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Temperature states on gauge groups

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ABSTRACT. — We introduce the notion of a temperature state on the Kac-Moody extension of the infinite dimensional Lie group Map ($\mathbb{R}$, U, (N)) and on its subgroups. We show that for Map ($\mathbb{R}$, U (N)), by utilising earlier work of one of us (A.L.C.) with S.N.M. Ruijsenaars, these temperature states are associated with type III_1 factor representations of the group. In particular this may be interpreted as yielding type III_1 factor representations of Kac-Moody algebras. In the general setting of KMS states on twisted group C*-algebras we address the question of uniqueness of these temperature states and obtain both a general formula for such states and a criterion for uniqueness. We find that uniqueness holds for states on Map ($\mathbb{R}$, U (N)) but fails for certain subgroups leading to other possibilities which have a natural physical interpretation in terms of Bose-Einstein condensation.

RÉSUMÉ. — Nous introduisons une notion d'état en température sur l'extension de Kac-Moody du groupe de Lie de dimension infinie Map($\mathbb{R}$, U (N)), et sur ses sous groupes. Nous montrons en utilisant un résultat antérieur de l'un d'entre nous (A.L.C.) et S.N.M. Ruijsenaars que pour Map ($\mathbb{R}$, U (N)) ces états en température sont associés à des représentations factorielles de type III_1 du groupe. Ceci peut s'interpréter comme la construction de représentations factorielles de type III_1 des algèbres de Kac-Moody. Dans le contexte général des états KMS sur les « twisted group C* algebras » nous obtenons une formule générale pour ces états et un critère d'unicité. Nous montrons l'unicité pour les états sur Map (ℝ, U (N)) mais qu'elle est fausse pour certain sous groupes ce qui conduit à d'autres possibilités ayant une interprétation physique naturelle en terme de condensation de Bose-Einstein.

1. INTRODUCTION

In this paper we are concerned with certain projective representations of the gauge groups of $1+1$-dimensional quantum field theory. These are the infinite dimensional Lie groups Map (ℝ, G) of maps from ℝ into a compact Lie group G which, for simplicity we take to be U (N).

At the "infinitesimal level" these afford representations of affine Lie algebras (*cf.* [6], [17]). In the present paper we concentrate on representations suggested by 'quantum field theory at finite temperature'. (Our concerns here however have little to do with heuristic work on the subject cf Dolan and Jackiw [14].) Specifically we show that on a central extension of Map (ℝ, G) (the "Kac-Moody" extension) there is a naturally defined positive definite function satisfying the KMS condition for the translation action of ℝ on Map (ℝ, G). This positive definite function we call a temperature state on the gauge group. The representation associated with this state generates a type III_1 factor. Such representations do not seem to be accessible from the purely algebraic theory of Kac-Moody algebras, nor do they fit easily into the framework of 'positive energy' representations considered by Segal [31] (although they are closely related to them).

The technical tools needed to establish these facts are drawn from recent work of one of us with S.N.M. Ruijsenaars [6]. There a 'strong-form' of the boson-fermion correspondence of quantum field theory is established and exploited to prove the existence of hyperfinite type III_1 factor representations of Map (ℝ, U (N)) associated with the theory of free massive Dirac fermions.

To explain what is meant by the term 'strong form' we need to digress slightly. The basic idea of the boson-fermion correspondence in 1-space dimension is that, given a representation of the canonical anticommutation relations or CAR (*i. e.* of the fermion algebra) in which the local gauge group $\mathcal{G}$ is implementable, one obtains, by restricting to those maps in $\mathcal{G}$

which take their values in maximal torus, a representation (of the Weyl form) of the canonical commutation relations (CCR). The operators representing the Lie algebra can thus be interpreted as boson fields. This is the easy half of the correspondence. On the other hand physicists have for a long time written fermion fields (*i.e.* CAR generators) as formal functions of boson fields.

To make mathematical sense of these formal expressions is difficult. It was Garland [17] and Segal [29], [30] who recognized the role of vortex operators from string theory and of Kac-Moody algebras in this connection. In fact Segal's viewpoint [31] is that much of the quantum field theory of solvable models in two space-time dimensions is the representation theory of infinite dimensional groups. This latter view also provided the starting point for us (*cf.* [8], [10]-[12]). The idea then is that one consider particular gauge group elements called 'blips' written γ_ε. These depend on the real parameter ε in such a way that they become singular as $\varepsilon \to 0$ but also such that there is a constant c_ε with $c_\varepsilon \Gamma(\gamma_\varepsilon)$ (where Γ is the representation of the gauge group) converging in a certain sense to a fermion field.

This convergence is delicate. It was not discussed in [16] while in [10], [30] where Map (S^1, G) is considered, different methods are applied which fail for Map$(\mathbb{R}, G)$. Following the method of [6] however we can establish strong convergence on a dense domain of our approximate fermion fields giving our 'strong form' of the boson-fermion correspondence.

In keeping with this group theoretical approach we investigate in section 4 the question of uniqueness of the temperature states we have constructed on the central extension of the gauge group. We obtain a classification of KMS states on a class of abstract twisted group C*-algebras with identity which includes the case of loop groups (*cf.* [10]) and the gauge groups of this paper. In particular we 'explain' the non-uniqueness of KMS states on the CCR algebra and show how the group theoretic considerations lead naturally to the KMS-states associated with Bose-Einstein condensation. This particular case of our analysis has been studied previously [28].

2. TEMPERATURE STATES

2.1. Gauge action on the Fermion algebra

We let H denote the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^N)$ $(N=1, 2, \ldots)$ and $\mathscr{A}(H)$ denote the C*-algebra of the canonical anticommutation relations (CAR) over H (often referred to as the fermion algebra) generated by the annihilation and creation operators $\{a(g), a(g)^* \mid g \in H\}$ satisfying

$$a(g)\,a(g') + a(g')\,a(g) = 0$$
$$a(g)^*\,a(g') + a(g')\,a(g)^* = \langle g', g \rangle_H 1.$$

The time evolution on H is given by the one parameter group

$$r_t g(x) = g(x+t); \qquad g \in H, \quad t \in \mathbb{R}. \tag{2.1}$$

There is correspondingly an automorphism group $t \to \tau_t$ of $\mathscr{A}(H)$ with

$$\tau_t a(g) = a(r_t g) \tag{2.2}$$

and a (τ, β)-KMS or temperature state ω_β, $\beta \in [0, \infty]$ specified by (*cf.* [5]):

$$\omega_\beta(a(g)^* a(h)) = \langle h, A_\beta g \rangle_H \tag{2.3}$$

where A_β is defined by its action on the Fourier transform by

$$(A_\beta g)\hat{}\,(k) = e^{-\beta k}(1 + e^{-\beta k})^{-1} \hat{g}(k);$$

with $g(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{g}(k)\, e^{ikx}\, dk.$

(Note: ω_β is often referred to as a quasifree state [5].)

For technical reasons which are explained later we take our gauge group to consist of functions $\varphi : \mathbb{R} \to U(N)$, acting by multiplication operators on H and such that the $\mathbb{C}$-valued functions

$$x \mapsto \langle u, (\varphi(x) - 1)\, v \rangle \tag{2.4}$$

for $u, v \in \mathbb{C}^N$ lie in the Sobolev space $W^{1,2}(\mathbb{R})$; (1 = identity operator on $\mathbb{C}^N$). We write $\mathscr{G}$ for the group of such maps. We will discuss the role of functions $\varphi : \mathbb{R} \to U(N)$ which are constant separately. Writing the action of $\varphi \in \mathscr{G}$ on $g \in H$ as $g \mapsto \varphi . g$ we note that each such φ defines an automorphism of $\mathscr{A}(H)$ by

$$a(g) \mapsto a(\varphi . g). \tag{2.5}$$

We are interested in the representation of $\mathscr{G}$ which arises when we try to implement the action (2.5) on the G.N.S. Hilbert space corresponding to the state ω_β on $\mathscr{A}(H)$. In order to understand this we introduce the usual 'doubling up trick' by considering the projection

$$P_\beta = \begin{bmatrix} A_\beta & A_\beta^{1/2}(1 - A_\beta)^{1/2} \\ A_\beta^{1/2}(1 - A_\beta)^{1/2} & 1 - A_\beta \end{bmatrix}$$

On H⊕H, we think of $\mathscr{A}(H)$ as the subalgebra $\mathscr{A}(H \oplus (0))$ of the CAR algebra $\mathscr{A}(H \oplus H)$ over H⊕H. Then the state ω_{P_β} on $\mathscr{A}(H \oplus H)$ defined by P_β (*cf.* [5]) restricts on $\mathscr{A}(H)$ to the state ω_β.

Let $\mathscr{F}$ denote the Hilbert space corresponding to ω_{P_β}. It may be chosen independently of β and to coincide with the Dirac fermion Fock space which is the G.N.S. Hilbert space for ω_{P_∞} where

$$(P_\infty \hat{g})(k) = \begin{bmatrix} \theta(-k) & 0 \\ 0 & \theta(k) \end{bmatrix} \hat{g}(k), \qquad g \in H \oplus H. \tag{2.7}$$

and $\theta(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$. The representation π_{P_β} of $\mathscr{A}(H \oplus H)$ acting on $\mathscr{F}$ restricts on $\mathscr{A}(H)$ to the representation (which we denote π_β) corresponding to the state ω_β (*see* [1] and [26] for more details on this). Moreover the G.N.S. cyclic vector $\Omega_\beta \in \mathscr{F}$ for π_{P_β} is cyclic and separating for $\pi_\beta(A(H))''$ and ω_β is a (τ, β)-KMS state (*cf.* [5]).

Acting on $H \oplus H$ by multiplication-operators is the group $\tilde{\mathscr{G}}$ consisting of all functions $\tilde{\varphi} : \mathbb{R} \to U(N) \oplus U(N)$ where $\tilde{\varphi} = \varphi_1 \oplus \varphi_2$ with $\varphi_1, \varphi_2 \in \mathscr{G}$. We think of $\mathscr{G}$ as embedded in $\tilde{\mathscr{G}}$ under the map

$$\varphi \to \varphi \oplus 1, \qquad \varphi \in \mathscr{G} \tag{2.8}$$

and hence think of $\mathscr{G}$ as acting on $H \oplus H$ via this embedding. Note that $\mathscr{G}$ acts by automorphisms of $\mathscr{A}(H \oplus H)$ via

$$a(g) \to a(\tilde{\varphi} . g), \qquad g \in H \oplus H, \quad \tilde{\varphi} \in \tilde{\mathscr{G}}. \tag{2.9}$$

Noting that the spectrum of A_β is continuous for $\beta \neq 0, \infty$ it follows that $\pi_\beta(\mathscr{A}(H))''$ is a type III_1 factor. For readers unfamiliar with the theory of quasifree states this may need some explanation which we now briefly give.

Firstly A_β has continuous spectrum because the generator h of r_t does and this means that the Connes spectrum of the group $t \to \Gamma(\tau_t)$ implementing τ_t on $\mathscr{F}$ coincides with that of h, *i.e.* it is $\mathbb{R}$. But one knows that π_β is a factor and hence by definition $\pi_\beta(\mathscr{A}(H))''$ is type III_1 (*cf.* [25] for more detail). Note that as usual the automorphism of $\mathscr{A}(H \oplus H)$ defined by $\tilde{\varphi} \in \tilde{\mathscr{G}}$ is said to be implemented in τ_{P_β} if there exists a unitary operator: $\Gamma_\beta(\tilde{\varphi}) : \mathscr{F} \to \mathscr{F}$ with

$$\Gamma_\beta(\tilde{\varphi}) \pi_{P_\beta}(a(g)) \Gamma_\beta(\tilde{\varphi})^{-1} = \pi_{P_\beta}(a(\tilde{\varphi} g)). \tag{2.10}$$

The necessary and sufficient condition for this is that

$$P_\beta \tilde{\varphi} (1 - P_\beta) + (1 - P_\beta) \tilde{\varphi} P_\beta$$

be Hilbert-Schmidt and this latter requirement is guaranteed by the fact that each $\varphi \in \mathscr{G}$ satisfies (2.4).

Since π_{P_β} is irreducible each $\Gamma_\beta(\tilde{\varphi})$ is determined by (2.10) up to a phase factor, so that $\tilde{\varphi} \to \Gamma_\beta(\tilde{\varphi})$ defines a projective representation of $\tilde{\mathscr{G}}$. When $\tilde{\varphi} = \varphi \oplus 1$ we will write $\Gamma_\beta(\varphi)$ for an operator satisfying (2.10). To show how the phase factor can be fixed we pass to the corresponding projective representation of the Lie algebra $\tilde{\mathscr{L}}$ of $\tilde{\mathscr{G}}$. The latter consists of maps $\tilde{f} : \mathbb{R} \to u(N) \oplus u(N)$ with $\tilde{f} = f_1 \oplus f_2$ and

$$x \mapsto \langle u, f_j(x) v \rangle \in W^{1,2}(\mathbb{R}), \qquad u, v \in \mathbb{C}^N, \quad j = 1, 2 \tag{2.11}$$

[here $u(\mathrm{N}) \equiv$ Lie algebra of U(N) which we take to consist of $n \times n$ hermitian matrices in accord with the usual physics conventions.]

Thus:

$$\tilde{\varphi} = \exp if_1 \oplus \exp if_2 \in \tilde{\mathscr{G}}.$$

We may complexify $\tilde{\mathscr{L}}$ and consider maps into $gl(\mathrm{N}) \oplus gl(\mathrm{N})$, where $gl(\mathrm{N})$ is the Lie algebra of $\mathrm{GL}(\mathrm{N}, \mathbb{C})$ satisfying (2.11). Denote the complexified Lie algebra by $\tilde{\mathscr{L}}_{\mathbb{C}}$. Then in section 2 of [6] (*cf.* also [22]) it is proved that for each $\tilde{f} \in \tilde{\mathscr{L}}_{\mathbb{C}}$ there is an operator $\mathrm{J}(\tilde{f})$ on $\mathscr{F}$ (unbounded in general) such that

$$[J(\tilde{f}), \pi_\beta(a(g))^*] = \pi_\beta(a(\tilde{f}.g)^*), \qquad g \in \mathrm{H} \oplus \mathrm{H} \tag{2.12}$$

and

$$\langle \Omega_\beta, \mathrm{J}(\tilde{f}) \Omega_\beta \rangle = 0 \tag{2.13}$$

where $\tilde{f}.g$ denotes the action of $\tilde{f}$ by multiplication on g. (Note that $\mathrm{J}(\tilde{f})$ is denoted $d\tilde{\Gamma}(\tilde{f})$ in [6]). Then (2.12) implies that for $\tilde{f} \in \tilde{\mathscr{L}}$ we may choose $\Gamma_\beta(\tilde{\varphi}) = \exp i\mathrm{J}(\tilde{f})$ (since $\mathrm{J}(\tilde{f})$ is essentially self-adjoint on a certain domain in $\mathscr{F}$ which includes Ω_β [6], [22]). This fixes the phase of $\Gamma_\beta(\tilde{\varphi})$ for those $\tilde{\varphi}$ of the form $\tilde{\varphi} = e^{i\tilde{f}}$ however there is no simple way of choosing the phase of $\Gamma_\beta(\tilde{\varphi})$ for general $\tilde{\varphi}$.

