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Lie-point symmetries in bifurcation problems

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ABSTRACT. — We extend the classical theorem (by Sattinger) relating linear symmetries of the full problem and of the bifurcation equation to the case of Lie-point, *i.e.* possibly nonlinear, symmetries. The symmetry of dynamical systems (ODEs) depending on a parameter is discussed in detail. We also discuss relation with Poincaré normal form, and a connection between bifurcation and “exceptional” symmetry algebras, and how to extend the present results. We give examples of applications to simple dynamical systems, including the case in which nonlinear symmetry of the original problem enforces a linear symmetry of the bifurcation equation.

RÉSUMÉ. — On donne une extension du théorème de Sattinger, qui relie les symétries linéaires de l'équation de bifurcation à celle de l'équation originaire, pour le cas de symétrie du type Lie-point, donc aussi non linéaires. Les symétries des systèmes dynamiques dépendantes d'un paramètre sont analysées en détail. On considère aussi les relations avec les formes normales, et une liaison entre la présence de points de bifurcation et d'algèbres de symétrie «exceptionnelles», et comment envisager une

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extension des résultats présentés ici. On présente aussi des exemples d'applications de nos résultats à des systèmes dynamiques simples, montrant aussi comme une symétrie non linéaire peut être à l'origine d'une symétrie linéaire de l'équation de bifurcation.

INTRODUCTION

Although the original idea goes back to S. Lie, an increasing interest has been received in recent years by the role of extended nonlinear symmetries in the study of differential equations ([Ovs], [Olv], [BC], [BK], [SW]).

In this paper, we want to analyze the application of this method to time-evolution differential equations (DE): more precisely, our attention will be mainly focused on DE arising from bifurcation problems, in which one is interested in the appearance of nontrivial stationary and/or periodic solutions when a "control parameter" crosses certain critical values.

As well known, the "standard" approach to symmetry theory, *i. e.* linear group representation theory, has been widely applied, with remarkable success, to symmetric (or "covariant") bifurcation problems (*see e. g.* [Sat79], [Van], [Sat83], [GSS], [Gae90a], [CK], and references therein). We believe that the success of the linear theory makes it interesting, and possibly useful, our attempt of extending these ideas and methods to the case of nonlinear Lie-point (LP) symmetries.

After a presentation (suitably oriented in view of the foregoing applications) of the typical techniques of nonlinear symmetries (section 1), and a short statement of the typical bifurcation problems (section 2), section 3 is devoted to show that the usual projection methods, allowing a reduction in the dimensionality of the problem, actually "preserves" the symmetry properties of the problem even in the case of nonlinear symmetries. This result, well known in the case of linear symmetries ([Sat79], [Sat83], [GSS]), and first extended to general LP symmetries in [Gae89] is here extended under a quite general point of view, and the underlying algebraic and geometrical settings are carefully investigated.

In section 4 we construct and analyze the different types of nonlinear symmetry generators, and in particular the Lie-point time independent (LPTI) ones, which are the most relevant in the case of autonomous

systems. We analyze the interplay between the properties of these generators and the presence of bifurcating stationary and/or periodic solutions.

There is also a close relationship between the LPTI symmetries and the existence of “canonical coordinates” [BK], and the possibility of reducing the problem into normal (or linear) form according to the classical Poincaré procedure [Arn]: this aspect is considered in section 5. In section 6 some further algebraic features of the problem are considered. Finally, in section 7, the possibility of extending the above results to more general problems and in particular to partial differential equations is briefly considered.

We provide throughout the paper a series of remarks and examples. The examples in this paper are rather simple, and are given essentially in order to clarify and illustrate the algebraic formalism. In other publications, we will give other examples and applications of this approach, e. g. to an extension of the “equivariant branching lemma” [Cic], to periodic solutions of nonlinear dynamical systems [Gae90b], and to problems connected with gauge symmetries [Gae91].

It is actually known that, in general, finding the whole algebra of LP symmetries admitted by a given system of DE may be a very difficult task, although it can be found by a completely algorithmic procedure, and this can also be implemented by computer programs ([Win], CHW). Nevertheless, let us emphasize that our results may be used even if not all of the LP algebra is explicitly known.

Equations, figures, remarks, examples, etc. are numbered separately in each section; reference e. g. to equation (1) means equation (1) in the same section, while if referring to equations in other sections, we denote them e. g. by equation (1. 1).

We would like to thank prof. Tanizhmani for a number of discussions and interesting questions, and proff. Gawedzki, Cartier, Libermann, Kossman-Schwarzback, Françoise, Gazeau and Winternitz for inviting us to report on this work in seminars and conferences. G. G. would also like to thank prof. R. Seneor for his invitation in the Centre de Physique Théorique of the École Polytechnique and prof. L. Michel for his invitation in I.H.E.S. and for stimulating discussions.

1. JET SPACE, GEOMETRY AND SYMMETRY OF DIFFERENTIAL EQUATIONS

In this section, we aim to fix some ideas and notation about jet space, geometrical approach to DE (differential equations), and the symmetry of DE and their solutions. Readers already familiar with these topics and with the principles of bifurcation theory could go directly to section 3.

We will necessarily be quite sketchy, and consider only nondegenerate cases, dealing with the kind of situations we want to consider in the following. On the other end, we will set a framework more general than what would be explicitly needed (*see* section 3 below), in view of the generalizations to be discussed in section 8.

For a comprehensive introduction to jets, we refer to the original series of papers by Ehresmann [Ehr]; an introduction is also contained in [GG] and [Olv]; for symmetries of DE and solutions, and the use of these, *see* [Olv], [BK], [Ovs].

We will in general consider DE in q independent variables ($q=1$ for ODE, $q>1$ for PDE) and p dependent ones; dependent variables will be denoted by u_α , $\alpha=1, \dots, p$, and their space will be denoted by $U \simeq \mathbb{R}^p$; independent variables will be denoted by x_i , $i=1, \dots, q$, and their space will be denoted by $X \simeq \mathbb{R}^q$. If dealing with ODE ($q=1$), the independent variable will be the time and denoted often by t , its space being sometimes denoted by T .

We will denote the space $X \times U$ by M

$$M = X \times U \quad (1)$$

so that a function $f(x): X \rightarrow U$ will define a *graph* Γ_f in M

$$\Gamma_f = \{ (x, u) \in X \times U / u = f(x) \} \quad (2)$$

We define *prolongations* of the space M by considering the direct product of this by the space of (partial) derivatives of u 's. Let $U_{[n]}$ be the space of n -th order (partial) derivatives of u_α 's with respect to x_i 's; then we define

$$U^{(n)} = U \times U_{[1]} \times \dots \times U_{[n]} \quad (3)$$

$$M^{(n)} = X \times U^{(n)} \quad (4)$$

$M^{(n)}$ is also called the jet space of order n for M , or the n -th jet space of M .

Example 1. — If $X = \mathbb{R}^1$, $U^{(1)}$ is just the tangent bundle TU ; this corresponds to the space of (applied) tangent vectors in U .

Remark 1. — Exactly in the same way as we can look at a tangent vector as an equivalence class of tangent curves, we can look at an n -jet (a point in the jet space of order n) as an equivalence class of curves with tangency of order n .

Remark 2. — $M^{(n)}$ has a natural multi-fibered structure. M can be seen as a fiber bundle with basis X , fiber U and projection $\pi^{(0)}: (x, u) \rightarrow x$; $M^{(1)}$ can be seen as a fiber bundle with basis M , fiber $U_{[1]}$ and projection $\pi^{(1)}: (x, u, u_x) \rightarrow (x, u)$, and so on.

If we are dealing with a DE of order n , it is natural to consider the n -th jet space $M^{(n)}$. In this space, one looks at $u, \frac{\partial u}{\partial x}, \dots$ as independent

(unrelated) variables; to take into account the relations which actually exist among them, one introduces a *contact structure* and a *jet connection*, as it will be shortly explained in the following.

Now, a DE of order n in M can be considered as an algebraic equation in $M^{(n)}$, which defines a manifold S in $M^{(n)}$.

Explicitly, for a (system of) DE

$$\Delta(x, u^{(n)}) = 0 \quad (5)$$

one has that Δ is a function

$$\Delta: M^{(n)} \rightarrow \mathbb{R}^k \quad (6)$$

(where k is the number of equations in the system), and (5) identifies a manifold $S_\Delta \subset M^{(n)}$ which is the zero level set of Δ :

$$S_\Delta = \Delta^{-1}(0) = \{ (x, u^{(n)}) \in M^{(n)} / \Delta(x, u^{(n)}) = 0 \} \quad (7)$$

This is called the *solution manifold* for Δ ; we will identify equations having the same solution manifold.

Let us now consider a function

$$f: X \rightarrow U \quad (8)$$

As we have remarked above, this identifies a surface $\Gamma_f \subset M$, but it does also implicitly define a surface $\Gamma_f^{(n)} \subset M^{(n)}$. Infacts, once f is given, its (partial) derivatives of any order are given as well.

Let us denote by \mathcal{J} a multiindex j_1, \dots, j_q ; then we write $\partial_{\mathcal{J}} f = \partial^{|\mathcal{J}|} f / \partial x_1^{j_1} \dots \partial x_q^{j_q}$, with $j_i \geq 0$, $|\mathcal{J}| = \sum_{i=1}^q j_i$.

With this, $\Gamma_f^{(n)}$ is given by

$$\Gamma_f^{(n)} = \{ (x, u^{(n)}) \in M^{(n)} / u_{\mathcal{J}} = \partial_{\mathcal{J}} f \} \quad (9)$$

where $u_{\mathcal{J}} = \partial_{\mathcal{J}} u \in u_{[n]}$.

Remark 3. – Notice that $\Gamma_f^{(n)} \equiv \Gamma_{f^{(n)}}$.

The surface (curve if $X = \mathbb{R}^1$, *i.e.* for ODE) $\Gamma_f^{(n)}$ is called the *lift* of $\Gamma_f \subset M$ to $M^{(n)}$, coherently with the picture of $M^{(n)}$ as a multi-fibered space over M .

The reader familiar with differential geometry will have noticed that this corresponds to giving the natural connection in jet space. In physicists' notation, if we move along the curve $x(\theta) \in X$, and $f(\theta) = f(x(\theta))$, with

$$\left. \begin{aligned} \delta x &= \xi, & x' &= x + \varepsilon \xi \\ \delta f &= \varphi \equiv \frac{\partial f}{\partial x} \delta x, & f' &= f + \varepsilon \varphi \end{aligned} \right\} \quad (10)$$

then it follows

$$\delta f_x = \Phi_{(x)} = \frac{\partial^2 f}{\partial x^2} \delta x, \quad f'_x = f_x + \varepsilon \Phi_{(x)} \tag{11}$$

and so on.

