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# Symmetries and constants of the motion for dynamics in implicit form

by

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ABSTRACT. — Variational principles applied to singular Lagrangians give rise to equations of motion in implicit form.

In the present paper we analyze the concepts of symmetries and constants of the motion for such differential equations.

RÉSUMÉ. — Les principes variationaux appliqués à des lagrangiennes singulières donnent lieu à équations du mouvement en forme implicite.

Dans le présent article nous analysons les concepts des symétries et constantes du mouvement pour ces équations différentielles.

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## INTRODUCTION

Many dynamical systems in Physics are described by singular Lagrangians in the sense of Dirac and Bergmann ([12], [13], [2], [7], [22]). This class includes all dynamical systems associated with gauge theories. In the usual Euler-Lagrange approach these theories give rise to dynamical systems (differential equations) which are implicit, *i.e.* they are not in a normal form and, because of the singularity (degeneracy) of the Lagrangian, they cannot be put in a normal form in an intrinsic way.

In the past, one of us ([23], [26], [27], [28], [19]) introduced a geometrical formalism for dealing directly with these dynamical systems in their implicit form. As a matter of fact this formalism allows for the usual Lagrangian, Hamiltonian and Hamilton-Jacobi treatment of these systems directly in their implicit form.

In this paper we wish to extend this approach by including also symmetries and constants of the motion in this setting.

The paper is organized as follows:

In section one we establish notation and recall some definitions and properties of the geometrical structure of tangent bundles. Then we briefly recall the definition of dynamical systems in their implicit form, define symmetries and constants of the motion for these systems, and prove some useful properties.

In section two we add the symplectic structure to the carrier space in order to be able to deal with Hamiltonian dynamical systems. We consider again symmetries and constants of the motion in this context and show that the usual relation between symmetries and constants of the motion is present.

For geometric formulations of dynamics we refer the reader to recent texts ([1], [18], [16]).

## SECTION 1

### 1.1. Notation and derivations of forms

For the geometry of differential manifolds and tensor bundles we refer to standard text-books (for example [9]).

Let P be a differential manifold. We denote by TP the tangent bundle over P, equipped with the projection $\tau_P : TP \rightarrow P$. We recall that a tangent vector in TP is an equivalence class of smooth curves in P. Each curve in this class is called an integral curve of the vector. The equivalence class of a curve $\gamma : \mathbf{R} \rightarrow P$ is denoted by $t\gamma(0)$. We define the tangent prolongation

$t\gamma : \mathbf{R} \to \mathrm{TP}$ of a curve $\gamma$ by $t\gamma(s) = t\gamma_s(0)$, where $\gamma_s : \mathbf{R} \to \mathrm{P}$ is defined by $\gamma_s(r) = \gamma(s+r)$. One of the properties of the tangent prolongation is expressed by the equality $\langle t\gamma(s), df \rangle = \frac{d}{ds}(f \circ \gamma)$, where $f$ is a function on P.

Let P and R be two differential manifolds. Each differentiable mapping $\varphi : \mathrm{P} \to \mathrm{R}$ induces the tangent mapping $\mathrm{T}\varphi : \mathrm{TP} \to \mathrm{TR}$ defined by $\mathrm{T}\varphi(v) = t(\varphi \circ \gamma)(0)$, where $\gamma$ is an integral curve of $v$. As a consequence of the definition of the tangent prolongation, we have

$$\mathrm{T}\varphi(t\gamma(s)) = t(\varphi \circ \gamma)(s).$$

For more details on the geometry of tangent bundles we refer to [32], [10], [30], [20].

We use symbols $\mathscr{F}(\mathrm{P})$, $\mathscr{X}(\mathrm{P})$ and $\Lambda(\mathrm{P})$ to denote the algebra of smooth functions, the Lie algebra of vector fields and the exterior algebra of differential forms on P respectively. The space of differential $k$-forms will be denoted by $\Lambda^k(\mathrm{P})$. The space $\Lambda^0(\mathrm{P})$ is the same as $\mathscr{F}(\mathrm{P})$. The pull-back mapping induced by a differentiable mapping $\varphi : \mathrm{P} \to \mathrm{R}$ is denoted by $\varphi^* : \Lambda(\mathrm{R}) \to \Lambda(\mathrm{P})$.

The Lie algebra of vector fields can be extended to a graded Lie algebra of derivations on $\Lambda(\mathrm{P})$. Here we follow the approach of Frölicher and Nijenhuis [14]; *see* also Klein [15].

A linear mapping $a : \Lambda(\mathrm{P}) \to \Lambda(\mathrm{P})$ is called a derivation of $\Lambda(\mathrm{P})$ of degree $r$ if $a\mu \in \Lambda^{k+r}(\mathrm{P})$ and $a(\mu \wedge \nu) = a\mu \wedge \nu + (-1)^{kr}\mu \wedge a\nu$, where $k$ is the degree of $\mu$.

The commutator $[a, b] = ab - (-1)^{rs} ba$ of two derivations of degrees $r$ and $s$ respectively is a derivation of degree $r+s$.

The exterior differential $d$ is a derivation of $\Lambda(\mathrm{P})$ of degree 1.

A derivation $a$ of $\Lambda(\mathrm{P})$ is said to be of type $i_*$ if $af = 0$ for each function $f \in \Lambda^0(\mathrm{P})$. A derivation $a$ of $\Lambda(\mathrm{P})$ is said to be of type $d_*$ if $[a, d] = 0$. The exterior differential $d$ is a derivation of type $d_*$ and the commutator $[a, d]$ is a derivation of type $d_*$ for any derivation $a$. If $a$ is a derivation of type $d_*$ then there is an unique derivation $b$ of type $i_*$ such that $a = [b, d]$.

A derivation of $\Lambda(\mathrm{P})$ is characterized by its action on $\Lambda^0(\mathrm{P})$ and $d\Lambda^0(\mathrm{P}) \subset \Lambda^1(\mathrm{P})$. A derivation of type $d_*$ is characterized by its action on $\Lambda^0(\mathrm{P})$.

For each vector field $\mathrm{X} \in \mathscr{X}(\mathrm{P})$ there is a derivation $i_\mathrm{X}$ of degree $-1$ and type $i_*$ characterized by $i_\mathrm{X}\mu = \langle \mathrm{X}, \mu \rangle$ for each $\mu \in \Lambda^1(\mathrm{P})$. The derivation $d_\mathrm{X} = [i_\mathrm{X}, d]$ of degree 0 coincides with the well known Lie derivative of differential forms with respect to X, usually denoted by $\mathrm{L}_\mathrm{X}$. The equation $[d_\mathrm{X}, d_\mathrm{Y}] = d_{[\mathrm{X}, \mathrm{Y}]}$ establishes a relation between the Lie bracket $[\mathrm{X}, \mathrm{Y}]$ of vector fields X and Y and the commutator of the derivations of type $d_*$ associated with these fields.