We also have the formula ([6], [22]) for $\tilde{f} \in \tilde{\mathscr{L}}$:

$$\begin{aligned}[\mathrm{J}(\tilde{f}_1), \mathrm{J}(\tilde{f}_2)] &= \mathrm{J}([\tilde{f}_1, \tilde{f}_2]) + 2i \operatorname{Im} \langle \Omega_\beta, \mathrm{J}(\tilde{f}_1)\mathrm{J}(\tilde{f}_2)\Omega_\beta \rangle.\mathbf{1} \\ &= \mathrm{J}([\tilde{f}_1, \tilde{f}_2]) + 2i \operatorname{Im} \operatorname{tr}(\mathrm{P}_\beta \tilde{f}_1 (1 - \mathrm{P}_\beta) \tilde{f}_2 \mathrm{P}_\beta).\mathbf{1}\end{aligned} \tag{2.14}$$

($\mathbf{1}$ = identity operator on $\mathscr{F}$ and tr denotes the trace of a trace class operator.)

This reduces, for the special case of $\tilde{f}_j = f_j \oplus 0$ $j = 1, 2$ and $f_j \in \mathscr{L}$ (the Lie algebra of $\mathscr{G}$) to

$$[\mathrm{J}(f_1), \mathrm{J}(f_2)] = \mathrm{J}([f_1, f_2]) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \operatorname{Tr}\left(\frac{df_1}{dx}(x) f_2(x)\right) dx \tag{2.15}$$

(Tr denotes matrix trace).

One should recognize (2.15) as defining the central extension of $\mathscr{L}$ arising in the theory of affine Lie algebras (*cf.* [17], [29]). We summarise this state of affairs as

PROPOSITION 2.1. — *The map $f \to \mathrm{J}(f)$, $f \in \mathscr{L}$, determines a representation of the derived affine Lie algebra which is the central extension of $\mathscr{L}$ defined by the 2-cocycle $\dot{\sigma}$ where*

$$\dot{\sigma}(f_1, f_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}\left(\frac{df_1}{dx}(x) f_2(x)\right) dx.$$

Remark 2.2. — Notice that (2.15) is independent of β. We shall show later in a special case that the 2-cocycle on $\mathscr{G}$ determined by the projective representation $\varphi \rightarrow \Gamma_\beta(\varphi)$ may be chosen to be independent of β. The relation (2.15) establishes this for the connected component of the identity of $\mathcal{G}$. It may seem surprising that the quasifree states of the fermion algebra defined in [6] and ω_β all give rise to the same 2-cocycle on $\mathcal{L}$. We note that for finite β this is a special case of the following general proposition.

PROPOSITION 2.3. – *Let* π_{A_1} *and* π_{A_2} *be two quasifree representations of* $\mathcal{A}(H)$ *for which the* $\mathcal{G}$*-action is implemented. Then the* 2*-cocycles* $\dot{\sigma}_1$ *and* $\dot{\sigma}_2$ *on* $\mathcal{L}$ *defined by*

$$\dot{\sigma}_j(f_1, f_2) = 2\,i\,\mathrm{Im}\,\mathrm{tr}\,(P_{A_j} f_1 (1 - P_{A_j}) f_2 P_{A_j}), \qquad f_1, f_2 \in \mathcal{L}$$

for $j = 1, 2$ *are cohomologous whenever* $P_{A_1} f P_{A_1} - P_{A_2} f P_{A_2}$ *is trace class for all* $f \in \mathcal{L}$.

Proof. – We have

$$\begin{aligned}\dot{\sigma}_1 (f_1, f_2) - \dot{\sigma}_2 (f_1, f_2) &= \mathrm{tr}\,(P_{A_1} [f_1, f_2] P_{A_1} - P_{A_2} [f_1, f_2] P_{A_2}) \\ &\quad + \mathrm{tr}\,(- P_{A_1} f_1 P_{A_1} f_2 P_{A_1} + P_{A_2} f_1 P_{A_2} f_2 P_{A_2} \\ &\qquad + P_{A_1} f_2 P_{A_1} f_1 P_{A_1} - P_{A_2} f_2 P_{A_2} f_1 P_{A_2}) \qquad (2.16)\end{aligned}$$

by writing

$$\begin{aligned}P_{A_1} f_1 P_{A_1} f_2 P_{A_1} - P_{A_2} f_1 P_{A_2} f_2 P_{A_2} &= P_{A_1} f_1 P_{A_1} (P_{A_1} f_2 f_{A_1} - P_{A_2} f_2 P_{A_2}) \\ &\quad + (P_{A_1} f_1 P_{A_1} - P_{A_2} f_1 P_{A_2}) P_{A_2} f_2 P_{A_2}\end{aligned}$$

and using cyclicity of the trace the second term in (2.16) vanishes. Thus

$$\dot{\sigma}_1 (f_1, f_2) - \dot{\sigma}_2 (f_1, f_2) = \mathrm{tr}\,(P_{A_1} [f_1, f_2] P_{A_1} - P_{A_2} [f_1, f_2] P_{A_2}) \qquad (2.17)$$

and the functional on the right hand side of (2.17) is a coboundary as required.

Remarks. – While the preceding proposition is formulated in the setting of Lie algebra cohomology it seems likely that the most natural viewpoint is that of Connes' cyclic cohomology. Following [7], let $\mathcal{B}$ be a subalgebra of the (Banach) algebra of operators X on H with $PX - XP \in$ Hilbert Schmidt operators, where P is any orthogonal projection on H. Then the pair (H, P) determines a 2-summable Fredholm module for $\mathcal{B}$ as in [13]. Now let $\rho(X) = PXP$ and regard ρ as a homomorphism modulo the trace class operators. Then ρ defines a cyclic 1-cocycle φ by

$$\varphi(X^0, X^1) = \mathrm{tr}\,(\rho(X^0 X^1) - \rho(X^0)\rho(X^1)) - \mathrm{tr}\,(\rho(X^1 X^0) - \rho(X^1)\rho(X^0))$$

for $X^0, X^1 \in \mathcal{B}$ (*cf.* proposition 7.4 of [13] and [3,6.11]). It is then easy to see that if P′ is a second projection with $P'X - XP'$ Hilbert-Schmidt for all $X \in \mathcal{B}$ then P′ also defines a homomorphism modulo the trace class, say ρ' with $\rho'(X) = P'XP'$.

Let φ' denote the corresponding cyclic 1-cocycle, then φ and φ' are cohomologous whenever $\rho(X)-\rho'(X)$ is trace class for all $X \in \mathscr{B}$. In the latter case the map $X^0, X^1 \to (\rho-\rho')([X^0, X^1])$ is a cyclic 1-coboundary.

In the context above, of a family of projections depending on the parameter β, one can argue from the homotopy invariance of cyclic cohomology to the conclusion that the cyclic cocycle defined by the map $\rho_\beta : X \to P_\beta X P_\beta$ is independent of β in $(0, \infty]$.

We conclude this section with a comment on the constant maps from $\mathbb{R}$ to $U(N)$. If φ is such a map then it is easy to see that $\varphi \oplus 1$ is not implementable in π_{P_β} as an automorphism of $\mathscr{A}(H \oplus H)$.

However $\varphi \oplus \varphi$ is implementable (it commutes with P_β) and consequently if one is to represent the constant maps by operators on $\mathscr{F}$ they have to be treated in a different fashion. For this reason we do not discuss their action in the sequel. Nevertheless they indicate that a precise characterisation of the class of maps $\varphi : \mathbb{R} \to U(N)$ which are implementable is a subtle question. Our choice of $\mathscr{G}$ is dictated by the fact that this is the largest class for which technicalities are minimised.

2.2. The canonical commutation relations

From (2.14) we notice that on restriction to the subalgebra $\mathscr{T} \oplus \mathscr{T}$ where $\mathscr{T}$ consists of functions on L taking their values in the Lie algebra of a maximal torus of $u(N)$ one has

$$[J(\tilde{f}), J(\tilde{f})] = 2i \operatorname{Im} \operatorname{tr}(P_\beta \tilde{f}(1-P_\beta)\tilde{f}' P_\beta). \mathbf{1} \tag{2.18}$$

By taking the standard basis of $\mathbb{C}^N$ denoted by $\{e_j \mid j=1, \ldots, N\}$ we may take the maximal torus with respect to this basis to be spanned by

$$\{E_j = \operatorname{diag}(0, \ldots, \overset{j}{1}, \ldots, 0) \mid j=1, \ldots, N\}.$$

Then if

$$\tilde{f}(x) = f_1(x) E_j \oplus f_2(x) E_j, \qquad \tilde{f}'(x) = f'_1(x) E_k \oplus f'_2(x) E_k,$$

where, with our choice of Lie algebra, all functions are $\mathbb{R}$-valued, one finds

$$\operatorname{tr}(P_\beta \tilde{f}(1-P_\beta)\tilde{f}' P_\beta) = \frac{1}{2\pi} \delta_{jk} \int_{-\infty}^{\infty} |p| (\tilde{f}\,\hat{}\,(p))^* R(p) \tilde{f}'\,\hat{}\,(p)\, dp \tag{2.19}$$

where

$$f^{(\prime)}\,\hat{}\,(p) = \begin{bmatrix} f_1^{(\prime)}\,\hat{}\,(p) \\ f_2^{(\prime)}\,\hat{}\,(p) \end{bmatrix}$$

and

$$R(p)=\varepsilon(p)\begin{bmatrix}(1-e^{-\beta p})^{-1} & e^{-\beta p/2}(1-e^{-\beta p})^{-1}\\ e^{-\beta p/2}(1-e^{-\beta p})^{-1} & e^{-\beta p}(1-e^{-\beta p})^{-1}\end{bmatrix} \tag{2.20}$$

with

$$\varepsilon(p)=\theta(p)-\theta(-p).$$

From this one has

$$[J(\tilde{f}), J(\tilde{f}')]=\frac{\delta_{jk}}{2\pi}\int_{-\infty}^{\infty}|p|(\tilde{f}\,\hat{}\,(p))^* Q(p)\tilde{f}'\,\hat{}\,(p)\,dp \tag{2.21}$$

with

$$Q(p)=\begin{bmatrix}\varepsilon(p) & 0\\ 0 & \varepsilon(-p)\end{bmatrix}.$$

Now the right hand side of (2.21) defines a non-degenerate symplectic form on the subalgebra $\mathscr{T}\oplus\mathscr{T}$. Moreover (2.21) is recognisable as the canonical commutation relations (CCR) defined by this symplectic form (*see* [5] for a discussion of the CCR from this viewpoint). Notice finally that in the basis $\{E_j\}_{j=1}^{N}$ we may treat each subspace of $\mathscr{T}\oplus\mathscr{T}$ consisting of maps into the subspace of $u(N)\oplus u(N)$ spanned by $e_j\oplus 0$ and $0\oplus e_j$ independently of the others. Consequently for the purposes of the ensuing discussion in this section we restrict to the case $N=1$.

Thus henceforth $\tilde{\mathscr{L}}_1=L_1\oplus L_1$ denotes the space of $W^{1,2}$ maps from $\mathbb{R}$ into the diagonal real 2×2 matrices equipped with the symplectic form defined by the RHS of (2.21) (with $j=k$). We let $\tilde{f}\to J_\beta(\tilde{f})$ denote the representation of the CCR over $\tilde{\mathscr{L}}_1$ described above. Define a 2×2 matrix valued function on $\mathbb{R}\setminus\{0\}$ by

$$W(p)=\begin{bmatrix}T(p)^{1/2}\theta(-p)+S(p)^{1/2}\theta(p) & S(p)^{1/2}\theta(-p)+T(p)^{1/2}\theta(p)\\ S(p)^{1/2}\theta(-p)+T(p)^{1/2}\theta(p) & T(p)^{1/2}\theta(-p)+S(p)^{1/2}\theta(p)\end{bmatrix}$$

where

$$S(p)=\varepsilon(p)(1-e^{-\beta p})^{-1}$$
$$T(p)=\varepsilon(p)e^{-\beta p}(1-e^{-\beta p})^{-1}$$

Then for $p\neq 0$,

$$\left.\begin{aligned}Q(p)&=W(p)^* Q(p)W(p),\\ R(p)&=W(p)^* R_\infty(p)W(p)\end{aligned}\right\} \tag{2.22}$$

where $R_\infty(p)$ denotes $R(p)$ with $\beta=\infty$. Let C_∞ denote the complex structure on $\tilde{\mathscr{L}}_1$ given by

$$(C_\infty\tilde{f})\,\hat{}\,(p)=iQ(p)\tilde{f}\,\hat{}\,(p).$$

Let $\mathscr{D} \subset \tilde{\mathscr{L}}_1$ consist of functions whose Fourier transform vanishes on a neighbourhood of zero. Introduce the operator C_β on $\mathscr{D}$ of multiplication on the Fourier transform by

$$C_\beta(p) = W(p)^{-1} i Q(p) W(p). \tag{2.23}$$

Now we compute as in [6], [22] that for $\tilde{f} \in \tilde{\mathscr{L}}_1$

$$\langle \Omega_\beta, J_\beta(\tilde{f}) J_\beta(\tilde{f}) \Omega_\beta \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |p| (\tilde{f}\,\hat{}\,(p))^* R(p) \tilde{f}\,\hat{}\,(p)\, dp \tag{2.24}$$

Finally introduce creation and annihilation operators by:

$$\left.\begin{aligned} b(\tilde{f}) &= \frac{1}{2}(J_\beta(\tilde{f}) + i J_\beta(C_\beta \tilde{f})), \qquad \tilde{f} \in \mathscr{D} \\ b(\tilde{f})^* &= \frac{1}{2}(J_\beta(\tilde{f}) - i J_\beta(C_\beta \tilde{f})). \end{aligned}\right\} \tag{2.25}$$

Now we check from (2.24) and (2.23) that

$$\| b(\tilde{f}) \Omega_\beta \|^2 = 0 \tag{2.26}$$

so that $b(\tilde{f})\Omega_\beta = 0$. We also find from (2.25) that

$$[b(\tilde{f}), b(\tilde{f}')^*] = \langle \Omega_\beta, J_\beta(\tilde{f}) J_\beta(\tilde{f}') \Omega_\beta \rangle \mathbf{1} \tag{2.27}$$

and then from (2.26) and (2.27) standard arguments give

$$\langle \Omega_\beta, \exp i J_\beta(\tilde{f}) \Omega_\beta \rangle = \exp - \frac{1}{2} \langle \Omega_\beta, J_\beta(\tilde{f}) J_\beta(\tilde{f}) \Omega_\beta \rangle \tag{2.28}$$

One should recognise (2.28) as the generating functional for a Fock representation of the CCR. We now summarise the preceding as

PROPOSITION 2.6. — (i) *The cyclic representation* $\tilde{f} \to J_\beta(\tilde{f})$ *of the CCR over* $\mathscr{D}$ *generated from* Ω_β *is Fock with generating functional given by* (2.24) *and* (2.28).

(ii) *These representations, for different* β, *are all inequivalent.*

Proof. — Note that only (ii) does not follow immediately from the discussion preceding the proposition. However from the general theory of Fock representations (ii) follows easily using the simple observation that $C_\beta - C_{\beta'}$, is not Hilbert-Schmidt for $\beta \neq \beta'$ [32].

Remark. — One may interpret $W(p)$ as defining an operator W on $\mathscr{D}$. Then W is symplectic (*i.e.* preserves the RHS of (2.21)) and by (2.22) relates the representation of the CCR for finite β to that for $\beta = \infty$. It is not implemented however by virtue of the observation in the above proof.

