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Smoothing effect of small analytic solutions to nonlinear Schrödinger equations

by

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ABSTRACT. — We consider the initial value problem for nonlinear Schrödinger equations in $\mathbb{R}^n (\geqq 2)$:

$$(*) \qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = F(u, \nabla u, \bar{u}, \overline{\nabla u}), & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases}$$

where $F: \mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ is a polynomial of degree 3 satisfying

$$|F(u, \nabla u, \bar{u}, \overline{\nabla u})| \leqq C \cdot (|u| + |\nabla u|)^3$$

and

$$F(\omega u, \omega \nabla u, \overline{\omega u}, \overline{\omega \nabla u}) = \omega F(u, \nabla u, \bar{u}, \overline{\nabla u}),$$

for any complex number ω with $|\omega| = 1$. It is shown that global solutions of $(*)$ have a smoothing property.

RÉSUMÉ. — Nous considérons l'équation d'évolution de Schrödinger non linéaire dans $\mathbb{R}^n (n \geqq 2)$:

$$(*) \qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = F(u, \nabla u, \bar{u}, \overline{\nabla u}), & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases}$$

où $F: \mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ est un polynôme de degré 3 tel que

$$|F(u, \nabla u, \bar{u}, \overline{\nabla u})| \leqq C \cdot (|u| + |\nabla u|)^3$$

et

$$F(\omega u, \omega \nabla u, \overline{\omega u}, \overline{\omega \nabla u}) = \omega F(u, \nabla u, \bar{u}, \overline{\nabla u}),$$

pour tout nombre complexe ω avec $|\omega| = 1$. Nous montrons que les solutions globales de ($*$) ont la propriété de régularisation.

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1. INTRODUCTION

In this paper we consider the initial value problem for non-linear Schrödinger equations in $\mathbb{R}^n$ ($n \geqq 2$):

$$i \partial_t u + \frac{1}{2} \Delta u = F(u, \nabla u, \bar{u}, \overline{\nabla u}), \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n \tag{1.1}$$

$$u(0, x) = \varphi(x), \qquad x \in \mathbb{R}^n. \tag{1.2}$$

Here the nonlinear term $F : \mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ is a polynomial of degree 3 satisfying

$$\left.\begin{array}{r} |F(u, \nabla u, \bar{u}, \overline{\nabla u})| \leqq C.(|u| + |\nabla u|)^3, \\ F(\omega u, \omega \nabla u, \overline{\omega u}, \overline{\omega \nabla u}) = \omega F(u, \nabla u, \bar{u}, \overline{\nabla u}), \end{array}\right\} \tag{1.3}$$

for any complex number ω with $|\omega| = 1$, where $\bar{u}$ is the complex conjugate of u and ∇ stands for nabla with respect to x.

In [3] we proved that small analytic solutions of (1.1)-(1.2) exist globally in time if the initial function φ is analytic and sufficiently small. The purpose of this paper is twofold. One is to show that global analytic solutions of (1.1)-(1.2) have a smoothing property if φ satisfies certain analytical and exponential decaying conditions with respect to space variables. The other is to give a simple proof of an analogous result to Theorem 2 ([4]) in which we proved smoothing effects of solutions of (1.1)-(1.2) for the special nonlinearity $F = \pm |u|^2 u$.

Our strategy of the proof in this paper is to translate (1.1)-(1.2) into a system of nonlinear Schrödinger equations to which we can apply the previous methods developed in [2], [3], [5], and [6].

We note that smoothing properties for a class of nonlinear Schrödinger equations in the weighted Sobolev spaces were studied in [5] first (in the case of the usual Sobolev spaces, see [1], [9] and [10]).

We now state notations and function spaces used in this paper.