The following extension of derivations turns out to be useful to deal with the tangent bundle structure TP on P.

Let $\varphi : \mathrm{P} \to \mathrm{R}$ be a differentiable mapping. A linear mapping $a : \Lambda(\mathrm{R}) \to \Lambda(\mathrm{P})$ is called a $\varphi$-derivation from $\Lambda(\mathrm{R})$ to $\Lambda(\mathrm{P})$ of degree $r$ if $a\mu \in \Lambda^{k+r}(\mathrm{P})$ and $a(\mu \wedge \nu) = a\mu \wedge \varphi^* \nu + (-1)^{kr} \varphi^* \mu \wedge a\nu$, where $k$ is the degree of $\mu$.

A $\varphi$-derivation from $\Lambda(\mathrm{R})$ to $\Lambda(\mathrm{P})$ is characterized by its action on $\Lambda^0(\mathrm{R})$ and $\Lambda^1(\mathrm{R})$.

Let $a$ be a $\varphi$-derivation from $\Lambda(\mathrm{R})$ to $\Lambda(\mathrm{P})$ of degree $r$ and let $b$ denote a pair $(b_\mathrm{R}, b_\mathrm{P})$ of derivations of degree $s$ of the exterior algebras $\Lambda(\mathrm{R})$ and $\Lambda(\mathrm{P})$ respectively. If the relation $\varphi^* \circ b_\mathrm{R} = b_\mathrm{P} \circ \varphi^*$ is satisfied, then the commutator $[a, b] = ab_\mathrm{R} - (-1)^{rs} b_\mathrm{P} a$ is a $\varphi$-derivation from $\Lambda(\mathrm{R})$ to $\Lambda(\mathrm{P})$ of degree $r+s$.

A $\varphi$-derivation $a$ from $\Lambda(\mathrm{R})$ to $\Lambda(\mathrm{P})$ is said to be of type $i_*$ if $af = 0$ for each function $f \in \Lambda^0(\mathrm{R})$. The derivation $a$ is said to be of type $d_*$ if $[a, d] = 0$, where $d$ stands for the pair $(d, d)$ of exterior differentials in $\Lambda(\mathrm{R})$ and $\Lambda(\mathrm{P})$.

Properties of $\varphi$-derivations are similar to the properties of ordinary derivations of differential forms. For the complete theory of $\varphi$-derivations we refer the reader to [21].

A differentiable mapping $\mathrm{U} : \mathrm{P} \to \mathrm{TR}$ such that $\tau_\mathrm{R} \circ \mathrm{U} = \varphi$ is called an infinitesimal deformation of $\varphi$. For each infinitesimal deformation U of $\varphi$ we introduce a $\varphi$-derivation $i_\mathrm{U}$ from $\Lambda(\mathrm{R})$ to $\Lambda(\mathrm{P})$ of degree $-1$ and type $i_*$ characterized by $i_\mathrm{U}\mu = \mu \circ \mathrm{U}$, where $\mu$ is a 1-form on R. We denote by $d_\mathrm{U}$ the $\varphi$-derivation $[i_\mathrm{U}, d]$.

## 1.2. Derivations on tangent bundles

Let M be a manifold and TM its tangent bundle. We introduce a derivation which will turn out to be useful.

The identity mapping of TM is obviously an infinitesimal deformation of the projection $\tau_\mathrm{M} : \mathrm{TM} \to \mathrm{M}$. We denote this deformation by T and introduce the $\tau_\mathrm{M}$-derivations $i_\mathrm{T}$ and $d_\mathrm{T}$ from $\Lambda(\mathrm{M})$ to $\Lambda(\mathrm{TM})$ (*see* [24], [25]).

We have the following proposition:

PROPOSITION 1.1. — *The kernel of the operator* $d_\mathrm{T}$ *is the space of the locally constant functions on* M.

*Proof.* — If $f$ is a function on M, then $d_\mathrm{T} f(v) = (i_\mathrm{T} df)(v) = \langle v, df \rangle$ for each vector $v \in \mathrm{TM}$. Hence, if $d_\mathrm{T} f = 0$, $f$ is locally constant. If $\mu = \mu_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is a $k$-form, with $k > 0$, then

$$d_\mathrm{T}\mu = (d_\mathrm{T}\mu_{i_1 \ldots i_k})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \mu_{i_1 \ldots i_k} \sum_j dx^{i_1} \wedge \ldots \wedge d\dot{x}^{i_j} \wedge \ldots \wedge dx^{i_k}$$

is a combination of independent $k$-forms. It follows that $d_T \mu = 0$ implies $\mu = 0$. □

*Remark.* – We denote by $\tau_M^* \mathscr{F}(M)$ the subalgebra of the algebra $\mathscr{F}(TM)$ formed by pull-backs of functions on M. We denote by $d_T \mathscr{F}(M)$ the image of the mapping $d_T : \mathscr{F}(M) \to \mathscr{F}(TM)$. We note that the derivations of $\Lambda(TM)$ of type $d_*$ are completely determined by their action on $\tau_M^* \mathscr{F}(M)$ and $d_T \mathscr{F}(M)$.

Let X be a vector field on M and Y a vector field on TM. If $T\tau_M \circ Y = X \circ \tau_M$ then Y is called a lift of X to TM and Y is said to be $\tau_M$-projectable onto X.

Let $\varphi : \mathbf{R} \times M \to M$ be a one-parameter group of diffeomorphisms of M. The tangent lift of $\varphi$ is the one-parameter group $\varphi^T : \mathbf{R} \times TM \to TM$ defined by $\varphi^T(s, \cdot) = T\varphi_s$, where $\varphi_s$ denotes the mapping $\varphi(s, \cdot) : M \to M$.

Let $X \in \mathscr{X}(M)$ be a complete vector field and let $\varphi : \mathbf{R} \times M \to M$ be the flow of X. The tangent lift $X^T \in \mathscr{X}(TM)$ of X is defined as the infinitesimal generator of the tangent lift $\varphi^T$ of the group $\varphi$. This definition is easily extended to arbitrary fields by using local flows. The set $\mathscr{X}^T(TM)$ of all tangent lifts is a subalgebra of $\mathscr{X}(TM)$.