This operation of lifting is related to the contact structure mentioned above: as told before, we should look at $x_i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, \dots$ as independent variables. Then, for a given function $f: X \rightarrow U; f = (f^1, \dots, f^n)$, the requirement

$$u^\alpha = f^\alpha; \quad u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i}; \quad u_{ij}^\alpha = \frac{\partial u_i^\alpha}{\partial x_j} = \frac{\partial u_j^\alpha}{\partial x_i}; \quad \dots \tag{12}$$

corresponds to the assignment to each point of $M^{(n)}$ of an hyperplane. The field of hyperplanes so defined is a *contact structure* of $M^{(n)}$.

The lift $\Gamma_f^{(n)}$ of the graph Γ_f of a function $u=f(x)$ in M will then be tangent in any point of $\Gamma_f^{(n)}$ to this field of hyperplanes, and conversely any curve in $M^{(n)}$ everywhere tangent to the hyperplanes of the contact structure (this is also called a curve compatible with the contact structure) is the lift of a curve in M .

Therefore, a solution to the DE (5) is a function $f: X \rightarrow U$ whose lift lies in S_Δ

$$\Gamma_f^{(n)} \subset S_\Delta \iff \Delta(x, f^{(n)}) = 0 \tag{13}$$

or, conversely, is a curve in S_Δ which is compatible with the contact structure.

Example 2. – Let $X = \mathbb{R}^1 = U$, and $n=1$, with $\Delta(x, u^{(n)}) \equiv u_x - u$. The contact structure is depicted in Figure 1, while the solution manifold is shown in Figure 2. By considering the intersection of the planes of the contact structure with S_Δ , we get a direction field on S_Δ , which can be projected down to M by

$$\pi: (x, u, u_x) \rightarrow (x, u)$$

to get, obviously, the curves $u(x) = u_0 e^x$, as depicted in Figure 3.

Now that we have reached an understanding of the geometrical meaning (in jet space) of a DE and its solutions, we can discuss its Lie-point symmetries from a geometrical point of view.

Let

$$\eta = \xi^i(x, u) \partial_{x^i} + \varphi^\alpha(x, u) \partial_{u^\alpha} \equiv \xi \partial_x + \varphi \partial_u \tag{14}$$

be a tangent vector field on M , $\eta: M \rightarrow TM$ (in the case of ODE we will use also the notation $\eta = \tau \partial_t + \varphi \partial_u$). This is naturally lifted by the jet connection defined above to a vector field $\eta^{(n)}$ on $M^{(n)}$,

$$\eta^{(n)}: M^{(n)} \rightarrow TM^{(n)} \tag{15}$$

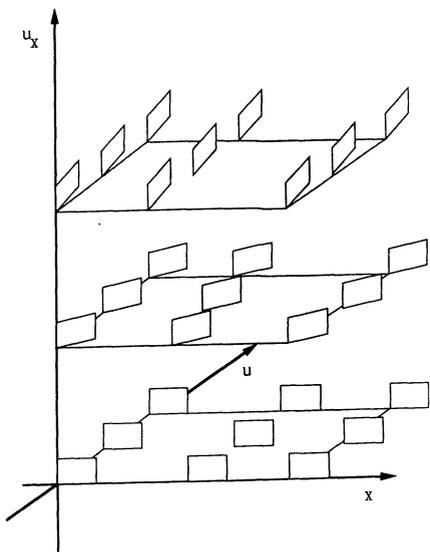


FIG. 1. — The contact structure.

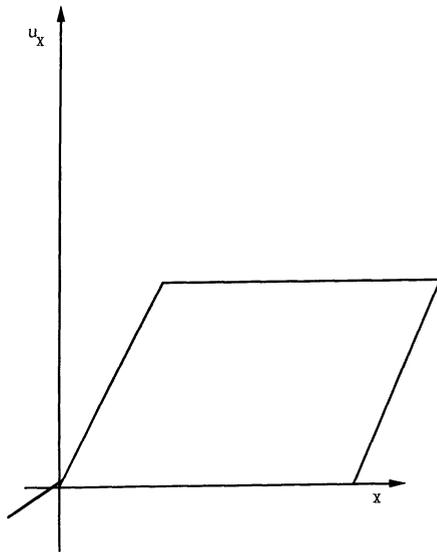


FIG. 2. — The solution manifold.

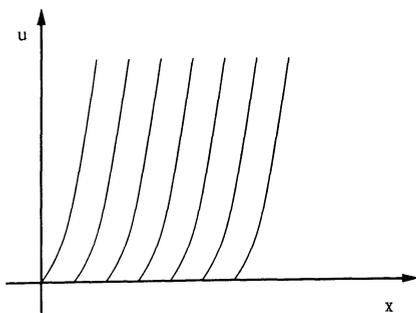


FIG. 3. — Solutions in the plane (x, u) .

If η is given by (14), then $\eta^{(n)}$ is given by

$$\eta^{(n)} = \xi \partial_x + \varphi \partial_u + \sum_{|\mathcal{J}|=1}^n \Phi^{(\mathcal{J})} \partial_{u_{\mathcal{J}}} \tag{16}$$

where

$$\Phi^{(\alpha, \mathcal{J})} = D_{\mathcal{J}} (\varphi^\alpha - \xi^i u_i^\alpha) + \xi^i u_{\mathcal{J}, i}^\alpha \tag{17}$$

and $D_{\mathcal{J}}$ the total derivative, *i. e.*

$$D_{x^i} = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ki}^\alpha \frac{\partial}{\partial u_k^\alpha} + \dots \tag{18}$$

We will not prove here (17), for which the reader is referred to [Olv], [BK]; this is also called the *prolongation formula* and provides an explicit form of the jet connection.

The vector field η induces a vector field $\tilde{\eta}$ in \mathcal{F} , the space of differentiable functions $f: X \rightarrow U$; if

$$I + \varepsilon\eta: \begin{cases} x \rightarrow x + \varepsilon\xi(x, u) = \tilde{x} \\ u \rightarrow u + \varepsilon\varphi(x, u) = \tilde{u} \end{cases} \quad (19)$$

then the action in \mathcal{F} is given by

$$I + \varepsilon\tilde{\eta}: f(x) \rightarrow f(x) + \varepsilon\Psi(x) = \tilde{f}(x) \quad (20)$$

where

$$\Psi(x) = \varphi(x, f(x)) - \xi(x, f(x)) \frac{\partial f(x)}{\partial x} \quad (21)$$

This is easily seen as follows: if $p \in M$ is a point in Γ_f , $p = (x, f(x))$, then $\tilde{p} = (I + \varepsilon\eta)p$ will be in $\Gamma_{\tilde{f}}$. But from (19) we get

$$\tilde{f}(\tilde{x}) = f(x) + \varepsilon\varphi(x, f(x)) \quad (22)$$

and given the first of (19) again, we have

$$\tilde{f}(\tilde{x}) = \tilde{f}(x) + \varepsilon\xi \frac{\partial f}{\partial x} \quad (23)$$

Putting this into (22), we get indeed (21).

Now, it is quite clear what we do mean by the Lie algebra of (point) symmetries of Δ or of f .

DEFINITION. — Given a DE $\Delta(x, u^{(n)}) = 0$, its symmetry algebra \mathcal{G}_Δ is the algebra of vector fields $\eta: M \rightarrow TM$ such that

$$\eta^{(n)}: S_\Delta \rightarrow TS_\Delta \quad (24)$$

or, equivalently, such that

$$\eta^{(n)} \Delta|_{S_\Delta} = 0 \quad (25)$$

DEFINITION. — Given a function $f: X \rightarrow U$, corresponding to a graph $\Gamma_f \subset M$, its symmetry algebra \mathcal{G}_f is the algebra of vector fields $\eta: M \rightarrow TM$ such that

$$\eta: \Gamma_f \rightarrow T\Gamma_f \quad (26)$$

or, equivalently, such that

$$\tilde{\eta}f = 0 \quad (27)$$

where $\tilde{\eta}$ is defined by (19), (20).

In other terms, one can consider the Lie groups $G_\Delta, G_f \subset \text{Diff}(M)$, defined by

$$G_\Delta = \{ \psi \in \text{Diff}(M) / \psi^{(n)} : S_\Delta \rightarrow S_\Delta \} \\ G_f = \{ \psi \in \text{Diff}(M) / \psi : \Gamma_f \rightarrow \Gamma_f \} \equiv \{ \psi \in \text{Diff}(M) / \psi^{(n)} : \Gamma_f^{(n)} \rightarrow \Gamma_f^{(n)} \} \quad (29)$$

where $\psi^{(n)}$ is the n -th prolongation of ψ . The Lie algebra of G_Δ is \mathcal{G}_Δ , that of G_f is \mathcal{G}_f .

Remark 4. – This \mathcal{G}_f corresponds to vector fields leaving the *graph* Γ_f of f unvaried, but by no means it is restricted to vector fields leaving *points* of Γ_f fixed. Consider, e.g., the vector field

$$\eta = \partial_t + \left(\frac{\partial f}{\partial t} \right) \partial_u$$

corresponding to time evolution of solutions,

$$(t, f(t)) \rightarrow (t + \varepsilon, f(t) + \varepsilon f'(t)) \simeq (t + \varepsilon, f(t + \varepsilon));$$

inserting $\xi = 1, \varphi = (\partial f / \partial t)$ into (21) we get $\Psi = 0$.

Under G_Δ , or better its n -th prolongation $G_\Delta^{(n)}$, solutions (*i.e.* surfaces $\Gamma_f^{(n)} \subset S_\Delta$) are transformed into – generally, different – solution. We will denote the subgroup of G_Δ which leaves $\Gamma_f^{(n)}$ invariant as G_f^Δ , and similarly for its Lie algebra \mathcal{G}_f^Δ

$$G_f^\Delta = G_\Delta \cap G_f; \quad \mathcal{G}_f^\Delta = \mathcal{G}_\Delta \cap \mathcal{G}_f \quad (30)$$

For a full discussion of the determination and the use of $G_\Delta, \mathcal{G}_\Delta$ in looking for solutions of DE, we refer to [Olv], [BK], [Ovs].

2. BIFURCATION PROBLEMS: GENERAL SETTING

This paper is devoted to point out some applications of the techniques and ideas presented in the above section to systems of time-evolution DE (or “dynamical systems”), which depend on some real parameter λ (in physical terms, a “control parameter”), as typically occurs in bifurcation theory.

