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A characterization of coherent states from varying Planck's constant

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ABSTRACT. – For a quantum system with one degree of freedom we introduce some families of states depending on the Planck's constant, that we call "classical states". We show that a definite class of these families allows to characterize the coherent states.

RÉSUMÉ. – Pour un système quantique à un degré de liberté, on introduit, sous le nom d'« états classiques », certaines familles d'états dépendant de la constante de Planck. On montre qu'une classe définie de ces familles permet de caractériser les états cohérents.

1. INTRODUCTION

The notion of coherent state in quantum Mechanics, at least in its elementary form, is generally associated with the description of classical situations. The current example, usually given in the one-dimensional case, stems from the behaviour of the coherent states under the dynamical law of the harmonic oscillator ([1], [2]). A more general argument, often considered as setting the "classical" character of these states, is found in their property of minimality with respect to the Heisenberg uncertainty relations. In that sense, coherent states are only "as classical as possible"

([3]-[5]), since, strictly speaking, every state in the Hilbert space must be interpreted from the general principles of quantum Mechanics.

The minimality property is yet a characteristic one in what any minimal state is a coherent state of some harmonic oscillator (Gaussian wave packet) [6]. More particularly, the coherent states of a *given* harmonic oscillator (that is, an oscillator whose mass and angular frequency are specified) may also be characterized by a property of minimum [7]. In accordance with these results, among the various generalizations of coherent states which have been devised, some of them were founded on the idea of minimizing a given uncertainty relation ([8], [9]).

The connection of coherent states with classical Mechanics is certainly made clearer in the works implying the vanishing of the Planck's constant. Typical results are those referring to a precise formulation of the Ehrenfest theorem ([10]-[12]). Coherent states there appear either under the form

$$|q,p\rangle = \exp\left[\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right]|0\rangle \tag{1}$$

or under the form

$$|\lambda/\sqrt{\hbar}\rangle = e^{-|\lambda|^2/2\hbar}\exp\left[\frac{\lambda}{\sqrt{\hbar}}a^+\right]|0\rangle \tag{2}$$

In these formulas $\hat{q}$ and $\hat{p}$ are the canonical operators, a the destruction operator of some given oscillator defined by

$$a = \frac{1}{\sqrt{2\hbar}}\left[\sqrt{\kappa}\,\hat{q} + \frac{i}{\sqrt{\kappa}}\hat{p}\right], \qquad \kappa = m\,\omega \tag{3}$$

and $|0\rangle$ the corresponding ground state. The states (1) and (2) are then equal under the following correspondence between their labels

$$\lambda = \sqrt{\frac{\kappa}{2}}\,q + \frac{i}{\sqrt{2\kappa}}\,p \tag{4}$$

The states (1) or (2) explicitly depend on $\hbar$. On account of that dependence a significative property is the existence of a finite limit when $\hbar \to 0$ of the expectation value on these states of any monomial in the canonical variables $\hat{q}$ and $\hat{p}$ [11]. This feature calls for the more general notion of a *family of states* $\psi(\hbar)$, depending on $\hbar$ and such that for any monomial $M(\hat{q}, \hat{p})$, the expectation value $\langle \psi(\hbar) | M(\hat{q}, \hat{p}) | \psi(\hbar) \rangle$ admits a limit when $\hbar$ goes to zero. We suggest to give the name of "classical states" to such families. Thus, a "classical state" would represent some kind of limit of quantum states.

The preceding condition is equivalent to require the existence of limits for the expectation values of the normal products

$\hbar^{(m+n)/2} \langle \psi(\hbar) | (a^+)^m a^n | \psi(\hbar) \rangle$, where m and n are any positive integers and a the operator defined by (3). In fact, as it is easily checked, for the states defined by (2) these expectation values are constant with respect to $\hbar$. Each family $\psi(\hbar)$ having that property is privileged in what the limits considered above can be calculated from any member of the family, in particular from the one corresponding to the actual value of $\hbar$ (1).

In that paper we intend to show that the property just mentioned essentially characterizes the coherent families (2). More precisely we shall prove that, under a regularity condition which will be stated later on, the conditions

$$\hbar^{(m+n)/2} \langle \psi(\hbar) | (a^+)^m a^n | \psi(\hbar) \rangle = \mathrm{C}_{m,n}, \qquad \forall (m,n), \quad \forall \hbar > 0, \tag{5}$$

in which the $\mathrm{C}_{m,n}$ are some constants, admit as only solutions the families of the type (2), up to a trivial factor. The proof would be trivial if the values of the $\mathrm{C}_{m,n}$ were *a priori* chosen equal to those given by the family (2). Here, however, the $\mathrm{C}_{m,n}$ are left undetermined at the start. Thus, the result expresses that the only constants which may appear in the right member of (5) are the values of the observables of a classical state.