2.3. Temperatures states on the gauge group

Let $\mathcal{M}$ denote the von Neumann algebra generated by $\{\Gamma_\beta(\varphi)\,|\,\varphi\in\mathcal{G}\}$ and let $\tilde{\mathcal{A}}(H\oplus 0)$ denote the von Neumann algebra generated by

$$\{\Gamma(-1)\,a(h\oplus 0)\,|\,h\in H\}.$$

Then the fact that

$$\tilde{\mathcal{A}}(H\oplus 0)=\mathcal{A}(0\oplus H)'$$

is well known (*cf.* discussion in [1]). Moreover we clearly have

$$\Gamma_\beta(\varphi)\in\mathcal{A}(0\oplus H)'$$

for all $\varphi\in\mathcal{G}$. Hence $\mathcal{M}\subseteqq\mathcal{A}(0\oplus H)'$. In the next section we prove equality of these algebras. For the moment we simply note that our automorphism group $t\to\tau_t$ is implemented in π_{P_β} by U_t where

$$U_t\,\pi_{P_\beta}(S)\,\Omega_\beta=\pi_{P_\beta}(\tau_t(S))\,\Omega_\beta,\qquad S\in\mathcal{A}(H\oplus 0). \tag{2.29}$$

Hence $U_t\,\Gamma_\beta(\varphi)\,U_t^{-1}$ and $\Gamma_\beta(\varphi_t)$ both implement the automorphism $a(h)\to a(\varphi_t h)\,(h\in H)$, where $\varphi_t(x)=\varphi(x+t)$. Thus

$$U_t\,\Gamma_\beta(\varphi)\,U_t^{-1}=c(\varphi,\,t)\,\Gamma_\beta(\varphi_t) \tag{2.30}$$

with $c(\varphi,\,t)\in\mathbb{C}$ of modulus one. [Actually one may show that $c(\varphi,\,t)=1$.] Hence τ_t acts as an automorphism of the von Neumann algebra $\mathcal{M}$. This proves:

PROPOSITION 2.7. – *The state* $\omega_{\beta\,|\,\mathcal{M}}$ *is a* $(\tau,\,\beta)$-KMS *state on the algebra* $\mathcal{M}$.

Hence the map

$$\varphi\to\langle\,\Omega_\beta,\,\Gamma_\beta(\varphi)\,\Omega_\beta\,\rangle,\qquad \varphi\in\mathcal{G} \tag{2.31}$$

provides an example of what we will henceforth refer to as a temprature state on $\mathcal{G}$. We write $\omega_\beta(\varphi)$ for the right-hand side of (2.31). The rest of this subsection is devoted to a discussion of the properties of the map $\varphi\to\omega_\beta(\varphi)$.

As a corollary of proposition 2.6 one has

$$\omega_\beta(\varphi)=\exp\left\{-\frac{1}{4\pi}\int_{-\infty}^{\infty}p\,\mathrm{tr}\,(\hat{f}^*(p)\,\hat{f}(p))\,(1-e^{-\beta p})^{-1}\,dp\right\}$$

for those φ of the form e^{if} with $f\in\mathcal{T}$ (i.e. f takes its values in the diagonal $N\times N$ matrices). We supplement this with

PROPOSITION 2.8. – *If* φ *does not lie in the connected component of the identity of* $\mathcal{G}$ *then* $\omega_\beta(\varphi)=0$.

Proof. – Consider firstly the case $\beta=\infty$. The connected components of $\mathcal{G}$ are labelled by the index map

$$i_\infty:\ \mathcal{G}\to\mathbb{Z},\qquad i_\infty(\varphi)=\text{Fredholm index}\ (P_\infty\,\varphi\,P_\infty)$$

(*see* [8]). (Note on the other hand that $P_\infty \varphi P_\infty$ is a matrix-valued Wiener-Hopf operator with Fredholm index equal to the winding number of the function $\theta \to \det(\varphi(\tan\theta/2))$ where $\theta \in S^1$, *cf.* [15].)

Introduce the index map

$$i_\beta(\varphi) = \text{Fredholm index } (P_\beta \varphi P_\beta).$$

It follows then that if $i_\beta(\varphi) = i_\infty(\varphi)$ for $\varphi \in \mathcal{G}$ we will have the connected components of $\mathcal{G}$ labelled by i_β and moreover since we know by [8] that $i_\beta(\varphi) \neq 0$ implies $\omega_\beta(\varphi) = 0$ this will prove the result.

Hence let U_β denote the operator

$$\begin{bmatrix} A_\beta^{1/2} A_\infty + (1-A_\beta)^{1/2}(1-A_\infty) & -(1-A_\beta)^{1/2} A_\infty + A_\beta^{1/2}(1-A_\infty) \\ -A_\beta^{1/2}(1-A_\infty) + (1-A_\beta)^{1/2} A_\infty & A_\beta^{1/2} A_\infty + (1-A_\beta)^{1/2}(1-A_\infty) \end{bmatrix}$$

Then $P_\beta = U_\beta P_\infty U_\beta^*$ and the operator U_β depends strongly continuously on $\beta \in (0, \infty]$. Thus the map

$$\beta \to P_\infty U_\beta^* \varphi U_\beta P_\infty$$

is continuous in the strong operator topology. On the other hand the map $\beta \to P_\infty U_\beta^* \varphi U_\beta (1-P_\infty)$ is continuous in the Hilbert-Schmidt topology whenever $\beta \to P_\beta \varphi (1-P_\beta)$ is continuous in the Hilbert-Schmidt topology. Now the latter statement is easy to prove except at $\beta = \infty$ and so consider

$$\| P_\beta \varphi (1-P_\beta) - P_\infty \varphi (1-P_\infty) \|_{H.S.} \leqq \| P_\beta (\varphi - 1)(P_\beta - P_\infty) \|_{H.S.} + \| (P_\beta - P_\infty)(\varphi - 1)(1-P_\infty) \|_{H.S.} \quad (2.32)$$

Now $P_\beta - P_\infty$ is the operator of multiplication by a 2×2 matrix valued function (in the Fourier transform) whose entries are all in $L^2(\mathbb{R})$. Also the matrix elements of $\varphi - 1$ are in $L^2(\mathbb{R})$. It follows by [27] (lemma X 1.20) that $(\varphi - 1)(P_\beta - P_\infty)$ is Hilbert-Schmidt and

$$\| (\varphi - 1)(P_\beta - P_\infty) \|_{H.S.} \leqq \text{Const. } \| \alpha_\beta \|_{L^2}$$

where α_β is the L^2 function:

$$\alpha_\beta(p) = e^{-\beta |p|/2} (1 + e^{-\beta |p|})^{-1}.$$

Clearly $\alpha_\beta \to 0$ in L^2 norm as $\beta \to \infty$ proving that

$$\beta \to \| (\varphi - 1)(P_\beta - P_\infty) \|_{H.S.}$$

is continuous and so from (2.32) that $\beta \to P_\beta \varphi (1-P_\beta)$ is continuous in the Hilbert-Schmidt topology for $\beta \to \infty$. Thus the map $\beta \to U_\beta^* \varphi U_\beta$ depends continuously on β when we equip $\mathcal{G}$ with the topology introduced in [9], namely we say $\varphi_n \to \varphi$ in $\mathcal{G}$ whenever $P_\infty \varphi_n P_\infty \to P_\infty \varphi P_\infty$ in the strong operator topology and

$$P_\infty \varphi_n (1-P_\infty) + (1-P_\infty) \varphi_n P_\infty \to P_\infty \varphi (1-P_\infty) + (1-P_\infty) \varphi P_\infty$$

in Hilbert-Schmidt norm. Now i_∞ is continuous in this topology [8] and so

$$i_\beta(\varphi)=i_\infty(U_\beta^*\varphi U_\beta)=i_\infty(\varphi)$$

as required.

3. THE CONVERGENCE ARGUMENT

3.1. Statement of results

In this subsection we state the first main theorem of the paper. The proof, which is technical and depends on arguments of [6], we defer to appendix 1. As we indicated in the introduction we are interested in a rigorous proof of the boson-fermion correspondence of quantum field theory (see [6] for a discussion and references to the physics literature). To do this we need to choose a special family of gauge group elements.

We introduce the 'standard kinks'

$$\gamma_{r,j}^{\varepsilon}(x)=\exp i\eta_r^{\varepsilon}(x)E_j; \qquad j=1, 2, \ldots, N \tag{3.1}$$

with $\eta_r^{\varepsilon}(x)=\pi+2\arctan\left(\frac{x-r}{\varepsilon}\right)$ and note that $\mathcal{G}$ is generated by functions of the form exp if with $f\in\mathcal{L}$ and the standard kinks. Consider the operator

$$\psi_{j,\varepsilon}(x)=\frac{1}{\sqrt{4\pi\varepsilon}}\Gamma_\beta(-1)\Gamma_\beta(\gamma_{x,j}^{\varepsilon})^* \tag{3.2}$$

where the phase of $\Gamma_\beta(\gamma_{x,j}^{\varepsilon})^*$ is fixed as in equation (4.50) of [6]. Now introduce the operator

$$\psi_{j,\varepsilon}(g)=\int\psi_{j,\varepsilon}(x)g(x)\,dx \tag{3.3}$$

where $g:\mathbb{R}\to\mathbb{C}$ has $\hat{g}$ smooth and of compact support. Denote by $\mathcal{D}$ the subspace of $\mathcal{F}$ spanned by vectors of the form

$$\pi_{P_\beta}(a^*(g_1)\ldots a^*(g_n)a(h_m)\ldots a(h_1))\Omega_\beta$$

where g_i, h_j are C^∞ with their Fourier transforms having compact support.

PROPOSITION 3.1:

$$\lim_{\varepsilon\to 0}\psi_{j,\varepsilon}^{(*)}(g)F=\pi_{P_\beta}(a(g_j\oplus 0)^{(*)})F=\pi_\beta(a(g_j)^{(*)})F \tag{3.4}$$

for $F\in\mathcal{D}$ *where* $g_j(x)=(0, \ldots, g(x), \ldots 0)$ (*jth position*).

The proof of this result is the same as that for the case of massive Dirac fermions considered in [6]. Consequently to avoid excessive repetition of technicalities we give a sketch of the argument in appendix 1 which should be read in conjunction with [6] to obtain a full discussion.

THEOREM 3.2. — *The projective representation* $\varphi \to \Gamma_\beta(\varphi)$ *of* $\mathcal{G}$ *generates a hyperfinite type* III_1 *factor* $\mathcal{M}$ *with* $\mathcal{M} = \mathcal{A}((0)\oplus H)'$.

Proof. — Following the proof of theorem 4.8 of [6] we have from proposition 3.1 that if A is an operator which commutes with $\Gamma_\beta(\varphi)$ for all $\varphi \in \mathcal{G}$ then $A \in \tilde{\mathcal{A}}(H\oplus(0))'$. Thus $\mathcal{M}' \subseteqq \tilde{\mathcal{A}}(H\oplus(0))'$ implying that $\mathcal{M} \supseteqq \mathcal{A}(0\oplus H)'$. We have already noted in section 2.3 the reverse inclusion hence the result.

Remark. — If G_0 is a subgroup of U(N) and $\mathcal{G}_0$ denotes the subgroup of $\mathcal{G}$ consisting of functions taking their values in G_0 then we can consider the subspace $\mathcal{F}_0$ of $\mathcal{F}$ generated from Ω_β by the action of $\{\Gamma_\beta(\varphi) \mid \varphi \in \mathcal{G}_0\}$. If $\mathcal{M}_0$ is the von Neumann algebra generated by $\{\Gamma_\beta(\varphi) \mid \varphi \in \mathcal{G}_0\}$ acting on $\mathcal{F}_0$ then Ω_β is cyclic and separating for $\mathcal{M}_0$. Moreover the modular automorphism group $t \to \tau_t$ leaves $\mathcal{M}_0$ invariant and so $\omega_{\beta \mid \mathcal{M}_0}$ is a (τ, β)-KMS state and hence, according to our terminology, a temperature state on $\mathcal{G}_0$. It should be possible in particular cases (e.g. G = SU(N)) to determine the type of $\mathcal{M}_0$, however we have not done so.

3.2. The 2-cocycle

In the subsequent section we are interested in the question of uniqueness of temperature states on gauge groups. For this we need to know firstly that the 2-cocycle on $\mathcal{T}$ can be chosen to the β-independent. We concentrate on the case N = 1 noting that the same method works for general N and also for the representations of the U(1) gauge group arising from the massive Dirac representation of the CAR (*i.e.* it proves that the cocycle is independent of the fermion mass parameter thus affirming a conjecture in [6]).

Let $\psi_1, \psi_2 \in \mathcal{G}$ with respective winding numbers w_1, w_2. Write $\psi_j = (\gamma_r^1)^{\omega_j}\varphi_j\,(j=1, 2)$ with φ_j in the identity component $\mathcal{G}_0$ of $\mathcal{G}$. Assume that some choice of phase for $\Gamma_\beta(\psi_j)$ has been made consistent with that implied by (2.13) for $\mathcal{G}_0$ and that of the previous section for the 'kinks'. Then we have

$$\Gamma_\beta(\psi_1)\Gamma_\beta(\psi_2) = \sigma(\psi_1, \psi_2)\Gamma_\beta(\psi_1\psi_2)$$

with σ a 2-cocycle on $\mathcal{G}$. The 2-cocycle relations for σ show that $\psi_1, \psi_2 \to \tilde{\sigma}(\psi_1, \psi_2) = \sigma(\psi_1, \psi_2)/\sigma(\psi_2, \psi_1)$ is a bicharacter on $\mathcal{G}$. Hence

$$\begin{aligned}\tilde{\sigma}(\psi_1, \psi_2) &= \tilde{\sigma}(\varphi_1, \varphi_2)\,\tilde{\sigma}(\gamma_r^1, \varphi_2)^{w_1}\,\tilde{\sigma}(\varphi_1, \gamma_r^1)^{w_2}\,\tilde{\sigma}(\gamma_r^1, \gamma_r^1)^{w_1 w_2}\\ &= \tilde{\sigma}(\varphi_1, \varphi_2)\,\tilde{\sigma}(\gamma_r^1, \varphi_2)^{w_1}\,\tilde{\sigma}(\gamma_r^1, \varphi_1)^{-w_2}\end{aligned}$$

Since we already have $\tilde{\sigma}(\varphi_1, \varphi_2) = \tilde{\sigma}_\infty(\varphi_1, \varphi_2)$, where $\tilde{\sigma}_\infty$ denotes the 2-cocycle when $\beta = \infty$, then it suffices to prove

$$\tilde{\sigma}(\gamma_r^1, \varphi) = \tilde{\sigma}_\infty(\gamma_r^1, \varphi),$$

for all $\varphi \in \mathscr{G}_0$ from which it will follow that $\tilde{\sigma} = \tilde{\sigma}_\infty$ and hence that σ and σ_∞ are cohomologous ([4], [18]).

