ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 58, nº 3 (1993), p. 323-341

<http://www.numdam.org/item?id=AIHPA_1993__58_3_323_0>

Ann. Inst. Henri Poincaré,

Vol. 58, n° 3, 1993, p. 323-341. *Physique théorique*

Quantization of the orientation preserving automorphisms of the torus

by

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ABSTRACT. – The quantization of any hyperbolic symplectomorphism of the 2-torus is described by the finite dimensional irreducible representations of its naturally associated Weyl algebra. Furthermore the even part of the spectrum of the quantum propagator is characterized in terms of the orbits of the symplectomorphism.

RÉSUMÉ. – Nous décrivons la quantification de tous les symplectomorphismes hyperboliques du 2-tore au moyen d'une représentation irréductible de l'algèbre de Weyl naturellement associée. De plus, nous caractérisons la partie paire du spectre du propagateur quantique en terme des orbites du symplectomorphisme.

0. INTRODUCTION

In this article we study the quantum mechanics of a particular family of discrete dynamical systems, namely the hyperbolic symplectomorphisms of the 2-torus.

M. Degli Esposti would like to thank J. Bellissard, S. Graffi, A. Knauf and R. Seiler for suggestions and helpful discussions. The author would also like to thank the Technische Universität Berlin for its hospitality and the Akademie der Wissenschaften zu Berlin for its financial support.

This problem has been first considered by Berry and Hannay [5] to investigate the quantum behaviour of the simplest examples of dynamical systems with positive entropy. Any hyperbolic matrix $A \in SL(2, \mathbf{Z})$ indeed defines, by standard action on $\mathbf{T}^n$, the easiest example of Anosov system ([1], [4]). This example is however "generic" in a topological sense because any Anosov system on $\mathbf{T}^n$ is topologically equivalent to a linear action [2]. Moreover any such system contains a dense set of closed periodic orbits. This fact plays an important role in the quantum context as it will be seen below.

The most relevant mathematical problems in this context are:

1) The determination of the quantization prescription.

2) The definition of the quantum evolution corresponding to any given automorphism of the torus.

3) The connection between the quantum evolution and the "classical" one, *i.e.*, with the orbits of the symplectomorphism.

The original approach of Berry and Hannay [5] consists essentially in solving just problem 2) by explicit construction of the "propagator". The periodicity of phase space requires "h" to be the reciprocal of an integer; on the other hand the definition of the quantum evolution (based on the "commutativity between quantization and classical evolution" because of the interpretation of the symplectomorphism as a "time one" linear map of an Hamiltonian flow) allows to realize the propagator as a unitary operator in $\mathbf{L}^2(\mathbf{T}^1, \mu)$ only for the subclass of 2×2 hyperbolic matrices in $SL(2, \mathbf{Z})$ of the form:

$$\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \quad \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}$$

(in the language of [5] only the automorphisms of the above form are "quantized").

Here we first show that the natural framework for problem 1) consists in the representations of the Weyl *-algebras already used in the quantum Hall effect [10], [11]: more specifically, in the irreducible representations in the Hilbert space $\mathbf{L}^2(\mathbf{T}^1, \mu)$ where μ is an atomic measure on the circle. These algebras are indexed by the rational values of h. Unlike the standard Weyl algebra over $\mathbb{R}^2$, where by the Stone-Von Neumann theorem there is a unique (up to unitary equivalences) infinite-dimensional representation, it is known that for any fixed rational value of h there are infinitely many inequivalent finite dimensional representations (if $h = \frac{1}{N}$ then the dimension is N) and for our purpose we will identify this finite dimensional space with $\mathbf{L}^2(\mathbf{T}^1, \mu)$ where μ has support on a finite number of points.

The quantum dynamics associated to any given map is simply defined by requiring the commutativity between the evolution and the natural algebra automorphism induced by the map itself (a related but different problem is the study of the properties of the algebraic dynamical system defined by the Weyl algebra together with the automorphism without considering any particular representation: *see* [12]). In this way the Berry-Hannay restriction is removed: we prove that for any automorphism of the torus and for any $h \in \mathbb{Q}$ there is a well defined finite dimensional representation (depending on the map but not on h) on which the quantum propagator is defined as a unitary operator in $L^2(\mathbf{T}^1, \mu)$. In the language of [5], all maps can be quantized. (For a partial result in this direction see [6]).

As far as problem [3] is concerned, we will supplement the existing literature ([5], [7], [8]), with the construction of the even eigenvectors of the propagator as finite linear combinations of exponentials by relating it to the classical orbits in the following way: we identify the family $\{\Lambda_k\}$ of subsets of the torous corresponding to the linear lagrangian submanifolds. To each element Λ_k of this family we associate the wave function $\Psi_k = \exp(ih^{-1} S_k)$, where S_k is the quadratic (discrete analog of the) generating function of Λ_k. If $W_k(q, p)$ denotes the Wigner function of Ψ_k, then its support is proved to belong to Λ_k and the even eigenvectors can be constructed as finite linear combinations of the Ψ_k. A critical property entering in the construction of the eigenvectors is that the quantum evolution is equivalent to a permutation of the set $\{\Lambda_k\}_k$ under the classical map. In particular, this allows to prove the conjecture of Eckhardt [7] relating the length of the "quantum cycle" to the periods of the automorphism.

The problem of the representations of the Weyl algebras is described in the next section together with the construction of the quantum propagator; in section 2 we describe the construction of the eigenvectors of the propagator using the classical dynamics.

1. QUANTIZATION OF AN AUTOMORPHISM $A \in SL(2, \mathbf{Z})$

The standard quantum mechanics is based on the representation of the Heisenberg relations:

$$[\hat{q}_j, \hat{p}_k] = i\hbar\,\delta^j_k, \qquad j, k = 1, 2, \ldots, n \tag{1}$$

in the space $L^2(\mathbf{R}^n)$ with the usual Lebesgue measure.