Let us state the problem in the following standard form. Let

$$u \equiv (u_1, u_2, \dots, u_n) \equiv u(t) \in \mathbb{R}^n$$

be a real time-dependent vector, $\lambda \in \mathbb{R}$, and consider the system of ODE

$$\frac{du}{dt} = G(\lambda, u) \quad (1)$$

where $G \equiv (G_1, G_2, \dots, G_n)$ and $G_i : \Lambda \times U \rightarrow \mathbb{R}^n$ are given smooth (e.g. analytical) functions defined in a neighbourhood $\Lambda \times U$ of the origin in

$\mathbf{R} \times \mathbf{R}^n$. As usual in bifurcation theory, we assume the existence of a stationary “trivial” solution u_0 of (1), *i. e.*

$$G(\lambda, u_0) = 0, \quad \forall \lambda \in \Lambda \quad (2)$$

where, with no loss of generality, we can put

$$u_0 \equiv 0. \quad (3)$$

Assume now that the linear part of G

$$L(\lambda) = \partial_u G(\lambda, 0) \quad (4)$$

at $\lambda = \lambda_0$ (we can put $\lambda_0 = 0$) has some eigenvalue $\sigma_0 = \sigma(\lambda_0)$ with $\operatorname{Re} \sigma(\lambda_0) = 0$ (“critical eigenvalue”): the classical bifurcation problem amounts to looking for nonzero solutions of (1) branching from $\lambda_0 = 0$. We will be concerned only with continuous branches of solutions, either stationary ($\dot{u} = 0$), or periodic (Hopf bifurcation) tending to zero when $\lambda \rightarrow \lambda_0 = 0$. For sufficient conditions ensuring the existence of such branches of solutions, *i. e.* that a bifurcation takes place, we refer to [CH], [IJ], [Sat79], [GH], [Rue89].

It would be possible to generalize (1) supposing that $u(t)$ belongs to a infinite dimensional function space (e. g. a Banach or Hilbert space): in this case, one would have to assume $L_0 = L(\lambda_0)$ a zero-index Fredholm operator with the noncritical part of its spectrum lying at finite distance from the imaginary axis at $\lambda = \lambda_0$; another (simpler) generalization is to consider λ as a multiparameter, $\lambda \in \mathbf{R}^p$. For sake of simplicity, we will not investigate this possibility. Similarly, we will assume that the critical eigenvalues of $L_0 = L(\lambda_0)$ are semisimple, in such a way that, denoting by $\mathcal{N} = \mathbf{R}^k$ the subspace spanned by the corresponding eigenvectors, and \mathcal{R} the range of L_0 , one may simply decompose

$$\mathbf{R}^n = \mathcal{N} + \mathcal{R} \quad (5)$$

and use standard Lyapunov-Schmidt projection procedure ([Sat73], [Sat79], [IJ]) in order to convert the original equation (1) to the reduced form

$$F(\lambda, v) = 0, \quad v \in \mathbf{R}^k \quad (6')$$

as usual (some detail of this reduction will be recalled in the next section).

Another procedure commonly used to reduce the dimensionality of the original problem (1) is based on Center Manifold technique ([Rue89], [GH], [HPS]). Here, one has to assume that all noncritical eigenvalues σ_i have $\operatorname{Re} \sigma_i < 0$. Since the Center Manifold is a (local) invariant manifold, one is allowed to consider the restriction of the problem to it, in the form

$$\frac{dw}{dt} = F(\lambda, w), \quad w \in \mathbf{M}^k \quad (6'')$$

(where M^k is the center manifold, tangent in 0 to \mathcal{N}); the asymptotic solutions to the full problem (1) are obtained as solutions of this restricted problem ([Rue89], [GH]).

It is a well known result ([Sat79], [Sat83], [GSS]) that, if the original problem (1) has some symmetry property ("covariance") under a *linear* representation $T(g)$ of some group Γ , *i. e.* if

$$G(\lambda, T(g)u) = T(g)G(\lambda, u), \quad \forall g \in \Gamma \quad (7)$$

then the same property is inherited by the reduced problem (6), through the reduced representation operating in the critical subspace, which is necessarily an invariant subspace for the linear representation T . In the next section, we shall show that an analogous result holds for the Lie point time-independent symmetries, and point out the geometrical and algebraic settings underlying this result.

3. LIE-POINT SYMMETRIES AND BIFURCATION

In this section, we extend the results contained in [Gae89] concerning the relation of the (Lie-Point) symmetries of the original equation and of the corresponding bifurcation equation.

We will discuss fully the case in which one is interested in bifurcation of stationary solutions, so that we can think of the bifurcation equation as obtained by means of a Lyapounov-Schmidt reduction ([IJ], [Sat73], [Sat79]).

For the extension to the case of Hopf bifurcation, and to the case the bifurcation equation is obtained by a center manifold reduction, we will present some short remarks in section 7.

Let

$$\dot{u} = G(\lambda, u), \quad \lambda \in \Lambda \simeq \mathbb{R}; \quad u \in U \subseteq \mathbb{R}^n \quad (1)$$

be the original problem, and as usual

$$G(\lambda, u_0) = 0, \quad \forall \lambda \quad (2)$$

$$L(\lambda) \equiv G_u(\lambda, u_0); \quad L_0 = L(\lambda_0) \quad (3)$$

where λ_0 is the bifurcation point (with no loss of generality, we can put $\lambda_0 = 0$ and $u_0 \equiv 0$); let

$$\left. \begin{aligned} \mathcal{N} &= \text{Ker } L_0 \simeq \mathbb{R}^k \\ \mathcal{R} &= \text{Ran } L_0 \simeq \mathbb{R}^{n-k} \end{aligned} \right\} \quad (4)$$

where we have supposed L_0 has semisimple eigenvalues.

Remark 1. — Now L_0 is an $n \times n$ real matrix, such that we can block diagonalize it and have

$$\mathcal{R}^\perp = \mathcal{N} \tag{5}$$

which we will assume in general for ease of notation. Notice that it would suffice that the *critical* eigenvalues of L_0 are semisimple.

Remark 2. — In the diagonalization we can be forced to pass to a complex matrix; in particular this will always be the case when we are in the presence of an Hopf bifurcation ([IJ], [Sat79], [CH], [MM]), due to the pair $(\pm i)$ of complex conjugate eigenvalues with nonzero imaginary part which are responsible for the bifurcation.

Remark 3. — For bifurcation of stationary solutions the critical eigenvalues are real, so that we can perform a partial change of basis, taking as $\tilde{u}_1, \dots, \tilde{u}_k$ eigenvectors corresponding to the critical eigenvalues, and choose for $i = k + 1, \dots, n$ real \tilde{u}_i such that $(\tilde{u}_i, \tilde{u}_j) = 0, \forall j = 1, \dots, k$. In the new basis, L_0 is still a real matrix, it is in $(k \oplus (n - k))$ block form, and the k -dimensional block is diagonalized.

Now, let P and Q be projection operators,

$$\left. \begin{aligned} P: \mathbb{R}^n &\rightarrow \mathcal{N} \\ Q: \mathbb{R}^n &\rightarrow \mathcal{R} \end{aligned} \right\} \tag{6}$$

and let us take, as usual,

$$v = Pu; \quad w = (I - P)u \tag{7}$$

Let us correspondingly consider the equations

$$QG(\lambda, v + w) \equiv H(\lambda, v, w) = 0 \tag{8}$$

$$(I - Q)G(\lambda, v + w) \equiv \mathcal{F}(\lambda, v, w) = 0 \tag{9}$$

Now we solve (8) for w as a function of λ and v

$$w = h(\lambda, v) \tag{10}$$

This is locally a one to one function, so it identifies a (local) manifold $W \subset \Lambda \times \mathbb{R}^n$ which can be mapped to [a neighbourhood of $(\lambda_0, Pu_0) = (0, 0) \subset] \Lambda \times \mathcal{N}$.

We can then consider the restriction of (9) to this manifold; this means considering

$$\mathcal{F}(\lambda, v, h(\lambda, v)) = 0 \tag{11}$$

and the function \mathcal{F} on the manifold W can be seen as a function on the space $\Lambda \times \mathcal{N}$ from which W is lifted, *i. e.*

$$\mathcal{F}(\lambda, v, h(\lambda, v)) \equiv F(\lambda, v) \tag{12}$$

or, we have the commutative diagram

$$\begin{array}{ccc}
 W & \xrightarrow{\mathcal{F}} & \mathbb{R}^k \\
 \uparrow I \times h & & \nearrow F \\
 \Lambda \times \mathcal{N} & &
 \end{array} \tag{13}$$

For later reference, we denote by ρ the operator which restricts functions defined on $\Lambda \times \mathbb{R}^n$ to the manifold W ,

$$\rho f(\lambda, v, w) = f(\lambda, v, h(\lambda, v)), \quad f: \Lambda \times \mathbb{R}^n \rightarrow \mathbb{R}^m \tag{14}$$

$$\rho: \mathcal{C}^\infty(\Lambda \times \mathbb{R}^n, \mathbb{R}^m) \rightarrow \mathcal{C}^\infty(W, \mathbb{R}^m) \tag{15}$$

and \mathcal{H} the operator which lifts a point in $\Lambda \times \mathcal{N}$ to the corresponding point in W , $\mathcal{H} = I \times h$,

$$\left. \begin{array}{l}
 \mathcal{H}: (\lambda, v) \rightarrow (\lambda, v, h(\lambda, v)) \\
 \mathcal{H}: \Lambda \times \mathcal{N} \rightarrow W \subset \Lambda \times \mathcal{N} \times \mathcal{N}^\perp
 \end{array} \right\} \tag{16}$$

Remark 4. — ρ and \mathcal{H} are defined only in a neighbourhood of (λ_0, v_0) , $v_0 = P u_0$ [notice that $h(\lambda_0, v_0) = 0$], since h itself is defined only locally, and W is a local manifold.

Now, let us consider the case in which (1) admits some nontrivial Lie-point symmetry $\eta \in \mathcal{D}iff(\Lambda \times \mathbb{R}^n)$, the algebra of differentiable vector fields on $(\Lambda \times \mathbb{R}^n)$; let \mathcal{G}_1 be its (Lie-point) symmetry algebra, and let $\{\eta_1, \dots, \eta_d\}$ be a basis of it

$$\eta_i \in \mathcal{G}_1; \quad \eta_i^{(1)}[\dot{u} - G(\lambda, u)] = 0 \tag{17}$$

$$[\eta_i, \eta_j] = c_{ij}^k \eta_k, \quad i, j, k = 1, \dots, d \tag{18}$$

We will in general write a vector field $\eta \in \mathcal{D}iff(\Lambda \times \mathbb{R}^n)$ as

$$\eta = \varphi^i(\lambda, u) \partial_{u^i} + v(\lambda, u) \partial_\lambda \equiv \varphi \partial_u + v \partial_\lambda \tag{19}$$

Remark 5. — Equation (1) can — and must — be supplemented with the equation

$$\dot{\lambda} = 0 \tag{20}$$

In this way, λ is on the same footing as the u variables, and we have a normal ODE in \mathbb{R}^{n+1} , to which usual symmetry criteria [Olv] can be applied. Anyway, we find it more convenient to maintain the notational distinction between λ and the u 's.

Remark 6. — According to the general procedures, in (19) we should have a dependence of φ and v on the time as well, and moreover a supplementary term $\tau(\lambda, u, t) \partial_t$. Since we are looking for stationary solutions, *i.e.* solutions of an equation, $G(\lambda, u) = 0$, in which time does not enter, it is quite natural to restrict our attention to η of the form (19): infacts, for stationary solutions (and equations), ∂_t is a trivial symmetry,

and a dependence of $\eta \in \mathcal{G}_1$ on t could only be equivalent to a time-dependent change of basis. We will return in later sections to the subject of “trivial” symmetries and on the role of time in symmetries.

