sum_{n=0}^{\infty} \kappa^n (K_n(z))_{\mathcal{Q}_J^*}\right)+\dots. \qquad (8.2)$$

Using the notation $\alpha=(\alpha_1, \dots, \alpha_l)$ to indicate an l-tuple of nonnegative integers and $|\alpha|=\sum_{i=1}^{l} \alpha_i$, we rearrange the above to yield

$$\sum_{n=0}^{\infty} \kappa^n \sum_{l=1}^{\infty} (-1)^{l+1} \sum_{\substack{|\alpha|=n\\ \alpha_i \geqq 0}} (K_{\alpha_1}(z))_{\mathcal{Q}_J^*} V_{\mathcal{Q}_J} \dots V_{\mathcal{Q}_J} (K_{\alpha_l}(z))_{\mathcal{Q}_J^*}. \qquad (8.3)$$

It turns out that the $(K_0(z))_{\mathcal{Q}_J^*}$'s may be isolated and summed neatly. Since

$$(K_0(z))_{\mathcal{Q}_J^*}=((H_+^0-z)^{-1}P_+)_{\mathcal{Q}_J^*}=(H_+^0-z)^{-1}_{\mathcal{Q}_{J_+}^*}P_+, \qquad (8.4)$$

using Corollary 5.3 gives

$$(K_0(z))_{\mathscr{D}_J^*}\left(\sum_{s=0}^{\infty}(V_{\mathscr{D}_J}(K_0(z))_{\mathscr{D}_J^*})^s\right)=(H_+-z)^{-1}_{\mathscr{D}_{J_+}^*}P_+ \tag{8.5}$$

and we may write

$$\begin{aligned} L\equiv\sum_{s=0}^{\infty}(-(K_0(z))_{\mathscr{D}_J^*}V_{\mathscr{D}_J})^s&=1-(K_0(z))_{\mathscr{D}_J^*}\left(\sum_{s=0}^{\infty}(V_{\mathscr{D}_J}(K_0(z))_{\mathscr{D}_J^*})^s\right)V_{\mathscr{D}_J}\\ &=1-(H_+-z)^{-1}_{\mathscr{D}_{J_+}^*}(V_+)_{\mathscr{D}_{J_+}}P_+, \end{aligned} \tag{8.6}$$

$$N\equiv\sum_{s=0}^{\infty}(-V_{\mathscr{D}_J}(K_0(z))_{\mathscr{D}_J^*})^s=1-(V_+)_{\mathscr{D}_{J_+}}(H_+-z)^{-1}_{\mathscr{D}_{J_+}^*}P_+.$$

We note that $V_{\mathscr{D}_J}L=NV_{\mathscr{D}_J}$. With these definitions (8.3) becomes

$$(H_+-z)^{-1}_{\mathscr{D}_{J_+}^*}P_++\sum_{n=1}^{\infty}\kappa^n\sum_{l=1}^{n}(-1)^{l+1}\sum_{\substack{|\alpha|=n\\ \alpha_i\geqq 0}}L(K_{(\alpha)}(z))_{\mathscr{D}_J^*}N \tag{8.7}$$

where for simplicity of expression we have defined

$$(K_{(\alpha)}(z))_{\mathscr{D}_J^*}=(K_{\alpha_1}(z))_{\mathscr{D}_J^*}NV_{\mathscr{D}_J}\dots NV_{\mathscr{D}_J}(K_{\alpha_l}(z))_{\mathscr{D}_J^*}. \tag{8.8}$$

Equation (8.7) is the power series expansion we desire. We will now show that when this series converges, it is the resolvent of the extended Dirac operator (6.13).

