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Integrable dynamical systems obtained by duplication

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ABSTRACT. – New integrable dynamical systems are generated from known ones, by using an argument of symmetry; and their properties are exhibited.

RÉSUMÉ. – Nous construisons des nouveaux systèmes dynamiques intégrables à partir de cas déjà connus en utilisant un argument de symétrie. Nous décrivons aussi leurs propriétés.

1. INTRODUCTION

Almost twenty years ago it was pointed out, in the context of nonrelativistic quantum mechanics, that the one-dimensional N-body problem

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with interparticle inverse-cube forces, and possibly in addition an external (or, equivalently, interparticle) harmonic interaction, is solvable [1]. Such a system is characterized by the hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + \lambda^2 q_j^2) + \frac{1}{2}g^2 \sum_{j=1}^{N}\sum_{k=1, k\neq j}^{N} v(q_j - q_k), \qquad (1.1)$$

with

$$v(x) = x^{-2}. \qquad (1.2a)$$

Subsequently J. Moser has shown that this system (with $\lambda = 0$), in the context of classical (rather than quantum) mechanics, is completely integrable [2]; and soon thereafter it has been shown [3] that this property is also possessed by the hamiltonian (1.1), with $\lambda = 0$, in the more general case

$$v(x) = \mathcal{P}(ax), \qquad (1.2b)$$

where $\mathcal{P}$ denotes the Wierstrass' elliptic function [of which (1.2a), as well as

$$v(x) = [\sin(ax)]^{-2}, \qquad (1.2c)$$
$$v(x) = [\sinh(ax)]^{-2}, \qquad (1.2d)$$

are special cases]. These findings have been followed by various extensions and applications (including the proof that the dynamical system characterized by the hamiltonian (1.1) with (1.2a) and $\lambda \neq 0$ is also integrable); they have opened an entire research field, that has been reviewed about ten years ago by M. A. Olshanetsky and A. M. Perelomov [4], and more recently by A. M. Perelomov [5]. Here we limit ourselves to recall that the system characterized by the hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + \lambda^2 q_j^2) + \sum_{j=1}^{N}[\alpha^2 v(q_j) + \beta^2 v(2q_j)] + \frac{1}{2}g^2 \sum_{j=1}^{N}\sum_{k=1, k\neq j}^{N}[v(q_j - q_k) + v(q_j + q_k)], \qquad (1.3)$$

has also been shown to be completely integrable, with $v(x)$ taking any one of the determinations (1.2a, b, c, d) if $\lambda = 0$, or the determination (1.2a) if $\lambda \neq 0$; and α, β arbitrary constants. The integrability of the dynamical system characterized by this hamiltonian originates from the possibility to replace the original many-body hamiltonian (1.1), whose potential energy part contains a sum over the interparticle distances $q_j - q_k$, by a (more general if less physical) analogous hamiltonian, whose potential energy part consists of an analogous sum, but extending instead over the root systems associated with semisimple Lie algebras [4]. It is, however, well known that analogous (albeit slightly less general) results can also be obtained directly from the many-body hamiltonian (1.1) via a process of "duplication" based on an argument of symmetry, as reviewed in the following Section.

The process of "duplication" had been hitherto confined to the real axis (*see*, however, [6], [7]). The main contribution of this paper is to point out that, by extending the "duplication" process to the complex plane, new (real) integrable systems can be obtained. The most interesting instance of such systems, on which our presentation will be mainly focussed, is characterized by the following (real) equations of motions

$$\left.\begin{aligned}&\ddot{x}_j+\lambda^2 x_j-\gamma^2 x_j^{-3}=g^2\sum_{k=1,\,k\neq j}^{m} f_e(x_j,x_k)+g^2\sum_{k=1}^{n} f_d(x_j,\xi_k),\\ &\qquad j=1,2,\ldots,m,\end{aligned}\right\}\qquad(1.4a)$$

$$\left.\begin{aligned}&\ddot{\xi}_j+\lambda^2 \xi_j-\gamma^2 \xi_j^{-3}=g^2\sum_{k=1,\,k\neq j}^{n} f_e(\xi_j,\xi_k)+g^2\sum_{k=1}^{m} f_d(\xi_j,x_k),\\ &\qquad j=1,2,\ldots,n,\end{aligned}\right\}\qquad(1.4b)$$

with λ and γ arbitrary (real) constants and

$$f_e(a,b)=2a(a^2+3b^2)(a^2-b^2)^{-3},\qquad(1.4c)$$

$$f_d(a,b)=2a(a^2-3b^2)(a^2+b^2)^{-3}=f_e(a,ib).\qquad(1.4d)$$

Clearly this dynamical system can be interpreted as describing the classical evolution of $m+n$ nonrelativistic one-dimensional unit-mass particles, m of one kind and n of another, with the force f_e acting between *equal* particles and the force f_d between *different* particles. Note that these forces depend on the coordinates of the two interacting particles not merely via their difference.