Remark 7. — Actually some restrictions on (19) are possible and in order. We will discuss them in the next section.

Let us now consider the restriction of these η_i 's to W

$$\zeta = d\rho \eta_i \quad (21)$$

We will decompose ζ_i into a tangent and a normal (to W) part:

$$\zeta_i = \zeta_i^T + \zeta_i^N \quad (22)$$

Remark 8. — ζ_i^T and ζ_i^N are sections, respectively, of the tangent and normal bundles of W , TW and NW .

Now, let us consider the stationary solution manifold S_G for (1):

$$S_G = \{ (\lambda, v, w) / G(\lambda, v + w) = 0 \} \subset \Lambda \times \mathbb{R}^n \quad (23)$$

The bifurcation theorem ensures (*see* section 6) that for λ in a neighbourhood of λ_0 , S_G can be decomposed as

$$S_G = (S_G^0 \cup S_G^b) \cup S_G^\alpha \quad (24)$$

where \cup represents disjoint union, S_G^α represents “big” solutions,

$$S_G^0 = \lambda \times u_0 \subset \Lambda \times \mathbb{R}^n \quad (25)$$

and

$$S_G^b \subseteq W \quad (26)$$

Moreover,

$$S_G^0 \cap S_G^b = (\lambda_0, u_0) = S_G^0 \cup W \quad (27)$$

(this follows from general bifurcation construction [CH], [IJ], [Sat73], [Sat79], *see* section 2).

Let us denote S_W the intersection of S_G and W . From the above, it follows

$$S_W \equiv S_G \cap W = S_G^b \cup S_G^0 \quad (28)$$

Analogously, let us define the stationary solution manifold S_F for (11), *i.e.* for the bifurcation equation

$$F(\lambda, v) = 0 \quad (29)$$

This is

$$S_F = \{ (\lambda, v) / F(\lambda, v) = 0 \} \subseteq \Lambda \times \mathcal{N} \quad (30)$$

and it is again decomposed as

$$S_F = (S_F^0 \cup S_F^b) \cup S_F^\alpha \quad (31)$$

Remark 9. — Relations similar to (24)-(28) hold again. We will not bother the reader by discussing them.

If we consider the lifts of S_F^0 and S_F^b to W , we have

$$\mathcal{H} S_F^0 = S_F^0 \tag{32}$$

$$\mathcal{H} S_F^b = S_G^b \subset W \tag{33}$$

The first of these merely reflects the fact that $(\lambda, u_0) \in W$, while the second one is again a translation in geometrical terms of the bifurcation theorem.

Let us now go back to (22) and notice that

LEMMA I:

$$\zeta_i^N |_{S_W} = 0 \tag{34}$$

Proof. – By definition, $\eta \in \mathcal{G}_1$ is such that

$$\left. \begin{aligned} \eta: S_G &\rightarrow TS_G \\ x &\rightarrow \eta(x) \in T_x S_G \end{aligned} \right\} \tag{35}$$

On the other end, we have that $S_G \subseteq W$, and $S_W \cap S_G = S_W$, so that necessarily

$$\eta: S_W \rightarrow TS_W \subset TW \quad \bullet \tag{36}$$

From this, it follows that ζ_i^T represent symmetries of the restriction of $G(\lambda, u) = 0$ to W , *i. e.* of (11).

Remark 9. – For any algebraic equation $Q(x) = 0$, $Q: R^n \rightarrow R^m$, we have always (trivial) symmetries of the form $\eta = f^i(Q(x)) \partial_{x^i}$, with f an arbitrary function $f: R^m \rightarrow R^n$ satisfying $f(0) = 0$; these correspond to vector fields which vanish on the solution manifold $S_Q = \{x \in R^n / Q(x) = 0\}$.

Remark 10. – From the definition of symmetry of an (algebraic) equation, it is clear that one can always add a v. f. of the above type to any η in the symmetry algebra \mathcal{G}_Q of the equation without changing the symmetry encoded in \mathcal{G}_Q and η . This suggest that we should be free to substitute ζ_i by ζ_i^T with no harm. This will be made precise in the following.

Remark 11. – For an equation $Lu = 0$, with L a (partial) differential operator, remarks 9 and 10 do not apply, as now $\eta(u) = f(Lu) \partial_u$ would correspond to a *generalized* vector field [Olv]. The same applies to higher order ODE; it is only for first order ODE that in this way we get Lie point vector fields.

It is quite obvious that, given (18), the ζ_i 's as well generate an algebra, with the same structure constants as the one generated by the η_i 's:

LEMMA II:

$$[\zeta_i, \zeta_j] = c_{ij}^k \zeta_k, \quad i, j, k = 1, \dots, d \tag{37}$$

Notice anyway that even if the η_i 's, $i = 1, \dots, d$ were linearly independent, this does not need to be true for the ζ_i 's: infacts, e. g., two of the

η_i 's could happen to be parallel at a given point $x \in W$, or even on the whole of W , but not on the whole \mathbb{R}^n , as e. g. the v. f. $\{\partial_x, y \partial_x, y^2 \partial_x\}$ on any line $y = \text{Const.}$ of $\mathbb{R}^2 = \{(x, y)\}$.

From remark 10 we would roughly expect that the algebrae generated by the ζ_i 's and by the ζ_i^T 's are equivalent. This is infacts the case.

LEMMA III. — *If the η_i 's, $i = 1, \dots, d$, satisfy (18), then the ζ_i^T 's defined as above satisfy*

$$[\zeta_i^T, \zeta_j^T] = c_{ij}^k \zeta_k^T \quad (38)$$

Proof. — Consider a chart \tilde{A} in W , and a neighbourhood A of \tilde{A} in \mathbb{R}^n , $A \subset \mathbb{R}^n$; $A \cup W = \tilde{A}$; in A choose coordinates $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$ such that $\tilde{A} = \{(x, y) | y = 0\}$. If in A η_i is written

$$\eta_i = \sum_{s=1}^k h_i^{(s)}(x, y) \partial_{x^s} + \sum_{s=k+1}^n h_i^{(s)}(x, y) \partial_{y^s} \quad (39)$$

then on \tilde{A} ζ_i is written

$$\zeta_i = \sum_{s=1}^k f_i^s(x) \partial_{x^s} + \sum_{s=k+1}^n g_i^s(x) \partial_{y^s} \quad (40)$$

where

$$\left. \begin{aligned} f_i^s(x) &= h_i^s(x, 0) & (s = 1, \dots, k); \\ g_i^s(x) &= h_i^s(x, 0) & (s = k+1, \dots, n) \end{aligned} \right\} \quad (41)$$

We will write for short (40) in the form

$$\zeta_i = f_i(x) \partial_x + g_i(x) \partial_y \quad (42)$$

Then the decomposition (22) is simply

$$\zeta_i^T = f_i(x) \partial_x; \quad \zeta_i^N = g_i(x) \partial_y \quad (43)$$

Let us now consider $[\zeta_i, \zeta_j]$ using the notation (42):

$$\begin{aligned} [\zeta_i, \zeta_j] &= [f_i(x) \partial_x + g_i(x) \partial_y, f_j(x) \partial_x + g_j(x) \partial_y] \\ &= (f_i \partial_x f_j - f_j \partial_x f_i) \partial_x + (f_i \partial_x g_j - f_j \partial_x g_i) \partial_y = [\zeta_i^T, \zeta_j^T] + \Phi \partial_y \end{aligned} \quad (44)$$

If now we introduce the projection operator ω , which associates to a v. f. on W with values in $T(\Lambda \times \mathbb{R}^n)$ its component in TW ,

$$\omega \zeta = \zeta^T \quad (45)$$

then (44) reads

$$[\omega \zeta_i, \omega \zeta_j] = \omega [\zeta_i, \zeta_j] \quad (46)$$

Now we just notice that from (37) it follows that

$$\omega [\zeta_i, \zeta_j] = c_{ij}^k \omega \zeta_k \quad (47)$$

and we have

$$[\omega_{\zeta_i}^{\zeta_j}, \omega_{\zeta_j}^{\zeta_k}] = c_{ij}^k \omega_{\zeta_k}^{\zeta_i} \quad (48)$$

In the previous notation, this just reads

$$[\zeta_i^T, \zeta_j^T] = c_{ij}^k \zeta_k^T \quad (49)$$

and completes the proof. ●

If we introduce the restriction operator ρ , which restricts v. f. defined in \mathbb{R}^n to the manifold W ,

$$\rho \cdot \eta = \eta|_W \quad (50)$$

we have for the v. f. η_i 's

$$\eta_i \xrightarrow{\rho} \zeta_i \xrightarrow{\omega} \zeta_i^T \quad (51)$$

(we will denote ζ_i^T by χ_i in the following), and at the algebra level

$$\mathcal{G}_{(1)} \xrightarrow{\rho} \mathcal{G}_W \xrightarrow{\omega} \mathcal{G}_W^T \quad (52)$$

In this language, lemmas II and III reads

LEMMA IV. — *The algebras \mathcal{G}_1 and Γ_W are homeomorphic; the algebras \mathcal{G}_W and \mathcal{G}_W^T are isomorphic. In other words, ρ is an algebra homeomorphism, ω an algebra isomorphism.*

From the above discussion it also follows that, denoting by $\mathcal{G}_{\mathcal{F}}$ the symmetry algebra of $\mathcal{F} = 0$, equation (11), one has the

LEMMA V:

$$\mathcal{G}_W^T \subseteq \mathcal{G}_{\mathcal{F}} \quad (53)$$

Remark 12. — We stress that one has no *a priori* reason to expect the equality sign to hold in (53): in fact, it is not difficult to think of a v. f. η_* such that $\eta_* : S_W \rightarrow TS_W$ but does not in general satisfy $\eta_* : S_G \rightarrow TS_G$, *i. e.* such that it does not satisfy $\eta_* : S_G^{\alpha} \rightarrow TS_G^{\alpha}$ [see equation (24)].

Remark 13. — The discussion conducted up to now did actually use only the decomposition (24) of S_G , and (26), (28) in order to ensure (36). It therefore applies in more general cases than that of bifurcation problems.

Remark 14. — In the case of bifurcation problems, we expect the degeneracy of critical eigenvalues is fully due to the symmetry, or it could be removed by a small perturbation of the equation [Rue73]. This means that $\chi_i(x)$, $i=1, \dots, d$ do span $T_x W$ in general for any $x \in W$, and any $\chi \in \mathcal{G}_{\mathcal{F}}$ is a function of χ_1, \dots, χ_d , *i. e.* the equality sign applies in (53).

Remark 15. — This is perhaps an appropriate point to stress that all our discussion does not consider *discrete* symmetries of (1), (11), or (29),

which too are possible. For the relevance of these in connection with the reduction from (1) to (29), see [GMS].