If $X = X^i \dfrac{\partial}{\partial x^i}$ is the local expression of a field X, then the local expression of the tangent lift is

$$X^T = (\tau_M^* X^i) \frac{\partial}{\partial x^i} + (d_T X^i) \frac{\partial}{\partial \dot{x}^i}.$$

It can be shown that the tangent lift $X^T$ of a vector field $X \in \mathscr{X}(TM)$ is the unique lift $Y \in \mathscr{X}(TM)$ of X satisfying:

$$[(d_Y, d_X), d_T] = 0. \tag{1}$$

*Proof.* – In order to prove this statement it is sufficient to show that $(d_Y d_T - d_T d_X) f = 0$, for all functions $f \in \mathscr{F}(M)$, implies that Y is the tangent lift of X. In local coordinates, with $Y = (\tau_M^* X^i) \dfrac{\partial}{\partial x^i} + Y^i \dfrac{\partial}{\partial \dot{x}^i}$, $X^i \in \mathscr{F}(M)$, $Y^i \in \mathscr{F}(TM)$, we have:

$$\begin{aligned}(d_Y d_T - d_T d_X) f &= d_Y \left( \frac{\partial f}{\partial x^i} \dot{x}^i \right) - d_T \left( \frac{\partial f}{\partial x^j} X^j \right) \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} X^j \dot{x}^i + \frac{\partial f}{\partial x^i} Y^i - \frac{\partial^2 f}{\partial x^i \partial x^j} X^j \dot{x}^i - \frac{\partial f}{\partial x^i} \frac{\partial X^i}{\partial x^j} \dot{x}^j \\ &= \frac{\partial f}{\partial x^i} \left( Y^i - \frac{\partial X^i}{\partial x^j} \dot{x}^j \right).\end{aligned}$$

If $(d_Y d_T - d_T d_X) f = 0$, then $Y^i = \dfrac{\partial X^i}{\partial x^j} \dot{x}^j$ or $Y = X^T$. □

The following statements are equivalent to the earlier characterization of the tangent lift:

$$[(i_{X^T}, i_X), d_T] = 0 \tag{2}$$
$$[(d_{X^T}, d_X), i_T] = 0 \tag{3}$$

*Proof.* — It is easy to derive these relations from (1). In $d\Lambda^0(M)$, we find:

$$i_{X^T} d_T (df) = i_{X^T} dd_T f = d_{X^T} d_T f = d_T d_X f = d_T i_X (df)$$
$$d_{X^T} i_T (df) = d_{X^T} d_T f = d_T d_X f = i_T dd_X f = i_T d_X (df). \quad \square$$

Having introduced the basic structures of the tangent bundle, we are now prepared to define implicit differential equations on M.

### 1.3. Implicit differential equations

DEFINITION 1.1. — A *first order differential equation* (*f.o.d.e.*) on a differential manifold M is a submanifold E of the tangent bundle TM.

DEFINITION 1.2. — A differentiable curve $\gamma : I \subset \mathbf{R} \to M$ is said to be a *solution* of a differential equation $E \subset TM$ if $\mathrm{Im}(t\gamma) \subset E$.

A solution of E is also called an *integral curve* of E.

DEFINITION 1.3. — A differential equation $E \subset TM$ is said to be *integrable* if for each $v \in E$ there is a solution $\gamma : I \to M$ of E such that $t\gamma(0) = v$.

DEFINITION 1.4. — A differential equation $E \subset TM$ is said to be *explicit* if there is a vector field $\Gamma : M \to TM$ such that $E = \mathrm{Im}(\Gamma)$.

Differential equations which are not explicit are said to be *implicit*. Explicit differential equations are integrable.

For a discussion of integrability of implicit differential equations and related problems we refer to [3], [4], [5], [6].

We give simple examples of integrable and non-integrable implicit differential equations.

*Example* 1:

$$M = \mathbf{R}^2, \qquad TM = \mathbf{R}^4$$
$$E = \{ (x, y; \dot{x}, \dot{y}) \vdash \dot{x}^2 + \dot{y}^2 + x^2 + y^2 = 1 \}$$

E is a 3-dimensional submanifold; this means that it cannot be put in normal form, not even locally, for this would require E to be 2-dimensional. E is integrable, indeed $\tau_M(E)$ = 2-dimensional disk of radius 1; for $x^2 + y^2 < 1$ there is no problem to exhibit integral curves. For $x^2 + y^2 = 1$, $\dot{x} = \dot{y} = 0$, thus integral curves are simply constant curves.

*Example* 2:

$$\mathrm{M}=\mathbf{R}^2, \qquad \mathrm{TM}=\mathbf{R}^4$$
$$\mathrm{E}=\{(x,0;0,\dot{y})\in\mathbf{R}^4\}$$

Here E is 2-dimensional. This differential equation is clearly non integrable, indeed there is no way to get a vector in the direction of $y$ by differentiating a curve along $x$.

*Example* 3:

$$\mathrm{M}=\mathbf{R}, \qquad \mathrm{TM}=\mathbf{R}^2$$
$$\mathrm{E}=\{(x,\dot{x})\vdash \dot{x}^2=f^2(x)\}$$

Here E has the right dimension to be the image of a vector field on $\mathbf{R}$; however $\dot{x}=\pm f(x)$ are two possible vector fields on $\mathbf{R}$ with images in E. This system is clearly integrable.

*Example* 4:

$$\mathrm{M}=\mathrm{S}^1=\{(x,y)\in\mathbf{R}^2\vdash x^2+y^2=1\}, \qquad \mathrm{TM}=\mathrm{S}^1\times\mathbf{R}$$
$$\varepsilon:\mathbf{R}\to\mathrm{S}^1\times\mathbf{R}:\varphi\mapsto(\cos\varphi,\sin\varphi,\varphi), \qquad \mathrm{E}=\operatorname{Im}\varepsilon$$

This equation is integrable. Its solutions are motions in a circle $\mathrm{S}^1$ with constant acceleration. For each $\varphi\in\mathrm{S}^1$ there is an infinity of solutions $\gamma$ such that $\gamma(0)=\varphi$. This is due to the fact that E is a helix, which is only locally the image of a vector field.

This brief introduction to implicit differential equations permits us to approach the problem of symmetries and constants of the motion for this class of differential equations.

## 1.4. Symmetries and constants of the motion for implicit differential equations

DEFINITION 1.5. – A differentiable function $f:\mathrm{M}\to\mathbf{R}$ is said to be a *constant of the motion* for a differential equation $\mathrm{E}\subset\mathrm{TM}$ if for each solution $\gamma:\mathrm{I}\to\mathrm{M}$ of E the composition $f\circ\gamma$ is a constant function. In other words a constant of the motion is a function constant along any solution of E.