It is easily seen that these equations are associated, in the standard manner, with the hamiltonian

$$\begin{aligned}\mathrm{H}(\mathbf{x},\mathbf{y};\xi,\eta)=&\frac{1}{2}\sum_{j=1}^{m}(y_j^2+\lambda^2x_j^2+\gamma^2x_j^{-2})\\ &-\frac{1}{2}\sum_{j=1}^{n}(\eta_j^2+\lambda^2\xi_j^2+\gamma^2\xi_j^{-2})\\ &+g^2\sum_{j=1}^{m}\sum_{k=1,\,k\neq j}^{m}\mathrm{V}_e(x_j,x_k)-g^2\sum_{j=1}^{n}\sum_{k=1,\,k\neq j}^{n}\mathrm{V}_e(\xi_j,\xi_k)\\ &+g^2\sum_{j=1}^{m}\sum_{k=1}^{n}\mathrm{V}_d(x_j,\xi_k),\end{aligned}\qquad(1.4e)$$

where

$$\mathrm{V}_e(a,b)=(a^2+b^2)(a^2-b^2)^{-2},\qquad(1.4f)$$

$$\mathrm{V}_d(a,b)=(a^2-b^2)(a^2+b^2)^{-2}=\mathrm{V}_e(a,ib),\qquad(1.4g)$$

so that

$$f_{e,d}(a,b) = -\partial V_{e,d}(a,b)/\partial a. \tag{1.4h}$$

Note the negative sign in front of the "kinetic energy term", as well as the "potential energy term", for the particles of second kind, in the hamiltonian (1.4 *e*). Of course a special case is that in which only one type of particles is present (say, $n=0$).

Below we show that this dynamical system is completely integrable, and moreover that, if $\lambda \neq 0$, all its orbits are periodic with the same period $2\pi/\lambda$, independently of the values of the (real) constants g and γ, and of course of the initial conditions; while if $\lambda = 0$ (in which case the motion is unbounded, with the particles incoming from infinity in the remote past and escaping to infinity in the remote future), the interaction gives rise to a scattering process whose final outcome is merely a reversal of the asymptotic velocities of each particle, so that particles incoming from the left (right) in the remote past return to the left (right) in the remote future, with the same asymptotic velocity (in modulus-opposite in sign) in the remote future that each of them had in the remote past.

Another interesting dynamical system, which we also show below to be integrable and to behave in an analogous fashion to the system described above, is characterized by the following equations of motion:

$$\left.\begin{aligned} &\ddot{x}_j + \lambda^2 x_j - \alpha^2 x_j^{-3} = g^2 \sum_{k=1,\, k\neq j}^{m} h_e(x_j, x_k) + g^2 \sum_{k=1}^{n} h_d(x_j, \xi_k), \\ &\qquad j = 1, 2, \ldots, m, \end{aligned}\right\} \tag{1.5a}$$

$$\left.\begin{aligned} &\ddot{\xi}_j + \lambda^2 \xi_j + \alpha^2 \xi_j^{-3} = -g^2 \sum_{k=1,\, k\neq j}^{n} h_e(\xi_j, \xi_k) - g^2 \sum_{k=1}^{m} h_d(\xi_j, x_k), \\ &\qquad j = 1, 2, \ldots, n, \end{aligned}\right\} \tag{1.5b}$$

with λ and α arbitrary (real) constants and

$$h_e(a,b) = 8\,a\,(a^8 + 12\,a^4 b^4 + 3\,b^8)(a^4 - b^4)^{-3}, \tag{1.5c}$$

$$h_d(a,b) = 8\,a\,(a^8 - 12\,a^4 b^4 + 3\,b^8)(a^4 + b^4)^{-3} = h_e(a, b e^{i\pi/4}). \tag{1.5d}$$

Note the differences in sign among (1.5 *a*) and (1.5 *b*). Of course in this case as well one may restrict attention to systems with only one kind of particles, namely set $m=0$ or $n=0$.

As mentioned above, these results are obtained via a process of generation of new integrable systems from known ones, that we have called "duplication". This process (which may have various twists; see below) can be used in several other situations as well. We treat below those presented above as examples to illustrate the general method.

Let us end this Section by pointing out that both systems, (1.4) and (1.5), described above, have the property to be invariant under the transformation that changes the sign of any coordinate (leaving the others unchanged). Thus, without loss of generality, one can actually restrict the study of these systems to the case when all the coordinates are positive, *i.e.* all the particles are on the positive real axis ($x_j>0$, $\xi_j>0$).