Notation and function spaces. – We let $L^p(\mathbb{R}^n)=\{f(x); f(x)$ is measurable on $\mathbb{R}^n$, $\|f\|_{L^p}<\infty\}$, where $\|f\|_{L^p}=\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p}$ if $1\leqq p<\infty$ and $\|f\|_{L^\infty}=\text{ess.sup}\{|f(x)|; x\in\mathbb{R}^n\}$ if $p=\infty$, and we let $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}$, $J_x^\alpha=J_{x_1}^{\alpha_1}\ldots J_{x_n}^{\alpha_n}$, where $\alpha=(\alpha_1,\ldots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$ is a multi-index and $J_{x_j}^{\alpha_j}=(x_j+it\,\partial_{x_j})^{\alpha_j}$. We denote by $\mathscr{F}$ and $\mathscr{F}^{-1}$ the Fourier transform and inverse, respectively. For each $r>0$ we denote $S(r)$ the strip $\{z; -r<\Im z_j<r; 1\leqq j\leqq n\}$ in the complex plane $\mathbb{C}^n$. For $x\in\mathbb{R}^n$, if a complex-valued function $f(x)$ has an analytic continuation to $S(r)$, then we denote this by the same letter $f(z)$ and if $g(z)$ is an analytic function on $S(r)$, then we denote the restriction of $g(z)$ to the real axis by the same letter $g(x)$.

We let

$$AL^2_\infty(r)=\{f(z); f(z)\text{ is analytic on }S(r), \|f\|_{AL^2_\infty(r)}<\infty\},$$
$$AL^2_{\infty,2}(r)=\{f(z); f(z)\text{ is analytic on }S(r), \|f\|_{AL^2_{\infty,2}(r)}<\infty\},$$

and B^m be the same function space as that defined in [3], p. 724, where

$$\|f\|_{AL^2_\infty(r)}=\sup_{y\in(-r,r)^n}\|f(\cdot+iy)\|^2_{L^2},$$

$$\|f\|_{AL^2_{\infty,2}(r)}=\sum_{l=1}^n\sup_{y\setminus y_l\in(-r,r)^n}\int_{-r}^r\|\partial_{z_l}f(\cdot+iy)\|^2_{L^2}\,dy_l.$$

Constants will be denoted by $C_j\,(j=1,2,\ldots)$. For a multi-index $k=(k_1,\ldots,k_n)$ $(k_j=0,1; j=1,2,\ldots,n)$, we let

$$\exp\left(-\sum_{j=1}^n(-1)^{k_j}x_j\right)\varphi(x)=\exp(-(-1)^k\cdot x)\,\varphi(x)=\Phi_k(x),$$

and

$$K=\{k\in\mathbb{R}^n; k_j=0,1, j=1,2,\ldots,n\}.$$

We denote by $[s]$ the largest integer which is less than or equal to s.

Remark 1:

$$2^n\sum_{k\in K}\exp(-(-1)^k\cdot x)=\prod_{j=1}^n\cosh x_j.$$

We state our results in this paper.

THEOREM 1. – *We assume that* $\varphi(x)$ *has an analytic continuation to* $S(r)$ *and*

$$\sum_{|\alpha|+|\beta|\leqq n+3}\left(\left\|\partial_z^\alpha z^\beta\left(\prod_{j=1}^n\cosh z_j\right)\varphi\right\|^2_{AL^2_\infty(r)}+\left\|\partial_z^\alpha z^\beta\left(\prod_{j=1}^n\cosh z_j\right)\varphi\right\|^2_{AL^2_{\infty,2}(r)}\right)$$

is sufficiently small. Then there exists a unique global solution $u(t,x)$ *of* (1.1)-(1.2) *such that* $u(t,x)$ *has an analytic continuation to* $S(r') \cup S(|t|)$ *and* $\exp(-iz^2/2t)\,u(t) \in AL^2_\infty(|t|)$, *where* $r' < r$.

Remark 2. — From the fact that $\exp(-iz^2/2t)\,u(t) \in AL^2_\infty(|t|)$ it is clear that the analytical domain of solutions increase with time. This implies the smoothing effect of solutions to (1.1)-(1.2).