PROPOSITION 1.2. – *If* $\mathrm{E}\subset\mathrm{TM}$ *is integrable, then a function* $f:\mathrm{M}\to\mathbf{R}$ *is a constant of the motion for* E *if and only if* $\langle v, df\rangle=0$ *for each* $v\in\mathrm{E}$.

*Proof.* – Let $f$ be a constant of the motion. For each $v\in\mathrm{E}$ there is a solution $\gamma$ of E such that $t\gamma(0)=v$. Hence,

$$\langle v, df\rangle=\langle t\gamma(0), df\rangle=\frac{d}{ds}(f\circ\gamma)\big|_{s=0}=0.$$

Conversely, if $f$ is a function on M such that $\langle v, df \rangle = 0$ for each $v \in \mathrm{E}$ and if $\gamma$ is a solution of E then $\frac{d}{ds}(f \circ \gamma) = \langle t\gamma(s), df \rangle = 0$. Hence, $f$ is a constant of the motion. □

The property $\langle v, df \rangle = 0$ for each $v \in \mathrm{E}$, used to characterize a constant of the motion for an integrable system E, is equivalent to $d_{\mathrm{T}} f|_{\mathrm{E}} = 0$. The application of this criterion to a function $f$ does not require the integrability of E. We will refer to a function $f : \mathrm{M} \to \mathbf{R}$ with this property as a constant of the motion for $\mathrm{E} \subset \mathrm{TM}$ even if the integrability of E has not been established.

DEFINITION 1.6. — A diffeomorphism $\varphi : \mathrm{M} \to \mathrm{M}$ is said to be a *symmetry* of a differential equation $\mathrm{E} \subset \mathrm{TM}$ if the composition $\varphi \circ \gamma$ of $\varphi$ with a solution $\gamma : \mathrm{I} \to \mathrm{M}$ of E is again a solution of E.

PROPOSITION 1.3. — *If* $\mathrm{E} \subset \mathrm{TM}$ *is integrable, then a diffeomorphism* $\varphi : \mathrm{M} \to \mathrm{M}$ *is a symmetry of* E *if and only if* $\mathrm{T}\varphi(\mathrm{E}) = \mathrm{E}$.

*Proof.* — Let $\varphi$ be a symmetry. For any $v \in \mathrm{E}$ we consider a solution $\gamma : \mathrm{I} \to \mathrm{M}$ such that $t\gamma(0) = v$. Since $\varphi \circ \gamma$ is a solution,

$$\mathrm{T}\varphi(v) = \mathrm{T}\varphi(t\gamma(0)) = t(\varphi \circ \gamma)(0) \in \mathrm{E}.$$

It follows that $\mathrm{T}\varphi(\mathrm{E}) \subset \mathrm{E}$. By applying the same reasoning to $\varphi^{-1}$ we obtain $\mathrm{E} \subset \mathrm{T}\varphi(\mathrm{E})$. Hence $\mathrm{T}\varphi(\mathrm{E}) = \mathrm{E}$.

Conversely, if $\mathrm{T}\varphi(\mathrm{E}) = \mathrm{E}$ and $\gamma : \mathrm{I} \to \mathrm{M}$ is a solution of E, then $t(\varphi \circ \gamma)(s) = \mathrm{T}\varphi(t\gamma(s)) \in \mathrm{E}$. Hence $\varphi \circ \gamma$ is a solution. □

In analogy with the terminology adopted for constants of the motion, we refer to a diffeomorphism $\varphi : \mathrm{M} \to \mathrm{M}$ such that $\mathrm{T}\varphi(\mathrm{E}) = \mathrm{E}$ as a symmetry of $\mathrm{E} \subset \mathrm{TM}$ even if the integrability of E has not been established.

DEFINITION 1.7. — A vector field $\mathrm{X} : \mathrm{M} \to \mathrm{TM}$ is said to be an *infinitesimal symmetry* of $\mathrm{E} \subset \mathrm{TM}$ if the local diffeomorphisms of M belonging to the local one-parameter groups generated by X are symmetries of E.

PROPOSITION 1.4. — *A vector field* $\mathrm{X} : \mathrm{M} \to \mathrm{TM}$ *is an infinitesimal symmetry of an integrable differential equation* $\mathrm{E} \subset \mathrm{TM}$ *if and only if* $\mathrm{X}^{\mathrm{T}}(\mathrm{E}) \subset \mathrm{TE}$.

*Proof.* — The proof is similar to that of proposition 1.3 and is based on the fact that $\mathrm{X}^{\mathrm{T}}$ is the infinitesimal generator of $\varphi^{\mathrm{T}}$, if X is the infinitesimal generator of $\varphi$. □

The criterion $\mathrm{X}^{\mathrm{T}}(\mathrm{E}) \subset \mathrm{TE}$ can be applied to a vector field $\mathrm{X} : \mathrm{M} \to \mathrm{TM}$ even if the integrability of E has not been established. We refer to a vector field X which satisfies this criterion as an infinitesimal symmetry of E. We note that $\mathrm{X}^{\mathrm{T}}(\mathrm{E}) \subset \mathrm{TE}$ simply means that the vector field $\mathrm{X}^{\mathrm{T}}$ is tangent to E.

This inclusion is obviously equivalent to the inclusion:

$$L_{X^T}\mathscr{F}_E \subset \mathscr{F}_E$$

where $\mathscr{F}_E$ denoes the ideal $\mathscr{F}_E = \{f \in \mathscr{F}(TM) \vdash f|_E = 0\}$.

It is frequently convenient to characterize $\mathscr{F}_E$ as generated locally by a set of independent functions $\phi^i \in \mathscr{F}_E$, in the style of the original works of Dirac ([12], [13]), If the submanifold E is characterized by such a set of functions, infinitesimal symmetries of E satisfy:

$$L_{X^T}\phi^i = A^i_j \phi^j$$

with $A^i_j \in \mathscr{F}(TM)$.

From the definition of symmetries of E it follows easily that if $\varphi_1$ and $\varphi_2$ are symmetries for E, the composition $\varphi_1 \circ \varphi_2$ is also a symmetry. Thus symmetries for E are a subgroup of Diff (M). Similarly, infinitesimal symmetries for E are a Lie subalgebra.

## 1.5. Infinitesimal symmetries and constants of the motion for differential equations in normal form

The case of dynamical systems in normal form is more familiar. For this reason we give explicit proofs of proposition 1.2 and 1.4 in the case of explicit differential equations.

We consider images of vector fields for reasons of simplicity; however what we are going to say is equally valid for sets E which are unions of images of local vector fields. An example of this more general situation was given in the example 4 of section 1.3.