The operators $\hat{q}_k$, $\hat{p}_k$ are the infinitesimal generators of the Weyl algebra defined abstractly as the $*$-algebra generated by the $2n$ parameter family of unitary operator $(T(x), x \in \mathbf{R}^{2n})$ acting on a Hilbert space such that:

$$T(x)\,T(y) = e^{i\pi h \omega(x, y)}\,T(x+y), \qquad x, y \in \mathbf{R}^{2n} \tag{2}$$

where $\omega(x, y)$ is the usual symplectic form, e.g. if $n=1$:

$$\omega(x, y)=x_1 y_2-x_2 y_1$$

In the case of $\mathbf{R}^{2n}$ we have only one (up to unitary equivalence) irreducible representation of the Weyl group given by the standard Schrödinger representation (infinite dimensional) on $\mathbf{L}^2(\mathbf{R}^n, d\mu)$, where μ is the usual Lebesgue measure and where $\mathrm{T}(x)$ is the translation in phase space *i.e.* (w.l.g. $n=1$)

$$\mathrm{T}(x=(q, p))=e^{i\pi hqp}\,\mathrm{T}_q\,\mathrm{T}_p$$

where $\mathrm{T}_q=e^{iq\hat{p}}$ and $\mathrm{T}_p=e^{ip\hat{q}}$.

In the Schrödinger representation the operators $\hat{q}$ and $\hat{p}$ are realized by multiplication and differentiation, respectively, and to each smooth rapidly decreasing function we associate an operator via the so called Weyl transform:

$$\hat{f}=\int_{\mathbb{R}^{2n}} \tilde{f}(\xi, \eta) \exp\{2 i\pi(\xi.\hat{q}+\eta.\hat{p})\}\, d\xi\, d\eta$$

where:

$$\tilde{f}(\xi, \eta)=\int_{\mathbb{R}^{2n}} f(q, p) \exp\{-2 i\pi(\xi.q+\eta.p)\}\, dq\, dp$$

Now let us consider the case of a hyperbolic (*i.e.*, $|\mathrm{Tr}(\mathrm{A})|>2$), area preserving [*i.e.*, $\det(\mathrm{A})=1$] automorphism $\mathrm{A}\in\mathrm{SL}(2, \mathbf{Z})$ of the 2-torus. This dynamical system can be realized in a natural way as a commutative $*$-algebra with an authomorphism as follows (notation as in [12]):

$\mathbf{L}_\infty(\mathbf{T}^2)$ is the commutative $*$-algebra with a trace given by the integral w.r.t. the Lebesgue measure and the algebra automorphism is the one induced by A, namely, $\forall f\in\mathbf{L}_\infty(\mathbf{T}^2)$:

$$(\alpha f)(x)=f(\mathrm{A}x) \tag{3}$$

$\forall n\in\mathbf{Z}^2$ let $\mathrm{W}_0(n)$ be the element of the $*$-algebra given by:

$$\mathrm{W}_0(n)(x)=\exp(2 i\pi n.x)$$

where $n.x=n_1 x_1+n_2 x_2$. Then if

$$\varphi(x)=\sum_{n\in\mathbf{Z}^2} a_n \exp(2 i\pi n.x)=\sum_{n\in\mathbf{Z}^2} a_n \mathrm{W}_0(n)$$

we have

$$(\alpha\varphi)=\sum_{n\in\mathbf{Z}^2} a_n \mathrm{W}_0(\mathrm{A}^t n)$$

The quantum picture is the non commutative deformation of the previous $*$-algebra *i.e.* we define as in [12], [13]:

DEFINITION 1. – *The Weyl algebra $\mathcal{A}_\hbar$ is the $*$-algebra generated by*

$$\mathcal{W}_\hbar=\{\mathrm{W}_\hbar(n)\}_{n\in\mathbf{Z}^2}$$

where:

(i) $W_\hbar(n)^* = W_\hbar(-n)$

(ii) $W_\hbar(n)\, W_\hbar(m) = e^{i\pi h\omega(n, m)}\, W_\hbar(n+m)$

Moreover the algebra automorphism is defined in the following way:

$$(\alpha W_\hbar)(n) = W_\hbar(A^t n)$$

This is the algebra which arises in the Quantum Hall effect where the parameter $\hbar$ is proportional to the product of the magnetic field and the Planck constant ([10], [11]). Set

$$\left.\begin{aligned} w_1 &= W_\hbar((1, 0)) \\ w_2 &= W_\hbar((0, 1)) \end{aligned}\right\} \tag{4}$$

Then, if $n = (n_1, n_2)$, we have:

$$\left.\begin{aligned} W_\hbar(n) &= e^{i\pi h n_1 n_2}\, w_2^{n_2}\, w_1^{n_1} \\ w_1\, w_2 &= e^{i 2h\pi}\, w_2\, w_1. \end{aligned}\right\} \tag{5}$$

Moreover we immediately get:

$$[W_\hbar(n), W_\hbar(m)] = 2\, i.\sin\{\pi\, h\, \omega(n, m)\}.W_\hbar(n+m) \tag{6}$$

Now $\forall n \in \mathbf{Z}^2$ and for any fixed $h \in \mathbb{R}^+$ set:

$$\Omega_{n, h} = \left\{ m \in \mathbf{Z}^2 : \omega(n, m) = \frac{k}{h},\ k \in \mathbf{Z} \right\} \tag{7}$$

Note that if $h \notin \mathbb{Q}$ then

$$\Omega_{n, h} = \Omega_n = \{ m \in \mathbf{Z}^2 : \omega(n, m) = 0 \}$$

and if $h = \frac{p}{q}$ with $(p, q) = 1$ then

$$\Omega_{n, h} = \{ m \in \mathbf{Z}^2 : \omega(n, m) = l.q,\ l \in \mathbf{Z} \}$$

Let

$$C(W_\hbar(n)) = \{ m \in \mathbf{Z}^2 : [W_\hbar(n), W_\hbar(m)] = 0 \} \tag{8}$$

be the center of $W_\hbar(n)$ in $\mathscr{W}_\hbar$ and

$$C(\mathscr{W}_\hbar) = \bigcap_{n \in \mathbf{Z}^2} C(W_\hbar(n)) \tag{9}$$

be the center of $\mathscr{W}_\hbar$. If $h \notin \mathbb{Q}$ we have:

$$C(\mathscr{W}_\hbar) = \{ m \in \mathbf{Z}^2 : \omega(n, m) = 0,\ \forall n \in \mathbf{Z}^2 \} = 0$$

on the other hand if $h \in \mathbb{Q}$ with $h = \frac{p}{q}$, $(p, q) = 1$ we have:

$$C(\mathscr{W}_\hbar) = \bigcap_{n \in \mathbf{Z}^2} \{ m : \omega(n, m) = l.q,\ l \in \mathbf{Z} \} = W_\hbar(q.\mathbf{Z}^2)$$

i.e. we have recovered the following well known result:

LEMMA 1. – *If* $h=\frac{p}{q}\in\mathbb{Q}$ *then*

$$\mathit{Center}(\mathscr{A}_h)=\text{Algebraicspan}_{\mathbb{C}}\{w_1^q, w_2^q\}.$$

Otherwise

$$\text{Center}(\mathscr{A}_h)=\mathbb{C}\,.\,\text{Id}$$

Now we want to identify those particular representations which realise our algebra in the unitary operators on the Hilbert space $\mathbf{L}^2(\mathbf{T}^1, \mu)$ for some atomic measure μ. Writing $\mathbf{T}^2=\mathrm{S}^1\times\mathrm{S}^1$ these representations allow us to interpret, in the spirit of the original Berry-Hannay approach [5], the Hilbert space functions as the analog of the usual wave functions defined on configuration space and the Fourier transformation yields the usual momentum space representation. Most importantly, this allows to relate in the natural way the quantum evolution to the standard action of the automorphism of the torus (the product measure $\mu\times\mu$ on $\mathbf{T}^2$ is invariant under the classical map).

To this end, let us define the appropriate finite dimensional Hilbert spaces, assuming w.l.g. $h=\frac{1}{\mathrm{N}}$.

DEFINITION 2. – $\forall h=\frac{1}{\mathrm{N}}$ let μ *be the atomic measure on the circle given by*

$$\mu(x):=\frac{1}{\mathrm{N}}\sum_{l=0}^{N-1}\delta\left(x-\frac{l}{\mathrm{N}}\right)$$

Remark that the vectors $|k\rangle=\Psi_k(x)=\delta^x_{k/\mathrm{N}}$ for $k=0, 1, \ldots, \mathrm{N}-1$ form a basis for the Hilbert space $\mathbf{L}^2(\mathbf{T}^1, \mu)$. The inner product between two vectors $f, g\in\mathbf{L}^2(\mathbf{T}^1, \mu)$ is of course given by:

$$(f, g):=\frac{1}{\mathrm{N}}\sum_{l=0}^{N-1}\bar{f}\left(\frac{l}{\mathrm{N}}\right)g\left(\frac{l}{\mathrm{N}}\right) \tag{10}$$

Furthermore on $\mathbf{L}^2(\mathbf{T}^1, \mu)$ we have the action of the unitary Fourier transformation defined by:

$$(\mathbf{F}_{\mathrm{N}}\Psi)_m:=\frac{1}{\mathrm{N}^{1/2}}\sum_{n=0}^{N-1}\exp\left(\frac{-2i\pi mn}{\mathrm{N}}\right)\Psi_n \tag{11}$$

Using the map

$$\mathrm{U}:\quad \mathbf{Q}_{\mathrm{N}}\to\mathbf{Z}_{\mathrm{N}}$$

defined by

$$\mathrm{U}(x):=\mathrm{N}\,.\,x$$

where

$$Q_N = \{0, 1/N, 2/N, \ldots, (N-1)/N\}$$

and

$$\mathbf{Z}_N := \mathbf{Z}/N\mathbf{Z} = \{0, 1, 2, \ldots, N-1\}$$

we can identify $\mathbf{L}^2(\mathbf{T}^1, \mu)$ with $L^2(\mathbf{Z}_N, \mu)$ where $\Psi \in L^2(\mathbf{Z}_N, \mu)$ is a vector in $\mathbb{C}^n$:

$$\Psi = (\Psi_0, \ldots, \Psi_{N-1})$$

Now let us define a family of $*$-algebra of unitary operators which will classify the irreducible representations of $\mathcal{W}_h\left(h = \frac{1}{N}\right)$.

DEFINITION 3. – $\forall \theta = (\theta_1, \theta_2) \in \mathbf{T}^2$ set

$$\left.\begin{aligned} t_1 | l \rangle &= \exp\left(\frac{2i\pi(\theta_1 + l)}{N}\right) | l \rangle; \qquad t_1^N = \exp(2i\pi\theta_1).\mathbf{I} \\ t_2 | l \rangle &= \exp\left(\frac{2i\pi\theta_2}{N}\right) | l+1 \rangle; \qquad t_2^N = \exp(2i\pi\theta_2).\mathbf{I} \end{aligned}\right\} \tag{12}$$

Therefore:

$$t_1 t_2 = \exp\left(\frac{2i\pi}{N}\right) t_2 t_1 \tag{13}$$

and the Weyl operators are:

$$T(k) = \exp\left(\frac{i\pi k_1 k_2}{N}\right) t_2^{k_2} t_1^{k_1}, \qquad \forall k = (k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}$$

From (12) and (13) we immediately get the validity of the group law characterizing the Weyl algebra:

$$T(m) T(k) = \exp\left(\frac{i\pi}{N} \omega(m, k)\right) T(k+m) \tag{14}$$

$\forall k = (k_1, k_2)$, $m = (m_1, m_2) \in \mathbf{Z} \times \mathbf{Z}$, and also the explicit expression of the matrix elements:

$$\langle m | T(k) | l \rangle = \delta_m^{l+k_2} . \exp\left\{\frac{i\pi}{N}[k_1(k_2 + 2l) + 2\langle \theta, k \rangle]\right\} \tag{15}$$

where:

$$\langle \theta, k \rangle := \theta_1 . k_1 + \theta_2 . k_2$$

DEFINITION 4. – $\forall \theta \in \mathbf{T}^2$ *and for* $h = 1/N$, *let* π_θ *be the* $*$-representation:

$$\pi_\theta : \mathcal{W}_h \to U(\mathbf{L}^2(\mathbf{T}^1, \mu))$$

completely determined by its action on the generators:

$$\pi_\theta(w_1) = t_1$$
$$\pi_\theta(w_2) = t_2$$

Then we have the following known result, implicitly contained e.g. in [13] whose proof we describe for convenience of exposition (*see* also [14]).

THEOREM 1. — 1) π_θ *is an irreducible* $*$*-representation of* $\mathcal{W}_h$.
2) π_θ *is unitarily equivalent to* $\pi_{\tilde{\theta}}$ *if and only if* $\theta=\tilde{\theta}$.