THEOREM 2. — *We assume that* $F = \pm |u|^2 u$ *and*

$$\sum_{|\alpha| \leq [n/2]+1} \left(\int \prod_{j-1}^{n} \cosh(2x_j) \{ |\partial_x^\alpha \varphi(x)|^2 + |x^\alpha \varphi(x)|^2 \} dx \right)$$

is sufficiently small. Then there exists a unique global solution $u(t,x)$ *of* (1.1)-(1.2) *such that* $u(t,x)$ *has an analytic continuation to* $S(|t|)$ *and* $\exp(-iz^2/2t)\,u(t) \in AL^2_\infty(|t|)$.

Remark 3. — In [4], Theorem 2, S. Saitoh and the author obtained the result similar to Theorem 2. However the above mentioned assumption on φ is more natural than that of [4], Theorem 2.

2. PROOF OF THEOREM 1

We consider the system of Schrödinger equations:

$$i\partial_t u + \frac{1}{2}\Delta u = F(v, \nabla v, \bar{v}, \overline{\nabla v}), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1}$$

$$-i\partial_t \bar{u} + \frac{1}{2}\Delta \bar{u} = \overline{F(v, \nabla v, \bar{v}, \overline{\nabla v})}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.2}$$

$$u(0,x) = \varphi(x), \qquad \bar{u}(0,x) = \bar{\varphi}(x), \qquad x \in \mathbb{R}^n. \tag{2.3}$$

We translate (2.1)-(2.3) into another system of equations and apply the previous result ([3], Theorem 1) to it. For any smooth function w, we put

$$W_k = P_k w = \Lambda \mathscr{F}^{-1} \exp((-1)^k t \cdot \xi) \mathscr{F} \Lambda^{-1} w = \exp((-1)^k J_x) w,$$
$$W_k^* = P_k^* w = \Lambda^{-1} \mathscr{F}^{-1} \exp((-1)^k t \cdot \xi) \mathscr{F} \Lambda \bar{w} = \exp((-1)^k \bar{J}_x) \bar{w},$$

where

$$\Lambda = \Lambda(t,x) = \exp(i|x|^2/2t)$$

and

$$(-1)^k J_x = \sum_{j=1}^{n} (-1)^{k_j} J_{x_j}, \qquad J_{x_j} = x_j + it\partial_{x_j}.$$

For any v such that V_k, $V_k^* \in B^{n+3}$, we shall show that solutions u and $\bar{u}$ of (2.1)-(2.3) satisfy the following system of Schrödinger equations:

$$i\partial_t U_k + \frac{1}{2}\Delta U_k = F(V_k, (\nabla - (-1)^k) V_k, V_k^*, (\nabla + (-1)^k) V_k^*), \quad (2.4)$$

$$-i\partial_t U_k^* + \frac{1}{2}\Delta U_k^* = \overline{F(\bar{V}_k^*, (\nabla + (-1)^k) \overline{V_k^*}, \bar{V}_k, (\nabla - (-1)^k) \bar{V}_k)}, \quad (2.5)$$

$$U_k(0, x) = \Phi_k(x), \qquad U_k^*(0, x) = \bar{\Phi}_k(x). \quad (2.6)$$

The reason why we must consider the system of equations (2.4)-(2.6) is that in general $|W_k|$ is not equal to $|W_k^*|$. We prove (2.4) and (2.6) only since the proof of (2.5) is the same as that of (2.4). We prove (2.4) first. Applying P_k to both sides of (2.1), we have

$$i\partial_t U_k + \frac{1}{2}\Delta U_k = P_k F(v, \nabla v, \bar{v}, \overline{\nabla v}), \quad (2.7)$$

where we have used the fact that

$$\left[i\partial_t + \frac{1}{2}\Delta, P_k\right] = \left(i\partial_t + \frac{1}{2}\Delta\right)P_k - P_k\left(i\partial_t + \frac{1}{2}\Delta\right) = 0.$$