In local coordinates:

$$\Gamma = \Gamma^i(m)\frac{\partial}{\partial x^i}$$

and the submanifold $E \subset TM$ can be defined as:

$$E = \operatorname{Im}\Gamma = \{(m, v) \in TM \vdash \phi^i := \dot{x}^i - \Gamma^i(m) = 0\}$$

where $\{\phi^i\}$ is a generating set for E.

Explicitly, an infinitesimal symmetry $X = A^i\frac{\partial}{\partial x^i}$, $A^i \in \mathscr{F}(M)$ for $\Gamma$ satisfies:

$$Z = [\Gamma, X] = 0.$$

In coordinates:

$$Z^i = \Gamma^j\frac{\partial A^i}{\partial x^j} - A^j\frac{\partial \Gamma^i}{\partial x^j} = 0$$

According to prop. 1.4, an infinitesimal symmetry of E is a vector field satisfying

$$L_{X^T} \phi^i = A^i_j \phi^j$$

where $A^i_j \in \mathscr{F}(TM)$.

The tangent lift of a generic vector field X is given by:

$$X^T = A^i \frac{\partial}{\partial x^i} + \frac{\partial A^i}{\partial x^j} \dot{x}^j \frac{\partial}{\partial \dot{x}^i}$$

Then:

$$L_{X^T} \phi^i - A^i_j \phi^j = i_{X^T}(d\dot{x}^i - d\Gamma^i) - A^i_j \phi^j = \frac{\partial A^i}{\partial x^j} \dot{x}^j - \frac{\partial \Gamma^i}{\partial x^j} A^j - A^i_j \phi^j$$
$$= \frac{\partial A^i}{\partial x^j} (\dot{x}^j - \Gamma^j) - A^i_j \phi^j + Z^i = Z^i$$

It follows that $X \in \mathscr{X}(M)$ is an infinitesimal symmetry for $\Gamma$ iff its tangent lift $X^T \in \mathscr{X}(TM)$ is tangent to E.

Concerning constants of the motion, if we consider the vector field $\Gamma$ as a map $\Gamma : M \to TM$, we have:

$$d_T f|_E = \Gamma^* (d_T f) = \Gamma^i \frac{\partial f}{\partial x^i} = L_\Gamma f$$

Thus a function $f \in \mathscr{F}(M)$ is a constant of the motion for $\Gamma$ iff $d_T f|_E = 0$.

Summarizing the above discussion we have, for $E = \operatorname{Im} \Gamma$:

$$X \in \mathscr{X}(M): \quad [\Gamma, X] = 0 \quad \Leftrightarrow \quad X^T \text{ tangent to } E \subset TM$$
$$f \in \mathscr{F}(M): \quad L_\Gamma f = 0 \quad \Leftrightarrow \quad d_T f|_E = 0$$

Until now we have considered generic first order differential equations. To deal with Hamiltonian or Lagrangian equations in implicit form we need to further qualify M to carry a cotangent bundle structure.

## SECTION 2

### 2.1. The tangent bundle over the phase space

We choose the manifold M to be a cotangent bundle, let us say $M = T^* Q$. We describe the geometrical structure of the space $TT^* Q$ and prove some useful properties.

As it is well known, the canonical structure of the cotangent bundle consists of:

(i) a projection map $\pi_Q : T^* Q \to Q$

(ii) a canonical 1-form $\theta_0 = p_i \, dq^i$

(iii) a symplectic structure $\omega_0 = d\theta_0 = dp_i \wedge dq^i$

To each vector field $Z \in \mathscr{X}(Q)$ there corresponds a vector field $Z^*$ on $T^*Q$ characterized by $L_{Z^*}\theta_0 = 0$ and called the canonical lift of Z.

If the manifold M is a cotangent bundle, the structure of this bundle induces additional structure in the tangent bundle $TM = TT^*Q$. This additional structure includes a canonical isomorphism from $TT^*Q$ to $T^*TQ$, which, in local coordinates $(q^i, p_j, \dot{q}^k, \dot{p}_l)$ of $TT^*Q$, is given by:

$$\alpha : (q^i, p_j, \dot{q}^k, \dot{p}_l) \mapsto (q^i, \dot{q}^k, \dot{p}_l, p_j)$$

Other objects belonging to this additional structure are the 1-forms:

$$\begin{aligned} \theta_1 &:= d_T \theta_0 = \dot{p}_i \, dq^i + p_i \, d\dot{q}^i, \\ \theta_2 &:= i_T \omega_0 = \dot{p}_i \, dq^i - \dot{q}^i \, dp_i \end{aligned}$$

and the 2-form:

$$\omega := d_T \omega_0 = d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i.$$

The following relations

$$\begin{aligned} \theta_2 &= \theta_1 - d(p_i \dot{q}^i) \\ \omega &= d\theta_1 = d\theta_2 \end{aligned}$$

are satisfied. The manifold $TT^*Q$ with the 2-form $\omega$ is a symplectic manifold.

For an intrinsic construction of the isomorphism $\alpha$ *see* [23], [27], [28], [30], [11], [8]. Intrinsic definitions of forms $\theta_1$, $\theta_2$ and $\omega$ can be found in [23], [26], [27], [28], [29].

From the definition of $\omega$ and the property (1), *i.e.* $L_{X^T} d_T = d_T L_X$, it follows that

$$L_{X^T} \omega = L_{X^T} d_T \omega_0 = d_T L_X \omega_0.$$

Since $d_T : \Lambda^k(T^*Q) \to \Lambda^k(TT^*Q)$ is a monomorphism for $k > 0$ (*see* prop. 1.1), we find:

$$L_{X^T} \omega = 0 \quad \Leftrightarrow \quad L_X \omega_0 = 0. \tag{4}$$

As a consequence of the geometric structure of $TT^*Q$, we have the following

PROPOSITION 2.1. – The tangent lift $X^T$ of a (locally) Hamiltonian vector field $X \in \mathscr{X}(T^*Q)$ preserves the 1-form $\theta_2$:

$$L_{X^T} \theta_2 = 0$$

and is globally Hamiltonian with respect to the symplectic structure $\omega$. If X is globally Hamiltonian and $i_X \omega_0 = -df$, then $i_{X^T} \omega = -d(d_T f)$.