Proof. — Let us first define a family of projections, to be used also later on:

$$\left.\begin{aligned} &Q_k=\frac{1}{N}\sum_{s=0}^{N-1}\exp\left(-\frac{2i\pi s(k+\theta_1)}{N}\right)t_1^s, \qquad k\in\{0, 1, 2, \ldots, N-1\} \\ &Q_k|m\rangle=\delta_k^m|k\rangle, \qquad \forall k, m\in\{0, 1, 2, \ldots, N-1\}\end{aligned}\right\} \tag{16}$$

In order to prove (1) it suffices to show that each vector is cyclic [13] *i.e.*, that $\forall\Psi, \Phi\in L^2(T^1, \mu)$, $\Psi, \Phi\neq 0$, there exists $w\in\mathcal{W}_h$ such that $\pi_\theta(w)\Psi=\Phi$. Set:

$$\Psi=\sum_{s=0}^{N-1}a_s|s\rangle$$

and assume $a_k\neq 0$ for some k; then, by definition:

$$\left[a_k^{-1}.\exp\left(-\frac{2i\pi\theta_2(p-k)}{N}\right).Q_p.t_2^{p-k}\right]\Psi=|p\rangle$$

$\forall p=0, \ldots, N-1$ and this proves irreducibility (we remark that the assertion could be alternatively proved by application of the Burnside Theorem [13] looking at the commutant of the algebra). In order to prove (2) assume that there exists a unitary operator U such that:

$$U\pi_\theta U^{-1}=\pi_{\tilde{\theta}}$$

then in particular, for $j=1, 2$:

$$U\pi_\theta(w_j^N)U^{-1}=\pi_{\tilde{\theta}}(w_j^N)$$

and (2) follows immediately.

Let us now proceed to identify the quantum evolution using the automorphism of the algebra. Let $A\in SL(2. \mathbf{Z})$ with

$$A=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \tag{17}$$

Fix $h=\frac{1}{N}$ and look for a unitary operator U_A which commutes with the automorphism of the algebra: *i.e.*, we require

$$U_A^*\pi_\theta(W_h(n))U_A=\pi_\theta(W_h(A^t n))$$

or, equivalently:

$$U_A^*(\theta)T(k)U_A(\theta)=T(A^t(k)), \qquad \forall k=(k_1, k_2)\in\mathbf{Z}\times\mathbf{Z} \tag{18}$$

In other words the Weyl operator should transform under U_A in the same way as the corresponding phase space function under the classical map. The periodicity of the generators [equation (12)] immediately yields, by (18), that the following condition has to be satisfied:

$$T(A^t(e_j))^N = \exp(2i\pi\theta_j).\mathbf{I}, \qquad j=1, 2 \tag{19}$$

where

$$e_1 := (1, 0), \qquad e_2 := (0, 1)$$

This condition restricts the possible representations and we get the following result (for a preliminary version *see* [6]).

THEOREM. – *Let* $A \in SL(2, \mathbf{Z})$, N *be a prime number and assume* [*in the notation of* (17)] $b \neq 0$ (*mod* N). *Then there is a well defined unitary operator* $U_A(\theta)$ *such that* (18) *is satisfied. More precisely:*

(1) *For any given automorphism* A, *all admissible representations are labeled by all* $(\theta_1, \theta_2) \in \mathbf{T}^2$ *such that*

$$(A-\mathbf{I})\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \frac{1}{2}N\begin{pmatrix}a.b\\ c.d\end{pmatrix} + \begin{pmatrix}m_1\\ m_2\end{pmatrix}, \qquad m_1, m_2 \in \mathbf{Z} \tag{20}$$

(2) *The matrix elements of* $U_A(\theta)$ *on the basis* $|k\rangle$ *admit the following expression*

$$\langle m | U_A(\theta) | k \rangle = (N)^{-(1/2)}.\exp\frac{-i\pi b^{-1}}{N}[dm^2 + ak^2 - 2km]$$
$$\times \exp\frac{-2i\pi}{N}[\theta_1 b^{-1}(dm + ak - m - k) + \theta_2(k-m)] \tag{21}$$

Proof. – We have

$$A^t\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = (a, b)$$

and by (14)

$$T(x)^k = T(kx)$$

i.e.

$$T\left(A^t\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\right)^N = \exp(i\pi abN)\, t_2^{Nb}\, t_1^{Na}$$
$$= \exp(i\pi ab\,N).\exp[2i\pi(b\theta_2 + a\theta_1)].\mathbf{I}$$
$$= \exp(2i\pi\theta_1).\mathbf{I} \tag{22}$$

In the same way:

$$T\left(A^t\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\right)^N$$
$$= \exp(i\pi cdN).\exp[2i\pi(d\theta_2 + c\theta_1)].\mathbf{I} = \exp(2i\pi\theta_2).\mathbf{I} \tag{23}$$

and equation (20) immediately follows.

In order to find the expression for the propagator let us the family of projections Q_k defined in (16). First note that equation (18) for $k=(0, 0)$ implies that U_A is unitary. Furthermore $\forall k, l, m \in \mathbf{Z}_N$ we have

$$\begin{aligned}
\langle m | U_A^* Q_k U_A | l \rangle
&= \langle m | U_A^* | k \rangle \langle k | U_A | l \rangle \\
&= \langle m | U_A^* \frac{1}{N} \sum_{s=0}^{N-1} \exp\left[-\frac{2 i \pi s (k+\theta_1)}{N} \right] T((s, 0)) U_A | l \rangle \\
&= \frac{1}{N} \sum_{s=0}^{N-1} \exp\left[-\frac{2 i \pi s (k+\theta_1)}{N} \right] \langle m | T(as, bs) | l \rangle \\
&= \frac{1}{N} \sum_{s=0}^{N-1} \exp\left[-\frac{2 i \pi s (k+\theta_1)}{N} \right] \delta_m^{l+bs} \exp \frac{i\pi}{N} \\
&\qquad [as(bs+2l)+2\langle \theta, (as, bs) \rangle] \quad \blacksquare
\end{aligned}$$

where the third equality follows from (18) and the last one from (15). Set now $s=b^{-1}(m-l) \in \mathbf{Z}_N$. Then:

$$\begin{aligned}
\langle m | U_A^* | k \rangle \langle k | U_A | l \rangle
&= \frac{1}{N} . \exp - \frac{2 i \pi}{N} [b^{-1}(m-l)(k+\theta_1)] \\
&\times \exp \frac{i\pi}{N} [ab^{-1}(m-l)(m+l)+2\langle \theta, (ab^{-1}(m-l), (m-l)) \rangle] \quad (24)
\end{aligned}$$