2. THE DUPLICATION PROCESS

Let us start from the (integrable) dynamical system characterized by the hamiltonian (1.1), namely by the equations of motion

$$\ddot{q}_j+\lambda^2 q_j=-g^2 \sum_{k=1,\, k\neq j}^{N} v'(q_j-q_k), \qquad j=1,2,\ldots,N, \tag{2.1}$$

with $v(x)$ given by (1.2 *a*) [actually, for $\lambda=0$, the dynamical system (2.1) is integrable for any one of the 4 determination (1.2 *a, b, c, d*) of $v(x)$; in the following we will concentrate on the determination (1.2 *a*), that yields more interesting results].

There is a well-known process that allows to obtain a new integrable system from this. Consider the case with an even number of particles, $N=2n$, and note that an initial configuration that is symmetrical around the origin remains symmetrical under the flow (2.1) [since $v(x)$ is an even function], so that one can set, compatibly with (2.1),

$$q_j(t)=x_j(t),\, q_{j+n}(t)=-x_j(t), \qquad j=1,2,\ldots,n, \tag{2.2a}$$

$$\left.\begin{aligned}\dot{q}_j(t)=p_j(t)=y_j(t)=\dot{x}_j(t),\, \dot{q}_{j+n}(t)=p_{j+n}(t)=-y_j(t)=-\dot{x}_j(t),\\ j=1,2,\ldots,n.\end{aligned}\right\} \tag{2.2b}$$

There thus result for $x_j(t)$ the equations of motion

$$\left.\begin{aligned}\ddot{x}_j+\lambda^2 x_j+g^2 v'(2x_j)=-g^2 \sum_{k=1,\, k\neq j}^{n} [v'(x_j-x_k)+v'(x_j+x_k)],\\ j=1,2,\ldots,n,\end{aligned}\right\} \tag{2.3a}$$

that may indeed be obtained in the standard manner from the hamiltonian

$$\begin{aligned}H(\mathbf{x},\mathbf{y})=\frac{1}{2}\sum_{j=1}^{n} [y_j^2+\lambda^2 x_j^2+g^2 v'(2x_j)]\\ +\frac{1}{2}g^2 \sum_{j=1}^{n}\ \sum_{k=1,\, k\neq j}^{n} [v(x_j-x_k)+v(x_j+x_k)].\end{aligned} \tag{2.3b}$$

Let us moreover note that if, in the duplication process, we add μ (static) particles at the origin, we get the equations of motion

$$\ddot{x}_j+\lambda^2 x_j+\mu g^2 v'(x_j)+g^2 v'(2x_j)$$
$$=-g^2\sum_{k=1,\,k\neq j}^{n}[v'(x_j-x_k)+v'(x_j+x_k)],\qquad j=1,2,\ldots,n,\qquad(2.4a)$$

corresponding to the hamiltonian

$$\mathrm{H}(\mathbf{x},\mathbf{y})=\frac{1}{2}\sum_{j=1}^{n}[y_j^2+\lambda^2 x_j^2+2\mu g^2 v(x_j)+g^2 v(2x_j)]$$
$$+\frac{1}{2}g^2\sum_{j=1}^{n}\sum_{k=1,\,k\neq j}^{n}[v(x_j-x_k)+v(x_j+x_k)].\qquad(2.4b)$$

But this is not a new system, being clearly the special case of (1.3) corresponding to $\alpha^2=\mu g^2$ and $\beta^2=\frac{1}{2}g^2$.

What seems less known (*see*, however, [6], [7]) is the possibility to generate a new model by putting *two* configurations of particles, one on the *real* axis and the other on the *imaginary* axis. One has moreover the option to add some (static) particles at the origin.

In this manner, as we presently show, one can manufacture an integrable dynamical system featuring two different types of "particles". Let us emphasize that this is a different kind of trick from that which generates two types of particles by an appropriate shift of their coordinates, as explained in [3], and studied in [8] in the case $v(x)=[\sinh(ax)]^{-2}$.

So we start again from the system (2.1), but now with $2m$ particles in a configuration symmetrical around the origin on the *real* axis, $2n$ particles in a configuration symmetrcial around the origin on the *imaginary* axis, and μ static particles at the origin; that is, we set $\mathrm{N}=2(m+n)+\mu$ and

$$q_j(t)=x_j(t),\; q_{j+m+n}(t)=-x_j(t),\qquad j=1,2,\ldots,m,\qquad(2.5a)$$

$$\left.\begin{aligned}\dot{q}_j(t)&=p_j(t)=y_j(t)=\dot{x}_j(t),\\ \dot{q}_{j+m+n}(t)&=p_{j+m+n}(t)=-y_j(t)=-\dot{x}_j(t),\qquad j=1,2,\ldots,m,\end{aligned}\right\}\qquad(2.5b)$$