*Proof.* – From the definition of $\theta_2$ and the property $L_{X^T} i_T = i_T L_X$ [*see* (3)], we find that, if $L_X \omega_0 = 0$, then

$$L_{X^T} \theta_2 = L_{X^T} i_T \omega_0 = i_T L_X \omega_0 = 0.$$

Further

$$i_{X^T}\omega + d(i_{X^T}\theta_2) = i_{X^T}d\theta_2 + di_{X^T}\theta_2 = L_{X^T}\theta_2 = 0$$

Hence $X^T$ is globally Hamiltonian. Moreover:

$$\begin{aligned} i_{X^T}\omega = i_{X^T}d_T\omega_0 = d_T i_X\omega_0 \\ = i_T di_X\omega_0 + di_T i_X\omega_0 = i_T L_X\omega_0 - d(-i_T i_X\omega_0) = -dF, \end{aligned}$$

where $F = -i_T i_X \omega_0$. We have used the relation (2) and the assumption that X is locally Hamiltonian. When $i_X\omega_0 = -df$, we find $F = d_T f$. □

We remark that a tangent lift preserving $\theta_2$ is not necessarily the lift of an Hamiltonian vector field. A simple counterexample is provided by a vector field $X = A^i\frac{\partial}{\partial q^i} + B_i\frac{\partial}{\partial p_i}$, with $\frac{\partial A^i}{\partial q^j} = 0$, $\frac{\partial B_i}{\partial p_j} = 0$, which is not in general Hamiltonian, in spite of satisfying $L_{X^T}\theta_2 = 0$.

Symmetries of differential equations in $T^*Q$ are frequently point transformations. Similarly, infinitesimal symmetries are usually canonical lifts of vector fields on Q. Let $Z = Z^i(q)\frac{\partial}{\partial q^i}$ be a vector field on Q. Then:

$$Z^* = Z^i\frac{\partial}{\partial q^i} - \left(p_j\frac{\partial Z^j}{\partial q^i}\right)\frac{\partial}{\partial p_i}$$

$$Z^{*T} = Z^i\frac{\partial}{\partial q^i} - \left(p_j\frac{\partial Z^j}{\partial q^i}\right)\frac{\partial}{\partial p_i} + \left(\frac{\partial Z^i}{\partial q^j}\dot{q}^j\right)\frac{\partial}{\partial \dot{q}^i} - \left(\dot{p}_j\frac{\partial Z^j}{\partial q^i} + p_j\frac{\partial^2 Z^j}{\partial q^i\partial q^k}\dot{q}^k\right)\frac{\partial}{\partial \dot{p}_i}$$

We note that $Z^*$ is an Hamiltonian vector field on $T^*Q$ as a direct consequence of the definition of the canonical lift. It follows that its tangent lift preserves $\theta_2$.

### 2.2. Generalized Hamiltonian systems

We recall a few basic facts about symplectic manifolds. We refer the reader for further details to [1], [18], [16], [31].

Let $(P, \omega)$ be a symplectic manifold. At each point $p$ of a submanifold $N \subset P$ we consider the tangent space $T_pN \subset T_pP$ and the symplectic polar $(T_pN)^\S = \{ w \in T_pP \vdash \langle w \wedge u, w \rangle = 0, \forall u \in T_pN \}$. In terms of these two spaces we have the following definitions:

(*a*) N is said to be isotropic at $p$ if $T_pN \subset (T_pN)^\S$.

(*b*) N is said to be coisotropic at $p$ if $T_pN \supset (T_pN)^\S$.

(*c*) N is said to be Lagrangian at $p$ if it is isotropic and coisotropic: $T_pN = (T_pN)^\S$.

In an equivalent way we can characterize Lagrangian submanifolds as follows.

DEFINITION 2.1. — A submanifold $N \subset P$ is called a Lagrangian submanifold of $(P, \omega)$ if $\omega|_N = 0$ and $\dim N = 1/2 \dim P$.

DEFINITION 2.2. — A first order differential equation which is a Lagrangian submanifold of $(TP, d_T\,\omega)$ is called a *generalized Hamiltonian system.*

For more details *see* [19].

Let $\Gamma_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}$ be the Hamiltonian vector field on $T^*Q$ associated with the Hamiltonian $H \in \mathscr{F}(T^*Q)$. The image E of $\Gamma_H$, described by the equations

$$\dot{q}^i = \partial H/\partial p_i$$
$$\dot{p}_j = -\partial H/\partial q^j$$

is both an integrable differential equation and a Lagrangian submanifold. Hence it is an integrable generalized Hamiltonian system. It is even an ordinary Hamiltonian system. E can be characterized in an equivalent way as the set of all points in $TT^*Q$ on which the forms $\theta_2$ and $(-\tau^*_{T^*Q}\,dH)$ coincide. In this case we say that E is generated by the function $-H(q,p)$ in the sense of [23], [29].

Also a Lagrangian function $\mathscr{L} \in \mathscr{F}(TQ)$ leads to a generalized Hamiltonian system $E \subset TT^*Q$. It is described by the equations

$$\dot{p}_i = \partial\mathscr{L}/\partial q^i$$
$$p_i = \partial\mathscr{L}/\partial \dot{q}^i.$$

Also in this case E can be characterized as the set of all points in $TT^*Q$ on which the 1-forms $\theta_1$ and $(\pi^*_{TQ}\,d\mathscr{L})$ coincide.

A generalized Hamiltonian system obtained from a Lagrangian is not necessarily an ordinary Hamiltonian system and is not necessarily integrable. An algorithm to extract an integrable part of a generalized Hamiltonian system derived from a Lagrangian function is found in the original papers on constrained Hamiltonian systems [12], [13], [2], [7].

### 2.3. Infinitesimal symmetries and constants of the motion for Hamiltonian systems

The symplectic structure of a symplectic manifold $(P, \omega)$ provides a connection between infinitesimal symmetries and constants of the motion. We recall that if $f \in \mathscr{F}(P)$ is a constant of the motion of an ordinary Hamiltonian system then the associated Hamiltonian vector field $X \in \mathscr{X}(P)$ is an infinitesimal symmetry. The converse is not always true as is seen from the following example.

Let us consider a particle in a constant gravitational field, whose Lagrangian is

$$\mathscr{L} = (1/2)\,m\,(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - kx_3.$$

The Hamiltonian function for this system is

$$H=(1/2m)(p_1^2+p_2^2+p_3^2)+kx_3$$

and the dynamical vector field $\Gamma_H$ is given by:

$$\Gamma_H=\frac{p_1}{m}\frac{\partial}{\partial x_1}+\frac{p_2}{m}\frac{\partial}{\partial x_2}+\frac{p_3}{m}\frac{\partial}{\partial x_3}-k\frac{\partial}{\partial p_3}$$

A well known symmetry for this system is the translation along $x_3$-axis, generated by the vector field $X=\frac{\partial}{\partial x_3}$. $X^*=\frac{\partial}{\partial x_3}$ is an infinitesimal symmetry for $\Gamma_H$, since $[X^*,\Gamma_H]=0$, but the Hamiltonian function $f=p_3$ associated with $X^*$ via $\omega_0$ is not a constant of the motion:

$$L_{\Gamma_H}f=L_{\Gamma_H}p_3=-k\neq 0.$$

As we have seen in Section 1 the natural setting for the analysis of implicit differential equations is the geometry of objects lifted to the tangent bundle. For this reason we will translate the above described relations between infinitesimal symmetries and constants of the motion into relations among the lifted geometrical objects introduced earlier on the tangent bundle $TT^*Q$, *i. e.* the tangent lift $X^T$, its Hamiltonian function $F=d_T f$ and the derived symplectic structure $\omega=d_T\omega_0$.