In the same way, using:

$$U_A(\theta) T(k) U_A^*(\theta) = T((A^t)^{-1}(k)), \qquad \forall k=(k_1, k_2) \in \mathbf{Z}_N$$

we obtain:

$$\begin{aligned}
\langle m | U_A Q_k U_A^* | l \rangle = \langle m | U_A | k \rangle \langle k | U_A^* | l \rangle
&= \frac{1}{N} . \exp - \frac{2 i \pi}{N} [b^{-1}(l-m)(k+\theta_1)] \\
&\times \exp \frac{i\pi}{N} [db^{-1}(l-m)(m+l)+2\langle \theta, (db^{-1}(l-m), (m-l)) \rangle] \quad (25)
\end{aligned}$$

From (24) and (25) we immediately have:

$$\begin{aligned}
\frac{\langle m | U_A | k \rangle \langle k | U_A^* | q \rangle}{\langle k | U_A^* | q \rangle \langle q | U_A | l \rangle} = \frac{\langle m | U_A | k \rangle}{\langle q | U_A | l \rangle}
&= \exp \frac{-2 i \pi b^{-1}}{N} [-km+ql+\theta_1(l+q-m-k)] \\
&\times \exp \frac{i \pi b^{-1}}{N} [-dm^2-ak^2+dq^2+al^2] \\
&\times \exp \frac{2 i\pi}{N} [\theta_1 b^{-1}(dq-dm-ak+al)-\theta_2(-l-m+k+q)] \quad (26)
\end{aligned}$$

Using (24) and (26) we obtain the expression (21) (up to a constant phase factor)

$$\langle m|\mathrm{U_A}|k\rangle=(\mathrm{N})^{-(1/2)}.\exp\frac{-i\pi b^{-1}}{\mathrm{N}}[dm^2+ak^2-2km]$$
$$\times\exp\frac{-2i\pi}{\mathrm{N}}[\theta_1 b^{-1}(dm+ak-m-k)+\theta_2(k-m)] \quad (27)$$

Remark. – If there are two even terms on one of the two diagonals then we can choose $\theta=(0, 0)$ and we recover the expression obtained by Berry and Hannay [5]. Because of the condition $\det(\mathrm{A})=1$, the only other possible case is when there is only one even term. Assume for example that ab is even and cd is odd with b and a not simultaneously even; then an easy computation shows that, $\forall m_1, m_2\in\mathbf{Z}$:

$$\theta_1=\frac{1}{a+d-2}\left\{(1-d)m_1+b\left(\frac{1}{2}+m_2\right)\right\}(\mathrm{mod}\ 1)$$
$$\theta_2=\frac{1}{a+d-2}\left\{cm_1+(1-a)\left(\frac{1}{2}+m_2\right)\right\}(\mathrm{mod}\ 1)$$

satisfies the previous condition and does not depend on N; in particular:

$$\theta_1=\frac{b}{2(a+d-2)} \quad (\mathrm{mod}\ 1)$$
$$\theta_2=\frac{1-a}{2(a+d-2)} \quad (\mathrm{mod}\ 1)$$

is a solution. For the Arnold cat map ($b=c=d=1, a=2$) we get $\theta=\left(\frac{1}{2}, \frac{1}{2}\right)$.

2. EIGENVECTORS AND EIGENVALUES OF A FAMILY OF HYPERBOLIC MAPS

Consider for the sake of simplicity just the family of hyperbolic maps originally studied by Berry and Hannay:

$$\mathrm{A}_m=\begin{pmatrix}2m & 1\\ 4m^2-1 & 2m\end{pmatrix} \quad (28)$$

($m\neq 0$). For this family it immediately follows from (20) that we can choose $\theta=(0, 0)$. Hence, as already remarked, the propagator reduces to the Hannay-Berry form, namely:

$$(\mathrm{U_A}\Psi)_n=\sum_{k=0}^{N-1}\mathrm{U_A}(n, k)\Psi_k \quad (29)$$

where:

$$U_A(n, k) = \left(\frac{i}{N}\right)^{1/2} \exp\left[\frac{2 i\pi}{N}(mn^2 - nk + mk^2)\right] \tag{30}$$

Let us begin the construction sketched in the introduction by fixing some notations (we restrict our considerations to the Berry-Hannay case $\theta = (0, 0)$ only to avoid the introduction of a too cumbersome notation: the generalization to any $\theta \neq (0, 0)$ is straightforward).

Set:

$$\tilde{\mathbf{Z}}_N = \mathbf{Z}_N \cup \{\infty\}$$
$$P_N = Q_N = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$$
$$L_N = Q_N \times P_N$$

and let a, N be two integer numbers with N prime; then $(a/N) := a^{-1}$ is by definition the unique integer in $\mathbf{Z}_N$ such that:

$$a . (a/N) = a . a^{-1} = 1 \pmod N$$

DEFINITION 5. – *Let p be a prime number and $a \in \mathbf{Z}$; then the Legendre symbol $\left(\frac{a}{p}\right)$ is defined as follows* [9]:

$$\left(\frac{a}{p}\right) = \begin{cases} +1 \text{ if there exists } m \in \mathbf{Z} \text{ such that: } m^2 = a \pmod p \\ -1 \text{ otherwise} \end{cases}$$

The *Jacobi symbol* is the generalization of this to arbitrary n:

$$\left(\frac{a}{n}\right) = \prod \left(\frac{a}{p_i}\right)$$

where the p_k's are the prime factors of n. Let us also recall the product laws:

$$\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{b}{n}\right)$$
$$\left(\frac{a}{nm}\right) = \left(\frac{a}{n}\right) . \left(\frac{a}{m}\right)$$

and the Gauss reciprocity theorem:

$$\left(\frac{a}{n}\right) . \left(\frac{n}{a}\right) = (-1)^{((a-1)(n-1)/4}$$

We denote by $\mathbf{L}_N \subset \mathbf{T}^2$the $N \times N$ subgroup of the 2-torus given by:

$$\mathbf{L}_N := \{(q, p) \in \mathbf{T}^2 \mid Nq, Np \in \mathbf{Z}\} = Q_N \times P_N$$

We also denote by A_N the restriction of A to this invariant sublattice. Note that $\mathbf{L}_N$ is a vector space over the field $\mathbf{Z}_N$ if N is prime.

Since A_N acts as a permutation of a finite set, it must have a period.