Given the local isomorphism between W and $\Lambda \times \mathcal{N}$ (we will from now on denote $\Lambda \times \mathcal{N}$ by \mathcal{M}), it is natural to think of *projecting* the tangent v.f. χ_i 's on W to tangent v.f. on \mathcal{M} ; this should give an isomorphism of algebrae, as in fact it does.

Let us introduce the invertible operator θ , associating to a v.f. $\chi: W \rightarrow TW$ a v.f. $\beta: \mathcal{M} \rightarrow T\mathcal{M}$ by the natural projection; *i.e.* θ acts on $W \subset \Lambda \times \mathbb{R}^n$ as $I \times P$, where P is the projection introduced in (6), and on $TW \subset T\Lambda \times TR^n$ as

$$\theta = I \times dP \tag{54}$$

Remark 16. — θ is the inverse of $d\mathcal{H}$, with \mathcal{H} defined in (16).

We consider then the v.f. χ_i 's obtained as above, and associate to them the v.f. $\beta_i: \mathcal{M} \rightarrow T\mathcal{M}$,

$$\theta_{\chi_i} = \beta_i, \quad i = 1, \dots, d \tag{55}$$

From the invertibility of θ and the fact W is locally a smooth regular manifold, it follows that, if $\mathcal{G}_F^{(e)}$ is the algebra generated by the β_i 's (the e stands for inherited),

$$\theta \mathcal{G}_W^T = \mathcal{G}_F^{(e)} \tag{55'}$$

then we have

LEMMA VI. — *The algebrae \mathcal{G}_F and $\mathcal{G}_{\mathcal{F}}$ are isomorphic.*

One has moreover that, given (18), (37), (38) and (55), then the key lemma follows:

LEMMA VII:

$$[\beta_i, \beta_j] = c_{ij}^k \beta_k \tag{56}$$

Proof. — The v.f. $\chi_i = \zeta_i^T$ can be written as

$$\chi_i = \varphi_i(\lambda, v, h(\lambda, v)) \partial_v + \psi_i(\lambda, v, h(\lambda, v)) \partial_w + v_i(\lambda, v, h(\lambda, v)) \partial_\lambda \tag{57}$$

The condition $\chi_i: W \rightarrow TW$ implies (in physicists' notation)

$$\delta w = h_v \delta v + h_\lambda \delta \lambda$$

or more precisely

$$\psi(\lambda, v, h(\lambda, v)) = \frac{\partial h}{\partial v} \varphi(\lambda, v, h(\lambda, v)) + \frac{\partial h}{\partial \lambda} v(\lambda, v, h(\lambda, v)) \tag{58}$$

In the notation of (57), β_i is given by

$$\beta_i = \theta_{\chi_i} = \bar{\varphi}_i(\lambda, v) \partial_v + \bar{v}_i(\lambda, v) \partial_\lambda \tag{59}$$

where

$$\bar{\varphi}_i(\lambda, v) \equiv \varphi_i(\lambda, v, h(\lambda, v)); \quad \bar{v}_i(\lambda, v) \equiv v_i(\lambda, v, h(\lambda, v)) \tag{60}$$

We stress that the $\bar{\varphi}, \bar{v}$ are functions of (λ, v) ,

$$\left. \begin{aligned} \bar{\varphi}_i &: \mathcal{M} \rightarrow \mathbb{R}^k \\ \bar{v}_i &: \mathcal{M} \rightarrow \mathbb{R} \end{aligned} \right\} \tag{61}$$

while the φ, v are functions of (λ, v, w) ,

$$\left. \begin{aligned} \bar{\varphi}_i &: \Lambda \times \mathbb{R}^n \rightarrow \mathbb{R}^k \\ \bar{v}_i &: \Lambda \times \mathbb{R}^n \rightarrow \mathbb{R} \end{aligned} \right\} \tag{62}$$

independently of the fact that in (57) only their restriction to $W \subset \Lambda \times \mathbb{R}^n$ does appear.

Therefore, from (60) it follows

$$\left. \begin{aligned} \partial_v \bar{\varphi}_i &= (\partial_v + h_v \partial_w) \varphi_i \\ \partial_\lambda \bar{\varphi}_i &= (\partial_\lambda + h_\lambda \partial_w) \varphi_i \\ \partial_v \bar{v}_i &= (\partial_v + h_v \partial_w) v_i \\ \partial_\lambda \bar{v}_i &= (\partial_\lambda + h_\lambda \partial_w) v_i \end{aligned} \right\} \tag{63}$$

Let us now consider $\{\beta_i, \beta_j\}$. We have

$$\begin{aligned} [\beta_i, \beta_j] &= [(\bar{\varphi}_i \partial_v \bar{\varphi}_j - \bar{\varphi}_j \partial_v \bar{\varphi}_i) + (\bar{v}_i \partial_\lambda \bar{\varphi}_j - \bar{v}_j \partial_\lambda \bar{\varphi}_i)] \partial_v \\ &\quad + [\bar{v}_i \partial_\lambda \bar{v}_j - \bar{v}_j \partial_\lambda \bar{v}_i] + (\bar{\varphi}_i \partial_v \bar{v}_j - \bar{\varphi}_j \partial_v \bar{v}_i) \partial_\lambda \end{aligned} \tag{64}$$

which, using (60) and (63), becomes

$$\begin{aligned} [\beta_i, \beta_j] &= [(\bar{\varphi}_i (\partial_v + h_v \partial_w) \varphi_j - \bar{\varphi}_j (\partial_v + h_v \partial_w) \varphi_i) \\ &\quad + (\bar{v}_i (\partial_\lambda + h_\lambda \partial_w) \varphi_j - \bar{v}_j (\partial_\lambda + h_\lambda \partial_w) \varphi_i)] \partial_v \\ &\quad + [\bar{v}_i (\partial_\lambda + h_\lambda \partial_w) v_j - \bar{v}_j (\partial_\lambda + h_\lambda \partial_w) v_i \\ &\quad + (\bar{\varphi}_i (\partial_v + h_v \partial_w) v_j - \bar{\varphi}_j (\partial_v + h_v \partial_w) v_i)] \partial_\lambda \end{aligned} \tag{65}$$

If now we use (58), (57), this can be rewritten as

$$[\beta_i, \beta_j] = (\chi_i \cdot \varphi_j - \chi_j \cdot \varphi_i) \partial_v + (\chi_i \cdot v_j - \chi_j \cdot v_i) \partial_\lambda \tag{66}$$

which can also be written as

$$[\chi_i, \chi_j] - [(\chi_i \Psi_j) - (\chi_j \Psi_i)] \partial_w \tag{67}$$

From (49) we have

$$[\beta_i, \beta_j] = c_{ij}^k \chi_k - \Xi_{ij} \partial_w \tag{68}$$

which, applying θ to both sides, gives explicitly

$$[\beta_i, \beta_j] = c_{ij}^k \beta_k \tag{69}$$

and completes the proof. ●

Remark 17. – Such a detailed computation was actually not needed, but we have preferred to include it for completeness.

Remark 18. – In (68) one would clearly have

$$c_{ij}^k \Psi_k - \Xi_{ij} = 0$$

The notation $\mathcal{G}_F^{(e)}$ in (55') could suggest that this is (part of) the symmetry algebra of the bifurcation equation (29). This is in fact the case, as it follows from remark 16 and (13). If we denote by \mathcal{G}_F the symmetry algebra of the bifurcation equation (19), we then have

LEMMA VIII:

$$\mathcal{G}_F^{(e)} = \theta \cdot \omega \cdot \rho \cdot \mathcal{G}_1 \subseteq \mathcal{G}_F \tag{70}$$

Remark 19. – This can be recast in the language of solution manifolds as follows: the solution manifold $S_F \subset \mathcal{M}$ of (29) satisfies

$$S_F = PS_{\mathcal{F}}$$

where P is defined in (6). By (54) it is obvious that

$$\chi: S_{\mathcal{F}} \rightarrow TS_{\mathcal{F}} \Rightarrow \beta: S_F \rightarrow TS_F$$

where $\beta = \theta_{\chi}$.

From the previous lemmas it follows

LEMMA IX. – *The symmetry algebra of the bifurcation equation (29) is a subalgebra of the symmetry algebra of the full equation (1).*

It may be worth summarizing the results of this section in the

PROPOSITION. – *Given the full equation $A(\lambda, u) = 0$ and its bifurcation equation $F(\lambda, v) = 0$, let \mathcal{G}_1 be the symmetry algebra of the full equation, and let \mathcal{G}_1 be spanned by $\{\eta_1, \dots, \eta_d\}$. Then the algebra $\mathcal{G}_F^{(e)} = \theta \cdot \omega \cdot \rho \mathcal{G}_1$, spanned by the (in general, not independent) vector fields $\{\beta_1, \dots, \beta_d\}$, $\beta_i = (\theta \cdot \omega \cdot \rho) \eta_i$, is contained in (generically, is equal to) the symmetry algebra \mathcal{G}_F of the bifurcation equation. The composed operator $\Phi = \theta \cdot \omega \cdot \rho$ is an algebra homeomorphism; $\mathcal{G}_F^{(e)}$ is homeomorphic to \mathcal{G}_1 and isomorphic to one of its subalgebras, and therefore such is generically also G_F . The structure constants of $\mathcal{G}_F^{(e)}$ in the basis $\{\beta_i\}$ are mutated from those of \mathcal{G}_1 in the basis $\{\eta_i\}$, i. e. $[\eta_i, \eta_j] = c_{ij}^k \eta_k \Rightarrow [\beta_i, \beta_j] = c_{ij}^k \beta_k$. \diamond*

It can also be worth reexpressing the above results in a less abstract setting. Using coordinates v, w in V, V^\perp , rewrite η as

$$\eta = \xi(\lambda, v, w) \partial_v + \psi(\lambda, v, w) \partial_w + \nu(\lambda, v, w) \partial_\lambda \tag{71}$$

Then we have in explicit terms

$$\zeta \equiv \eta|_w = \xi(\lambda, v, h(\lambda, v)) \partial_v + \psi(\lambda, v, h(\lambda, v)) \partial_w + \nu(\lambda, v, h(\lambda, v)) \partial_\lambda \tag{72}$$

and

$$\beta \equiv \bar{\xi}(\lambda, v) \partial_v + \bar{\nu}(\lambda, v) \partial_\lambda = \xi(\lambda, v, h(\lambda, v)) \partial_v + \nu(\lambda, v, h(\lambda, v)) \partial_\lambda \tag{73}$$

Example. – We think it is worth considering in detail a concrete, although elementary, example. We look for stationary solutions of

$$\left. \begin{aligned} \dot{x} &= \lambda x - xy \equiv X^{(1)}(\lambda, x, y) \\ \dot{y} &= -y + x^2 \equiv X^{(2)}(\lambda, x, y) \end{aligned} \right\} \tag{74}$$

bifurcating from $(x, y) = (0, 0)$ at $\lambda = \lambda_0 = 0$. Here $x \in V = \mathbb{R}^1$, $y \in V^\perp = \mathbb{R}^1$.