More precisely, let E be the image of the Hamiltonian vector field $\Gamma_H$. If the function $F=d_T f\in\mathcal{F}(TT^*Q)$ satisfies $F|_E=0$, then the Hamiltonian vector field $X^T$ associated to F is tangent to E. However if $X^T$ is tangent to E, then $F|_E$ is constant but not necessarily zero.

In the example described earlier, $X^*$ is an Hamiltonian vector field whose tangent lift is obviously tangent to $E=\operatorname{Im}\Gamma_H$. As expected, the function $d_T f=\dot{p}_3$, associated with $X^{*T}=\frac{\partial}{\partial x_3}$ via $\omega$, satisfies

$$d_T f|_E=\dot{p}_3|_E=-k\neq 0.$$

Having defined a generalized Hamiltonian system, we may further qualify our infinitesimal symmetries to be canonical.

Definition 2.2. — An infinitesimal symmetry $X:T^*Q\to TT^*Q$ of a generalized Hamiltonian system $E\subset TT^*Q$ is said to be canonical if $L_X\omega_0=0$.

We show that the relations between canonical infinitesimal symmetries and constants of the motion described above for ordinary Hamiltonian systems are present also in the case of generalized Hamiltonian systems.

Proposition 2.2. — *Let* $f\in\mathcal{F}(T^*Q)$ *be a constant of the motion for a generalized Hamiltonian system* $E\subset TT^*Q$, *and let* $X\in\mathcal{X}(T^*Q)$ *be the Hamiltonian vector field associated with f.*

*Then* X *is a canonical infinitesimal symmetry of* E.

*Proof.* — By proposition 2.1, $X^T$ is globally Hamiltonian. The assumption $F|_E=0$ implies $(i_{X^T}\omega)|_E=0$. This relation means that for each $v\in E$, we have $\langle X^T(v)\wedge w,\omega\rangle=0$ $\forall w\in T_v E$. Thus $X^T(v)\in(T_v E)^\S$. Since E is Lagrangian, $(T_v E)^\S=T_v E$. Hence $X^T(v)\in T_v E$ for each $v\in E$, *i.e.* X is a canonical infinitesimal symmetry of E. □

PROPOSITION 2.3. — *Let* $X\in\mathscr{X}(T^*Q)$ *be a canonical infinitesimal symmetry of a generalized Hamiltonian system* $E\subset TT^*Q$. *Then the Hamiltonian function of* X, $f\in\mathscr{F}(T^*Q)$, *satisfies* $d_T f|_E=\text{Const}$.

*Proof.* — Since X is Hamiltonian, $X^T$ is globally Hamiltonian and its Hamiltonian function is $d_T f$. The assumption that E is Lagrangian implies $\omega|_E=0$. Since $X^T$ is tangent to E, we have also $(i_{X^T}\omega)|_E=0$. Hence $d(d_T f)|_E=0$, *i.e.* $d_T f|_E=\text{Const}$. □

### 2.4. Implicit systems defined by Lagrangian functions

As seen before, if our system of differential equations is explicit, the submanifold E is the image of a global section $\Gamma_H$ of the tangent bundle $\tau_{T^*Q}:TT^*Q\to T^*Q$.

However when, due to the singularity of the Lagrangian, the equations are truly implicit, *i.e.* they cannot be put in normal form, the submanifold E is not the image of a vector field, not even locally.

One of the possible consequences of the singularity of the Lagrangian is the presence of constraints. The mechanism of the appearance of constraints is clarified by the following construction, which is the geometric version of the construction described in section 2.2.

Given a Lagrangian $\mathscr{L}\in\mathscr{F}(TQ)$, we can associate with it a submanifold of $T^*TQ$, the image of the differential $d\mathscr{L}$. This submanifold, in turn, can be mapped into $TT^*Q$ via the inverse of the isomorphism (*see* [23], [26]):

$$\alpha:\quad TT^*Q\to T^*TQ.$$

As a matter of fact, if the Lagrangian is regular, the submanifold obtained in $TT^*Q$ is the image of a global vector field $\Gamma_H$. But in the singular case the situation is quite different.

For regular Lagrangians, the submanifold obtained in $TT^*Q$ projects via $\tau_{T^*Q}$ on the whole $T^*Q$; on the contrary, in the singular case, the projection of this submanifold is usually only a part of $T^*Q$, and more precisely it coincides exactly with the image of TQ via the Legendre map $F\mathscr{L}$ it is the submanifold in $T^*Q$ defined by the primary constraints (in the theory of Dirac and Bergmann).

It is instructive to follow the construction described above in local coordinates:

$$(x,\dot{x})\in \mathrm{TQ} \overset{d\mathscr{L}}{\mapsto} \left(x,\dot{x},\frac{\partial \mathscr{L}}{\partial x},\frac{\partial \mathscr{L}}{\partial \dot{x}}\right)\in \mathrm{T^*TQ}$$
$$\overset{\alpha^{-1}}{\mapsto} \left(x,\frac{\partial \mathscr{L}}{\partial \dot{x}},\dot{x},\frac{\partial \mathscr{L}}{\partial x}\right)\in \mathrm{TT^*Q}$$
$$\overset{\tau_{\mathrm{T^*Q}}}{\mapsto} \left(x,\frac{\partial \mathscr{L}}{\partial \dot{x}}\right)\in \mathrm{T^*Q}$$

The following diagram illustrates the situation:

$$\begin{array}{ccc} \mathrm{T^*TQ} & \xrightarrow{\alpha^{-1}} & \mathrm{TT^*Q} \\ d\mathscr{L}\uparrow & & \downarrow \tau_{\mathrm{T^*Q}} \\ \mathrm{TQ} & \xrightarrow{\mathrm{F}\mathscr{L}} & \mathrm{T^*Q} \end{array}$$