Write

$$S\varphi(r) = \tilde{\sigma}(\gamma_r^1, \varphi)\,\tilde{\sigma}_\infty(\gamma_r^1, \varphi)^{-1}$$

and use the bicharacter property of $\tilde{\sigma}$ and $\tilde{\sigma}_\infty$ to obtain

$$\tilde{\sigma}(\gamma_r^\varepsilon, \varphi)\,\tilde{\sigma}_\infty(\gamma_r^\varepsilon, \varphi)^{-1} = \tilde{\sigma}(\gamma_r^1, \varphi)\,\tilde{\sigma}(\bar{\gamma}_r^1\gamma_r^\varepsilon, \varphi)\,\tilde{\sigma}_\infty(\gamma_r^1, \varphi)^{-1}\,\tilde{\sigma}_\infty(\bar{\gamma}_r^1\gamma_r^\varepsilon, \varphi)^{-1} = S\varphi(r)$$

since $\tilde{\sigma}$ and $\tilde{\sigma}_\infty$ agree on $\mathscr{G}_0$. Now

$$\Gamma_\beta(\varphi)\,\Gamma_\beta(\gamma_r^\varepsilon)\,\Gamma_\beta(\varphi)^* = \tilde{\sigma}(\gamma_r^\varepsilon, \varphi)^{-1}\,\Gamma_\beta(\gamma_r^\varepsilon) = (S\varphi(r).\tilde{\sigma}_\infty(\gamma_r^\varepsilon, \varphi))^{-1}\,\Gamma_\beta(\gamma_r^\varepsilon) \quad (3.5)$$

By direct calculation we see that if $\varphi = \exp if$ with $\hat{f}$ of compact support:

$$\tilde{\sigma}_\infty(\gamma_r^\varepsilon, \varphi) = \exp\left\{-\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\varepsilon|p|}e^{ipr}\hat{f}(p)\,dp\right\} = (1 + \varphi_\varepsilon(r))\,\varphi(r)^{-1}$$

where

$$\|\varphi_\varepsilon\|_\infty = O(\varepsilon) \quad (3.6)$$

Substituting in (3.5)

$$\varphi(r)\,\Gamma_\beta(\varphi)^*\,\Gamma_\beta(\gamma_r^\varepsilon)\,\Gamma_\beta(\varphi) = S\varphi(r)\,(1 + \varphi_\varepsilon(r))\,\Gamma_\beta(\gamma_r^\varepsilon).$$

Now $\Gamma(-1)$ commutes with $\Gamma_\beta(\varphi)$ so letting g be in L^2 we obtain

$$\Gamma_\beta(\varphi)^*\,\psi_\varepsilon(\varphi g)^*\,\Gamma_\beta(\varphi) = \psi_\varepsilon(S\varphi . g)^* + \psi_\varepsilon(S\varphi . \varphi_\varepsilon . g)^*$$

Now let $\xi, \eta \in \mathscr{D}$ and consider

$$\langle \xi, \psi_\varepsilon(\varphi g)^*\,\Gamma_\beta(\varphi)\,\eta \rangle = \langle \xi, \Gamma_\beta(\varphi)\,\psi_\varepsilon(S\varphi . g)^*\,\eta \rangle + \langle \xi, \Gamma_\beta(\varphi)\,\psi_\varepsilon(S\varphi . \varphi_\varepsilon . g)^*\,\eta \rangle \quad (3.7)$$

From the discussion in Appendix 2 we have the limits:

$$\lim_{\varepsilon \to 0} \langle \xi, \psi_\varepsilon(\varphi g)^*\,\Gamma_\beta(\varphi)\,\eta \rangle = \langle \pi_\beta(a(\varphi g))\,\xi, \Gamma_\beta(\varphi)\,\eta \rangle = \langle \xi, \Gamma_\beta(\varphi)\,\pi_\beta(\varphi)\,\pi_\beta(a(g))^*\,\eta \rangle$$

and

$$\lim_{\varepsilon \to 0} \langle \xi, \Gamma_\beta(\varphi)\,\psi_\varepsilon(S\varphi . g)^*\,\eta \rangle = \langle \xi, \Gamma_\beta(\varphi)\,\pi_\beta(a(S\varphi . g)^*)\,\eta \rangle.$$

For the last term in (3.7) we have the estimate

$$\|\psi_\varepsilon(S\varphi . \varphi_\varepsilon . g)^*\,\eta\| = O(\sqrt{\varepsilon})$$

which combined with (3.6) and the definition of $\psi_\varepsilon(S\varphi.\varphi_\varepsilon.g)^*$ gives

$$\lim_{\varepsilon\to 0}\langle \xi, \Gamma_\beta(\varphi)\psi_\varepsilon(S\varphi.\varphi_\varepsilon.g)^*\eta\rangle=0.$$

This then gives, when combined with the preceding limits, the relation

$$\langle \xi, \Gamma_\beta(\varphi)\pi_\beta(a(g)^*)\eta\rangle=\langle \xi, \Gamma_\beta(\varphi)\pi_\beta(a(S\varphi.g))^*\eta\rangle$$

for all $\xi, \eta\in\mathscr{D}$. Now $\Gamma_\beta(\varphi)$ is unitary and $\mathscr{D}$ is dense so we deduce that $\pi_\beta(a(g))=\pi_\beta(a(S\varphi.g))$; this relation holds for a dense set of $g\in\mathscr{H}$ and so $S\varphi=1$. Finally this last equality holds for general $f\in\mathscr{L}$ using continuity of $f\to\langle \xi, \Gamma_\beta(e^{if})\eta\rangle$ in the $W^{1,2}$ norm on $\mathscr{L}$.

3.3. Correlation Functions

In our previous work [10] on temperature states on loop groups we found that the boson-fermion correspondence as (embodied in a convergence result such as proposition 3.1) gave rise to theta function identities. They were obtained by computing

$$\omega_\beta(a(g_n)^*\ldots a(g_1)^*a(h_1)\ldots a(h_m))$$

in two ways, firstly using the definition of the temperature state on the CAR algebra and secondly by using the loop group elements to approximate the CAR elements and then exploiting the explicit formula for the temperature state on the loop group.

The same procedure applied here does not yield interesting identities except as a method of calculating determinants. In fact the correlations calculated in terms of the temperature state on the gauge group are just products of gamma functions and an elementary relation between the gamma function and the hyperbolic functions yields the relation:

$$\lim_{\underline{\varepsilon}, \underline{\varepsilon}' \to 0} \omega_\beta (\psi'_{\varepsilon_N}(r'_N) \ldots \psi_{\varepsilon'_1}(r'_1) \psi_{\varepsilon_1}(r_1)^* \ldots \psi_{\varepsilon_N}(r_N)^*) = (2\beta)^{-N} \frac{\prod_{j<k} \sinh(\pi/\beta)(r_j - r_k) \sinh(\pi/\beta)(r'_j - r'_k)}{\prod_{j,k} \sinh(\pi/\beta)(r'_j - r_k)}$$

($\underline{\varepsilon}^{(\prime)} = (\varepsilon_1^{(\prime)}, \ldots, \varepsilon_N^{(\prime)})$) which is exactly the expected result.

4. UNIQUENESS OF TEMPERATURE STATES AND APPLICATIONS

4.1. Uniqueness

In Section 2 we saw that the KMS state of the CAR algebra A (H) gave rise to a KMS state ω_β of the algebra $\mathcal{M}$ generated by a σ-representation Γ_β of the gauge group G, and in Section 3 we showed how to reconstruct the CAR algebra from ω_β. However, in order to have a completely satisfactory correspondence between the fermion and boson theories we should like to know that ω_β is the only (τ, β)-KMS state of $\mathcal{M}$. (That is certainly true for the corresponding boson theory on S^1, [10].)

This uniqueness question is of sufficiently wide interest that we shall consider it more generally for KMS states associated with automorphisms of twisted group algebras. The strategy will be as follows. We let G be a (possibly infinite dimensional) abelian Lie group with a continuous 2-cocycle (or multiplier) σ. (It will be convenient to introduce the notation:

$$(S\sigma) = \{\eta \in G : \tilde{\sigma}(\xi, \eta) = 1 \;\forall \xi \in S\}$$

where S is any subset of G. $(S\sigma)$ is always a closed subset of G.) The twisted group algebra M (G, σ) consists of the finite linear combination of δ-functions on G multiplied by the rule

$$\delta_\varepsilon * \delta_\eta = \sigma(\xi, \eta)\, \delta_{\xi\eta},$$

and with the involution

$$\delta_\xi^* = \sigma(\xi, \xi^{-1})^{-1}\, \delta_{\xi^{-1}}.$$

(This amounts to constructing the twisted group algebra of G as a discrete group. It can be given a C*-norm and completed in the usual way, but we shall follow [5] in using the KMS condition on a dense subalgebra only, so that the completion is not needed.)

In appendix 3 we show that, for the examples considered in this section, any 1-parameter group of automorphisms of G, say $\{\tau_t\}$, which preserves the cohomology class of σ induces automorphisms also denoted $\{\tau_t\}$ of $M(G, \sigma)$. We can look for non-degenerate states ω of $M(G, \sigma)$ which satisfy the (τ, β)-KMS condition. Now any state is defined by its values on individual δ-functions; abbreviating $\omega(\delta_\xi)$ to $\omega(\xi)$ we may regard ω as a σ-positive function on G. (The topology of G has so far played no role but we shall henceforth assume that ω is regular, that is continuous on every one-parameter subgroup.) Under suitable conditions on σ the KMS-condition provides a functional equation for ω, (Prop. 4.1). This forces ω to vanish outside the set $(F\sigma)$ where F is the fixed point set of $\{\tau_t\}$, (Prop. 4.2). On the other hand it also gives an explicit formula for ω on a large subgroup of $(F\sigma)$, denoted by $\Phi^\perp$, (Prop. 4.4).

The problem of determining ω is therefore reduced to finding any non-vanishing extensions of the formula beyond $\Phi^\perp$. We shall tackle this by showing that the KMS state itself determines a natural pseudometric on $\Phi^\perp$, (4.5-4.8), and that the support of ω (as a function on G) must lie in a completion $G_c \cap (F\sigma)$ of $\Phi^\perp$, (4.9). If $G_c \cap (F\sigma)$ coincides with $\Phi^\perp$, the set on which the state was already known then ω is unique. (This is the case for the group G of Sections 2 and 3, as we shall see in Section 4.2.) In general, however, ω can be extended from $\Phi^\perp$ to $G_c \cap (F\sigma)$ in a variety of ways, which are characterised in Theorem 4.13.

To prepare notation for what follows let us write $\varphi_t = \tau_t \varphi$ for $\varphi \in G$, $t \in \mathbb{R}$. We shall assume that σ extends to a multiplier, also denoted σ, on the semi-direct product $\mathbb{R} \circledS G$. (This is automatic for the examples considered here as we show in appendix 3.) Then there is an $\mathbb{R}$-action on the twisted group algebra $M(G, \sigma)$ given by

$$\delta_\varphi \to \tilde{\sigma}(\varphi_t, t)\,\delta_{\varphi_t}$$

where

$$\tilde{\sigma}(\varphi_t, t) = \sigma(t, \varphi_t)/\sigma(\varphi, t). \tag{4.1}$$

(and $\sigma(t, \varphi) \equiv \sigma((t, 1), (e, \varphi))$ where e is the identity of G).

Since ω is assumed to be a (τ, β)-KMS state, the KMS condition implies that for each pair $\varphi, \psi \in G$, the function

$$t \to \omega(\psi\varphi_t)\,\sigma(\psi, \varphi_t)\,\tilde{\sigma}(\varphi_t, t)$$

admits an analytic continuation to $t = s + i\beta$ and there equals

$$\omega(\varphi_s \psi)\,\sigma(\varphi_s, \psi)\,\tilde{\sigma}(\varphi_s, s).$$

Replacing φ_s by φ we see that we may as well take $s=0$. Substituting $\varphi^{-1}\psi$ in place of ψ we then obtain the condition that it must be possible to analytically continue the function

$$t \to \omega(\varphi^{-1}\psi\varphi_t)\,\sigma(\varphi^{-1}\psi, \varphi_t)\,\tilde{\sigma}(\varphi_t, t)/\sigma(\varphi, \varphi^{-1}\psi)$$

to $t=i\beta$, and that its value there must be $\omega(\psi)$. We write this symbolically as

$$\omega(\varphi^{-1}\psi\varphi_t)\,\sigma(\varphi^{-1}\psi,\,\varphi_t)\,\tilde{\sigma}(\varphi_t,\,t)/\sigma(\varphi,\,\varphi^{-1}\psi)\big|_{t=i\beta}=\omega(\psi).$$

Using the cocycle identity for σ we finally arrive at

$$\omega(\psi\varphi^{-1}\varphi_t)\,\tilde{\sigma}(\psi,\,\varphi)\,\sigma(\psi,\,\varphi^{-1}\varphi_t)\,\tilde{\sigma}(\varphi_t,\,t)/\sigma(\varphi,\,\varphi^{-1}\varphi_t)\big|_{t=i\beta}=\omega(\psi). \qquad (4.2)$$

A 1. Now in the case of the Weyl algebra the analytic continuation can be done explicitly by complexifying the underlying vector space and extending the action of $\mathbb{R}$ to an action of $\mathbb{C}$. In the examples which interest us complexification is also possible: maps into a semi-simple Lie group K can be complexified to give maps into its complexification $K_{\mathbb{C}}$. We assume therefore that our abstract system has the same property: that G has a complexification $G_{\mathbb{C}}$ on which the $\mathbb{R}$-action defined by τ extends to an action of $\mathbb{C}$ and that σ extends analytically to a (non-unitary) cocycle on $\mathbb{C}$ Ⓢ $G_{\mathbb{C}}$. (The extension is sometimes possible only on a certain domain D. However, the arguments are unaffected provided that $G_{\mathbb{C}}$ is replaced by D throughout.)

PROPOSITION 4.1. – *Under the assumptions made about* G, τ *and* σ, ω *is a* $(\tau,\,\beta)$-*KMS state for the twisted group algebra if and only if*

$$\omega(\psi\varphi^{-1}\varphi_{i\beta})=\frac{\tilde{\sigma}(\varphi,\,\psi)\,\sigma(\varphi,\,\varphi^{-1}\varphi_{i\beta})}{\sigma(\psi,\,\varphi^{-1}\varphi_{i\beta})\,\tilde{\sigma}(\varphi_{i\beta},\,i\beta)}\,.\,\omega(\psi)$$

for all $\psi\in G$, $\varphi\in G_{\mathbb{C}}$.

Proof. – Since σ in (4.2) admits an analytic continuation in t, if ω is a KMS-state it must also admit such a continuation. Then the formula itself is just a rearrangement of (4.2) and so it certainly follows if ω is a KMS state. However under our assumptions each step in the argument can be reversed to show that the preceding formula for ω implies the KMS condition.

Proposition 4.1 provides a functional equation for ω whose solution we now derive. Tho this end we introduce the subgroup

$$F=\{\varphi\in G\,|\,\varphi_t=\varphi,\ \text{for all } t\in\mathbb{R}\}.$$

If $\varphi\in F_{\mathbb{C}}\subseteq G_{\mathbb{C}}$ then clearly, by analytic continuation the functional equation for ω gives

$$\omega(\psi)=\frac{\tilde{\sigma}(\varphi,\,\psi)}{\tilde{\sigma}(\varphi,\,i\beta)}\,\omega(\psi),\qquad \psi\in G,\quad \varphi\in F_{\mathbb{C}} \qquad (4.3)$$

A 2. In the systems of interest to us

$$\tilde{\sigma}(\varphi,\,t)=1 \qquad \text{for all}\quad \varphi\in F\quad \text{and}\quad t\in\mathbb{R},$$

and by analytical continuation

$$\tilde{\sigma}(\varphi, i\beta)=1 \qquad \text{for all} \quad \varphi\in F_{\mathbb{C}}.$$

To simplify the analysis we include this as an extra assumption. Then (4.3) becomes

$$\omega(\psi)=\tilde{\sigma}(\varphi, \psi)\,\omega(\psi), \qquad \psi\in G, \quad \varphi\in F_{\mathbb{C}}. \tag{4.4}$$

From this we deduce

PROPOSITION 4.2. — *The support,* supp(ω) *of* ω *is contained in the set* (F σ).