In section 2, we give some preliminaries and state the regularity condition under which the proof will be achieved.

In section 3, the problem is reduced to the solution of a linear partial differential equation constrained by some non linear conditions.

In section 4, the solution is obtained from a complete integral of a non linear partial differential equation deduced from the preceding one, and some remarks are added.

2. THE REGULARITY CONDITION

Let $\{|n\rangle\}$ be the usual basis of the oscillator eigenstates defined from the operator a [2]. With any $\psi = \sum_n \psi_n |n\rangle$ in the Hilbert space we associate the power series of the complex variable z given by

$$f_\psi(z) = \sum_n |\psi_n|^2 z^n \tag{6}$$

and with (ψ, ψ') we associate the series

$$f_{\psi,\psi'}(z) = \sum_n \bar{\psi}_n \psi'_n z^n \tag{7}$$

(1) That constancy property cannot hold for *all* expectation values since, for example, one has $\hbar a a^+ = \hbar\, a^+ a + \hbar$. Thus, the requirement of constancy for the normal products represents a maximum.

These series are absolutely convergent for $|z| \leqq 1$ and we have

$$f_{\psi}(1)=\|\psi\|^2, \qquad f_{\psi,\psi'}(1)=\langle\psi|\psi'\rangle \tag{8}$$

$$f_{\psi}(0)=|\psi_0|^2, \qquad f_{\psi,\psi'}(0)=\bar{\psi}_0\psi'_0 \tag{9}$$

Let ρ_{ψ} and $\rho_{\psi,\psi'}$ be their respective radii of convergence. From the Schwarz inequality

$$|f_{\psi,\psi'}(z)| \leqq f_{|\psi|,|\psi'|}(|z|) \leqq (f_{\psi}(|z|) f_{\psi'}(|z|))^{1/2}, \tag{10}$$

in which $|\psi|$ denotes the state with components $|\psi_n|$, we obtain

$$\rho_{\psi,\psi'} \geqq \inf(\rho_{\psi}, \rho_{\psi'}) \geqq 1 \tag{11}$$

Let us denote by D_{∞} the common domain of the monomials in a and a^+. If ψ and ψ' belong to D_{∞} it is easy to prove the relations ([2])

$$f_{a^k\psi}=f_{\psi}^{(k)}, \qquad f_{a^k\psi,\,a^k\psi'}=f_{\psi,\psi'}^{(k)} \tag{12}$$

from which results the equality $\rho_{a^k\psi}=\rho_{\psi}$. Let us also introduce the correlation coefficients

$$C_{m,n}(\psi)=\overline{C_{n,m}(\psi)}=\langle\psi|(a^+)^m a^n|\psi\rangle \tag{13}$$

From (8) and (9) we have

$$C_{m,n}(\psi)=f_{a^m\psi,\,a^n\psi}(1) \tag{14}$$

$$\bar{\psi}_m\psi_n=\frac{1}{\sqrt{m!\,n!}} f_{a^m\psi,\,a^n\psi}(0) \tag{15}$$

The $C_{m,n}(\psi)$ are obviously determined by the ψ_n. Conversely, if $\rho_{\psi}>1$, the $C_{m,n}(\psi)$ determine the functions $f_{a^m\psi,\,a^n\psi}$ in the neighbourhood of 1, on account of (12) and (14), and therefore at the origin; from (15) it then follows that the ψ_n are determined up to a common phase factor.

Now, the result we have in view will be attained by the solution of the following problem: find the families $\psi(\hbar)\in D_{\infty}$, $\hbar>0$, such that

$$C_{m,n}(\psi(\hbar))=C_{m,n}\hbar^{-(m+n)/2} \tag{16}$$

where the $C_{m,n}$ are some constants. To simplify the notation let us put $\rho_{\psi(\hbar)}=\rho_{\hbar}$. We have the following:

LEMMA. — *For any solution* $\psi(\hbar)$ *of* (16) *we have* $\rho_{\hbar}>1$ *for all* $\hbar$ *or* $\rho_{\hbar}=1$ *for all* $\hbar$.

Proof. — Let us assume that $\rho_{\mu}>1$ for some given value μ of $\hbar$. The formulas (12), (14) and (16) imply the relation

$$f_{\psi(\hbar)}^{(k)}(1)=\left(\frac{\mu}{\hbar}\right)^k f_{\psi(\mu)}^{(k)}(1)$$

([2]) As usual $f^{(k)}$ denotes the derivative of order k of f.

Therefore, the Taylor series of $f_{\psi(\hbar)}$ about 1 has a finite radius of convergence for any value of $\hbar$. Since the coefficients of the power series representing $f_{\psi(\hbar)}$ about the origin are positive, it follows that this function can be analytically continued in the neighbourhood of 1, and that we have $\rho_\hbar > 1$.