$$q_{j+m}(t)=i\xi_j(t),\; q_{j+2m+n}(t)=-i\xi_j(t),\qquad j=1,2,\ldots,n,\qquad(2.5c)$$

$$\left.\begin{aligned}\dot{q}_{j+m}(t)&=p_{j+m}(t)=i\eta_j(t)=i\dot{\xi}_j(t),\\ \dot{q}_{j+2m+n}(t)&=p_{j+2m+n}(t)=-i\eta_j(t)=-i\dot{\xi}_j(t),\\ j&=1,2,\ldots,n,\end{aligned}\right\}\qquad(2.5d)$$

$$q_{j+2m+2n}(t)=0,\qquad j=1,2,\ldots,\mu.\qquad(2.5e)$$

It is clear that these positions are compatible with (2.1), and they yield for the m "particles of the first type", of (real) coordinates x_j and (real) momenta (or, equivalently, velocities) y_j, the equations of motion

$$\ddot{x}_j+\lambda^2 x_j+\mu g^2 v'(x_j)+g^2 v'(2x_j) = -g^2 \sum_{k=1, k\neq j}^{n} [v'(x_j-x_k)+v'(x_j+x_k)] - g^2 \sum_{k=1}^{n} [v'(x_j-i\xi_k)+v'(x_j+i\xi_k)], \quad j=1,2,\ldots,m, \tag{2.6a}$$

and for the n "particles of the second type", of (real) coordinates ξ_j and (real) momenta (or velocities) η_j, the equations of motion

$$\ddot{\xi}_j+\lambda^2 \xi_j-i\mu g^2 v'(i\xi_j)-ig^2 v'(2i\xi_j) = ig^2 \sum_{k=1, k\neq j}^{n} [v'(i(\xi_j-\xi_k))+v'(i(\xi_j+\xi_k))] + ig^2 \sum_{k=1}^{m} [v'(i\xi_j-x_k)+v'(i\xi_j+x_k)], \quad j=1,2,\ldots,n. \tag{2.6b}$$

It is easily seen that these two systems of coupled second-order ODEs can be obtained from the following (real) Hamiltonian:

$$\begin{aligned} \mathrm{H}(\mathbf{x},\mathbf{y};\xi,\eta) &= \frac{1}{2}\sum_{j=1}^{m}(y_j^2+\lambda^2 x_j^2)-\frac{1}{2}\sum_{j=1}^{n}(\eta_j^2+\lambda^2\xi_j^2) \\ &+\frac{1}{2}g^2\left\{\sum_{j=1}^{m}[2\mu v(x_j)+v(2x_j]+\sum_{j=1}^{n}[2\mu v(i\xi_j)+v(2i\xi_j)]\right\} \\ &+\frac{1}{2}g^2\sum_{j=1}^{m}\sum_{k=1, k\neq j}^{m}[v(x_j-x_k)+v(x_j+x_k)] \\ &+\frac{1}{2}g^2\sum_{j=1}^{n}\sum_{k=1, k\neq j}^{n}[v(i(\xi_j-\xi_k))+v(i(\xi_j+\xi_k))] \\ &+g^2\sum_{j=1}^{m}\sum_{k=1}^{n}[v(x_j-i\xi_k)+v(x_j+i\xi_k)]. \end{aligned} \tag{2.6c}$$

These results imply of course that the Hamiltonian system (2.6), that as we have seen can be considered to describe the motion of two groups of particles of two different types, is completely integrable if $v(x)$ is given by any one of the determinations (1.2a, b, c, d) and $\lambda=0$, or if $v(x)$ is given by (1.2a) and λ is an arbitrary (real) constant. In particular, in the latter case, to which our treatment is actually limited (this restriction being instrumental to guarantee the reality of the coordinates x_j, ξ_j), this system coincides with (1.4), with

$$\gamma^2=\left(2\mu+\frac{1}{4}\right)g^2. \tag{2.7}$$

Note however that, while this derivation would seem to suggest that the system (1.4) is integrable only for the special values of γ^2 given by this formula with μ a nonnegative integer, in fact (1.4) is completely integrable for any arbitrary value of the constant γ^2 (as implied by the results of Section 4 below). In our discussion below we will however, for simplicity, restrict attention to the case when the constant γ^2 is positive.