It will turn out that the dual fixed point set is also a useful tool so let $\hat{G}$ denote the dual of G, on which $\mathbb{R}$ acts by transposing the action on G and set

$$\Phi=\{\chi\in\hat{G}\mid \chi_t=\chi \text{ for all } t\in\mathbb{R}\}.$$

Let $\Phi^{\perp}$ be the subgroup of G annihilated by all $\chi\in\Phi$.

PROPOSITION 4.3:

$$\{\varphi^{-1}\varphi_{i\beta}\in G\mid \varphi\in G_{\mathbb{C}}\}\subseteqq\Phi^{\perp}\subseteqq(F\sigma)_0$$

where $(F\sigma)_0$ *denotes the identity component in* (F σ).

Proof. — If $\chi\in\Phi$ then for all $\varphi\in G$, $t\in\mathbb{R}$:

$$\chi(\varphi^{-1}\varphi_t)=\chi(\varphi_t)/\chi(\varphi)=\chi_{-t}(\varphi)/\chi(\varphi)=1$$

so that

$$\{\varphi^{-1}\varphi_t\mid\varphi\in G\}\subseteqq\Phi^{\perp} \text{ for all } t\in\mathbb{R}.$$

By analytic continuation

$$\{\varphi^{-1}\varphi_{i\beta}\in G\mid\varphi\in G_{\mathbb{C}}\}\subseteqq\Phi^{\perp}.$$

Under the natural map $\varphi\to\tilde{\sigma}(\varphi, .)$ from G to $\hat{G}$, the subgroup F maps into Φ. Since the annihilator of the image is (F σ) we have

$$\Phi^{\perp}\subseteqq(F\sigma). \tag{4.5}$$

Finally note that, since the action of $\mathbb{R}$ cannot map elements from one connected component to another, $\varphi^{-1}\varphi_t$, which the above argument shows is in (F σ), always lies in the identity component $(F\sigma)_0$. Consequently, whenever $(F\sigma)_0$ is contained in the kernel of χ,

$$\chi_t(\varphi)=\chi(\varphi_{-t})\,\chi(\varphi^{-1}\varphi_{-t})^{-1}=\chi(\varphi)$$

for all $\varphi\in G$, $t\in\mathbb{R}$. Hence $\chi\in\Phi$ implying $\Phi^{\perp}\subseteqq(F\sigma)_0$.

Remark. — A 3. In all the examples we consider it is true that

$$\Phi^{\perp}=\{\varphi^{-1}\varphi_{i\beta}\in G\mid\varphi\in G_{\mathbb{C}}\}$$

and so we now add this to our list of assumptions. (An argument like that above shows that $\Phi^\perp$ is connected.) This brings about a remarkable simplification of proposition 4.1 which tells us that ω is determined on each coset of $\Phi^\perp$ by its value at a single point.

PROPOSITION 4.4. — *Under the above assumptions if* ω *is a KMS state on the twisted group* C*-*algebra of* G *then, writing any* $\xi\in\Phi^\perp$ *in the form* $\xi=\varphi^{-1}\varphi_{i\beta}$

$$\omega(\xi)=\sigma(\varphi,\xi)/\tilde{\sigma}(\varphi_{i\beta}, i\beta)$$

and $\omega(\psi\xi)=\tilde{\sigma}(\varphi,\psi)\,\omega(\psi)\,\omega(\xi)/\sigma(\psi,\xi)$. *Conversely if values* $\omega(\psi)$ *for one point* ψ *in each* $\Phi^\perp$ *coset are given and these two equations are used to define* ω, *then* ω *defines a KMS state provided it is positive.*

Proof. — If ω is a KMS state then taking $\psi=1$ in proposition 4.1 gives the first equation. The second equation follows by substituting into the general form of proposition 4.1. For the converse we need to consider first the case where $\psi=\eta=\theta^{-1}\theta_{i\beta}$. The equation defining ω on $\Phi^\perp$ then gives

$$\omega(\xi\eta)=\sigma(\theta\varphi,\xi\eta)/\tilde{\sigma}((\theta\varphi)_{i\beta}, i\beta).$$

Hence

$$\begin{aligned}
\frac{\omega(\xi\eta)}{\omega(\xi)\,\omega(\eta)} &= \frac{\sigma(\theta\varphi,\xi\eta)}{\sigma(\varphi,\xi)\,\sigma(\theta,\eta)}\cdot\frac{\tilde{\sigma}(\varphi_{i\beta}, i\beta)\,\tilde{\sigma}(\theta_{i\beta}, i\beta)}{\tilde{\sigma}((\theta\varphi)_{i\beta}, i\beta)}\\
&= \frac{\sigma(\theta\varphi,\xi\eta)}{\sigma(\varphi,\xi)\,\sigma(\theta,\eta)}\cdot\frac{\sigma(\theta,\varphi)}{\sigma(\theta_{i\beta},\varphi_{i\beta})}\\
&= \sigma(\theta,\varphi)\,\sigma(\theta\varphi,\xi\eta)/\sigma(\varphi,\xi)\,\sigma(\theta,\eta)\,\sigma(\theta\eta,\varphi\xi)\\
&= \sigma(\theta,\varphi\xi\eta)\,\sigma(\varphi,\xi\eta)/\sigma(\varphi,\xi)\,\sigma(\theta,\eta\varphi\xi)\,\sigma(\eta,\varphi\xi)\\
&= \sigma(\varphi,\xi)\,\sigma(\varphi\xi,\eta)/\sigma(\xi,\eta)\,\sigma(\varphi,\xi)\,\sigma(\eta,\varphi\xi)\\
&= \tilde{\sigma}(\varphi\xi,\eta)/\sigma(\xi,\eta)=\tilde{\sigma}(\varphi,\eta)/\sigma(\eta,\xi).
\end{aligned}$$

So we have

$$\omega(\eta\xi)=\tilde{\sigma}(\varphi,\eta)\,\omega(\eta)\,\omega(\xi)/\sigma(\eta,\xi) \tag{4.6}$$

Consider now the case where $\psi=\lambda\eta$ for λ a coset representative at which ω is known. By definition:

$$\begin{aligned}
\omega(\lambda\eta\xi) &= \tilde{\sigma}(\theta\varphi,\lambda)\,\omega(\lambda)\,\omega(\eta\xi)/\sigma(\lambda,\eta\xi)\\
&= \frac{\tilde{\sigma}(\theta\varphi,\lambda)}{\sigma(\lambda,\eta\xi)}\omega(\lambda)\cdot\frac{\tilde{\sigma}(\varphi,\eta)}{\sigma(\eta,\xi)}\omega(\eta)\,\omega(\xi)\\
&= \tilde{\sigma}(\varphi,\lambda\eta)\,\tilde{\sigma}(\theta,\lambda)\,\omega(\lambda)\,\omega(\eta)\,\omega(\xi)/\sigma(\lambda\eta,\xi)\,\sigma(\lambda,\eta)\\
&= \tilde{\sigma}(\varphi,\psi)\,\omega(\lambda\eta)\,\omega(\xi)/\sigma(\psi,\xi)\\
&= \tilde{\sigma}(\varphi,\psi)\,\sigma(\varphi,\xi)\,\omega(\lambda\eta)/\sigma(\psi,\xi)\,\tilde{\sigma}(\varphi_{i\beta}, i\beta)
\end{aligned}$$

which is precisely the condition of proposition 4.1.

Remark. — While positivity is a more delicate matter it can be shown that ω is automatically hermitian on $\Phi^{\perp}$. To derive the hermitian property in general the value of ω at the coset representative must be chosen carefully. Positivity imposes a still stronger constraint which is enough to specify the support of ω completely. To this end we record some technical results.

LEMMA 4.5. — *Let* H *be a group and let* $\Delta : \mathrm{H} \to [0, \infty)$ *satisfy*

(i) $\Delta(\xi^n) = |n| \Delta(\xi)$,

(ii) $\Delta(\xi\eta) \leqslant \Delta(\xi) + \Delta(\eta)$,

for all $\xi, \eta \in \mathrm{H}$, $n \in \mathbb{Z}$. *Then* $d(\xi, \eta) = \Delta(\xi\eta^{-1})$ *defines a pseudometric on* H *which is invariant under right translation.*

Proof. — Clearly for any $\xi \in \mathrm{H}$, $d(\xi, \xi) = \Delta(1) = \Delta(\xi^0) = 0$ by (i). Also for $\xi, \eta \in \mathrm{H}$,

$$d(\xi, \eta) = \Delta(\xi\eta^{-1}) = \Delta((\eta\xi^{-1})^{-1}) = \Delta(\eta\xi^{-1}) \text{ by (i).}$$

Finally for $\xi, \eta, \zeta \in \mathrm{H}$,

$$d(\xi, \zeta) = \Delta(\xi\zeta^{-1}) = \Delta(\xi, \eta^{-1}\eta\zeta^{-1})$$
$$\leqq \Delta(\xi\eta^{-1}) + \Delta(\eta\zeta^{-1}) = d(\xi, \eta) + d(\eta, \zeta).$$

LEMMA 4.6. — *For all* $\xi, \eta \in \Phi^{\perp}$, $|\omega(\xi^p)| = |\omega(\xi)|^{p^2}$ *and*

$$\ln|\omega(\xi)| \,.\, \ln|\omega(\eta)| \geqq \left[\frac{1}{2}\ln|\tilde{\sigma}(\varphi, \eta)|\right]^2.$$

Proof. — Setting $\eta = \xi^{-1}$ in (4.6) we obtain

$$\tilde{\sigma}(\varphi, \xi) = \omega(\xi^{-1})\,\omega(\xi)/\sigma(\xi^{-1}, \xi) = |\omega(\xi)|^2.$$

Hence

$$|\omega(\xi^p)| = |\tilde{\sigma}(\varphi^p, \xi^p)|^{1/2} = |\omega(\xi)|^{p^2} \tag{4.7}$$

for any $p \in \mathbb{Z}$. More generally (4.6) gives

$$|\omega(\xi\eta)| = |\tilde{\sigma}(\varphi, \eta)| \,.\, |\omega(\xi)| \,.\, |\omega(\eta)|$$

so that for any $p, q \in \mathbb{Z}$

$$|\tilde{\sigma}(\varphi, \eta)|^{pq} |\omega(\xi)|^{p^2} |\omega(\eta)|^{q^2} = |\omega(\xi^p\eta^q)| \leqq 1.$$

Taking logarithms we see that for $q \neq 0$

$$(p/q)^2 \ln|\omega(\xi)| + \left(\frac{p}{q}\right)\ln|\tilde{\sigma}(\varphi, \eta)| + \ln|\omega(\eta)| \leqq 0.$$

The inequality now follows by considering the discriminant of this quadratic in (p/q).

COROLLARY 4.7. — $d(\xi, \eta) = [-\ln|\omega(\xi\eta^{-1})|]^{1/2}$ *defines an invariant pseudometric on* $\Phi^{\perp}$.

Proof. — By Lemma 4.6

$$[-\ln|\omega(\xi^n)|]^{1/2}=[-n^2\ln|\omega(\xi)|]^{1/2}=|n|[-\ln|\omega(\xi)|]^{1/2}.$$

Also

$$\{[-\ln|\omega(\xi)|]^{1/2}+[-\ln|\omega(\eta)|]^{1/2}\}^2 \geqq -\ln|\omega(\xi)|-\ln|\omega(\eta)|-\ln|\tilde{\sigma}(\varphi,\eta)|=-\ln|\omega(\xi\eta)|.$$

So $\Delta(\xi)=[-\ln|\omega(\xi)|]^{1/2}$ satisfies the conditions of lemma 4.5 proving the result.

Remark. — If ω were a Fock state on a vector group then d would just be the Hilbert space metric used to define ω.

In general we can now restrict ω in terms of d.

PROPOSITION 4.8. — *For any* $\psi\in G$, $\omega(\psi)=0$ *unless the map taking* $\xi\in\Phi^\perp$ *to* $|\tilde{\sigma}(\varphi,\psi)|$ (*using the notation of proposition* 4.4) *is well defined and continuous with respect to the pseudometric topology on* $\Phi^\perp$.

Proof. — The pseudometric is translation invariant so the given map is continuous if and only if it is continuous at the identity. Now if $\xi\to|\tilde{\sigma}(\varphi,\psi)|$ is not continuous at $\xi=1$ then for some $M>1$ and any $L\in(0,1)$ there exists $\xi\in\Phi^\perp$ such that $|\omega(\xi)|>L$ but

$$|\tilde{\sigma}(\varphi,\psi)|\notin(M^{-1},M).$$

(This is just an exponentiated version of the continuity condition.)

By inverting ξ if necessary we can assume $|\tilde{\sigma}(\varphi,\psi)|>M$. Now, given any $K>0$ choose a positive integer n with $M^n>2K$ and pick $\xi\in\Phi^\perp$ such that $|\omega(\xi)|>2^{-1/n^2}$ but $|\tilde{\sigma}(\varphi,\psi)|>M$.

$$1\geqq|\omega(\psi\xi^n)|=|\tilde{\sigma}(\varphi,\xi)|^n|\omega(\xi)|^{n^2}|\omega(\psi)|>M^n\,.\,\frac{1}{2}\,.\,|\omega(\psi)|>K|\omega(\psi)|.$$

But then $|\omega(\psi)|<K^{-1}$ for any positive K proving the result.

Introduce the notation G_c for the set of $\psi\in G$ such that $\tilde{\sigma}(\varphi,\psi)=1$ for all $\varphi\in G_{\mathbb{C}}$ satisfying $\varphi=\varphi_{i\beta}$ and such that the map $\xi\to|\tilde{\sigma}(\varphi,\psi)|$ does define a continuous function on $\Phi^\perp$, G_c is clearly a subgroup of F_σ. We shall also henceforth make the simplifying assumption that $\Phi^\perp$ is divisible (for locally compact groups this would be a consequence of connectedness).

COROLLARY 4.9. — Supp $(\omega)\subseteqq G_c$.

Proof. — This follows from proposition 4.8.

Remark. — In the examples we consider later there are no further restrictions on supp (ω) so this seems to be the best possible result.

The pseudometric on $\Phi^\perp$ often has an extension to the whole of G_c. We note therefore the following boundedness condition for continuity.

PROPOSITION 4.10. — *For each d-continuous homomorphism* λ *from* $\Phi^\perp$ *into* $\mathbb{R}^+$ *there exists* $L \in (0, \infty)$ *such that* $|\ln \lambda(\xi)| \leqq L[-\ln|\omega(\xi)|]^{1/2}$. *Conversely if there exists* L *for which this inequality holds then* λ *is continuous.*

Proof. — If λ is continuous we can find, for any $\varepsilon > 0$, a δ such that $[-\ln|\omega(\xi)|^{1/2}] < \delta$ implies that $|\ln \lambda(\xi)| < \varepsilon$. If now $[-\ln|\omega(\xi)|]^{1/2} \geqq \delta$ then choose n a positive integer such that

$$\frac{1}{2} n\delta \leqq [-\ln|\omega(\xi)|]^{1/2} < n\delta.$$

Since $\Phi^\perp$ is assumed divisible we may choose $\zeta \in \Phi^\perp$ satisfying $\zeta^n = \xi$. Then by lemma 4.6

$$[-\ln|\omega(\zeta)|]^{1/2} = \frac{1}{n}[-\ln|\omega(\xi)|]^{1/2} < \delta$$

so

$$|\ln \lambda(\xi)| = n|\ln \lambda(\zeta)| < n\varepsilon < \frac{2}{\delta}\varepsilon[-\ln|\omega(\xi)|]^{1/2}$$

If $[-\ln|\omega(\xi)|]^{1/2} < \delta$ choose n a positive integer such that

$$\delta/2n < [\ln|\omega(\xi)|]^{1/2} < \delta/n$$

Then

$$|\ln \lambda(\xi)| = n^{-1}|\ln \lambda(\xi^n)| < \varepsilon/n < \frac{2}{\delta}\varepsilon[-\ln|\omega(\xi)|]^{1/2}$$

So we can take $L = 2\varepsilon/\delta$. The converse is straightforward.