Q.E.D.

From now we only consider the regular case for which $\rho_\hbar > 1$ for all $\hbar$.

3. THE CONSTRAINED PARTIAL DIFFERENTIAL EQUATION

Let μ be a particular value of $\hbar$, and let

$$c_n = \sqrt{n!}\,\psi_n(\mu) \tag{17}$$

and

$$\varphi_{m,n}(x) = f_{a^m\psi(\mu),\, a^n\psi(\mu)}(x) = \sum_k \frac{x^k}{k!}\bar{c}_{m+k}\, c_{n+k} \tag{18}$$

where x is a real variable. From (12), (14) and (16), and with the help of the Taylor series, we easily find the relation

$$f_{a^m\psi(\hbar),\, a^n\psi(\hbar)}(1+z) = \left(\frac{\mu}{\hbar}\right)^{(m+n)/2} f_{a^m\psi(\mu),\, a^n\psi(\mu)}\left(1+\frac{\mu}{\hbar}z\right)$$

valid for all z such that $1+(\mu/\hbar)|z| < \rho_\mu$. For the values of $\hbar$ satisfying the condition $1+(\mu/\hbar) < \rho_\mu$ we can take $z=-1$ in the preceding equation, thus obtaining, by (15),

$$\overline{\psi_m(\hbar)}\,\psi_n(\hbar) = \frac{1}{\sqrt{m!\,n!}}\left(\frac{\mu}{\hbar}\right)^{(m+n)/2} \varphi_{m,n}\left(1-\frac{\mu}{\hbar}\right) \tag{19}$$

These equations allow to calculate the components $\psi_n(\hbar)$. They imply, however, some compatibility conditions, namely

$$\varphi_{m,n}\,\varphi_{0,0} = \varphi_{m,0}\,\varphi_{0,n} \tag{20}$$

which, by analytic continuation, must be valid in the whole interval $|x| < \rho_\mu$.

The equations (20) represent some contraints on the coefficients c_n. To render them tractable let us introduce the generating function

$$\varphi(x, u, v) = \sum_{m,n} \varphi_{m,n}(x)\frac{u^m}{m!}\,\frac{v^n}{n!} \tag{21}$$

in which u and v are real variables. With the help of the Schwarz inequality (10) and of the Cauchy inequalities [13] written for $f_{|\psi(\mu)|}$, it is not difficult to show that, with (18), the right member of (21) becomes a series in three variables absolutely convergent for $|x| < \rho_\mu$ and for all (u, v). The conditions (20) then become

$$\varphi(x, u, v)\,\varphi(x, 0, 0) = \varphi(x, u, 0)\,\varphi(x, 0, v) \tag{22}$$

Moreover, from the relations $\bar{\varphi}_{m,n} = \varphi_{n,m}$ and $\varphi^{(1)}_{m,n} = \varphi_{m+1,n+1}$ we respectively deduce

$$\overline{\varphi(x, u, v)} = \varphi(x, v, u) \tag{23}$$

$$\frac{\partial}{\partial x}\varphi(x, u, v) = \frac{\partial}{\partial u}\frac{\partial}{\partial v}\varphi(x, u, v) \tag{24}$$

so that we have to solve the linear partial differential equation (24) together with the non linear constraints (22) and (23).

The last equations are furthermore transformed by successively putting $f(x,u) = \varphi(x, u, 0)$ then $g(x, u) = \operatorname{Log} f(x, u)$ (the logarithm being defined by its principal value in the neighbourhood of a point of the form $(x, u) = (x_0, 0)$, $0 < x_0 < \rho_\mu$, at which f takes a positive value). We then find the following expression of $\varphi(x, u, v)$

$$\varphi(x, u, v) = f(x, u)\,\overline{f(x, v)}/f(x, 0) \tag{25}$$

the equations for g being

$$\overline{g(x, 0)} = g(x, 0) \tag{26}$$

$$\partial_x g(x, u) + \partial_x \overline{g(x, v)} - \partial_x g(x,0) = \partial_u g(x, u)\,\partial_v \overline{g(x, v)} \tag{27}$$

The next section is devoted to the solution of these equations.