3. SYSTEMS OF "MOLECULES" AND ITERATED DUPLICATIONS

In the case when $v(x)$ is given by (1.2*a*), it is easily seen that the equations of motion (2.6*a*, *b*) for $x_j(t)$ and $\xi_j(t)$ become identical, so that it is consistent to set $m=n$ and, as a special configuration (compatible with the motion),

$$x_j(t)=\xi_j(t), \qquad \dot{x}_j(t)=y_j(t)=\eta_j(t)=\dot{\xi}_j(t), \qquad j=1,2,\ldots,n. \quad (3.1)$$

Note that, in this configuration, the hamiltonian (2.6*c*) becomes a (vanishing) constant. On the other hand the equations of motion (2.6*a*, *b*) remain valid, and they read as follows:

$$\ddot{x}_j+\lambda^2 x_j-\alpha^2 x_j^{-3}=g^2 \sum_{k=1,\, k\neq j}^{m} h(x_j, x_k), \qquad j=1,2,\ldots,m, \quad (3.2a)$$

with

$$\alpha^2=g^2\left(2\mu+\frac{1}{4}\right) \quad (3.2b)$$

and

$$h(a,b)=8\,a\,(a^8+12\,a^4 b^4+3\,b^8)\,(a^4-b^4)^{-3}. \quad (3.2c)$$

This m-body system is therefore another example of integrable problem; we refer to it, in the title of this section, as a system of "molecules", since each particle may be considered as made up of two tightly bound different particles of the previous system, whose coordinates actually coincide [*see* (3.1)].

Let us note that, as in the case discussed in the preceding Section, while (3.2*b*) seems to imply a limitation on the permitted range of values for α^2 (arising from the condition that μ be a nonnegative integer), the treatment given below implies that the complete integrability of the models discussed in this Section actually holds for any arbitrary value of α^2 (although for simplicity we will in the following generally assume α^2 to be positive).

As already mentioned in the Introduction, without loss of generality we can hereafter assume that all the coordinates x_j are positive (the equations are clearly invariant under the transformation $x_j \to -x_j$, even if performed only for the j-th coordinate; indeed, they are also invariant under $x_j \to \pm i x_j$). In any case, as implied by (3.2 *a*) [and the positivity of α^2; *see* (3.2 *b*)], the dynamics prevents $x_j(t)$ from changing sign throughout its time evolution (the singular repulsive force $\alpha^2 x^{-3}$ keeps the particles away from the origin); moreover the singular repulsive pair force (3.2 *c*) prevents the particles from crossing each other.

Clearly the system (3.2) implies the conditions $x_j \neq \pm x_k$ (as well as $x_j \neq \pm i x_k$); the possibility to "duplicate" this system in the manner of the previous Section is thereby excluded. It is however possible to perform a different kind of "duplication", by replacing m with $m+n$ and then setting

$$x_j(t) = x_j(t), \qquad j = 1, 2, \ldots, m, \tag{3.3 a}$$

$$x_{j+m}(t) = \exp(i\pi/4)\,\xi_j(t), \qquad j = 1, 2, \ldots, n, \tag{3.3 b}$$

of course with x_j, for $j = 1, 2, \ldots, m$, as well as $\xi_j(t)$, for $j = 1, 2, \ldots, n$, being *real* coordinates. Then in place of the system (3.2) one gets the more general system characterized by the following equations of motion:

$$\ddot{x}_j + \lambda^2 x_j - \alpha^2 x_j^{-3} = g^2 \sum_{k=1,\, k\neq j}^{m} h_e(x_j, x_k) + g^2 \sum_{k=1}^{n} h_d(x_j, \xi_k), \tag{3.4 a}$$

$$\ddot{\xi}_1 + \lambda^2 \xi_j + \alpha^2 \xi_j^{-3} = -g^2 \sum_{k=1,\, k\neq j}^{n} h_e(\xi_j, \xi_k) - g^2 \sum_{k=1}^{m} h_d(\xi_j, x_k), \tag{3.4 b}$$

$$h_e(a, b) = h(a, b) = 8\,a\,(a^8 + 12\,a^4 b^4 + 3\,b^8)\,(a^4 - b^4)^{-3}, \tag{3.4 c}$$

$$h_d(a, b) = h(a, b\exp(i\pi/4)) = 8\,a\,(a^8 - 12\,a^4 b^4 + 3\,b^8)\,(a^4 + b^4)^{-3}. \tag{3.4 d}$$

This can be described as a system of $m+n$ one-dimensional unit-mass classical particles, m of one type and n of another; the previous system is of course the special case of this corresponding to $n=0$; and another (different) system, involving again only one type of particles, is obtained by setting instead $m=0$. But note that, while as pointed out above the particles of first type are repelled by the origin and also repel each other, the particles of second type are instead attracted to the origin (with a force that becomes infinite at the origin), as well as pairwise among themselves (again, with a force that becomes infinite at zero distance). Hence, if particles of the second type are present, the system, in spite of its completely integrable character, may give rise to a singular behaviour (collapse) at a finite time. Note moreover that the different behaviour of the two types of particles exclude the possibility to generate yet another dynamical system by the trick of putting together molecules, as described above. Indeed it appears that the models described thus far exhaust the range of possible "integrable many-body (classical nonrelativistic) models on the line" that can be manufactured by this kind of tricks (as mentioned above, one may also try to apply such tricks with the more general functions (1.2 *b, c, d*), rather than (1.2 *a*); but the resulting results are not sufficiently interesting to warrant reporting here).