COROLLARY 4.11. — $G_c \supseteqq \Phi^\perp$.

Proof. — For $\eta \in \Phi^\perp$ lemma 4.6 gives

$$\left|\frac{1}{2}\ln|\tilde{\sigma}(\varphi, \eta)|\right| \leqq [-\ln|\omega(\xi)|]^{1/2}[-\ln|\omega(\eta)|]^{1/2} \tag{4.8}$$

so $\xi \to |\tilde{\sigma}(\varphi, \eta)|$ is continuous by the converse part of Proposition 4.10.

As we are assuming that $\Phi^\perp$ is divisible we may define for $\psi \in G_c$

$$\Delta(\psi) = \inf\{L \in (0, \infty) : |\ln|\tilde{\sigma}(\varphi, \psi)|| \leqq 2L[-\ln|\omega(\xi)|]^{1/2}\}$$

PROPOSITION 4.12. — *The function* $\delta(\psi, \chi) = \Delta(\psi\chi^{-1})$ *defines a pseudometric on* G^c *extending* d *on* $\Phi^\perp$.

Proof:

$$|\ln|\tilde{\sigma}(\varphi, \psi^n)|| = |n \ln|\tilde{\sigma}(\varphi, \psi)||$$

whence it follows that $\Delta(\psi^n) = |n|\Delta(\psi)$. Also

$$|\ln|\tilde{\sigma}(\varphi, \psi\chi)|| = |\ln|\tilde{\sigma}(\varphi, \psi)| + \ln|\tilde{\sigma}(\varphi, \chi)||$$

So that

$$\Delta(\psi\chi) \leqq \Delta(\psi) + \Delta(\chi)$$

The fact that δ is a pseudometric follows from lemma 4.5. From (4.8) we see immediately that

$$\Delta(\eta) \leqq [-\ln|\omega(\eta)|]^{1/2}$$

On the other hand the equation preceding (4.7) gives

$$|\ln|\tilde{\sigma}(\varphi, \xi)|| = 2|\ln|\omega(\xi)||$$

whence $\Delta(\xi) \geqq [-\ln|\omega(\xi)|]^{1/2}$. We see therefore that for $\xi \in \Phi^\perp$

$$\Delta(\xi) = [-\ln|\omega(\xi)|]^{1/2}$$

which implies that δ and d agree on $\Phi^\perp$.

Before giving the main theorem characterising KMS states we need two more assumptions:

A 4. $\Phi^\perp$ is δ-dense in $G_c \cap (F\sigma)_0$.

A 5. $\tilde{\sigma}$ is δ-continuous.

Remark. – By definition $|\tilde{\sigma}|$ is continuous and in the vector space case that is sufficient to ensure continuity of $\tilde{\sigma}$ itself.

LEMMA 4.13. – *Under the assumptions A 1-4 the pairing*

$$(\varphi^{-1}\varphi_{i\beta}, \eta) \mapsto \tilde{\sigma}(\varphi_{i\beta}, \eta)^{-1}$$

extends to a δ-continuous bicharacter β on $G_c \cap (F\sigma)_0$. The function ω defined on $\Phi^\perp$ by Proposition 4.4 extends to a δ-continuous function ω_0 on $G_c \cap (F\sigma)_0$ satisfying

$$\omega_0(\xi^{-1}\kappa) = \sigma(\xi, \xi^{-1}\kappa)\,\beta(\xi, \kappa)\,\overline{\omega_0(\xi)}\,\omega_0(\kappa)$$

for all ξ and κ in $G_c \cap (F\sigma)_0$.

Proof. – We have

$$\tilde{\sigma}(\varphi_{i\beta}, \eta)^{-1} = \tilde{\sigma}(\eta, \xi)\,\tilde{\sigma}(\varphi, \eta)^{-1}$$

and $\tilde{\sigma}(\varphi, \eta)$ defines a d-continuous bicharacter in

$$(\xi, \eta) \in \Phi^\perp \times (G_c \cap (F\sigma)_0).$$

It therefore extends to a bicharacter β as required. We note that by continuity $\beta(\eta, \eta) \in [1, \infty)$ for all $\eta \in G_c \cap (F\sigma)_0$. (The lower bound follows from the fact that in $\Phi^\perp$ we have the identity $1 = \beta(\eta, \eta)|\omega(\eta)|^2$.)

The construction of ω_0 proceeds by a Zorn's lemma argument. We consider pairs (H, ω) consisting of a subgroup $H \supseteq \Phi^\perp$ and an ω satisfying the desired identity for all ξ and κ in H. These can be ordered by $(H, \omega) \prec (\tilde{H}, \tilde{\omega})$ when $H \subseteq \tilde{H}$ and $\tilde{\omega}$ restricts to ω on H. Each chain (H_j, ω_j) has an upper bound $(\bigcup H_j, \omega)$ with $\omega = \omega_j$ on H_j. We can therefore find a maximal element (H, ω_0). If $H \subset G_c \cap (F\sigma)_0$ then we choose

$\eta \in (G_c \cap (F\sigma)_0) \setminus H$. Now choose $\omega_0(\eta)$ so that $|\omega_0(\eta)|^2 = \beta(\eta, \eta)^{-1}$,

$$\omega_0(\eta^s) = \beta(\eta, \eta)^{-s(s-1)/2} \omega_0(\eta)^s \Big/ \prod_1^{s-1} \sigma(\eta, \eta^j),$$

for $s > 0$, and

$$\omega_0(\eta^s) = \beta(\eta, \eta)^{-s(s-1)/2} \omega_0(\eta)^s \prod_1^{-s} \sigma(\eta, \eta^{-j}),$$

for $s < 0$. Finally set

$$\omega_0(\zeta^{-1}\eta^s) = \beta(\zeta, \eta)^s \sigma(\zeta, \zeta^{-1}\eta^s) \overline{\omega_0(\zeta)} \omega_0(\eta^s)$$

for $\zeta \in \Phi^\perp$. It may be checked that this defines ω_0 on the subgroup generated by η and H, in such a way that the equality still holds. This contradicts maximality so $H = G_c \cap (F\sigma)_0$. Since we have incorporated all the equations which were equivalent to it, the KMS condition will be satisfied by ω_0.

THEOREM 4.14. — *Under the assumptions A 1-4 the most general KMS state ω has the form $\omega \circ \rho$ where ρ is lifted from a function of positive type on $G_c \cap (F\sigma)_0/\Phi^\perp$. In particular, if $G_c \cap (F\sigma)_0 = \Phi^\perp$ and a KMS state exists then it is unique and is given by Proposition 4.4. If $G_c \cap (F\sigma)_0/\Phi^\perp$ is locally compact then the positive function can be written in the form*

$$\int_\Phi \chi(\psi)\, dm(\chi)$$

where m is probability measure on Φ.

Proof. — By Corollary 4.9, ω is supported in $G_c \cap (F\sigma)_0$ and by Proposition 4.4 the only freedom in the definition of ω is to define $\omega(\psi)$ on one point of each $\Phi^\perp$ coset. The most general KMS state must therefore be of the form $\omega \circ \rho$ is lifted from $G_c \cap (F\sigma)_0/\Phi^\perp$.

For any η and ξ in $G_c \cap (F\sigma)_0$ we therefore have:

$$\begin{aligned}\omega(\xi^{-1}\eta)/\sigma(\xi, \xi^{-1}\eta) &= \rho(\xi^{-1}\eta)\, \omega_0(\xi^{-1}\eta)/\sigma(\xi, \xi^{-1}\eta) \\ &= \rho(\xi^{-1}\eta)\, \beta(\xi, \eta)\, \overline{\omega_0(\xi)}\, \omega_0(\eta).\end{aligned}$$

the last simplification following from Lemma 4.13. Moreover, for any $\tilde{\xi} \in \xi\Phi^\perp$ and $\tilde{\eta} \in \eta\Phi^\perp$ the constancy of ρ on $\Phi^\perp$-cosets gives

$$\omega(\tilde{\xi}^{-1}\tilde{\eta})/\sigma(\tilde{\xi}, \tilde{\xi}^{-1}\tilde{\eta}) = \rho(\xi^{-1}\eta)\, \beta(\tilde{\xi}, \tilde{\eta})\, \overline{\omega_0(\tilde{\xi})}\, \omega_0(\tilde{\eta}).$$

We also note that the original definition of Δ means that for $\zeta \in \Phi^\perp$ we have $\omega_0(\zeta) = \exp(-\Delta(\zeta)^2)$ and continuity enables us to extend this to more general ζ and deduce that ω_0 never vanishes.

The positivity of ω means that given any $z_1, z_2, \ldots, z_k \in C$ and $\xi_1, \xi_2, \ldots, \xi_k \in G_c \cap (F\sigma)_0$ we have

$$0 \leq \sum \overline{(z_i/\omega_0(\xi_i))}\,(z_j/\omega_0(\xi_j))\, \omega(\xi_i^{-1}\xi_j)/\sigma(\xi_i, \xi_i^{-1}\xi_j).$$

Using our earlier identity we see that if $\tilde{\xi}_j \in \xi_j \Phi^\perp$ this yields

$$0 \leqq \sum \bar{z}_i z_j \beta(\tilde{\xi}_i, \tilde{\xi}_j) \rho(\xi_i^{-1} \xi_j).$$

By the density of $\Phi^\perp$, for any $\xi \in G_c \cap (F\sigma)_0$ we can find a sequence $\{\xi^{(n)}\}$ of elements of $\Phi^\perp$ which converge to ξ in the δ-topology. We now set $\tilde{\xi}_1 = (\xi_1^{(n)})^{-1} \xi_1 \in \xi_1 \Phi^\perp$, so that $\tilde{\xi}_1 \to 1$, and take $\tilde{\xi}_j = \xi_j$ for $j>1$. For $j \neq 1$ we have $\beta(\tilde{\xi}_1, \tilde{\xi}_j)$ and $\beta(\tilde{\xi}_j, \tilde{\xi}_1)$ converging to one as $n \to \infty$ and so $\beta(\tilde{\xi}_1, \tilde{\xi}_1) = |\omega_0(\tilde{\xi}_1)|^{-2}$. Proceeding to this limit, we see that

$$0 \leqq \sum \bar{z}_i z_j \beta(\tilde{\xi}_i, \tilde{\xi}_j) \rho(\xi_i^{-1} \xi_j).$$

where $\tilde{\xi}_1 = 1$ and $\tilde{\xi}_j = \xi_j$ for $j>1$. We now repeat the argument with $\tilde{\xi}_2$ running through a convergent sequence and so on to arrive at a situation in which all the $(\tilde{\xi})_j$ can be replaced by one. This gives

$$0 \leqq \sum \bar{z}_i z_j \rho(\xi_i^{-1} \xi_j).$$

and shows that ρ is a function of positive type.

Finally if $G_c \cap (F\sigma)_0/\Phi^\perp$ is locally compact we may use Bochner's theorem to represent ρ as an integral over its dual, which can be identified with $\Phi/(G_c \cap (F\sigma)_0^\perp$.

Remark. – This result extends the work of Rocca, Sirugue and Testard classifying KMS states for the canonical commutation relations, [28].

4.2. Examples

The theory of the preceding section can be illustrated conveniently by four interesting physical examples. (We omit details of checking the various assumptions however since these are straightforward.)

Example 1. – In [10] we considered the loop group $G = \mathrm{Map}(S^1, U(1))$ with the rotation action of $\mathbb{R}$ and multiplier

$$\sigma(e^{if}, e^{ig}) = \exp\left\{\frac{-i}{4\pi}\left[\int g\,df + g(0)(f(2\pi) - f(0))\right]\right\}$$

(*see* also [11], [16], [31]). In this case F is the subgroup of constant functions and $(F\sigma)$ is the identity component G_0 of G. The map $\varphi \to \varphi^{-1}\varphi_t$ maps G onto G_0 so that the invariant characters are those which annihilate G_0 and $\Phi^\perp = G_0$. Since $(F\sigma) = \Phi^\perp$ we see immediately in this case that the KMS state is unique.

Example 2. – More traditionally one would consider (instead of the group of example 1) $G = C^\infty(S^1, \mathbb{R})$ and again impose the rotation action of $\mathbb{R}$ and the multiplier

$$\sigma(f, g) = \exp\left(\frac{-i}{4\pi}\int g\,df\right)$$

F is again the subgroup of constant functions and now $(F\sigma)$ is the whole group. By considering the Fourier series of $f \in G$:

$$f(s) = \sum_n \hat{f}_n e^{ins}$$

we see that the invariant characters are those of the form

$$\chi_\alpha : \quad f \to \exp(i\alpha \hat{f}_0)$$

for some α. Thence

$$\Phi^\perp = \{ f \in G : \hat{f}_0 = 0 \}$$

The KMS state on $\Phi^\perp$ is (by calculation using proposition 4.4)

$$\omega(f) = \exp\left\{ \frac{-1}{4\pi} \sum_n \coth \beta n |\hat{f}_n|^2 \right\}$$

The pseudometric is therefore a weighted l^2 metric and one readily sees that $G_c = G$. The freedom in the KMS state is to multiply by an integral of $\chi_\alpha(\psi) = \exp(i\alpha\hat{\psi}_0)$, that is by a positive function of $\hat{\psi}_0$.

Example 3. — For the free bose gas we take $G = \mathscr{S}(\mathbb{R})$ (real-valued Schwartz functions) with the translation action of $\mathbb{R}$ and

$$\sigma(f, g) = \exp\left(\frac{-i}{4\pi} \int g df \right)$$

This is very similar to the previous example except that the constants do not lie in $\mathscr{S}(\mathbb{R})$ so that F is trivial and $(F\sigma) = G$. We can identify Φ with the constant distributions and

$$\chi_\alpha : \quad f \to \exp\left(\frac{i\alpha}{\sqrt{2\pi}} \int f(s)\, ds \right) = \exp(i\alpha \hat{f}(0))$$

where $\hat{f}$ denotes the Fourier transform of f. So,

$$\Phi^\perp = \{ f \in G : \hat{f}(0) = 0 \}.$$

Using proposition 4.4 we find that on $\Phi^\perp$

$$\omega(f) = \exp\left(\frac{-1}{4\pi} \int k \coth \beta k |\hat{f}(k)|^2 dk \right)$$

One can now check that $G_c = G$ and much as in the previous example the freedom in ω is the freedom to multiply by a function of $\hat{f}(0)$.

Extra factors of this kind have been introduced to describe Bose-Einstein condensation (see Araki-Woods [2], Lewis-Pule [20], Lewis [21]). Since only $\hat{f}(0)$ appears in the additional term this change only affects the zero energy mode and thus permits the physical interpretation of corresponding to macroscopic occupation of the ground state.