By (1.3) we see that the right hand side of (2.7) is rewritten as

$$\Lambda \mathscr{F}^{-1} \exp((-1)^k t \cdot \xi) \mathscr{F} \Lambda^{-1} F(\Lambda^{-1} v, \Lambda^{-1} \nabla v, \Lambda \bar{v}, \Lambda \overline{\nabla v}). \quad (2.8)$$

On the other hand, we have by [7], p. 99

$$\mathscr{F}^{-1} \exp((-1)^k t \cdot \xi) \mathscr{F} \Lambda^{-1} v = \Lambda^{-1}(t, z) v(t, z) \quad (2.9)$$

since $V_k \in B^{n+3}$, where $z = x - i(-1)^k t$. From (2.8), (2.9) and the assumption that F is a polynomial it follows that the right hand side of (2.7) is equal to

$$\Lambda F(\Lambda^{-1}(t, z) v(t, z), \Lambda^{-1}(t, z) \nabla_z v(t, z),$$
$$\Lambda^{-1}(t, z) \overline{v(t, \bar{z})}, \Lambda^{-1}(t, z) \overline{\nabla_{\bar{z}} v(t, \bar{z})}).$$

We again apply (2.9) to the above and use the homogeneous condition (1.3) to see that the right hand side of (2.7) is equal to

$$F(P_k v, P_k \nabla v, P_k^* \bar{v}, P_k^* \overline{\nabla v}). \quad (2.10)$$

A direct calculation yields

$$[P_k, \nabla] = -(-1)^k P_k, \qquad [P_k^*, \nabla] = (-1)^k P_k^*. \quad (2.11)$$

From (2.10) and (2.11) we obtain (2.4). We next prove (2.6). We have by (2.9)

$$U_k = P_k u = \exp\left(\frac{n}{2} it - (-1)^k \cdot x\right) u(t, x - i(-1)^k t).$$

This shows (2.6). Thus solutions of (2.1)-(2.3) satisfy (2.4)-(2.6). Though we do not treat a system of nonlinear Schrödinger equations in [3], the proof of Theorem 1 in [3] is applicable to our problem, since

$$\sum_{k\in K}\sum_{|\alpha|+|\beta|\leq n+3}(\|\partial_z^\alpha z^\beta \Phi_k\|^2_{AL^2_\infty(r)}+\|\partial_z^\alpha z^\beta \Phi_k\|^2_{AL^2_{\infty,2}(r)})$$

is equivalent to

$$\sum_{|\alpha|+|\beta|\leq n+3}\left(\left\|\partial_z^\alpha z^\beta\left(\prod_j^n \cosh z_j\right)\varphi\right\|^2_{AL^2_\infty(r)}+\left\|\partial_z^\alpha z^\beta\left(\prod_j^n \cosh z_j\right)\varphi\right\|^2_{AL^2_{\infty,2}(r)}\right).$$

Hence, in the same way as in the proof of Theorem 1 in [3], it follows that there exist unique solutions U_k and U_k^* which are in B^{n+3} and satisfy

$$i\partial_t U_k+\frac{1}{2}\Delta U_k=F(U_k,(\nabla-(-1)^k)U^k,U_k^*,(\nabla+(-1)^k)U_k^*),$$

$$-i\partial_t U_k^*+\frac{1}{2}\Delta U_k^*=\overline{F(\overline{U_k^*},(\nabla+(-1)^k)\overline{U_k^*},\overline{U_k},(\nabla-(-1)^k)\overline{U_k})},$$

$$U_k(0,x)=\Phi_k(x),\qquad U_k^*(0,x)=\bar{\Phi}_k(x).$$

Since $B^{n+3}\subset C(\mathbb{R};L^2(\mathbb{R}^n))$, we have U_k, $U_k^*\in L^2(\mathbb{R}^n)$ for any t. Therefore we obtain by Remark 1

$$\begin{aligned}\sum_{k\in K}\|U_k\|^2_{L^2}=\sum_{k\in K}\|U_k^*\|^2_{L^2}&\\ &=\sum_{k\in K}\int \exp((-1)^k 2t\cdot\xi)\,|\mathscr{F}\Lambda^{-1}u(t,\xi)|^2\,d\xi\\ &=2^n\int\prod_j^n\cosh(2t\xi_j)\,|\mathscr{F}\Lambda^{-1}u(t,\xi)|^2\,d\xi.\end{aligned}$$

From this equality and [3], Lemma 2.1, it follows that $\Lambda^{-1}u$ has an analytic continuation $\Lambda^{-1}(t,z)u(t,z)$ which belongs to $AL^2_\infty(|t|)$. This completes the proof of the theorem.