4. EXISTENCE OF LAX PAIRS FOR THE FLOWS

The systems described in the two previous Sections (and mentioned in the Introduction) are of course completely integrable, as implied by the way they have been obtained. In this Section we demonstrate this by explicitly providing the corresponding Lax representation, as well as their solution. However, for simplicity, we limit such an explicit treatment to the system (2.6) with (1.2 *a*) and (2.7) [or, equivalently, (1.4)].

We start with the $N \times N$ matrices (with $N = 2(m+n)$) associated [9] with the system (1.3) with (1.2 *a*), that read

$$\mathbf{L} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B} & -\mathbf{A} \end{pmatrix}, \tag{4.1 a}$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} \end{pmatrix}, \tag{4.1 b}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{F} & \mathbf{E} \end{pmatrix}, \tag{4.1 c}$$

with the $(N/2) \times (N/2)$ matrices **A, B, D, E, F** defined as follows:

$$\mathrm{A}_{jk} = \delta_{jk}\dot{q}_j + i(1-\delta_{jk})\,g/(q_j - q_k), \tag{4.2 a}$$

$$\mathrm{B}_{jk} = 2i\,\delta_{jk}\,\gamma/q_j + i(1-\delta_{jk})\,g/(q_j + q_k), \tag{4.2 b}$$

$$\mathrm{D}_{jk} = \delta_{jk}\,q_j, \tag{4.2 c}$$

$$\mathrm{E}_{jk} = i\,\delta_{jk}\left\{2^{-1/2}\gamma/q_j^2 + g\sum_{k=1,\,k\neq j}^{N/2}[(q_j+q_k)^{-2} + (q_j-q_k)^{-2}]\right\} - i(1-\delta_{jk})\,g/(q_j-q_k)^2, \tag{4.2 d}$$

$$\mathrm{F}_{jk} = 2^{-1/2}\,i\,\delta_{jk}\,\gamma/q_j^2 - i(1-\delta_{jk})\,g/(q_j+q_k)^2. \tag{4.2 e}$$

Here the indices j and k run from 1 to $m+n=N/2$, and of course $\delta_{jk}=1$ if $j=k$, $\delta_{jk}=0$ if $j\neq k$, while

$$\gamma^2 = \alpha^2 + \frac{1}{4}\beta^2. \tag{4.2 f}$$

The equations of motion corresponding to the integrable hamiltonian (1.3) can be recast, using these matrices, in the following (generalized) Lax form:

$$\dot{\mathbf{X}} = [\mathbf{X}, \mathbf{M}] + \mathbf{L}, \tag{4.3 a}$$

$$\dot{\mathbf{L}} = [\mathbf{L}, \mathbf{M}] - \lambda^2\,\mathbf{X}. \tag{4.3 b}$$

To get the equations of the system (1.4) [or, equivalently, (2.6) with (1.2 *a*)], it is now sufficient to perform the substitutions (2.5 *a*, *c*).

Note that the hamiltonian (1.4 *e*) is given, via (2.5 *b*) and (2.5 *d*), by the formula

$$H(\mathbf{x}, \mathbf{y}; \xi, \eta) = \frac{1}{2} \operatorname{tr} (\lambda^2 \mathbf{X}^2 + \mathbf{L}^2). \tag{4.4}$$

Let us recall that the generalized Lax equations (4.3 *a*) can be recast in the standard Lax form by the following trick. Let us introduce the two $N \times N$ matrices

$$\mathbf{Z} = \mathbf{L} + i\lambda \mathbf{X}, \tag{4.5 a}$$

$$\tilde{\mathbf{Z}} = \mathbf{L} - i\lambda \mathbf{X}. \tag{4.5 b}$$

It is then easily seen that (4.3) yield

$$\dot{\mathbf{Z}} = [\mathbf{Z}, \mathbf{M}] + i\lambda \mathbf{Z}, \tag{4.6 a}$$

$$\dot{\tilde{\mathbf{Z}}} = [\tilde{\mathbf{Z}}, \mathbf{M}] - i\lambda \tilde{\mathbf{Z}}, \tag{4.6 b}$$