Example 4. – Finally we consider the case where G is the subgroup $\mathcal{T}$ of $\mathcal{G}$ and σ as in the earlier sections of this paper with the $\mathbb{R}$-action given by translation. Once more F is trivial and $(F\sigma)=G$. Proceeding as in Example 1 we find that $\Phi^{\perp}=G_0$ the identity component of G. On $\Phi^{\perp}$ the state is again given by the formula

$$\omega(f)=\exp\left(\frac{-1}{4\pi}\int k\coth\beta k\,|\hat{f}(k)|^2\,dk\right)$$

However this time only elements of the identity component are continuous with respect to the pseudometric, so that $G_c=G_0$. (Note that the pseudometric is again clearly the restriction of an L^2 metric so that continuous linear functions must be given by appropriate L^2 functions. The standard kinks, which may be taken as $\Phi^{\perp}$ coset representatives, do not give L^2 functions so that they, and therefore the entire coset, cannot be in G_c.) This fact was of course known from proposition 2.8 but it is interesting to see how the group theoretic treatment arrives at the same conclusion. Since $\Phi^{\perp}=G_c\cap(F\sigma)$ the KMS state is unique.

4.3. Bose-Einstein Condensation

In the previous section we saw that the KMS states for $\mathcal{G}$ and for $C^{\infty}(S^1, U(1))$ are unique, although those for their identity components are not. In this section we shall consider more carefully the relationship between states on the full group and those on a subgroup.

PROPOSITION 4.15. – *Let* G, σ *be as in theorem* 4.14 *and suppose that* $F\subseteqq G_c$. *Let* G′ *be a closed* τ*-invariant subgroup of* G, *which contains* G_c, σ' *the restriction of* σ *to* G′ *and let* G'_c, Φ', F′ *be the subgroups of* G′ *defined in the corresponding way to the subgroups* G_c, Φ *and* F *in* G. *Then there is a natural projection from a subgroup of*

$$G'_c\cap(F'\sigma')/\Phi'^{\perp}$$

onto $G_c\cap(F\sigma)/\Phi^{\perp}$.

Proof. – We have $F'=F$ from which it follows that $(F'\sigma')=G'\cap(F\sigma)$. Identifying $\hat{G}/G'^{\perp}$ with $\hat{G}'$ we have

$$\Phi'=\{\chi\in\hat{G}:\chi_t\chi^{-1}\in G'^{\perp}\}\supseteqq\Phi$$

and so $\Phi'^{\perp}\subseteqq\Phi^{\perp}\cap G'=\Phi^{\perp}$ by (corollary 4.11). It therefore follows that

$$G'_c\supseteqq G_c\cap G'=G_c.$$

Taking quotients we see that $G_c/\Phi'^{\perp}\cap G_c$, which projects onto $G_c/\Phi^{\perp}$, is a subgroup of $G'_c/\Phi'^{\perp}$.

COROLLARY 4.16. — *Let Φ' satisfy the condition (A3). Under the conditions of proposition* 4.15, *if* G' *has a unique* (τ, β)-*KMS state then so does* G, *but not necessarily conversely.*

Proof. — Assumptions (A 1) and (A 2) automatically held in G'. The result therefore follows from theorem 4.14 since the proposition implies that if $G'_c \cap (F' \sigma')/\Phi'^{\perp}$ is trivial so is $G_c \cap (F \sigma)/\Phi^{\perp}$, but not conversely.

Remark. — Corollary 4.16 covers both the examples: $\mathscr{S}(\mathbb{R}) \subseteqq \mathscr{T}$ and $C^{\infty}(S^1, \mathbb{R}) \subseteqq C^{\infty}(S^1, U(1))$.

We shall now show how to construct families of KMS state on G' from a single KMS state ω on G. Firstly we also denote by ω the restriction of the state to G'. Conjugation by δ_γ, $\gamma \in G$, produces a new state ω_γ on G' via

$$\omega_\gamma(\varphi) = \tilde{\sigma}(\varphi, \gamma)\, \omega(\gamma^{-1} \varphi \gamma), \qquad \varphi \in G',$$

where, in a non-abelian group

$$\tilde{\sigma}(\varphi, \gamma) = \sigma(\varphi, \gamma)/\sigma(\gamma, \gamma^{-1} \varphi \gamma).$$

But, since G is abelian, this reduces to

$$\omega_\gamma(\varphi) = \tilde{\sigma}(\varphi, \gamma)\, \omega(\varphi).$$

If $\gamma \in (G' \sigma)$ then $\omega_\gamma = \omega$ and conversely. Now,

$$\omega_\gamma(\psi \varphi_t) = \tilde{\sigma}(\psi, \gamma)\, \tilde{\sigma}(\varphi_t, \gamma)\, \omega(\psi \varphi_t) = \tilde{\sigma}(\psi, \gamma)\, \tilde{\sigma}(\varphi, \gamma_{-t})\, \omega(\psi \varphi_t).$$

Suppose now that $\gamma_{-t} \gamma^{-1}$ lies in $(G' \sigma)$ for all $t \in \mathbb{R}$. Then

$$\omega_\gamma(\psi \varphi_t) = \tilde{\sigma}(\psi \varphi, \gamma)\, \omega(\psi \varphi_t).$$

PROPOSITION 4.17. — *If* ω *satisfies the KMS condition on* G, *and* $\gamma_t \gamma^{-1} \in (G' \sigma)$ *for all* $t \in \mathbb{R}$ *then* ω_γ *satisfies the KMS condition on* G'.

Proof:

$$\omega_\gamma(\psi \varphi_t)\big|_{t=i\beta} = \tilde{\sigma}(\psi \varphi, \gamma)\, \omega(\psi \varphi_t)\big|_{t=i\beta} = \tilde{\sigma}(\psi \varphi, \gamma)\, \omega(\varphi \psi) = \omega_\gamma(\varphi \psi)$$

from which the result follows.

Example. — The case where $G = \mathscr{T}$ and G' its identity component is not illustrative since then $(G' \sigma)$ is trivial and there are no γ's satisfying the condition of the proposition. However when $G = C^{\infty}(S^1, U(1))$ and G' its identity component then $(G' \sigma) = F$, the constant functions. The group elements γ for which $\gamma_{-t} \gamma^{-1}$ is constant are those in the subgroup generated by F and the functions $e_n(\theta) = e^{in\theta}$ $(n = \pm 1, \pm 2, \ldots)$.

If $\gamma = e_n$ and $\varphi(\theta) = \exp(if(\theta)) = \exp\left(i \sum_k \hat{f}_k e^{ik\theta}\right)$ then

$$\omega_\gamma(\varphi) = \tilde{\sigma}(\varphi, e_n)\, \omega(\varphi) = e^{in\hat{f}_0}\, \omega(\varphi) \tag{4.9}$$

In this way an infinite series of KMS states of G′ can be generated from a single KMS state of G. This can be carried still further by introducing an element R of the twisted group C*-algebra of G, such that $\omega(R^* R)=1$, for we can then modify the state ω to ω^R where

$$\omega^R(A)=\omega(R^* AR)$$

Taking $R=\sum_n R(n)\,\delta_{e_n}$ we obtain

$$\omega^R(\varphi)=\sum R(m)^* R(n)\,\sigma(\varphi, e_n)\,\omega(e_m^{-1}\varphi e_n)/\sigma(e_m, \varphi)$$

Since ω is supported in the identity component G′, the terms with $m\neq n$ vanish leaving

$$\omega^R(\varphi)=\sum_n |R(n)|^2\,\tilde{\sigma}(\varphi, e_n)\,\omega(\varphi)=\sum_n |R(n)|^2\,\omega_{e_n}(\varphi).$$

THEOREM 4.18. — *If* $R=\sum_n R(n)\,\delta_{e_n}$ and ω *is a* (τ, β)-KMS *state on* $C^\infty(S^1, U(1))$ *then* ω^R *defines a* (τ, β)-KMS *state on the identity component. Writing*

$$\rho(\alpha)=\sum_n |R(n)|^2\,e^{in\alpha}$$

we have

$$\omega^R(e^{if})=\rho(\hat{f}_0)\,\omega(e^{if}).$$

Proof. — Since the formula (4.10) expresses ω^R as a convex combination of KMS states it is clear that ω^R is itself KMS. Moreover,

$$\omega^R(e^{if})=\sum_n |R(n)|^2\,\omega_{e_n}(e^{if})=\sum_n |R(n)|^2\,e^{in\hat{f}_0}\,\omega(e^{if}) \quad \text{(by 4.9)}$$

$$=\rho(\hat{f}_0)\,\omega(e^{if}).$$

Remark. — The function ρ is of positive type normalised so that $\rho(0)=1$ and periodic. The general theory of section 4.2 example 2 tells us that multiplication by a function of $\hat{f}_0$ is the only freedom, and the positivity and normalisation are necessary conditions for a state. We deduce that every (τ, β)-KMS state on G′ can be expressed in the form ω^R. As noted in Section 4.2 ρ can be interpreted as a Bose-Einstein condensation term and the theorem shows how such terms can be generated by applying charge-raising operators $\Gamma(e_n)$ to the vacuum Ω_β.

We conclude this section with a comment on the effect such condensation terms have on the correlation functions for fermions constructed as limits of loop group representatives. Without going into too much detail we introduce some notation from [10].

Let $B(g)=\int B_\alpha \overline{g(\alpha)}\,d\alpha\ (g\in L^2(S^1))$ denote a typical generator of the CAR algebra over $L^2(S^1)$. Here B_α is an unsmeared fermion operator, that is a quadratic form, as in [10], [6]. This is the analogue of the formula $\psi(g)=\int_{-\infty}^{\infty}\psi(x)\overline{g(x)}\,dx$ when one works with functions $g\in L^2(\mathbb{R})$.

THEOREM 4.19. — *The correlation coefficients are given by*

$$\omega_\beta^R(B_{\alpha_1}^*\dots B_{\alpha_M}^*B_{\zeta_N}\dots B_{\zeta_1})$$
$$=\sum_k R(k-M)^*R(k-N)C_k(\alpha_1\dots\alpha_M,\zeta_1\dots\zeta_N)\,\theta_3\big(\sum_j(\alpha_j-\zeta_j)\big)\,\theta_3(0)^{-1}.$$
$$\prod_{r<s}(\theta_1(\alpha_r-\alpha_s)\,\theta_1(\zeta_s-\zeta_r))\big(\prod_{r,s}\theta_1(\alpha_r-\zeta_s)\big)^{-1}(-i\theta_1'(0))^{(1/2)(M+N)-(1/2)(M-N)^2}$$

where $C_k(\alpha_1,\dots,\alpha_M,\zeta_1\dots\zeta_N)$ *is a phase factor and we adopt the conventions of* [10] *regarding the distribution on the right hand side. In particular, if* $M=N$

$$\omega_\beta^R(B_{\alpha_1}^*\dots B_{\alpha_N}^*B_{\zeta_N}\dots B_{\zeta_1})=\rho\big(\sum_j(\alpha_j-\zeta_j)\big)\,\omega_\beta(B_{\alpha_1}^*\dots B_{\zeta_1})$$

Proof. — By definition

$$\omega_\beta^R(B_{\alpha_1}^*\dots B_{\zeta_1}^*)=\sum_{m,n}R(m)^*R(n)\,\omega_\beta(\Gamma(e_m)^*B_{\alpha_1}^*\dots B_{\zeta_1}\Gamma(e_n))$$

now

$$\Gamma(e_n)=\exp\left(-\frac{i\pi}{4}n(n-1)\right)\Gamma(e_1)^n=\exp\left(-\frac{i\pi}{4}n(n-1)\right)B_{0,0}^n$$

using Formula (2.12) and Section 3.1 of [10]. We can therefore write

$$\omega_\beta^R(B_{\alpha_1}^*\dots B_{\zeta_1})=\sum_{m,n}R(m)^*R(n)$$
$$\exp\frac{i\pi}{4}[m(m-1)-n(n-1)]\,\omega_\beta((B_{0,0}^m)^*B_{\alpha_1}^*\dots B_{\zeta_1}B_{0,0}^n)$$

Now $\omega_\beta((B_{0,0}^m)^*B_{\alpha_1}^*\dots B_{\zeta_1}B_{0,0}^n)$ is the limit of

$$\omega_\beta^R((B_{0,0}^*)^m B_{\alpha_1,\lambda_1}^*\dots B_{\zeta_1,\mu_1}B_{0,0}^n)$$

as the λ_j's and μ_j's tend to 1 (notation as in [10]).

However this correlation coefficient can be calculated directly as in [6], [10] and the formula then follows by rather laborious algebra.

When $M=N$ there is some simplification and we obtain

$$\omega_\beta^R(B_{\alpha_1}^*\dots B_{\zeta_1})=\sum_n|R(n)|^2\exp\big(in\sum_j(\alpha_j-\zeta_j)\big)\,\omega_\beta(B_{\alpha_1}^*\dots B_{\zeta_1})$$

using proposition 3.6 of [9], whence the second formula is immediate.

Remarks. — 1. Notice that while the even correlations change only slightly, the state ω^R is no longer an even function.

2. Just before submitting this manuscript we received a preprint by K. R. Parthasarathy and K. B. Sinha entitled "Boson-fermion relations in several dimensions". Using stochastic integrals they obtain a rather different type of boson-fermion correspondence relating non-zero temperature representations of the CAR and CCR. The connection with our approach is unclear.

APPENDIX 1: Proof of convergence (proposition 3.1)

The first step is to note that it is sufficient to consider the case $N=1$ as the general result follows immediately from that.

$$W_\beta=\begin{bmatrix} (1-A_\beta)^{1/2} & A_\beta^{1/2} \\ -A_\beta^{1/2} & (1-A_\beta)^{1/2} \end{bmatrix} \qquad (A\,1.1)$$

so that $W_\beta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}W_\beta^*=P_\beta$ and we have an isomorphism of $L^2(\mathbb{R})$ with $P_\beta(H\oplus H)$ via

$$h\to W_\beta\begin{bmatrix} 0 \\ h \end{bmatrix}=\begin{bmatrix} A_\beta^{1/2}h \\ (1-A_\beta)^{1/2}h \end{bmatrix},\qquad h\in L^2(\mathbb{R}). \qquad (A\,1.2)$$

Similarly $(1-P_\beta)(H\oplus H)$ is also isomorphic to $L^2(\mathbb{R})$ via

$$h\to W_\beta\begin{bmatrix} h \\ 0 \end{bmatrix}=\begin{bmatrix} (1-A_\beta)^{1/2}h \\ -A_\beta^{1/2}h \end{bmatrix},\qquad h\in L^2(\mathbb{R}).$$

Now let $Q_\pm$ be the projections onto the first and second summand resp. in $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ and so $Q_+=W_\beta^*(1-P_\beta)W$, $Q_-=W_\beta^*P_\beta W_\beta$. Introduce the notation $U'_{\delta\delta}=Q_\delta U Q_{\delta'}$, $\delta,\delta'=\pm$ for U a unitary operator on $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$.