Q.E.D.

3. PROOF OF THEOREM 2

We introduce the function space Σ_t^m:

$$\Sigma_t^m=\{f\in L^2(\mathbb{R}^n);\ \|f\|^2_{\Sigma_t^m}=\sum_{|\alpha|\leq m}(\|J_x^\alpha f\|^2_{L^2}+\|\partial_x^\alpha f\|^2_{L^2})<\infty\}.$$

Here we note that $J_x^\alpha=\Lambda(it\partial_x)^\alpha\Lambda^{-1}$. We give a useful lemma first which will be used to obtain the result.

LEMMA 3.1. — (a) *For any* $f\in\Sigma_t^m$ *with* $m\geq[n/2]+1$, *we have*

$$\|f\|_{L^\infty}\leq C_1\cdot(1+|t|)^{-n/2}\|f\|_{\Sigma_t^m}.$$

(b) *For any* $f_j \in \Sigma_t^m$ *with* $m \geqq [n/2]+1, j=1,2,3,$ *we have*

$$\| f_1 \bar{f}_2 f_3 \|_{\Sigma_t^m} \leqq C_2 \cdot (1+|t|)^{-n} \prod_{j=1}^{3} \| f_j \|_{\Sigma_t^m}.$$

Proof. – This lemma was already shown in [2], [3], [5] or [6] essentially. Hence we only give a sketch of proof. Part (*a*) follows from an easy application of Sobolev's inequality (for details, *see* [2], Corollary 1.3, [3], Lemma 2.2 and [6], Proposition 5). We prove Part (*b*). We have by [5], Lemma A.2 (*see* also [2], [6]).

$$\sum_{|\alpha| \leqq m} \| \partial_x^\alpha f_1 \bar{f}_2 f_3 \|_{L^2} \leqq C_3 \sum_{|\alpha| \leqq m} \prod_{j=1}^{3} \| f_j \|_{L^\infty}^{\beta_j(\alpha)} \| f_j \|_{\Sigma_t^m}^{1-\beta_j(\alpha)},$$

where $0 \leqq \beta_j(\alpha) \leqq 1$ and $\sum_{j=1}^{3} \beta_j(\alpha) = 2$. Applying Part (*a*) to this inequality, we obtain

$$\sum_{|\alpha| \leqq m} \| \partial_x^\alpha f_1 \bar{f}_2 f_3 \|_{L^2} \leqq C_4 \cdot (1+|t|)^{-n} \prod_{j=1}^{3} \| f_j \|_{\Sigma_t^m}. \tag{3.1}$$

Similarly, we have

$$\sum_{|\alpha| \leqq m} \| J_x^\alpha f_1 \bar{f}_2 f_3 \|_{L^2} \leqq C_5 \cdot (1+|t|)^{-n} \prod_{j=1}^{3} \| f_j \|_{\Sigma_t^m}. \tag{3.2}$$

From (3.1) and (3.2) Part (*b*) follows.

Q.E.D.

We now in a position to prove Theorem 2.