so that, by setting

$$\mathbf{W} = \mathbf{Z}\tilde{\mathbf{Z}}, \tag{4.7}$$

there obtains for $\mathbf{W}$ the standard Lax equation

$$\dot{\mathbf{W}} = [\mathbf{W}, \mathbf{M}]. \tag{4.8}$$

Of course a standard Lax equation is also satisfied by the matrix

$$\mathbf{Z}' = \exp(i\lambda t)\, \mathbf{Z}, \tag{4.9 a}$$

since clearly (4.6 *a*) implies

$$\dot{\mathbf{Z}}' = [\mathbf{Z}', \mathbf{M}]. \tag{4.9 b}$$

The Lax representations for the system (1.5) can be obtained in analogous manner, by using appropriate specializations of the variables, as suggested by the treatment of Section 3; of course the corresponding matrices will be of order $N \times N$ with $N = 4(m+n)$.

5. EXPLICIT SOLUTION IN THE $\lambda \neq 0$ CASE: PERIODICITY OF THE ORBITS

We now indicate how the system (1.4) can be explicitly solved, via the Lax representation given in the preceding Section; and we thereby prove the following

THEOREM 5.1. – *All the orbits of the hamiltonian system* (1.4) *are periodic, with period* $2\pi/\lambda$ (*independently of* g *and* γ).

This result was indeed expected, since it holds for the original system (1.1) with (1.2a), of which the models treated in this paper are after all merely special cases.

Of course in this Section the real constant λ is assumed not to vanish,

$$\lambda \neq 0. \tag{5.1}$$

Let us introduce the N $\times$ N matrix $\mathbf{U}(t)$ via the (matrix) Cauchy problem

$$\dot{\mathbf{U}}(t) = \mathbf{U}(t)\mathbf{M}(t), \tag{5.2a}$$

$$\mathbf{U}(0) = 1, \tag{5.2b}$$

where the N $\times$ N matrix $\mathbf{M}(t)$ is that defined in Section 4 [and of course $\mathrm{N} = 2(m+n)$].

We then set

$$\tilde{\mathbf{X}} = \mathbf{UXU}^{-1}, \tag{5.3a}$$

$$\tilde{\mathbf{L}} = \mathbf{ULU}^{-1}, \tag{5.3b}$$

where the matrices $\mathbf{X}$ and $\mathbf{L}$ are again those defined in Section 4.

It is then easily seen that (5.2a) and (4.3a) yield

$$\dot{\tilde{\mathbf{X}}} = \mathbf{ULU}^{-1}, \tag{5.4}$$

and then (5.2a), (4.3b) and (5.3a) yield

$$\ddot{\tilde{\mathbf{X}}} + \lambda^2 \tilde{\mathbf{X}} = 0. \tag{5.5a}$$

This equation can be explicitly integrated:

$$\tilde{\mathbf{X}}(t) = \tilde{\mathbf{X}}(0)\cos(\lambda t) + \dot{\tilde{\mathbf{X}}}(0)[\sin(\lambda t)]/\lambda, \tag{5.5b}$$

or, equivalently [*see* (5.2b), (5.3a) and (5.4)],

$$\tilde{\mathbf{X}}(t) = \mathbf{X}(0)\cos(\lambda t) + \mathbf{L}(0)[\sin(\lambda t)]/\lambda. \tag{5.5c}$$

This formula provides an explicit expression of the matrix $\tilde{\mathbf{X}}(t)$ in terms of the initial positions and velocities of the particles [*see* (4.1a, b), (4.2a, b, c) and (2.5$a, b, c\,d$)]; on the other hand the positions $x_j(t«$ and $\xi_j(t)$ of the particles at time t are just the eigenvalues of the matrix $\tilde{\mathbf{X}}(t)$ [*see* (5.3a), (4.1b), (4.2c) and (2.5a, c)].

It is thus seen that the solution of the problem is merely reduced to the computation of the (real and imaginary) eigenvalues of the matrix $\tilde{\mathbf{X}}(t)$, given, in terms of the initial data, by the explicit formula (5.5c).

Since the matrix $\tilde{\mathbf{X}}(t)$ is clearly periodic in time with period $2\pi/\lambda$ [*see* (5.5c)], the set of its eigenvalues is also periodic in time, with the same period; and this entails the periodicity of each one of the $x_j(t)$'s resp. $\xi_j(t)$'s (the real resp. imaginary eigenvalues), since their ordering on the (positive) real resp. imaginary axes cannot change throughout the motion (since the particles of each type cannot go through each other, as explained above).

It is left to the diligent reader to spell out the analogous technique of solution for the dynamical system (1.5), as well as to prove that Theorem 5.1 holds in this case as well (provided there is no collapse). The only difference from the treatment given just above is that the relevant matrices have now $4(m+n)$ [rather than $2(m+n)$] rows and columns, and in the appropriate identification of the quantities $q_j(t)$ and $\dot{q}_j(t)$, for $j=1,2,\ldots,2(m+n)$.