It is immediate to see that stationary solutions to (74) are given by

$$\left. \begin{aligned} (x, y) &= (0, 0), & \forall \lambda \\ (x, y) &= (\pm \sqrt{\lambda}, \lambda), & \lambda \geq 0 \end{aligned} \right\} \quad (75)$$

but let us anyway check our construction works correctly in this case.

Solving $X^{(2)} = 0$ gives

$$y = h(\lambda, x) \equiv x^2 \quad (76)$$

and the bifurcation equation is

$$F(\lambda, x) \equiv \lambda x - x^3 = 0 \quad (77)$$

Now, let us write as before

$$\eta = \xi \partial_x + \psi \partial_y + v \partial_\lambda \quad (78)$$

The action of this on X gives

$$\left. \begin{aligned} \eta \cdot X^{(1)} &= (\lambda - y) \xi - x \psi + x v \\ \eta \cdot X^{(2)} &= 2 x \xi - \psi \end{aligned} \right\} \quad (79)$$

and the condition to have that η generates a symmetry of $X=0$, *i. e.* that $\eta \cdot X|_{\mathcal{S}_X} = 0$, reads

$$\left. \begin{aligned} x(v - \psi) &= 0 \\ 2x\xi - \psi &= 0 \end{aligned} \right\} \quad (80)$$

i. e. the symmetries of $X=0$ are generated by

$$\eta = \xi(\partial_x + 2x\partial_y + 2x\partial_\lambda) \quad (81)$$

with $\xi = \xi(\lambda, x, y)$ an arbitrary function.

It is immediate to check that such a η leaves the manifold W defined by (76) invariant, and that (58) is satisfied.

According to our general discussion, the vector field

$$\beta = \bar{\xi}(\partial_x + 2x\partial_\lambda) \quad (82)$$

where $\bar{\xi} \equiv \xi(\lambda, x, h(\lambda, x))$, *i. e.* in this case with $\bar{\xi} = \bar{\xi}(\lambda, x)$ an arbitrary function, should generate a symmetry of the bifurcation equation (77).

Actually, by writing

$$\beta = \bar{\xi} \partial_x + \bar{v} \partial_\lambda \quad (83)$$

we see that

$$\beta \cdot F = (\lambda - 3x^2) \bar{\xi} + x \bar{v} \quad (84)$$

and on $\mathcal{S}_F = \{(\lambda, x) / \lambda = x^2\}$, *i. e.* for $F(\lambda, x) = 0$, the above reads

$$\beta \cdot F|_{\mathcal{S}_F} = -x(2x\bar{\xi} - \bar{v}) \quad (85)$$

I. e., our construction gave all the LPTI symmetries of the bifurcation equation.

A less trivial example will be given in the next section, dealing with a case in which the present setting is actually more powerful than the one based on linear symmetries alone.

It could also be useful to resume some of the (final and intermediate) results and constructions of this section in terms of diagrams, which we do here:

$$\begin{array}{ccccc}
 & & \mathcal{G}_1 & & \\
 & & \longrightarrow & & \\
 S_G & \xrightarrow{\quad} & TS_G & & \\
 \downarrow \rho & \searrow & \searrow d\rho & & \\
 I \times P & & S_w & \xrightarrow{\mathcal{G}_F} & TS_w \\
 \downarrow \mathcal{H} & \nearrow & \nearrow d\mathcal{H} & & \\
 S_F & \xrightarrow{\mathcal{G}_F} & TS_F & &
 \end{array} \tag{86}$$

$$\begin{array}{ccccc}
 & & G & & \\
 & & \longrightarrow & & \\
 \Lambda \times \mathcal{R}^n & \xrightarrow{\quad} & \mathcal{R}^s & & \\
 \downarrow \rho & \searrow & \searrow Q & & \\
 I \times P & & W & \xrightarrow{\mathcal{F}} & \mathcal{R}^k \\
 \downarrow \mathcal{H} & \nearrow & \nearrow I & & \\
 \mathcal{M} & \xrightarrow{F} & \mathcal{R}^k & &
 \end{array} \tag{87}$$

4. GENERAL SYMMETRIES FOR SYSTEMS OF ODE'S

Let us now state some general results concerning Lie point-symmetries (LP) for systems of time-evolution first-order ODE's, depending on a real parameter λ :

$$\frac{du}{dt} = f(\lambda, u), \quad u = u(t) \in \mathbb{R}^n \tag{1_1}$$

$$\frac{d\lambda}{dt} = 0, \quad \lambda \in \mathbb{R} \tag{1_2}$$

where f is a given smooth vector field; we consider also for a moment (until otherwise stated) the possibility that the system is nonautonomous:

$$f = f(\lambda, u, t) \tag{1'_1}$$

As explained in section 3, we have added the equation (1₂) for the parameter λ in order to apply, in a more convenient way, the general methods discussed in sections 1 and 3. The results below come essentially from a technique already used by Ovsjannikov [Ovs] (see also [Olv], [BK]), and suitably adapted to the present case.

Remark 1. — If the above system (1) is viewed as a bifurcation problem, all results from now on are true both if equation (1) describes the “original” problem [*i.e.* in the form of equation (2.1)], and if it describes the “reduced” problem (*i.e.* obtained after reduction through either Lyapunov-Schmidt projection or Center-Manifold procedure).

According to sections 1 and 3, we are looking now for the LP symmetries admitted by problem (1): the Lie generators of these symmetries are vector field operators of the form

$$\eta = \varphi \partial_u + \tau \partial_t + \nu \partial_\lambda \quad (\varphi \partial_u \equiv \varphi_i \partial_i) \quad (2)$$

where $\varphi_i = \varphi_i(\lambda, u, t), \dots; \partial_i \equiv \partial/\partial u_i$. It can be useful to remark that the usual case of linear symmetries would correspond within this formalism to restrict the operators η to the form

$$\eta = \varphi \partial_u \equiv \Phi_{ij} u_j \partial_i$$

where Φ is a constant matrix, *i.e.* to the case of linear diffeomorphisms.

The condition that the general operator (2) is a symmetry generator admitted by (1) takes explicitly the form

$$\partial_t \nu + f_i \partial_i \nu = 0 \quad (3_1)$$

$$\tau \partial_t f_j + \varphi_i \partial_i f_j + \nu \partial_\lambda f_j - \partial_i \varphi_j + f_j \partial_i \tau - f_i \partial_i \varphi_j + f_j f_i \partial_i \tau = 0 \quad (3_2)$$

Putting

$$\varphi_j = \tau f_j + \theta_j \quad (4)$$

one finds that equation (3₂) is transformed into

$$-\partial_i \theta_j + \theta_i \partial_i f_j - f_i \partial_i \theta_j + \nu \partial_\lambda f_j = 0 \quad (5)$$

Therefore, one can say:

PROPOSITION 1. — *Any system of ODE's (1) admits LP symmetry generators (2) of the following types:*

(i) $\nu = \theta_i = 0$, *i.e.*

$$\eta = \tau (f_i \partial_i + \partial_t)$$

for any arbitrary function $\tau(\lambda, u, t)$;

(ii) $\tau = 0$ and ν, θ_i satisfying (3₁) and (5).

If the system is autonomous, (3) are satisfied also by

(iii) $\tau = 1, \nu = \varphi_i = 0$, *i.e.* (as obvious)

$$\eta = \partial/\partial t \quad \diamond$$

We shall consider, from now on, only *autonomous* problems. For these, time-independent symmetry generators are of special interest: they satisfy

$$f_i \partial_i \nu = 0, \quad \theta_i \partial_i f_j - f_i \partial_i \theta_j + \nu \partial_\lambda f_j = 0 \quad (6)$$

which can be equivalently written in terms of a Lie commutator [Gae]

$$[\eta, F] = 0 \quad \text{where} \quad F \equiv f_i \partial_i \quad (6')$$

For autonomous systems, equations (6) admit the obvious solution $v=0$; $\theta_i=f_i$ *i.e.*

$$\eta = \eta_f \equiv f_i \partial_i \tag{7}$$

which is equivalent, once applied to a solution $\bar{u}(\lambda, t)$, to the time translation generator ∂_t ; in fact it produces the transformation

$$\bar{u}(\lambda, t) \rightarrow \bar{u} + \delta u = \bar{u} + \varepsilon f = \bar{u}(\lambda, t + \varepsilon) \tag{8}$$

But this gives no real information on the problem of finding new solutions, indeed the result that also $\bar{u}(\lambda, t + \varepsilon)$ is a solution is a trivial consequence of the property of the system of being autonomous. Similarly, symmetries of type (i) in Proposition 1 are not useful in this sense, in fact they satisfy

$$\tilde{\eta} \bar{u} \equiv 0 \tag{9}$$

[with the notations of (1.19-21) and (1.27)], so they belong to the algebra $\mathcal{G}_{\bar{u}}$ corresponding to the flow of the equation itself.

For what concerns the functions v , we can see that it is sufficient to consider only the two possibilities $v=0$ and e.g. $v=1$; we have in fact that if v, θ_i satisfy (3) and (5), also $v' = vN(v), \theta'_i = \theta_i N(v)$, for any smooth function $N(v)$, satisfy the same conditions.

Before discussing the properties of these symmetries, let us give a method for constructing all solutions of (5) (*cf.* [Ovs], [Olv], [BK]), *i.e.* all symmetries of type (ii) in Proposition 1. Let $y = y(\lambda, u, t)$ satisfy the following linear PDE

$$\partial_t y + f_i \partial_i y = 0 \tag{10}$$

Assume that one can find n functionally independent solutions y_1, \dots, y_n in some open domain; let us put (this choice is made in order to have agreement with the above choice for v)

$$y_{n+1} = \lambda \tag{11}$$

and let p_{jk} be defined by the linear system

$$p_{jk} \partial_i y_j = \delta_{ik}, \quad i, j, k = 1, \dots, n+1 \tag{12}$$

Then, it can be shown that

$$\eta_{0(j)} = p_{jk} \partial_k, \quad j, k = 1, \dots, n \tag{13}$$

and

$$\eta_\lambda = p_{n+1, k} \partial_k + \partial_\lambda \quad \text{where} \quad p_{n+1, k} = -p_{jk} \partial_\lambda y_j \tag{14}$$

are $n+1$ independent symmetry generators. In addition, if $\tilde{y} = \tilde{y}(\lambda, u, t)$ is any solution of (10), and η any symmetry, then also $\eta' = \tilde{y} \eta$ is a symmetry [this is true of course also for symmetries (i) and (ii) of Proposition 1].

In order to understand the physical meaning of these results, we can observe that (10) defines precisely the “integrals of motion” of (1): if

$\bar{u}(\lambda, u_0, t)$ is a solution of (1) with initial datum u_0 , then $y(\lambda, \bar{u}, t)$ is constant along \bar{u} . Therefore, η and η' have the same effect once applied to a solution. This fact allows us to consider η and η' as “identical” symmetries and, when enumerating the possible independent symmetries, to look only for truly different symmetries.