DEFINITION 6. – *The period* $n(N)$ *of* A_N *is the smallest positive integer* n *such that* $A^n = I \pmod N$.

The following lemma has been proved in [3].

LEMMA 2. – *Let* $A \in SL(2, \mathbf{Z})$ *be hyperbolic and* p *be a prime number with* $p > 2(\|A\|+1)$. *Then:*

$$n(p) \,\Big|\, \left(p - \left(\frac{(\operatorname{Tr} A)^2 - 4}{p}\right)\right)$$

Finally let us recall the following useful formula: if N is prime and a, $b \in \mathbf{N}$ then [9]:

$$\sum_{k=0}^{N-1} \exp\left[\frac{2i\pi}{N}(ak^2+bk)\right] = N^{1/2}\left(\frac{2a}{N}\right)\exp\left[\frac{-i\pi}{4}(N-1)\right].\exp\left[\frac{-2i\pi}{N}ab^2(2a/N)^2\right] \quad (31)$$

After these preliminaries we can go over to the characterization of the eigenvectors of U_A along the lines sketched in the Introduction. Formula (31) immediately implies that $\forall k \in \mathbf{N}$ we have (as in [5]):

$$U_A^k = e^{i2\pi\varphi_k} U_{A^k} \quad (32)$$

and if $n(N) := n$ is the period as in Definition 5, *i.e.* if

$$A^n = \mathbf{I} \pmod N$$

then:

$$U_A^n = e^{i2\pi\Phi}\mathbf{I} \quad (33)$$

where Φ is a constant factor (depending on N). This restricts the N eigenvalues of U to lie on the n possible sites:

$$\left\{\exp\left[\frac{2i\pi(m+\Phi)}{n}\right] : 0 \leq m \leq n-1\right\} \quad (34)$$

In general $n \neq N$, that is, there is no one-to-one correspondence between eigenvalues and sites. Typically there are both unoccupied and multiply occupied sites and this distribution follows the highly irregular behavior of $n(N)$ as function of N [5].

DEFINITION 6 [5]. – $\forall \Psi \in \mathbf{L}^2(\mathbf{T}^1, \mu)$ *let* $W_\Psi(q, p)$ *be the (discrete) Wigner function associated to* Ψ *(with the identification* $\mathbf{L}^2(\mathbf{T}^1, \mu) \cong L^2(\mathbf{Z}_N, \mu)$) *defined as follows:* $\forall (q, p) \in \mathbf{Z}_N \times \mathbf{Z}_N$

$$W_\Psi(q, p) := \sum_{q' \in \mathbf{Z}^N} \bar{\Psi}(q+q').\Psi(q-q').\exp\left(-\frac{2i\pi}{N}2pq'\right) \quad (35)$$

We are going to define in L_N certain sets of points which represent the discrete analog of linear Lagrangian subspaces (i.e., the lines $p=2kq+l$ for $(p, q)\in\mathbb{R}^2$).

DEFINITION 7. — $\forall (k, l)\in \tilde{\mathbf{Z}}_N \times Q_N$

$$\begin{aligned} \Lambda_{k,l} &:= \{(q, p) : p=2kq+l \ (\mathrm{mod}\ 1),\ q\in Q_N\}, \qquad k\neq\infty \\ \Lambda_{\infty,l} &:= \{(q, p) : q=l \ (\mathrm{mod}\ 1),\ p\in P_N\} \end{aligned} \tag{36}$$

Remark. — $k=\infty$ corresponds to the vertical "line" ("$q=$Const.").

To each set $\Lambda_{k,l}$ we can associate a "generating function", namely:

$$S_{k,l}(q)=kq^2+lq \qquad k\in\mathbf{Z}_N, \qquad q, l\in Q_N$$

The wave function naturally associated to each set $\Lambda_{k,l}$ is the exponential (in units $\hbar$) of its generating function, *i.e.*:

$$\Psi(q)=c.\exp i\hbar^{-1} S_{k,l}(q)=\exp i2\pi \mathrm{N} S_{k,l}(q)$$

where c is a normalization constant.

Because of the identification $\mathbf{L}^2(\mathbf{T}^1, \mu)\cong L^2(\mathbf{Z}_N, \mu)$ we have:

DEFINITION 8. — $\forall (k, l)\in \tilde{\mathbf{Z}}_N \times \mathbf{Z}_N$, $\forall q\in\mathbf{Z}_N$ set:

$$\Psi_{k,l}(q)=\begin{cases} \exp\left[\dfrac{2i\pi}{\mathrm{N}}(kq^2+lq)\right], & k\in\mathbf{Z}_N \\ (\mathrm{N})^{1/2}.\delta_l^q & k=\infty \end{cases} \tag{37}$$

A short computation based on (35) and (37) yields:

LEMMA 3. — $\forall (k, l)\in \tilde{\mathbf{Z}}_N \times \mathbf{Z}_N$:

$$\mathrm{Supp}(W_\Psi(q, p))=\Lambda_{k,l} \tag{38}$$

Next we describe how these "Lagrangian" subsets are mapped into reach other by the automorphism.

Denoting by x^{-1} the inverse of x in the field $\mathbf{Z}_N$ when N is a prime, and by $-x$ the number $\mathrm{N}-x$ for each $x\in\mathbf{Z}_N$ we have:

LEMMA 4. — *Let* $\mathrm{N}\geqq 3$ *be a prime number*, $(m\neq 0)$ *and let*

$$A:=A_m:=\begin{pmatrix} 2m & 1 \\ 4m^2-1 & 2m \end{pmatrix}$$

Then:

$$A_N(\Lambda_{(k,l)})=\Lambda_{k',l'}:=\Lambda_{\zeta(k,l)} \tag{39}$$

where

$$\zeta:\ \tilde{\mathbf{Z}}_N\times Q_N \to \tilde{\mathbf{Z}}_N\times Q_N$$

is defined as:

$$k'=\begin{cases} m-4^{-1}(m+k)^{-1}, & k\neq\infty,\ -m \\ \infty & k=-m \\ m & k=\infty \end{cases} \tag{40}$$

$$l' = \begin{cases} 2^{-1} l(m+k)^{-1}, & k \neq \infty, \ -m \\ l & k = -m \\ -l & k = \infty \end{cases} \tag{41}$$

Proof. — Consider first the case $k \neq \infty$, $k \neq -m$. Then:

$(x, y) := A_N(q, 2kq+l) = (2mq+2kq+l, 4m^2q - q + 4mkq + 2ml) \in L_N$

i.e. $q = 2^{-1}(m+k)^{-1}(x-l) \in L_N$, that is

$$y = 2k'x + l', \qquad \forall x \in Q_N$$

where k' and l' are given by (40) and (41). Moreover if $k = -m$ we have $\forall q \in Q_N$

$A(q, -2mq+l) = (2mq - 2mq + l, 4m^2 q - q - 4m^2 q + 2ml) = (l, 2ml - q)$

and if $k = \infty$ we get

$$A(l. p) = (2ml + p, 4m^2 l - l + 2mp), \qquad p \in P_N$$

which immediately implies the conclusion of the lemma. □

Concerning the action of the propagator on the functions $\Psi_{k,l}$ corresponding to the subsets $\Lambda_{k,l}$, we have the following:

LEMMA 5. — *Let* $N \geqq 3$ *be a prime number and* A *as before; then*

$$U_A \Psi_{k,l} = \exp i\sigma(k, l) \Psi_{k',l'} = \exp i\sigma(k, l) \Psi_{\zeta(k,l)} \tag{42}$$

where

$$\zeta : \tilde{\mathbf{Z}}_N \times \mathbf{Z}_N \to \tilde{\mathbf{Z}}_N \times \mathbf{Z}_N$$

is given as in the previous lemma and:

$$\sigma(k, l) = \begin{cases} -\frac{\pi}{4}N + \frac{\pi}{2}\left(\frac{2(m+k)}{N}\right) + \frac{2i\pi}{N}[-4^{-1}l^2(m+k)^{-1}], & k \neq \infty, -m \\ \frac{\pi}{4} + \frac{2i\pi}{N} ml^2, & k = -m, \infty \end{cases} \tag{43}$$

Remark. — Up to a phase, the quantum evolution acts on this set of functions as a permutation induced by the automorphism, *i.e.* by the "classical evolution".

Proof. — Consider first the case $k \neq \infty$, $-m$

$$\begin{aligned} (U_A \Psi_{k,l})(q_2) &= \sum_{q_1=0}^{N-1} \left(\frac{i}{N}\right)^{1/2} \exp\left[\frac{2i\pi}{N}(mq_1^2 - q_1 q_2 + mq_2^2)\right] . \exp\left[\frac{2i\pi}{N}(kq_1^2 + lq_1)\right] \\ &= \left(\frac{i}{N}\right)^{1/2} \exp\left[\frac{2i\pi}{N}(mq_2^2)\right] . \sum_{q_1=0}^{N-1} \exp\left[\frac{2i\pi}{N}((m+k)q_1^2 + (l-q_2)q_1)\right] \\ &= i^{1/2}\left(\frac{2(m+k)}{N}\right) . \exp\left[\frac{2i\pi}{N} mq_2^2\right] . \exp\left[-\frac{i\pi}{N}(N-1)\right]. \end{aligned}$$

$$\times \exp\left[-\frac{2i\pi}{N}(4^{-1}(m+k)^{-1}(l-q_2)^2)\right]$$
$$= i^{1/2}\left(\frac{2(m+k)}{N}\right).\exp\left[-\frac{i\pi}{4}(N-1)\right]\exp\left[-\frac{2i\pi}{N}(4^{-1}(m+k)^{-1}l^2)\right].$$
$$\times \exp\left\{\frac{2i\pi}{N}[(m-4^{-1}(m+k)^{-1})q_2^2+2^{-1}l(m+k)^{-1}q_2]\right\}$$

where the first equality follows from (29), (30) and the third one from (31). Now the result follows from the relation:

$$i^{1/2}\left(\frac{2(m+k)}{N}\right)\exp\left[-\frac{i\pi}{4}(N-1)\right]=\exp\left[i\left(-\frac{\pi}{4}N+\frac{\pi}{2}\left(\frac{2(m+k)}{N}\right)\right)\right]$$

Furthermore we immediately have:

$$(U_A\Psi_{-m,l})(q)=(iN)^{1/2}.\exp\left[\frac{2i\pi}{N}ml^2\right].\delta_q^l$$
$$(U_A\Psi_{\infty,l})(q)=(i)^{1/2}.\exp\left[\frac{2i\pi}{N}ml^2\right].\exp\left[\frac{2i\pi}{N}(mq^2-lq^2)\right]$$

and the lemma is proved. □

From the expression (40) of the map ζ defined in (39) it easily follows that $\tilde{Z}_N\times\{0\}$ is invariant under it, i. e.:

$$U_A\Psi_{k,0}:=U_A\Psi_k=U_A\Psi_{k',0}$$

Let us now proceed to the construction of the eigenvectors. A preliminary step is represented by the determination of the fixed points of the restriction of ζ to $\tilde{Z}_N\times\{0\}$, denoted once more by ζ by standard abuse of notation.

LEMMA 6. — *Let* $N\geq 3$ *be a prime integer, suppose*

$$\left(\frac{(\mathrm{Tr}\,A)^2-4}{N}\right)=\left(\frac{16m^2-4}{N}\right)=1,$$

i. e. *there exists* $q\in\mathbf{Z}_N$ *such that* $q^2=16m^2-4$ (mod N). *Then:*

$$k=\pm q.(4/N)\,(\mathrm{mod}\ N):=\pm 4^{-1}.q \tag{44}$$

are the only fixed points of the map ζ.

Proof. —

$$k=m-(m+k)(2m+k)/N)^2:=m-(m+k)^{-1}.4^{-1}$$
$$\Leftrightarrow\quad m-k=(m+k)^{-1}.4^{-1}$$
$$\Leftrightarrow\quad (m-k)(m+k)-4^{-1}=0\quad (\mathrm{mod}\ N)$$

$k=\pm q.(4/N)$ are two solutions of the third equation and since it admits no more than two solutions [9] they are the only ones. □

The fixed points obviously correspond to the two invariant one-dimensional eigenspaces of the automorphism.

Consider now the equivalence relation induced by the orbit of ζ:

$k_1 \sim k_2 \Leftrightarrow$ there exists a positive integer j such that $\zeta^j(k_1)=k_2$

Moreover let

$$\pi_\zeta : \ \tilde{\mathbf{Z}}_N \to \tilde{\mathbf{Z}}_N/\sim$$

be the usual projection that to each $k \in \tilde{\mathbf{Z}}_N$ associates the orbit

$$\{\zeta^j(k)\}_{j\in\mathbf{Z}} := [k]$$

In this case each orbit is finite and $\forall k_0 \in \tilde{\mathbf{Z}}_N$ there exists a positive integer l such that:

1) $\pi_\zeta^{-1}([k_0]) = \{k_0, k_1=\zeta(k_0), \ldots, k_{l-1}=\zeta^{l-1}(k_0)\}$
2) $\zeta^l(k_0)=k_0$
3) $k_j \neq k_0 \ \forall j=1, \ldots, l-1$.