Proof of Theorem 2. – We consider the system of nonlinear Schrödinger equations:

$$i \partial_t u + \frac{1}{2} \Delta u = \pm |u|^2 u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.3}$$

$$-i \partial_t \bar{u} + \frac{1}{2} \Delta \bar{u} = \pm |u|^2 \bar{u}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.4}$$

$$u(0,x) = \varphi(x), \qquad \bar{u}(0,x) = \bar{\varphi}(x), \qquad x \in \mathbb{R}^n. \tag{3.5}$$

We shall prove Theorem 2 by making use of the contraction mapping principle. For that purpose we prepare the function space:

$$E^m = \{ f(t,z) \in AL_\infty^2 (|t|);\ \| f \|_{E^m}^2 < \infty \},$$

where

$$\| f \|_{E^m}^2 = \sup_{t \in \mathbb{R}} \sum_{|\alpha| \leqq m} (\| \Lambda^{-1} J_z^\alpha f \|_{AL_\infty^2 (|t|)}^2 + \| \Lambda^{-1} \partial_x^\alpha f \|_{AL_\infty^2 (|t|)}^2),$$

We first prove that

$$\left.\begin{aligned}\sum_{|\alpha|\leq m} \|\Lambda^{-1} J_z^\alpha f\|_{AL^2_\infty(|t|)} &\cong \sum_{\substack{k\in K\\|\alpha|\leq m}} \int |J_x^\alpha P_k f|^2\,dx,\\ \sum_{|\alpha|\leq m} \|\Lambda^{-1} \partial_x^\alpha f\|_{AL^2_\infty(|t|)} &\cong \sum_{\substack{k\in K\\|\alpha|\leq m}} \int |\partial_x^\alpha P_k f|^2\,dx.\end{aligned}\right\} \tag{3.6}$$

Here $\cong$ means the two norms are equivalent to each other. The first relation of (3.6) is proved in the same way as in the proof of the second one, and so we only prove the second one. From [3], Lemma 2.1 (1), it follows that

$$\|\Lambda^{-1}\partial_x^\alpha f\|^2_{AL^2_\infty(|t|)} \cong \int \prod_j^n \cosh(2t\xi_j)\,|\mathscr{F}(\Lambda^{-1}\partial_x^\alpha f)(\xi)|^2\,d\xi. \tag{3.7}$$

We apply Remark 1 and the Plancherel theorem to the right hand side of (3.7) to obtain

$$\|\Lambda^{-1}\partial_x^\alpha f\|^2_{AL^2_\infty(|t|)} \cong \sum_{k\in K}\int |P_k\,\partial_x^\alpha f(x)|^2\,dx. \tag{3.8}$$

By (2.11) we see that

$$\sum_{\substack{k\in K\\|\alpha|\leq 1}} \int |P_k\,\partial_x^\alpha f(x)|^2\,dx \cong \sum_{\substack{k\in K\\|\alpha|\leq 1}} \int |\partial_x^\alpha P_k f(x)|^2\,dx.$$

We iterate this argument to obtain

$$\sum_{\substack{k\in K\\|\alpha|\leq m}} \int |P_k\,\partial_x^\alpha f(x)|^2\,dx \cong \sum_{\substack{k\in K\\|\alpha|\leq m}} \int |\partial_x^\alpha P_k f(x)|^2\,dx. \tag{3.9}$$

From (3.8) and (3.9) the second relation of (3.6) follows. By using (3.6) and the contraction mapping principle we prove Theorem 2. For that purpose, we consider the following system of Schrödinger equations:

$$i\partial_t U_k + \frac{1}{2}\Delta U_k = \pm V_k^2 V_k^*, \tag{3.10}$$

$$-i\partial_t U_k^* + \frac{1}{2}\Delta U_k^* = \pm V_k^{*2} V_k, \tag{3.11}$$

$$U_k(0,x) = \Phi_k(x), \qquad U_k^*(0,x) = \bar{\Phi}_x(x). \tag{3.12}$$

We put $\tilde{U}_k = \begin{pmatrix} U_k \\ U_k^* \end{pmatrix}$, $\tilde{V}_k = \begin{pmatrix} \pm V_k^2 V_k^* \\ \pm V_k^{*2} V_k \end{pmatrix}$, and define the map M by $\tilde{U}_k = M\tilde{V}_k$. We show M is a contraction mapping from $G_\rho^{[n/2]+1}$ to itself if ρ is sufficiently small, where