THEOREM 8.1. — *Let* $H^0(\kappa)$, V, *and* J *be as defined in Section* 6. *Assume that* z *and* $\kappa\neq 0$ *are such that* $\sum_{n=0}^{\infty}\kappa^n(K_n(z))_{\mathscr{D}_J^*}$ *converges in norm to* $((H^0(\kappa))_{\mathscr{D}_J}-z)^{-1}$. *Assume that the series* (8.7) *converges in norm to an operator* $R_{\mathscr{D}_J^*}(z,\kappa):\mathscr{D}_J^*\to\mathscr{D}_J$. *Then*

$$R_{\mathscr{D}_J^*}(z,\kappa)=((H^0(\kappa))_{\mathscr{D}_J}+V_{\mathscr{D}_J}-z)^{-1}. \tag{8.9}$$

Proof. – Using $(K_0(z))_{\mathscr{Q}_J^*} V_{\mathscr{Q}_J} L = 1 - L$, $(K_0(z))_{\mathscr{Q}_J^*} N = (H_+ - z)^{-1}_{\mathscr{Q}_{J_+}^*} P_+$, and $V_{\mathscr{Q}_J} L = N V_{\mathscr{Q}_J}$, we have

$$\left(\sum_{n=0}^{\infty} \kappa^n (K_n(z))_{\mathscr{Q}_J^*}\right) V_{\mathscr{Q}_J} \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} L (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$= (K_0(z))_{\mathscr{Q}_J^*} V_{\mathscr{Q}_J} \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} L (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$+ \left(\sum_{n=1}^{\infty} \kappa^n (K_n(z))_{\mathscr{Q}_J^*}\right) \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} N V_{\mathscr{Q}_J} (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$= (K_0(z))_{\mathscr{Q}_J^*} V_{\mathscr{Q}_J} L \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$+ \sum_{n=2}^{\infty} \kappa^n \sum_{l=2}^{n} (-1)^{l} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$= (1 - L) \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ a_i \geqq 1}} (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N \tag{8.10}$$

$$+ \left(\sum_{n=1}^{\infty} \kappa^n (K_n(z))_{\mathscr{Q}_J^*}\right) N - \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$= (((H^0(\kappa))_{\mathscr{Q}_J} - z)^{-1} - (K_0(z))_{\mathscr{Q}_J^*}) N$$

$$- \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} L (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$= ((H^0(\kappa))_{\mathscr{Q}_J} - z)^{-1} N - R_{\mathscr{Q}_J^*}(z, \kappa).$$

Thus,

$$V_{\mathscr{Q}_J} R_{\mathscr{Q}_J^*}(z, \kappa) = V_{\mathscr{Q}_J} (H_+ - z)^{-1}_{\mathscr{Q}_{J_+}^*} P_+$$

$$+ V_{\mathscr{Q}_J} \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} L (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N$$

$$= V_{\mathscr{Q}_J} (H_+ - z)^{-1}_{\mathscr{Q}_{J_+}^*} P_+ + N - ((H^0(\kappa))_{\mathscr{Q}_J} - z) R_{\mathscr{Q}_J^*}(z, \kappa)$$

$$= I - ((H^0(\kappa))_{\mathscr{Q}_J} - z) R_{\mathscr{Q}_J^*}(z, \kappa). \tag{8.11}$$

From this (8.9) follows immediately. □

COROLLARY 8.2. — *Under the conditions of Theorem* 8.1, *the Dirac operator* (6.14) *is closed and densely defined.*

Proof. — By Theorem 8.1, $(H^0(\kappa))_{\mathscr{Q}_J}+V_{\mathscr{Q}_J}$ is closed on $\mathscr{Q}_J^*$ and its resolvent operator is bounded as a map from $\mathscr{Q}_J^*$ to $\mathscr{Q}_J$. By Theorem 5.1 the Dirac operator (6.14) is closed and densely defined. □

We now turn to the question of when the series (8.7) converges in norm. Using (6.10), it is easy to see that

$$\| V_{\mathscr{Q}_J} \|_{\mathscr{Q}_J \to \mathscr{Q}_J^*} \leqq a, \tag{8.12}$$