4. SOLUTION

Instead of g we consider the following function on three variables (equal to Log φ)

$$G(x, u, v) = g(x, u) + \overline{g(x, v)} - g(x,0) \tag{28}$$

On account of (27) is satisfies the non linear equation

$$\partial_x G = \partial_u G \,.\, \partial_v G \tag{29}$$

The latter admits the following particular solution, depending on three complex parameters α, β, γ,

$$G_0(x, u, v;\ \alpha, \beta, \gamma) = \alpha u + \beta v + \alpha\beta x + \gamma \tag{30}$$

Actually, G_0 is a complete integral [14] of (29). More precisely, for any solution G of (29), there exists a mapping

$$(x, u, v) \rightarrow (\alpha(x, u, v),\ \beta(x, u, v),\ \gamma(x, u, v))$$

such that the following equations are satisfied

$$G = u\alpha + v\beta + x\alpha\beta + \gamma \tag{A}$$

$$\partial_u G = \alpha, \qquad \partial_v G = \beta, \qquad \partial_x G = \alpha\beta \tag{B}$$

Taking the derivatives of (A) and using (B) gives

$$\left.\begin{aligned} u\,\partial_x\alpha + v\,\partial_x\beta + x\,\partial_x(\alpha\beta) + \partial_x\gamma = 0 \\ u\,\partial_u\alpha + v\,\partial_u\beta + x\,\partial_u(\alpha\beta) + \partial_u\gamma = 0 \\ u\,\partial_v\alpha + v\,\partial_v\beta + x\,\partial_v(\alpha\beta) + \partial_v\gamma = 0 \end{aligned}\right\} \tag{C}$$

These equations can be considered as determining γ; then, they imply the integrability conditions

$$\partial_x\alpha = \partial_u(\alpha\beta), \qquad \partial_x\beta = \partial_v(\alpha\beta), \qquad \partial_u\beta = \partial_v\alpha \tag{D}$$

Let us now introduce the particular form (28) of G. The equations (B) show that α and β have the form

$$\alpha(x, u, v) = \xi(x, u), \qquad \beta(x, u, v) = \overline{\xi(x, v)} \tag{31}$$

and these equations become

$$\left.\begin{aligned} \partial_u g(x, u) = \xi(x, u) \\ \partial_x g(x, u) + \partial_x \overline{g(x, v)} - \partial_x g(x, 0) = \xi(x, u)\,\overline{\xi(x, v)} \end{aligned}\right\} \tag{B'}$$

Furthermore, the equations (D) come to the unique equation

$$\partial_x \xi(x, u) = \overline{\xi(x, v)}\,\partial_u \xi(x, u)$$

By differentiating this latter with respect to v we find $\partial_u \xi(x, u) = 0$, then $\partial_x \xi(x, u) = 0$. Therefore, the functions α and β are some constants, and $\beta = \bar{\alpha}$ from (31). The equations (C) then show that γ is also a constant.

Finally, the function g is easily determined from (B′) and the condition (26). We find $g(x, u) = \alpha u + |\alpha|^2 x + \gamma$, with γ real. Afterwards, we have $\varphi(x, u, v) = K\exp(\alpha u + \bar{\alpha} v + |\alpha|^2 x)$, where K is a positive constant. That expression, up to now defined in a neighbourhood of $(x_0, 0, 0)$, extends for all values of (x, u, v).

The expansion of φ supplies the $\varphi_{m,n}$ according to (21), then (18) gives the equations

$$\bar{c}_{m+k}\, c_{n+k} = K\,\alpha^m (\bar{\alpha})^n\, |\alpha|^{2k}$$

The general solution of these latter is $c_n = \sqrt{K}(\bar{\alpha})^n e^{i\theta}$, depending on an arbitrary phase factor. It follows that the convergence radius ρ_μ is infinite, as also $\rho_\hbar$ for all $\hbar$, since μ was arbitrarily chosen.

Lastly, the components $\psi_n(\hbar)$ are calculated from (19); by putting $\lambda = \bar{\alpha}\sqrt{\mu}$ we find

$$\psi_n(\hbar) = Ce^{i\theta(\hbar)} \cdot \frac{1}{\sqrt{n!}} (\lambda/\sqrt{\hbar})^n e^{-|\lambda|^2/2\hbar}$$

in which C is a positive constant and $\theta(\hbar)$ an arbitrary phase depending on $\hbar$. Up to the factor $C e^{i\theta(\hbar)}$, the corresponding state is the coherent state (2).

Thus, under the only regularity condition stated in section 2, the coherent families (2) are characterized by the condition of constancy (5). The consideration of a varying $\hbar$ was obviously a fundamental one. In that respect let us stress that the family (2) weakly converges to zero when $\hbar$ goes to zero, and does not correspond to a uniquely defined vector in the space of states. Its physical sense is contained in the $\hbar$-dependence of its terms rather than in any particular of them.

Let us add that replacing in (5) the state $\psi(\hbar)$ by the state (2) corresponding to another value κ' than the value κ of the parameter defining the operator a does not give a left-hand side constant, but admitting only a limit when $\hbar \to 0$. Generally, the families (2) associated with an arbitrary value of κ are "classical states" in the sense suggested in the introduction. A more general study of that notion will be the subject of another paper.

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