$$\mathrm{G}^m=\left\{\tilde{v}(t,x)=\begin{pmatrix} v_1(t,x) \\ v_2(t,x)\end{pmatrix};\ \|\tilde{v}\|_{\mathrm{G}^m}=\sup_{t\in\mathbb{R}}\sum_{j=1}^{2}\|v_j(t)\|_{\Sigma_t^m}^2<\infty\right\},$$

and G_ρ^m is a closed ball in G^m with radius $\rho>0$ and center at the origin. In what follows we let $m=[n/2]+1$. From (3.6), Remark 1 and $\mathrm{P}_k(0)=\exp(-(-1)^k\cdot x)$ it is clear that the assumption on φ given in the theorem is equivalent to the condition that

$$\rho^2=4\sum_{|\alpha|\leqq m}(\|\partial_x^\alpha\Phi_k\|_{\mathrm{L}^2}^2+\|x^\alpha\Phi_k\|_{\mathrm{L}^2}^2)$$

is sufficiently small for any $k\in\mathrm{K}$.

Multiplying both sides of (3.10) and (3.11) by $\bar{\mathrm{U}}_k$ and $\overline{\mathrm{U}_k^*}$ respectively, integrating with respect to x and t, we obtain

$$\|\tilde{\mathrm{U}}_k\|_{\mathrm{G}^m}\leqq\frac{\rho}{2}+\mathrm{C}_6\int_0^t(\|\mathrm{V}_k^{*2}\mathrm{V}_k(s)\|_{\Sigma_t^m}+\|\mathrm{V}_k^2\mathrm{V}_k^*(s)\|_{\Sigma_t^m})\,ds. \tag{3.13}$$

By using Lemma 3.1 it can be shown that the second term of the right hand side is estimated by

$$\mathrm{C}_7\int_0^t(1+|s|)^{-n}\|\tilde{\mathrm{V}}_k\|_{\mathrm{G}^m}^3\,ds. \tag{3.14}$$

From (3.13) and (3.14) it follows that if $\tilde{\mathrm{V}}_k\in\mathrm{G}_\rho^m$

$$\|\mathrm{M}\tilde{\mathrm{V}}_k\|_{\mathrm{G}^m}\leqq\frac{\rho}{2}+\mathrm{C}_8\,\rho^3.$$

Similarly, we have

$$\|\mathrm{M}\tilde{\mathrm{V}}_{k,1}-\mathrm{M}\tilde{\mathrm{V}}_{k,2}\|_{\mathrm{G}^m}\leqq\mathrm{C}_9\,\rho^2\,\|\tilde{\mathrm{V}}_{k,1}-\tilde{\mathrm{V}}_{k,2}\|_{\mathrm{G}^m},$$

where $\mathrm{M}\tilde{\mathrm{V}}_{k,1}$ and $\mathrm{M}\tilde{\mathrm{V}}_{k,2}$ are the solutions of (3.10)-(3.12) with the same initial data. Therefore M is a contraction mapping from G_ρ^m into itself if ρ is sufficiently small, and hence has a unique fixed point $\tilde{\mathrm{U}}_k=\begin{pmatrix}\mathrm{U}_k\\ \mathrm{U}_k^*\end{pmatrix}$ which belongs to G^m and satisfies

$$i\,\partial_t\mathrm{U}_k+\frac{1}{2}\Delta\mathrm{U}_k=\pm\,\mathrm{U}_k^2\,\mathrm{U}_k^*,$$
$$-i\,\partial_t\mathrm{U}_k^*+\frac{1}{2}\Delta\mathrm{U}_k^*=\pm\,\mathrm{U}_k^{*2}\,\mathrm{U}_k,$$
$$\mathrm{U}_k(0,x)=\Phi_k(x),\qquad \mathrm{U}_k^*(0,x)=\bar{\Phi}_x(x).$$

By (3.6) we see that there exists a unique solution u of (3.3)-(3.5) which belongs to E^m. This completes the proof of Theorem 2.

Q.E.D.

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