By Cor. 5.3 and the remark following it (or [F] Theorem 5.2), H_+ is self-adjoint and $D(H_+) \subseteqq D(\mathscr{D}) = D(J)$. Hence, for $z \notin \sigma(H_+)$, we have $\operatorname{Ran}((H_+ - z)^{-1}) = \operatorname{Ran}((H_+ - \bar{z})^{-1}) \subseteqq D(J)$ and by Lemma 3.4, $(H_+ - z)^{-1}_{\mathscr{Q}_{J_+}^*} : \mathscr{Q}_{J_+}^* \to \mathscr{Q}_{J_+}$ is bounded. If we let

$$e_4(z) = 1 + a \| (H_+ - z)^{-1}_{\mathscr{Q}_{J_+}^*} \|_{\mathscr{Q}_{J_+}^* \to \mathscr{Q}_{J_+}}, \tag{8.13}$$

then

$$\| L \|_{\mathscr{Q}_J} \leqq e_4(z) \qquad \text{and} \qquad \| N \|_{\mathscr{Q}_J^*} \leqq e_4(z). \tag{8.14}$$

Using the estimates (7.16), we have

$$\begin{aligned} &\| L (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N \|_{\mathscr{Q}_J^* \to \mathscr{Q}_J} \\ &\qquad \leqq \| L \|_{\mathscr{Q}_J} \| N \|_{\mathscr{Q}_J^*}^{l} \| V \|_{\mathscr{Q}_J \to \mathscr{Q}_J^*}^{l-1} \prod_{i=1}^{l} \| (K_{\alpha_i}(z))_{\mathscr{Q}_J^*} \|_{\mathscr{Q}_J^* \to \mathscr{Q}_J} \\ &\qquad \leqq a^{l-1} (e_4(z))^{l+1} \prod_{i=1}^{l} M(z) \left(\frac{|z|}{(2mc^2)^{1/2}} (e_3(z))^{1/2} \right)^{\alpha_i} \\ &\qquad = \frac{e_4(z)}{a} (M(z)\, a e_4(z))^l \left(\frac{|z|}{(2mc^2)^{1/2}} (e_3(z))^{1/2} \right)^{|\alpha|} \end{aligned} \tag{8.15}$$

Writing $h(z) = M(z)\, a e_4(z)$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} |\kappa|^n \sum_{l=1}^{n} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} \| L (K_{(\alpha)}(z))_{\mathscr{Q}_J^*} N \|_{\mathscr{Q}_J^* \to \mathscr{Q}_J} \\ &\qquad \leqq \sum_{n=1}^{\infty} |\kappa|^n \sum_{l=1}^{n} \sum_{\substack{|\alpha|=n \\ a_i \geqq 1}} \frac{e_4(z)}{a} (h(z))^l \\ &\qquad = \frac{e_4(z)}{\mathfrak{a}} \sum_{n=1}^{\infty} |\kappa|^n \left(\frac{|z|}{(2mc^2)^{1/2}} (e_3(z))^{1/2} \right)^n \sum_{l=1}^{n} \tau_l(n) (h(z))^l \\ &\qquad = \frac{e_4(z)}{a} \sum_{n=1}^{\infty} |\kappa|^n \left(\frac{|z|}{(2mc^2)^{1/2}} (e_3(z))^{1/2} \right)^n h(z) (1 + h(z))^{n-1} \end{aligned} \tag{8.16}$$

where $\tau_l(n)=\begin{pmatrix} n-1 \\ l-1 \end{pmatrix}$ is the number of ordered l-tuples of positive integers whose sum is n. This sum converges then if

$$|\kappa|\left(\frac{|z|}{(2mc^2)^{1/2}}(e_3(z))^{1/2}\right)(1+h(z))<1. \tag{8.17}$$

So for all $z\in\mathbb{C}\setminus(\sigma(H_+)\cup\sigma(H_+^0))$, there is a neighborhood of $\kappa=0$ in which the series converges.

Using this result we may show that the resolvent of the extended Dirac operator (6.13) is defined and holomorphic.