Remark 2. – Writing now the characteristic equations associated to (10)

$$dt = \frac{du_1}{f_1} = \dots = \frac{du_n}{f_n} \tag{15}$$

we see that one could – in principle – obtain just one integral of the form

$$y_1 = t - Y_1(\lambda, u) \tag{16}$$

and $n - 1$ conserved quantities y_2, \dots, y_n independent of time. Including $y_{n+1} = \lambda$, the procedure (12)-(14) then provides $n + 1$ symmetries all independent of time. It is clear that this construction is closely related to the problem of finding “canonical coordinates” [BK] for the system (1): assuming in fact y_1, \dots, y_n as new coordinates, one obtains

$$\left. \begin{array}{l} \dot{y}_1 = 1 \\ \dot{y}_i = 0 \quad \text{for } i = 2, \dots, n \end{array} \right\} \tag{17}$$

However, it is known that the existence of n functionally independent integrals y_i can in general be granted only locally, this point being related also to the existence of regions where the system behaves chaotically.

Summarizing, we have:

PROPOSITION 2. – *Given any solution $\bar{u}(\lambda, t)$ of (1), there are, generically, n independent Lie-point time-independent symmetry generators not transforming \bar{u} into itself; one further symmetry simply produces the transformation $\bar{u}(\lambda, t) \rightarrow \bar{u}(\lambda, t + \epsilon)$. All other symmetries can be obtained multiplying these by an integral of motion. It is possible to choose these $n + 1$ generators in such a way that only one has $v \neq 0$ and therefore involves changes in the parameter λ . \diamond .*

Let us comment on the meaning of these symmetries. Apart from the time translation (7), the $n - 1$ generators η_0 with $v = 0$ express the property of the vector field f_i of “being transformed under the symmetry as the variables u_i ”, i.e. a “generalized covariance” of equation (1) (with fixed λ), extending to LP symmetries the covariance property (2.7). Assume in fact that the functions θ_i in (4.6) are written in the form

$$\theta_i(\lambda, u) = \Theta_{ij} u_j + \psi_i(\lambda, u) \tag{18}$$

where Θ is a $n \times n$ matrix (possibly depending on λ) and $\psi_i(\lambda, u)$ are higher order terms. Then, condition (6) with $v = 0$ implies in particular

$$[\Theta, L] = 0 \tag{19}$$

where $L = L(\lambda) = \partial_u f(\lambda, 0)$. This is precisely the typical relation, well known in standard (via linear representations) covariant bifurcation theory ([Sat79], [Sat83], [GSS]): in the case of a Lie group of linear transformations, in fact, $\psi_i \equiv 0$ and Θ is the matrix representing the Lie generators of this group in \mathbb{R}^n , and (19) follows from (2.7).

In order to investigate the properties of the additional symmetry η_λ having $v=1$, let us consider first the particular class of solutions of (1) corresponding to bifurcations of non-trivial stationary solutions, $\dot{u}=0$. In this case, conditions (6) become

$$\theta_i \partial_i f_j = -v \partial_\lambda f_j, \quad f_i = 0 \tag{20}$$

Recalling that $f_j(\lambda, 0) = 0$ and then $\partial_\lambda f_j(\lambda, 0) = 0$ along the trivial branch S^0 where $u_0 = 0$, one obtains from (20) that along this branch, whenever

$$\det(\partial_i f_j(\lambda, 0)) \equiv \det(L(\lambda)) \neq 0$$

all symmetries have

$$\theta_i = 0 \tag{21}$$

and the only one remaining is

$$\eta_\lambda = \partial_\lambda \tag{21'}$$

which generates just the translation $\lambda \rightarrow \lambda + \varepsilon$ along the trivial branch S^0 . Then, the well known necessary condition $\det L(\lambda) = 0$ for the appearance of bifurcating nontrivial solutions naturally arises in this context.

Assume there is a continuous bifurcating branch, parametrized by a real parameter s :

$$f_j(\lambda(s), u(s)) = 0 \tag{22}$$

then

$$\partial_i f_j \frac{du_i}{ds} + \partial_\lambda f_j \frac{d\lambda}{ds} = 0 \tag{22'}$$

which is the same as (20) with the identification

$$\theta_i = du_i/ds, \quad v = d\lambda/ds \tag{23}$$

and this allows to interpret $\eta_\lambda = \theta_i \partial_i + v \partial_\lambda$ as the generator of translations along the bifurcating branch. More in detail, let $u_1 \neq 0, \lambda_1 \neq 0$ be a point in a bifurcating branch; if

$$\det(\partial_i f_j(\lambda_1, u_1)) \neq 0,$$

the above condition (22') uniquely determines θ_i in terms of v (we can fix e.g. $v=1$), and then the symmetry generating translations along the branch. If instead

$$\det(\partial_i f_j(\lambda_1, u_1)) = 0,$$

then other symmetries η_0 with $v=0$ are allowed, and they generate motions in the "plane" $\lambda = \lambda_1$, and correspond to the existence of a manifold S^b

with dimension > 1 , of bifurcating stationary solutions; these symmetries leave the intersection of S^b with the hyperplane $\lambda = \lambda_1$ invariant.

Similar results about the role of the generators η_0 and η_λ hold essentially unchanged also for the case of the Hopf bifurcation of periodic solutions. In agreement with Proposition 2, symmetries η_0 having $v=0$ (if any) will connect different bifurcating solutions with the same value of the parameter λ , whereas the symmetry η_λ with $v \neq 0$ will produce changes along the branching manifold, corresponding to changes of the parameter λ . This symmetry will connect a periodic solution with another similar (*i.e.* periodic) solution, but clearly in general with different period (as independently known from usual Hopf bifurcation theory). In order to take into account this fact, it is usual ([Sat79], [IJ]) to introduce a new real parameter ω and a new time variable t'

$$t' = \omega t \quad (24)$$

in such a way that all periodic solutions have period 2π in the rescaled time t' . As a consequence, we can introduce the new equation to the system (1)

$$\frac{d\omega}{dt} = 0 \quad (25)$$

which produces an additional time dependent symmetry generator of the form (with $\partial_\omega = \partial/\partial\omega$)

$$\eta_\omega = \eta_0 + \chi \partial_\omega \quad (26)$$

Using again the same procedure as for obtaining η_λ (12)-(14), and remembering the form (16) of the integral y_1 , we can obtain for the above generator η_ω the expression

$$\eta_0 = -p_{jk}(\partial_\omega h_j) \partial_k = tp_{1k} \partial_k; \quad \chi = \omega. \quad (27)$$

In order to clarify their action, it can be useful to give the explicit expression of the generators η_0 and η_λ in the following well known and very simple example [in which, for simplicity, the change of variable (24) is unnecessary, the frequency ω of the bifurcating solutions being in fact fixed].

Example 1. — With $u_1 = x$, $u_2 = y$, let

$$\left. \begin{aligned} \dot{x} &= \lambda x - \delta y - x(x^2 + y^2) \\ \dot{y} &= \lambda y + \delta x - y(x^2 + y^2) \end{aligned} \right\} \quad (28)$$

where δ is any constant (possibly 0). Apart from η_f and the $SO(2)$ symmetry generator

$$\eta_0 = x \partial_y - y \partial_x \quad (29)$$

the other symmetry generator, according to Proposition 2, is given by

$$\eta_\lambda = \left(\frac{1}{2\lambda} + \frac{\lambda - x^2 - y^2}{2\lambda^2} \ln \frac{x^2 + y^2}{\lambda - x^2 - y^2} \right) (x \partial_x + y \partial_y) + \partial_\lambda \quad (30)$$

Applying these generators to a generic solution, we obtain other solutions of (28), but once on the branch $u_0 = 0$ (and $\lambda \neq 0$) we obtain just

$$\eta_0 \equiv 0 \quad \text{and} \quad \eta_\lambda \equiv \partial_\lambda \quad (31)$$

as expected, and we remain on the trivial branch. When on the branch $\lambda = x^2 + y^2$, we have

$$\eta_\lambda = \frac{x}{2\lambda} \partial_x + \frac{y}{2\lambda} \partial_y + \partial_\lambda \quad (32)$$

and we see that we remain on the manifold S^b of bifurcating solutions applying to S^b both generators (32) and (29).

This example shows that the generator η_λ (32) becomes singular at the bifurcation point $u=0$, $\lambda=0$. Alternatively, recalling the arbitrariness in the function v , one could choose for η_λ , instead of (32), the form

$$\eta'_\lambda = x \partial_x + y \partial_y + 2\lambda \partial_\lambda \quad (32')$$

but with this choice $\eta'_\lambda \equiv 0$ at the bifurcation point. It is clear that this fact is generically true for any bifurcation point [roughly, η'_λ should give $\lambda \rightarrow \lambda + \varepsilon$, $u_0 \rightarrow u_0$ along the trivial solution $u = u_0 = 0$, and $\lambda \rightarrow \lambda + \varepsilon$, $u \rightarrow u + \delta u$, $\delta u \neq 0$ along the bifurcating branch, so that the unicity of the limit for $(\lambda, \mu) \rightarrow (0, 0)$ along the trivial and the bifurcating branch imposes $\eta'_\lambda = 0$].

It is interesting now to distinguish the two cases $\delta \neq 0$ and $\delta = 0$ in problem (28). If $\delta \neq 0$, we have periodic Hopf bifurcation on the manifold $\lambda = x^2 + y^2$, and, on this manifold, we obtain also

$$\eta_f \equiv \eta_0 = x \partial_y - y \partial_x. \quad (33)$$

Remark 3. — The above result (33), *i.e.* the identification of the generator η_f of time shift $t \rightarrow t + \varepsilon$ in periodic solutions with one of the other η symmetry generators [precisely with the $SO(2)$ generator] is actually true for any generic Hopf bifurcation: it is known in fact that in standard Hopf bifurcation the time shift $t \rightarrow t + \varepsilon$ is equivalent to a linear transformation in the \mathbb{R}^2 space of solutions (introducing normal forms (*see next section*) this is just a $SO(2)$ rotation [GS]).

Remark 4. — In agreement with a result given in [Wul], if the periodic bifurcating solution is (for fixed λ) a “limit cycle”, then all symmetries η_0 (but not η_λ) become trivial, *i.e.* either are zero or coincide with a multiple of η_f , on the bifurcating branch.

If instead $\delta=0$ in (28), we meet the case $\det(\partial_i f_j)=0$ on $\lambda=x^2+y^2$, and in fact a 2-dimensional manifold of stationary bifurcating solutions is present in this case.