Note incidentally that the fact that the structure of the set of eigenvalues of the matrix $\tilde{\mathbf{X}}(t)$ is preserved over time [for instance, the fact that, if $x_j(t)$ is an eigenvalue, also $-x_j(t)$ is] may be formulated as the following nontrivial, purely mathematical

REMARK. – *Let* $\mathbf{Z}$ *be any matrix, of rank* $\mathrm{N}=2n$, $\mathrm{N}=4n$ *or* $\mathrm{N}=8n$, *whose characteristic polynomial,*

$$p_{\mathrm{N}}(z)=\det[\mathbf{Z}-z\mathbf{1}] \tag{5.6}$$

has one of the following (special) forms:

$$p_{2n}(z)=\prod_{j=1}^{n}(z^2-z_j^2), \tag{5.7a}$$

$$p_{4n}(z)=\prod_{j=1}^{n}[(z^2-x_j^2)(z^2+y_j^2)], \tag{5.7b}$$

$$p_{4n}(z)=\prod_{j=1}^{n}(z^4-z_j^4), \tag{5.7c}$$

$$p_{8n}(z)=\prod_{j=1}^{n}[(z^4-x_j^4)(z^4+y_j^4)] \tag{5.7d}$$

where the quantities z_j *or* x_j, y_j *are real. Then there exists a (nonvanishing) matrix* $\mathbf{Q}$ *(independent of* t *and* λ*) such that the two-parameter set of matrices*

$$\mathbf{Z}(t,\lambda)=\mathbf{Z}\cos(\lambda t)+\mathbf{Q}[\sin(\lambda t)]/\lambda \tag{5.8}$$

all have characteristic polynomials with the same special structure (5.7) *(although with different eigenvalues, of course), for any arbitrary values of* λ *and* t.

For instance, as implied by the results above, for $\mathrm{N}=2n$

$$\mathbf{Z}=\mathbf{U}\,\mathrm{diag}(z_j,-z_j)\,\mathbf{U}^{-1}, \tag{5.9a}$$

$$\mathbf{Q}=\mathbf{U}\mathbf{R}\mathbf{U}^{-1}, \tag{5.9b}$$

with $\mathbf{U}$ any arbitrary invertible matrix of rank N and

$$\mathbf{R}=\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ -\mathbf{B} & -\mathbf{A}\end{pmatrix}, \tag{5.9c}$$

where the matrices **A** and **B**, of rank $n = N/2$, have the form

$$\mathrm{A}_{jk} = \delta_{jk} a_j + i(1-\delta_{ij}) b/(z_j - z_k), \qquad (5.9\,d)$$
$$\mathrm{B}_{jk} = \delta_{jk} c/z_j + i(1-\delta_{jk}) b/(z_j + z_k), \qquad (5.9\,e)$$

wth b, c and the n parameters a_j real but otherwise arbitrary.

It is left for the diligent reader to exhibit explicit examples in the other cases.

6. EXPLICIT SOLUTION IN THE $\lambda = 0$ CASE: OUTCOME OF THE SCATTERING PROCESS

The treatment given in the preceding Section remains applicable in the $\lambda = 0$ case, with obvious modifications: for instance (5.5 c), for $\lambda = 0$, reads

$$\tilde{\mathbf{X}}(t) = \mathbf{X}(0) + t\,\mathbf{L}(0). \qquad (6.1)$$

In this case the motion is, of course, unbounded; and, for both systems (1.4) and (1.5) (but, in the latter case, only if there occurs no collapse), there holds the following

THEOREM 6.1:

$$x_j(t) \xrightarrow[t \to \mp\infty]{} y_j(\mp\infty)\,t + x_j^{(\mp)} + O(t^{-1}), \qquad (6.2\,a)$$
$$\xi_j(t) \xrightarrow[t \to \mp\infty]{} \eta_j(\mp\infty)\,t + \xi_j^{(\mp)} + O(t^{-1}), \qquad (6.2\,b)$$
$$y_j(+\infty) = -y_j(-\infty), \qquad (6.3\,a)$$
$$\eta_j(+\infty) = -\eta_j(-\infty), \qquad (6.3\,b)$$
$$x_j^{(+)} = -x_j^{(-)}, \qquad (6.4\,a)$$
$$\xi_j^{(+)} = -\xi_j^{(-)}. \qquad (6.4\,b)$$

We omit an explicit proof of this result, since it is a straightforward consequence, via the "duplication" idea, of the analogous result for the prototype model; a result that was first proven, in the quantal 3-body case, by C. Marchioro [10], and was then extended to the quantal n-body case in [1], and to the classical case in [2].

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