THEOREM 8.3. – *Let $z\notin\sigma(H_+)\cup\sigma(H_+^0)$. Let*

$$R_{\mathcal{Q}_J^*}(z,\kappa)=\left\{\begin{array}{lll} ((H(\kappa))_{\mathcal{Q}_J}-z)^{-1} & \textit{if} & |\kappa|>0 \\ (H_+-z)^{-1}_{\mathcal{Q}_{J_+}^*}P_+ & \textit{if} & |\kappa|=0 \end{array}\right\} \tag{8.18}$$

Then $R_{\mathcal{Q}_J^*}(z,\kappa)$ is norm holomorphic in κ in a z-dependent neighborhood of $\kappa=0$.

Proof. – By Theorem 7.2, for any $z\notin\sigma(H_+^0)$ we may choose $|\kappa|$ small enough to give norm convergence of $\sum_{n=0}^{\infty}\kappa^n(K_n(z))_{\mathcal{Q}_J^*}$ to $((H^0(\kappa))_{\mathcal{Q}_J}-z)^{-1}$. The above calculation, together with Theorem 8.1, shows that for any $z\notin\sigma(H_+)\cup\sigma(H_+^0)$, we may choose $|\kappa|$ small enough that (8.7) converges in norm to $((H^0(\kappa))_{\mathcal{Q}_J}+V_{\mathcal{Q}_J}-z)^{-1}=((H(\kappa))_{\mathcal{Q}_J}-z)^{-1}$. That is, $(8.7)=R_{\mathcal{Q}_J^*}(z,\kappa)$ for $|\kappa|$ small enough, so our conclusion follows. □

A similar result then follows for the Dirac resolvent.

COROLLARY 8.4. – *Let $z\notin\sigma(H_+)\cup\sigma(H_+^0)$. Let*

$$R(z,\kappa)=\left\{\begin{array}{lll} (H(\kappa)-z)^{-1} & \textit{if} & |\kappa|>0 \\ (H_+-z)^{-1}P_+ & \textit{if} & |\kappa|=0 \end{array}\right\} \tag{8.19}$$

Then $R(z,\kappa)$ is norm holomorphic in κ in a z-dependent neighborhood of $\kappa=0$.

Proof. – We have $(H(\kappa)-z)^{-1}=((H(\kappa))_{\mathcal{Q}_J}-z)^{-1}|_{\mathcal{H}}$ and for $|\kappa|>0$ small enough, $((H(\kappa))_{\mathcal{Q}_J}-z)^{-1}=(8.7)$. Since norm convergence as a map from $\mathcal{Q}_J^*$ to $\mathcal{Q}_J$ implies norm convergence as a map from $\mathcal{H}$ to $\mathcal{H}$ [*cf.* (5.6)], for $|\kappa|>0$ small enough, we have

$$\begin{aligned}(H(\kappa)-z)^{-1}=(H_+-z)^{-1}P_++\sum_{n=1}^{\infty}\kappa^n\sum_{l=1}^{n}(-1)^{l+1}\\ \times\sum_{\substack{|\alpha|=n\\ \alpha_i\geq 1}}L(K_{(\alpha)}(z))_{\mathcal{Q}_J^*}(1-(V_+)_{\mathcal{Q}_{J_+}}(H_+-z)^{-1}P_+). \quad \square\end{aligned} \tag{8.20}$$

Remarks. — 1) Assume that $\kappa \in \mathbb{R}$. When (8.17) holds for some $z_0 \notin \mathbb{R}$, it holds also for $\bar{z}_0$. So if (8.17) holds for $\kappa \in \mathbb{R}$ and any $z \in \mathbb{C}$, by the remark following Theorem 5.1, $H(\kappa)$ is self-adjoint.