As in Eckhardt [7] we can state the following

DEFINITION 9. — $\pi_\zeta^{-1}([k_0])$ *is called a Quantum cycle and* l *the length of the cycle.*

Remark. — If a cycle has length l it means that after l iterations the subset Λ_k goes into itself and from the fact that $\mathrm{A_N}$ is the restriction to $\mathrm{L_N}$ of a map defined on the torus it follows that we can have either $\mathrm{A}^l(x)=x$ or $\mathrm{A}^l(x)=-x$, $\forall x \in \Lambda_k$. In fact suppose

$$\mathrm{A_N}\left(\frac{1}{\mathrm{N}}, \frac{2k}{\mathrm{N}}\right)=\left(\frac{n}{\mathrm{N}}, \frac{2kn}{\mathrm{N}}\right)\in \mathrm{L_N}$$

for some $n \in \mathbf{Z}_N$, (w. l. g. we can assume $l=1$). This gives:

$$\mathrm{A_N}\left(\frac{2}{\mathrm{N}}, \frac{4k}{\mathrm{N}}\right)=\left(\frac{2n}{\mathrm{N}}, \frac{4kn}{\mathrm{N}}\right)\in \mathrm{L_N}$$

If $n \neq \pm 1 \pmod{\mathrm{N}}$ then there exists $t \in (1, 2)$ such that $p=tn$ is an integer, *i.e.* $\mathrm{A}\left(\frac{t}{\mathrm{N}}, \frac{2kt}{\mathrm{N}}\right)=\left(\frac{p}{\mathrm{N}}, \frac{2kp}{\mathrm{N}}\right)\in \mathrm{L_N}$, which is impossible.

Now the following result, which gives a positive answer to the conjecture proposed in [7] relating the quantum cycle to the classical periods, is an immediate consequence of the above construction.

LEMMA 7. — *The following statements are equivalent:*

1) $-\mathbf{I}_N \in \{A_N, A_N^2, \ldots, A_N^{n-1}, \mathbf{I}_N\}$

2) $n(N) = 2q,\ q \in \mathbf{N}$

3) $A_N^q(\Lambda_k) = \Lambda_k,\ \forall k \in \mathbf{Z}$ *and* $A_N^q = -\mathbf{I}_N$

4) *Each quantum cycle (different from a fixed point) has length* $q = n(N)/2$

Equivalently:

1) $-\mathbf{I}_N \notin \{A_N, A_N^2, \ldots, A_N^{n-1}, \mathbf{I}_N\}$

2) $n(N) = 2q+1,\ q \in \mathbf{N}$

3) $A_N^n(\Lambda_k) = \Lambda_k,\ \forall k \in \mathbf{N}$ *and* $A_N^n = \mathbf{I}_N$

4) *Each quantum cycle (different from a fixed point) has length* $n(N)$

Let P the parity operator on $\mathbf{L}^2(\mathbf{T}^1, \mu)$ defined as

$$(P\Phi)(x) = \Phi(-x) = \Phi(N-x)$$

then, $U_A P = P U_A$ as it follows from the simmetry of the propagator [*see* (30)]

$$U_A(n, k) = U_A(-n, -k)$$

(which is related to the fact that the classical map A commute with the map $-\mathbf{I}$).

Because of this simmetry all the eigenstates of the propagator must be either odd or even under P. Using the quantum cycle we are now able to caracterize the even part of the spectrum, namely:

THEOREM 3. — *Let* $k_0 \in \tilde{\mathbf{Z}}_N$ *be such that* $k_0 \neq \zeta(k_0)$; *consider*

$$\pi_\zeta^{-1}([k_0]) = \{k_0, k_1, \ldots, k_{l-1}\}$$

and, for each $m = 0, 1, \ldots, l-1$, *set:*

$$\lambda_m := \frac{2m\pi}{l} + \frac{1}{l} \sum_{j=0}^{l-1} \sigma(\zeta^j(k_0)) \tag{45}$$

Let moreover $(\gamma_0, \gamma_1, \ldots, \gamma_{l-1})$ *be recursively defined as follows:*

$$\gamma_{j+1} = \gamma_j + \sigma(k_j) - \lambda_m, \qquad j = 0, 1, \ldots, l-2 \tag{46}$$

where γ_0 *is arbitrary. Then*

1) $\Phi = \sum_{j=0}^{l-1} e^{i\gamma_j} . \Psi_{\zeta^j(k_0)}$ *is an even eigenvector with eigenvalue* λ_m.

2) *Each cycle defines the same family of eigenvalues.*

Proof. — The relation $\Phi(-x) = \Phi(x)$ and $U_A \Phi = e^{i\lambda_m} \Phi$ follows immediately from equation (42) and equation (46). In order to prove (ii) note that $U_A^n \Phi = e^{i\varphi} \Phi$ where φ is a constant. Now assume that the cycles have length $l = n$. Then:

$$U_A^n \Psi_{k_0} = e^{i\sigma(k_0)} . U_A^{n-1} \Psi_{k_1} = \ldots = e^{i \sum_{j=0}^{l-1} \sigma(\zeta^j(k_0))} . \Psi_{k_0}$$

That is:

$$\sum_{j=0}^{l-1} \sigma(\zeta^j(k_0)) = \varphi \pmod{2\pi} \tag{47}$$

is constant on each cycle, *i.e.* $\{\lambda_m\}_{(m=0,1,\ldots,l-1)}$ is the same for each quantum cycle. If $l=n/2=q$ we have $(U_A^q \Phi)(x) = e^{i\beta}\Phi(-x)$ where β is a constant. This implies:

$$\sum_{j=0}^{l-1} \sigma(\zeta^j(k)) = \sum_{j=0}^{l-1} \sigma(\zeta^j(\tilde{k})) \pmod{2\pi} \tag{48}$$

where k, $\tilde{k}$ belong to two different orbits. □

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(*Manuscript received December* 26, 1991;
Revised version received March 16, 1992.)