Suppose that $U_{+-}+U_{-+}$ is Hilbert-Schmidt and

$$\ker U_{--}=\{\lambda e_-\,|\,\lambda\in\mathbb{C}\},\qquad \ker U_{--}^*=0$$

with $\|e_-\|=1$. Then

$$\ker U_{++}^*=\{\lambda U e_-\,|\,\lambda\in\mathbb{C}\},\qquad \ker U_{++}=0.$$

Hence the operators

$$E_-=Q_--U_{-+}U_{+-}^*=U_{--}U_{--}^* \qquad (A\,1.3)$$
$$E_+=U_{++}^*U_{++}=Q_+-U_{+-}^*U_{-+} \qquad (A\,1.4)$$

have bounded inverses as operators on $L^2(\mathbb{R})$. Finally we need the operator Z on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ obtained by setting

$$Z_{++} = -U_{++} E_+^{-1}, \qquad Z_{+-} = -U_{++} E_+^{-1} U_{+-}^* \tag{A 1.5}$$
$$Z_{-+} = -U_{--}^* E_-^{-1} U_{-+}, \qquad Z_{--} = -U_{--}^* E_-^{-1} \tag{A 1.6}$$

The operator Z enters into the definition of the operator $\Gamma(U)$ which implements the automorphism of the CAR over $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ defined by U in the representation determined by Q_-. Since we are simply following the argument of [6] section 3 we prove the analogues of the essential lemmas 3.4, 3.5 and 3.6 of [6]. We are interested in the case where

$$U = W_\beta^* \begin{pmatrix} \gamma_r^\varepsilon & 0 \\ 0 & 0 \end{pmatrix} W_\beta. \tag{A 1.7}$$

LEMMA A 1.1. — *With* U *given by* (*A* 1.7) *the operators* $U_{\delta\delta'}$ *are integral operators in* $L^2(\mathbb{R})$ *with kernels given by*

$$U_{\delta\delta} - Q_\delta = -2\varepsilon (1+e^{-\delta\beta p})^{-1/2} (1+e^{-\delta\beta q})^{-1/2} \theta(p-q) \times \exp[-i(p-q)r - \varepsilon(p-q)] \tag{A 1.8}$$
$$U_{\delta,-\delta} = -2\varepsilon e^{-\delta\beta q/2} (1+e^{-\delta\beta p})^{-1/2} (1+e^{-\delta\beta q})^{-1/2} \theta(p-q) \times \exp[-i(p-q)r - \varepsilon(p-q)] \tag{A 1.9}$$

($\delta = \pm$).

Proof. — This is a direct calculation based on the preceding definitions and the fact that

$$(\gamma_r^\varepsilon - 1)\hat{}\,(p) = -2\varepsilon \sqrt{2\pi}\, \theta(p)\, e^{ipr} e^{-\varepsilon p} \tag{A 1.10}$$

LEMMA A 1.2:

$$\ker U_{--} = \{\lambda e_- \mid \lambda \in \mathbb{C}\}, \qquad \ker U_{--}^* = \{0\}$$

where

$$e_-(p) = -K_\varepsilon(\beta)^{-1} e^{-ipr} (1+e^{-\beta p})^{-1/2} (1+e^{\beta p})^{-2\varepsilon/\beta} e^{-\varepsilon p} \tag{A 1.11}$$

with

$$K_\varepsilon(\beta) = \left\{ \Gamma\left(1+\frac{2\varepsilon}{\beta}\right) \Gamma\left(\frac{2\varepsilon}{\beta}\right) \Big/ \beta\, \Gamma\left(1+\frac{4\varepsilon}{\beta}\right) \right\}^{1/2} \tag{A 1.12}$$

(*where* Γ *denotes the gamma function*).

Proof. — Firstly solve the integral equation

$$(U_{--} - Q_-) g = -g$$

for $g \in L^2(\mathbb{R})$ by converting to the differential equation

$$\frac{d}{dp} \tilde{g}(p) = 2\varepsilon e^{-\beta p} (1+e^{-\beta p})^{-1} \tilde{g}(p) \tag{A 1.13}$$

where $\tilde{g}(p)=e^{\varepsilon p}(1+e^{\beta p})^{1/2} e^{ipr} \hat{g}(p)$. Now (A 1.13) has a unique (up to a constant factor) solution in $L^2(\mathbb{R})$. The corresponding equation arising from $(U^*_{--}-Q_-)g=-g$ has only the zero solution. Thus e_- is the solution of (3.18) normalised to have norm one and $K_\varepsilon(\beta)$ is the appropriate normalisation factor.

LEMMA A 1.3. — *The following estimates holds as* $\varepsilon \to 0$

$$K_\varepsilon(\beta)=(2\varepsilon)^{-1/2}+O(\varepsilon^{1/2}) \quad \text{(A 1.14)}$$

$$\|Z_{\delta,-\delta}\|_{\text{H.S.}}=O(\varepsilon) \quad \text{(A 1.15)}$$

$$\|Z_{++}+U_{++}\|=O(\varepsilon^2)=\|Z_{--}+U^*_{--}\| \quad \text{(A 1.16)}$$

$$\|(U_{++}-Q_+)h\|=O(\varepsilon^{1/2}) \quad \text{(A 1.17)}$$

$$\|(U_{--}-Q_-)h\|=O(\varepsilon) \quad \text{(A 1.18)}$$

where $h\in L^2(\mathbb{R})$ *with* $\hat{h}$ *of compact support.*

Proof. — These estimates all follow by direct calculations on the integral kernels given in lemma A 1.1. The arguments are the same as those in the proof of lemma 3.6 of [6] where analogous estimates are proved and so we omit them. Note that it is useful to have the explicit evaluation

$$\int_{-\infty}^{\infty} (1+e^{-\beta p})^{-1-4\varepsilon/\beta} e^{-2\varepsilon p}\,dp=K_\varepsilon(\beta)^2$$

and minor variations thereof.

Now we indicate how the previous lemmas combine to prove proposition 3.1. To explain in detail the convergence of

$$\psi^{(*)}_\varepsilon(g)\,F, \qquad F\in\mathscr{D}$$

as $\varepsilon\to 0$ we use the explicit formula for $\Gamma_\beta(\gamma^\varepsilon_r)$ which is given by equation (4.50) of [6]. To save a lengthy excursion into notation we will not repeat this formula here remarking only that it involves the operators $Z_{\delta,\delta'}$. The estimate (A 1.16) is used to replace Z_{++} and Z_{--} where they occur in the formula for $\Gamma_\beta(\gamma^\varepsilon_r)$ by $-U_{++}$ and $-U^*_{--}$. The estimate (A 1.15) shows that the terms involving $Z_{\delta,-\delta}$ converge to the identity operator so that combining this with (A 1.17) and (A 1.18) leads to the conclusion that $\psi^*_\varepsilon(g)\,F$ converges as $\varepsilon\to 0$ to the same limit as $\pi_{P_\beta}(a(f_\varepsilon)^*)\,F$ where

$$\hat{f}_\varepsilon(p)=\frac{1}{K_\varepsilon(\beta)\sqrt{2\varepsilon}}\begin{bmatrix}\hat{g}(p)\,e^{-\varepsilon p}\\ 0\end{bmatrix}$$

so that $f_\varepsilon\in H\oplus H$. However it is trivial that

$$\lim_{\varepsilon\to 0} f_\varepsilon=\begin{bmatrix} g\\ 0\end{bmatrix}$$

in H⊕H proving that

$$\pi_{\mathrm{P}_\beta}(a(f_\varepsilon)^*)\,\mathrm{F} \to \pi_{\mathrm{P}_\beta}\left(a\begin{pmatrix} g \\ 0 \end{pmatrix}\right)^* \mathrm{F} \equiv \pi_\beta(a(g)^*)\,\mathrm{F},$$

as $\varepsilon \to 0$. The argument for $\psi_\varepsilon((g)\,\mathrm{F}$ is similar.

APPENDIX 2: Here we explain briefly how the limit as $\varepsilon \to 0$ is taken in (3.7)

Careful examination of the proof of Lemma 4.7 of [6] reveals that the estimates we established in section 3.2 suffice to control the limit as $\varepsilon \to 0$ of the first two terms in (3.7). The only tricky point arises from the fact that (as in proposition 3.1) it would seem necessary to know that $\mathrm{S}\varphi.g$ and $\bar{\varphi}.g$ have Fourier transform with compact support. However in (3.7) we are considering weak convergence and for that one can relax this last constraint (which is fortunate since neither $\mathrm{S}\varphi.g$ nor $\bar{\varphi}.g$ have this support property).

To amplify a little on this point in the case of the second term we note that the worst possible terms which arise in proving convergence by the method of lemma 4.7 of [6] involve integrals of the form:

$$\frac{2\varepsilon}{\mathrm{K}_\varepsilon(\beta)\sqrt{2\varepsilon}} \int (\mathrm{S}\varphi.g)^{\wedge}(k-q+p)\,\mathrm{Z}^\varphi(p,l)\,\hat{g}_2(l)\,\hat{g}_1(q)\,\hat{h}_1(k)\,\theta(k-q) \times (1+e^{-\beta k})^{-1/2}(1+e^{-\beta q})^{-1/2}(1+e^{-\beta p})^{-1/2}(1+e^{-\beta p})^{-2\varepsilon/\beta} \times e^{-\varepsilon(k-q+p)}\,dp\,dq\,dk\,dl$$

where $\mathrm{Z}^\varphi(p,l)$ denotes the kernel of the Hilbert-Schmidt operator $\varphi_{-+}\varphi^{*-1}_{++}$ and g_1, g_2, h_1 are functions in $\mathrm{L}^2(\mathbb{R})$ whose Fourier transform has compact support. Now the $\sqrt{\varepsilon}$ in front of this integral will force the contribution of this term to be zero (as it should) provided we can establish that the integral is bounded independently of ε. Since $\mathrm{S}\varphi.g$ is in L^2 and $\mathrm{Z}^\varphi(p,l)$ is a Hilbert-Schmidt kernel it follows immediately that this is so.

APPENDIX 3

In Section 4.1 we assumed that the multiplier σ could be extended from G to $\mathbb{R}$ Ⓢ G. We shall now show that for an abelian Lie group G which is the product of a discrete group D and connected component G_0, this is automatically true when the $\mathbb{R}$-action preserves the class of the multiplier on G. (We also remark that if G/G_0 is a free abelian group then G is certainly a direct product. To see this we just pick elements of G which project onto the generators of G/G_0. These will generate a free subgroup D isomorphic to G/G_0 and such that $G = D \times G_0$.)

PROPOSITION. – *Let* G *be the direct product of a discrete abelian group* D *and a connected abelian Lie group* G_0 *(possibly infinite-dimensional). Let* σ *be a continuous multiplier on* G *and* τ_t *a one-parameter group of automorphisms of* G *which preserve* σ. *Then* σ *extends from* G *to a borel multiplier (also denoted by* σ*) on* $\mathbb{R} \circledS G$. *Setting* $\tilde{\sigma}(\varphi_t, t) = \sigma(\varphi_t, t)/\sigma(t, \varphi)$ *and* $\tau_t(\delta_\varphi) = \tilde{\sigma}(\varphi_t, t)\,\delta_{\varphi_t}$ *for* $\varphi \in G$ *induces an automorphism of* $M(G, \sigma)$.

Proof. – We first observe that, by [24] Theorem 8.1 and the subsequent discussion, the simply connected covering group of G_0 is isomorphic to the additive group of its Lie algebra, that is to a vector group V. Thus G_0 can be identified with V/L for some discrete subgroup L in V. Let us introduce the double cover $\tilde{G}_0 = V/2L$. We may lift σ to a multiplier (also denoted by σ) on $\tilde{G} = D \times \tilde{G}_0$, and we may similarly lift the $\mathbb{R}$-action to a continuous action on $\tilde{G}$.

From x, y in $\tilde{G}_0$ we obtain well-defined elements of $x^{1/2}$, $y^{1/2}$ in G_0 so that we may define

$$\tau(\xi x, \eta y) = \sigma(\xi, \eta)\,\tilde{\sigma}(\xi, y^{1/2})\,\tilde{\sigma}(x^{1/2}, \eta)\,\tilde{\sigma}(x, y^{1/2}) = \tilde{\sigma}(\xi x, \eta y);$$

τ is in the same class as σ. Thus as in proposition 1.1 of [18] there must exist a borel function λ on $\tilde{G}$ such that for all a, b in $\tilde{G}$

$$\sigma(a, b) = \frac{\lambda(a)\,\lambda(b)}{\lambda(ab)}\,\tau(a, b).$$

(Note that the discussion of [18] needs only the existence of multiplier representations of $\tilde{G}$.)

Since the $\mathbb{R}$-action on G preserves the multiplier class of σ it must leave $\tilde{\sigma}$ invariant. The last three factors in the formula for τ are therefore invariant under the action of $\mathbb{R}$. We also have

$$\left[\frac{\tau(\xi x, \eta y)}{\tau(\xi, \eta)}\right]^2 = [\tilde{\sigma}(\xi, y^{1/2})\,\tilde{\sigma}(x^{1/2}, \eta)\,\tilde{\sigma}(x^{1/2}, y)]^2 = \frac{\tilde{\sigma}(\xi x, \eta y)}{\tilde{\sigma}(\xi, \eta)}$$

Now the continuous $\mathbb{R}$-action can only change $\xi \in D$ to an element of the form $\xi_t = \xi . \tilde{\xi}_t$, where $\tilde{\xi}_t \in \tilde{G}_0$. So

$$\left[\frac{\tau(\xi_t, \eta_t)}{\tau(\xi, \eta)}\right]^2 = \frac{\tilde{\sigma}(\xi_t, \eta_t)}{\tilde{\sigma}(\xi, \eta)} = 1.$$

The ratio $\tau(\xi_t, \eta_t)/\tau(\xi, \eta)$ depends continuously on t and is 1 when $t = 0$, so we deduce that it is identically 1. Coupled with our previous observation we see that τ is invariant under the action of $\mathbb{R}$.

Thence for any $a, b \in \tilde{G}$, $t \in \mathbb{R}$

$$\frac{\sigma(a_t, b_t)}{\sigma(a, b)} = \frac{\lambda(a_t)\lambda(b_t)}{\lambda(a_t b_t)} \frac{\lambda(ab)}{\lambda(a)\lambda(b)} \frac{\tau(a_t, b_t)}{\tau(a, b)} = \frac{\lambda(a_t)}{\lambda(a)} \frac{\lambda(b_t)}{\lambda(b)} \Big/ \frac{\lambda(a_t b_t)}{\lambda(ab)}.$$

At first sight the function $\tilde{\sigma}(a, t) = \lambda(a_t)/\lambda(a)$ is a borel function on $\tilde{G} \times \mathbb{R}$. However, if $k \in \tilde{G}$ lies in the kernel of the projection onto G, we note that k is t-invariant and that $\sigma(a, k) = 1 = \sigma(k, a)$ for all $a \in \tilde{G}$ by construction. Thus

$$1 = \frac{\sigma(a_t, k_t)}{\sigma(a, k)} = \frac{\lambda(a_t)}{\lambda(a)} \Big/ \frac{\lambda(a_t k)}{\lambda(ak)}.$$

In other words, $\tilde{\sigma}(a_t k, t) = \dfrac{\lambda(a_t k)}{\lambda(ak)} = \dfrac{\lambda(a_t)}{\lambda(a)} = \tilde{\sigma}(a_t, t)$ and $\tilde{\sigma}(a_t, t)$ is lifted from a function on $G \times \mathbb{R}$. Since

$$\tilde{\sigma}(a, t)\,\tilde{\sigma}(a_t, s) = \tilde{\sigma}(a, t+s)$$

by a trivial calculation, the desired result now follows by using [23] Theorem 9.4 on the general form of multipliers on $\mathbb{R}$ Ⓢ G.

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