2) If we assume that $D(\mathcal{D}) \subseteq D(V)$, *i.e.* that V is relatively bounded with respect to $\mathcal{D}$, then the expansion for $(H(\kappa)-z)^{-1}$ may be slightly simplified. We have $V_{\mathcal{Q}_J}|_{D(\mathcal{D})} = V|_{D(\mathcal{D})}$ and in general we have $R_{\mathcal{Q}_J^*}|_{\mathcal{H}} = R$ when $R_{\mathcal{Q}_J^*}$ is the extension via (3.11) of a bounded operator R. Thus, we may drop the $\mathcal{Q}_J$'s and $\mathcal{Q}_J^*$'s which appear in (8.7). For this case, if we redefine

$$L = 1-(H_+-z)^{-1} V_+ P_+ \quad \text{and} \quad N = 1 - V_+ (H_+ - z)^{-1} P_+ \tag{8.21}$$

and define

$$K_{(\alpha)}(z) = K_{\alpha_1}(z)\, NV \dots NV\, K_{\alpha_l}(z) \tag{8.22}$$

then the expansion (8.20) becomes

$$(H(\kappa)-z)^{-1} = (H_+ - z)^{-1} P_+ + \sum_{n=1}^{\infty} \kappa^n \sum_{l=1}^{n} (-1)^{l+1} \sum_{\substack{|\alpha|=n \\ \alpha_i \geqq 1}} L K_{(\alpha)}(z) N. \tag{8.23}$$

This expansion is equivalent to the one given in [GGT], equation (2.34). One may use this expansion to show that the form sum is the same as the operator sum when V is $\mathcal{D}$-bounded.

3) Theorem 8.3 and Corollary 8.4 demonstrate that $R_{\mathcal{Q}_J^*}(., \kappa)$ and $R(., \kappa)$ are pseudoresolvents which are analytic at zero (*see* [GNP], [V1]). In particular, we may say that $H(\kappa)$ converges to $H_+ P_+$ in the norm pseudoresolvent sense (*see* [W], *see* also [V2] for a related notion of convergence).

4) If the free Dirac operator is defined with the rest mass added [rather than subtracted as in (6.5)], then

$$H^0(\kappa) = \frac{c}{\kappa} \mathcal{D} + \frac{2mc^2}{\kappa^2} P_+ \quad .$$

and an analogous proof shows that $(H(\kappa)-z)^{-1}$ is holomorphic in κ with an analogous expansion. The limiting operator ($\kappa = 0$) in this case is $(-H_- - z)^{-1} P_-$.

To end this section we note that we can derive the standard results on analyticity of eigenvalues, by following the proofs of [GGT] and [GNP] for the relatively bounded case. Using

$$(x, (K_n(z))_{\mathcal{Q}_J^*} Y) = \langle K_n(\bar{z}) x, Y \rangle \tag{8.24}$$

it is easy to show that

$$(K_{2n}(z))_{\mathcal{Q}_J^*} : \mathcal{Q}_{J\pm}^* \to \mathcal{Q}_{J\pm} \quad \text{and} \quad (K_{2n+1}(z))_{\mathcal{Q}_J^*} : \mathcal{Q}_{J\pm}^* \to \mathcal{Q}_{J\mp}. \tag{8.25}$$

Hence, even terms (κ^{2n}) in the expansions (8.7) and (8.23) are "even" (diagonal relative to $\mathcal{Q}_J^* = \mathcal{Q}_{J+}^* \oplus \mathcal{Q}_{J-}^*$) and odd terms are "odd" (off-diagonal). Using these facts one may follow [GGT] Section 2 and [GNP] Section III, substituting form sums for operators sums as appropriate, to obtain direct analogues of the results there. For completeness we restate the Corollary of Theorem III.1 of [GNP] as its generalizes to the current case:

COROLLARY 8.5. — *Let* E_0 *be an isolated eigenvalue of* H_+ *of finite multiplicity n, and assume that the conditions of Theorem* 8.1 *are satisfied. Then for* κ *sufficiently small there are r isolated eigenvalues of* $H(\kappa)$, *given by the functions* $E_1(\kappa), \ldots, E_r(\kappa)$ *which are analytic in* κ^2 *on a neighborhood of* $\kappa = 0$, *have multiplicities* $n_1, \ldots, n_r$ *such that* $n_1 + \ldots + n_r = n$ *and satisfy* $\lim_{\varkappa \to 0} E_j(\kappa) = E_0$ *for* $j = 1, \ldots, r$. *The corresponding eigenvectors may be chosen to depend analytically in* κ (*for* κ *small*) *and to have the form*

$$f_j(\kappa) = f_j^+(\kappa) + \kappa f_j^-(\kappa) \tag{8.26}$$

where $f_j^+(\kappa) \in \mathcal{H}^+$ *and* $f_j^-(\kappa) \in \mathcal{H}^-$, $f_j^+(\kappa)$ *and* $f_j^-(\kappa)$ *are analytic in* κ^2 (*for* κ *small*) *and* $\lim_{\varkappa \to 0} f_j^+(\kappa) \in D(H_+)$.

Remarks. — 4) Relativistic corrections to nondegenerate eigenvalues may be computed using the scheme given in [GGT]. Given the explicit expansion (8.7), relativistic corrections to the eigenvalues can, in principle, be computed to any order. If the general correction term can be found, error estimates should be possible.

9. CONVERGENCE ESTIMATES AND LOWER BOUNDS

We would like to demonstrate conditions under which the series (8.7) converges. It is of particular interest to know when the Pauli-Schrödinger resolvent $(H_+ - z)^{-1}$ is part of a holomorphic family of operators which includes the usual Dirac resolvent $(H(1) - z)^{-1}$. To simplify matters (and possibly to optimize), we will assume z to be on the negative real axis. We then have:

$$\left. \begin{aligned} & e_1(z, k) \leqq \frac{1}{2|z|}(k + \sqrt{k^2 + 2mc^2|z|}), \\ & e_2(z) \leqq 1, \qquad \text{and} \qquad e_3(z) \leqq \frac{1}{|z|}. \end{aligned} \right\} \tag{9.1}$$

Using (7.10), (7.11), (8.12), and the fact that $k=\frac{b}{a}$, we have

$$\| (V_+)_{\mathcal{Q}_{J_+}} (H^0_+ - z)^{-1}_{\mathcal{Q}^*_{J_+}} \|_{\mathcal{Q}^*_{J_+}} \leqq \frac{1}{2|z|} (b + \sqrt{b^2 + 2mc^2 a^2 |z|}) \tag{9.2}$$

Now, $\frac{1}{2|z|}(b+\sqrt{b^2+2mc^2a^2|z|})<1$ is equivalent to

$$|z| > \frac{1}{2} mc^2 a^2 + b. \tag{9.3}$$

So when (9.3) holds, Corollary 5.3 applies and from (5.15) it follows that

$$\begin{aligned} a \| (H_+ - z)^{-1}_{\mathcal{Q}^*_{J_+}} \|_{\mathcal{Q}^*_{J_+} \to \mathcal{Q}_{J_+}} &\leqq a \| (H^0_+ - z)^{-1}_{\mathcal{Q}^*_{J_+}} \|_{\mathcal{Q}^*_{J_+} \to \mathcal{Q}_{J_+}} \sum_{n=0}^{\infty} \| (V_+)_{\mathcal{Q}_{J_+}} (H^0_+ - z)^{-1}_{\mathcal{Q}^*_{J_+}} \|_{\mathcal{Q}^*_{J_+}} \\ &\leqq \frac{1}{2|z|} (b + \sqrt{b^2 + 2mc^2 a^2 |z|}) \\ &\qquad \times \left(1 - \frac{1}{2|z|} (b + \sqrt{b^2 + 2mc^2 a^2 |z|})\right)^{-1}. \end{aligned} \tag{9.4}$$