As a pedagogical aid, it is instructive to follow step by step the above construction for a very simple example; let us consider the relativistic free particle, whose Lagrangian, with the metric tensor $g_{\mu\nu}=\mathrm{diag}(-+++)$, is given by:

$$\mathscr{L}=-m(-\dot{x}^2)^{1/2}$$

We use the homogeneous Lagrangian in order to have a parametrization invariant action functional. Explicitly, we obtain:

$$\begin{array}{ccccccc} \mathrm{TQ} & \xrightarrow{d\mathscr{L}} & \mathrm{T^*TQ} & \xrightarrow{\alpha^{-1}} & \mathrm{TT^*Q} & \xrightarrow{\tau_{\mathrm{T^*Q}}} & \mathrm{T^*Q} \\ (x^\mu,\dot{x}^\mu) & \mapsto & \left(x^\mu,\dot{x}^\mu,0,\dfrac{m\dot{x}^\mu}{(-\dot{x}^2)^{1/2}}\right) & \mapsto & \left(x^\mu,\dfrac{m\dot{x}^\mu}{(-\dot{x}^2)^{1/2}},\dot{x}^\mu,0\right) & \mapsto & \left(x^\mu,\dfrac{m\dot{x}^\mu}{(-\dot{x}^2)^{1/2}}\right) \end{array}$$

The submanifold $\mathrm{E}\subset \mathrm{TT^*Q}$ can be described by:

$$\phi_\mu := p_\mu - \frac{m\dot{x}_\mu}{(-\dot{x}^2)^{1/2}} = 0$$
$$\phi_{\mu+n} := \dot{p}_\mu = 0$$

These differential equations cannot be represented by a vector field on $\mathrm{T^*Q}$; indeed, the projection of E only covers a closed submanifold of $\mathrm{T^*Q}$, namely the submanifold described by

$$p_\mu p^\mu + m^2 = 0$$

We can easily obtain also the infinitesimal symmetries and constants of the motion for the submanifold E. According to our definitions, a vector field $X \in \mathscr{X}(TT^* Q)$ given locally as:

$$X = A^\mu \frac{\partial}{\partial x^\mu} + B_\mu \frac{\partial}{\partial p_\mu} + d_T A^\mu \frac{\partial}{\partial \dot{x}^\mu} + d_T B_\mu \frac{\partial}{\partial \dot{p}_\mu}$$

is the tangent lift of an infinitesimal symmetry for E represented by $\{\phi_\mu, \phi_{\mu+n}\}$ if it satisfies:

$$L_X \phi_\mu |_E = B_\mu - m \frac{\partial}{\partial \dot{x}^\nu} \left( \frac{\dot{x}_\mu}{(-\dot{x}^2)^{1/2}} \right) d_T A^\nu |_E$$

$$= B_\mu + \frac{m}{(-\dot{x}^2)^{3/2}} (\dot{x}^2 g_{\mu\nu} - \dot{x}_\mu \dot{x}_\nu) d_T A^\nu |_E = 0$$

$$L_X \phi_{\mu+n} |_E = d_T B_\mu |_E = 0.$$

It is possible to solve these equations in full generality. However, we limit ourselves to list the well known symmetries associated with the Poincaré group.

First we can consider the generator of space-time translations:

$$Z = \alpha^\mu \frac{\partial}{\partial x^\mu}$$

which is obviously an infinitesimal symmetry of E; the associated Hamiltonian function

$$f = \alpha^\mu \frac{m \dot{x}_\mu}{(-\dot{x}^2)^{1/2}} = \alpha^\mu P_\mu$$

satisfies the condition:

$$d_T f |_E = \alpha^\mu \dot{p}_\mu |_E = 0$$

hence it is a constant of the motion for E.

Now we consider spatial rotations, generated by the vector fields:

$$J_i = \varepsilon_{ij}^k x^j \frac{\partial}{\partial x^k}$$

In this case the lift to $TT^* Q$ is given by:

$$W_i := (J_i^*)^T = \varepsilon_{ij}^k x^j \frac{\partial}{\partial x^k} + \varepsilon_{ik}^j p_j \frac{\partial}{\partial p_k} + \varepsilon_{ij}^k \dot{x}^j \frac{\partial}{\partial \dot{x}^k} + \varepsilon_{ik}^j \dot{p}_j \frac{\partial}{\partial \dot{p}_k}$$

and it satisfies:

$$L_{W_i} \left( p_k - \frac{m \dot{x}_k}{(-\dot{x}^2)^{1/2}} \right) \Bigg|_E = \varepsilon_{ik}^j p_j + \frac{m}{(-\dot{x}^2)^{3/2}} (\dot{x}^2 g_{kl} - \dot{x}_k \dot{x}_l) \varepsilon_{ij}^l \dot{x}^j \Bigg|_E$$

$$= \varepsilon_{ik}^j \left( p_j - \frac{m}{(-\dot{x}^2)^{1/2}} \dot{x}_j \right) \Bigg|_E = 0$$

$$L_{W_i} \dot{p}_k |_E = \varepsilon_{ik}^j \dot{p}_j |_E = 0$$

For the Hamiltonian functions:

$$f_i = \frac{m\,\varepsilon_{ij}^{k}\, x^j\, \dot{x}_k}{(-\dot{x}^2)^{1/2}} = \varepsilon_{ij}^{k}\, x^j\, p_k$$

we have:

$$d_{\mathrm{T}} f_i\,|_{\mathrm{E}} = \varepsilon_{ij}^{k} \frac{m\dot{x}^j\, \dot{x}_k}{(-\dot{x}^2)^{1/2}} + \varepsilon_{ij}^{k}\, x^j\, \dot{p}_k\,|_{\mathrm{E}} = \varepsilon_{ij}^{k}\, x^j\, \dot{p}_k\,|_{\mathrm{E}} = 0$$

In the same way we could analyze the boost generators:

$$\mathrm{K}_i = x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0}$$

and thus complete the list of symmetries associated with the Poincaré group.

### 2.5. Conclusions

The theory of dynamical systems described by explicit differential equations (vector fields) is well established. In particular, relations between symmetries and conservation laws are well known.

Dynamical systems which are not described by explicit equations were first studied by Dirac and Bergmann. Such systems, derived from singular Lagrangians, are common in theoretical physics. They are encountered in gauge theories and relativity. Systems of this kind are usually represented by families of explicit differential equations by working locally in appropriate charts and by "fixing gauges".

In this paper we have chosen a formulation in terms of implicit differential equations and have extended the concepts of symmetries and conservation laws to this formulation.

An analysis of constrained systems based on the geometry of $\mathrm{T}^*\mathrm{TQ}$ can be found in [17].

We have not yet dealt with the classification of constraints (primary, secondary, ..., first class and second class) and their relation with symmetries, neither we have described relations between the traditional formulation and the formulation in terms of implicit differential equations.

We have limited the discussion to first order differential equations on phase manifolds, leaving out description of dynamics by second order differential equations on configuration manifolds.

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