Hence,

$$e_4(z) \leqq \left(1 - \frac{1}{2|z|} (b + \sqrt{b^2 + 2mc^2 a^2 |z|})\right)^{-1}. \tag{9.5}$$

Assuming that $k < \sqrt{6mc^2|z|}$, or equivalently $b < \sqrt{6mc^2a^2|z|}$, then

$$M(z) = \frac{k}{2|z|} + \left(\frac{2mc^2}{|z|}\right)^{1/2}. \tag{9.6}$$

where $M(z)$ is given by (7.17). Then

$$\begin{aligned} h(z) &= M(z)\, a e_4(z) \\ &\leqq \left(\frac{b}{2|z|} + \left(\frac{2mc^2a^2}{|z|}\right)^{1/2}\right)\left(1 - \frac{1}{2|z|}(b + \sqrt{b^2 + 2mc^2a^2|z|})\right)^{-1}, \end{aligned} \tag{9.7}$$

and from (8.17) we have convergence of (8.7) if

$$|\kappa| \left(\frac{|z|}{2mc^2}\right)^{1/2} (1 + h(z)) < 1. \tag{9.8}$$

Condition (9.8) becomes simple enough to invite further analysis when $b=0$ (which we must approach as a limit given our definition of the scale of sapces). This case is of particular interest as Coulomb potentials fall into this class. In this case (9.8) may be rewritten as

$$a|\kappa|<\left(\frac{2mc^2a^2}{|z|}\right)^{1/2}\left(\frac{1-(1/2)(2mc^2a^z/|z|)^{1/2}}{1+(1/2)(2mc^2a^2/|z|)^{1/2}}\right) \tag{9.9}$$

The right side of (9.9) is maximized when $\left(\frac{2mc^2a^2}{|z|}\right)^{1/2}=2(\sqrt{2}-1)$, where it has the value $6-4\sqrt{2}\approx .3431$. That is, in this case if we choose $z=-\frac{mc^2a^2}{2(\sqrt{2}-1^2}$ and $a|\kappa|<6-4\sqrt{2}$, then the series (8.7) converges and the conditions of Theorem 8.1 hold.

This shows that if $a<6-4\sqrt{2}$, then $\kappa=1$ is inside the region of convergence of the power series. That is, the usual Dirac resolvent is linked to the Pauli-Schrödinger resolvent via a holomorphic family of operators. In terms of hydrogen-like atoms, this corresponds to a nuclear charge of $Z\leqq 47$.

Remarks. — 1) As remarked in Section 7, better estimates for $\|(K_n(z))_{\mathscr{D}_J^*}\|_{\mathscr{D}_J^*\to\mathscr{D}_J}$ are possible for $n>1$. A more sophisticated analysis utilizing such estimates should improve the condition on a.

2) Another consequence of (9.2) through (9.4) is that for

$$z<-\frac{1}{2}mc^2a^2-b,$$

we know that $z\notin\sigma(H_+)$. Hence, $-\frac{1}{2}mc^2a^2-b$ is a lower bound for H_+. For the Coulomb potential with a nuclear charge of Z, we have

$$|h_V[x]|\leqq\frac{\pi}{2}Z\,h_{c|\mathscr{D}|}[x] \qquad \forall x\in Q(\mathscr{D}) \tag{9.10}$$

(*see* [K], p. 308). Thus we may set $a=\frac{\pi}{2}Z$ and $b=0$, yielding a lower bound estimate of $-\left(\frac{\pi}{2}\right)^2\left(\frac{me^4Z^2}{2\hbar^2}\right)$. This estimate is only $\left(\frac{\pi}{2}\right)^2\approx 2.47$ times the actual ground state energy, as determined by solving the Schrödinger equation. We note that $\frac{\pi}{2}$ is the sharp estimate for (9.10) (*see* [L]).

ACKNOWLEDGEMENTS

The author would like to thank Prof. William Arveson and the Department of Mathematics at UC Berkeley for their hospitality during his stay at Berkeley.

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(*Manuscript received June* 10, 1992.)