Remark 5. — Notice, incidentally, that combining the symmetries of type (i) of Proposition 1 with the remaining η_0 , η_λ and η_f , one can obtain some interesting (time-dependent) symmetry generators: for the above example (28) with $\delta=0$, we can construct, e. g., the generator

$$\eta = \frac{x}{2\lambda} \partial_x + \frac{y}{2\lambda} \partial_y - \frac{t}{\lambda} \partial_t + \partial_\lambda$$

corresponding to the symmetry of the problem under the scaling transformation $x \rightarrow x \exp \varepsilon$, $y \rightarrow y \exp \varepsilon$, $t \rightarrow t \exp(-2\varepsilon)$, $\lambda \rightarrow \lambda \exp(2\varepsilon)$.

The main points of the above discussion can be summarized in the two following statements:

Remark 6. — Along the stationary solutions, one has clearly $\eta_f \equiv 0$. This fact, together with Remarks 3 and 4 above, give a characterization in terms of purely algebraic properties, concerning LP symmetries, of stationary and periodic bifurcations.

PROPOSITION 3. — *The submanifold of stationary bifurcating solutions, and that of periodic bifurcating solutions, are left invariant by the action of the $n+1$ symmetry generators given in Proposition 2.*

The following examples will show how these ideas may concretely work: the first one is actually rather simple, but allows us to provide the explicit expression in closed form of all its symmetry generators; the other example may be useful in order to show and stress that the method can be applied even if not all of the LP symmetry is known.

Example 2 [here and in the following example, $u \equiv (x, y, z) \in \mathbb{R}^3$]:

$$\left. \begin{aligned} \dot{x} &= xf(\lambda, z) - yg(\lambda, z) \\ \dot{y} &= yf(\lambda, z) + xg(\lambda, z) \\ \dot{z} &= zf(\lambda, z) + zh(\lambda, z) \end{aligned} \right\} \quad (34)$$

where f, g, h are given regular real functions. Apart from η_f [defined in (7)], this problem admits the following simple symmetries generated by

$$\begin{aligned} \eta_1 &= x \partial_y - y \partial_x \\ \eta_2 &= x \partial_x + y \partial_y \end{aligned}$$

which are independent of the explicit expression of the functions f, g, h . The generator η_λ can be obtained using (14) and can be written in the form

$$\eta_\lambda = -(\partial_\lambda \Phi) \eta_f + (\partial_\lambda \Psi_1) \eta_1 + (\partial_\lambda \Psi_2) \eta_2 + \partial_\lambda$$

where Φ is an integral function of $1/[z(f+h)]$ (integrated with respect to the variable z), Ψ_1 of $g/[z(f+h)]$, Ψ_2 of $f/[z(f+h)]$. It can be easily

verified that in the case $h \equiv 0$, the cones $x^2 + y^2 = \text{Const.} \times z^2$ are invariant manifolds under the flow of the dynamics. With $h \equiv 0$, assume that for $\lambda = \lambda^*$, $z = z^* \neq 0$ one has $f(\lambda^*, z^*) = 0$, $g(\lambda^*, z^*) \neq 0$; then all cones $x^2 + y^2 = \text{Const.} \times (z^*)^2$ contain a periodic solution (at the fixed level $z = z^*$) with period $2\pi/g(\lambda^*, z^*)$. Notice that, in agreement with Proposition 3, the above symmetries transform periodic solutions into periodic solutions: precisely, η_1 (just as η_f , according to the Remark 3) produces a “rotation” of the solution into itself, η_2 “dilates” the radius $r^2 = x^2 + y^2$ of the solution keeping $z = z^*$ fixed, and finally η_λ changes simultaneously λ and x, y, z . The z -axis contains instead a stationary solution $z = z^*$.

Example 3. – Putting $r^2 = x^2 + y^2$ and $w = z e^{-y}$, consider now

$$\left. \begin{aligned} \dot{x} &= x f(\lambda, r^2, w) - y g(\lambda, r^2, w) \\ \dot{y} &= y f(\lambda, r^2, w) + x g(\lambda, r^2, w) \\ \dot{z} &= y z f(\lambda, r^2, w) + x z g(\lambda, r^2, w) - e^y h(\lambda, r^2, w) \end{aligned} \right\} \quad (35)$$

where f, g, h are given regular real functions. Apart from η_f , a LP symmetry for problems of this form is

$$\eta_0 = x \partial_y - y \partial_x + x z \partial_z \quad (36)$$

Notice that

$$r^2 = x^2 + y^2 = \text{Const.}, \quad w = z e^{-y} = \text{Const.}$$

are invariant curves under η_0 . If e.g. $f = \lambda - r^2$, $h = z e^{-y}$, then Lyapunov-Schmidt reduction gives $z = 0$, and the reduced symmetry, evaluated according to Section 3, becomes the usual rotation around the z -axis: notice that the original problem (35) is *not* rotationally symmetric. We stress that in this case our construction gives some information about the form (symmetry) of the bifurcation equation which is not obtained from linear theory. In fact, if we consider the bifurcation equations constructed on the basis of symmetry data, no information can be provided by the linear theory, as no linear symmetry is present in the original problem, while considering the LP symmetry η_0 and its restriction $\beta_0 = x \partial_y - y \partial_x$ allows us to state that for any system of the form

$$\left. \begin{aligned} \dot{x} &= x(\lambda - r^2) - y g(\lambda, r^2, w) \\ \dot{y} &= y(\lambda - r^2) + x g(\lambda, r^2, w) \\ \dot{z} &= -z + x z g(\lambda, r^2, w) + y z(\lambda - r^2) \end{aligned} \right\}$$

the reduced 2-dimensional bifurcation equation is of the form

$$\left. \begin{aligned} \dot{x} &= x \alpha(\lambda, r^2) - y \beta(\lambda, r^2) \\ \dot{y} &= y \alpha(\lambda, r^2) + x \beta(\lambda, r^2). \end{aligned} \right\}$$

A different interesting situation occurs choosing, for instance,

$$f = \lambda - r^2, \quad h = \lambda z e^{-y} - r^4$$

now the whole \mathbf{R}^3 is the kernel of the linearized problem; in agreement with Remarks 3 and 6, and Proposition 3, the following bifurcating solution is found in this case

$$\lambda = r^2, \quad \lambda = z e^{-y},$$

which corresponds to a periodic solution if $g(\lambda, \lambda, \lambda) \neq 0$, and to a manifold of stationary solutions if $g(\lambda, \lambda, \lambda) = 0$.

Let us briefly mention here that there are other applications based on the introduction of LP symmetries of the given dynamical problem.

For instance, we can introduce an extension of the "equivariant bifurcation lemma" [GSS] to the case of LP symmetries. We do not give here the details (see [Cic], [Gae90b]); the argument is similar to that in the linear case, the main difference being that the role of the linear fixed subspaces under the symmetry subgroups is played in this case by manifolds. Similarly to the linear case, the main property of these manifolds is that of being flow-invariant, which allows a reduction of the original problem to a restricted one. For another application along the same lines, including an extension to gauge theories, see [Gae91].

5. LIE-POINT SYMMETRIES AND POINCARÉ NORMAL FORMS

In this section, we examine the close relationship existing between the classical problem of transforming the given dynamical system

$$\frac{du}{dt} = f(u) \quad (1)$$

in the Poincaré normal form [Arn] and the presence of LP symmetries. Since the equations of interest in this section do not involve the control parameter λ , we simply omit to indicate the dependence on it. First of all, we write the r. h. s. of equation (1) as a sum (or series) of different terms of order m , *i. e.*

$$f(u) = Lu + h(u) = Lu + h^{(2)}(u) + h^{(3)}(u) + \dots \quad (2)$$

where $h^{(m)}(u)$ is a n -vector field whose components are linear combinations of monomials $u^m = u_1^{m_1} u_2^{m_2} \dots u_n^{m_n}$, with $m = m_1 + m_2 + \dots + m_n \geq 2$, $m_i \geq 0$; we assume now that the functions $\theta_i(u)$ appearing in (4.4), (4.6) can be constructed as a formal series expansion

$$\theta_i = \Theta_{ij} u_j + \psi_i^{(2)}(u) + \psi_i^{(3)}(u) + \dots \quad (3)$$

where Θ is a constant $n \times n$ matrix, as introduced in Section 4, and $\psi^{(m)}(u)$ have the same meaning as the $h^{(m)}(u)$. The determining equation (4.6) for θ gives, separately for the various orders $m \geq 2$ [at the order $m = 1$, it gives

just condition (4. 19)]:

$$\left. \begin{aligned} (L \psi^{(2)})_k - (L u)_i \partial_i \psi_k^{(2)} &= (\Theta h^{(2)})_k - (\Theta u)_i \partial_i h_k^{(2)} \\ (L - (L u) \cdot \partial) \psi_k^{(3)} &= (\Theta - (\Theta u) \cdot \partial) \psi_k^{(3)} + \{h^{(2)}, \psi^{(2)}\}_k \\ (L - (L u) \cdot \partial) \psi_k^{(4)} &= (\Theta - (\Theta u) \cdot \partial) \psi_k^{(4)} + \{h^{(2)}, \psi^{(3)}\}_k + \{h^{(3)}, \psi^{(2)}\}_k \end{aligned} \right\} \quad (4)$$

(having introduced the shorthand notation

$$\{h^{(a)}, \psi^{(b)}\}_k = h_i^{(a)} \partial_i \psi_k^{(b)} - \psi_i^{(b)} \partial_i h_k^{(a)}$$

and so on. All equations in (4) have the form of "homological equations" (cf. [Arn])

$$D_L \psi^{(m)} = w^{(m)} \equiv D_{\Theta} h^{(m)} + \Sigma_{(m)} \{h^{(a)}, \psi^{(b)}\} \quad (4')$$

where at the r. h. s. of each order m the sum is extended to all possible brackets $\{h^{(a)}, \psi^{(b)}\}$ giving monomials of degree m , and D_L is the operator

$$D_L = (L - (L u)_i \partial_i) \quad (4'')$$

(and similarly for D_{Θ}).

Then, once a matrix Θ has been chosen in agreement with (4. 19), the r. h. s. of the first line in (4) is known, and we see that if the first p lines of (4) can be solved, then also the r. h. s. of the $(p+1)$ -th equation is known. But the l. h. s. are formally identical to those encountered performing the classical Poincaré procedure [Arn] for reducing the original problem (1) to normal or possibly linear form. Assume now that L can be diagonalized (this is not a restriction, cf. [Arn]), with eigenvalues σ_k and eigenvectors e_k ; denoting by $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)$ coordinates with respect to the basis e_k , then L also is diagonal in the space of vector-valued monomials $\tilde{u}^m e_k$, with eigenvalues

$$D_L \tilde{u}^m e_k = (\sigma_k - (\sigma, m)) \tilde{u}^m e_k \quad (5)$$

where $(\sigma, m) = \sigma_1 m_1 + \dots + \sigma_n m_n$, and the m -th equation of the system (4) splits into $N = \binom{n+m-1}{m}$ equations, N being the number of all monomials of degree m . Then, each one of these equations can be solved if the r. h. s. of (5) is never zero, *i. e.* if all the eigenvalues σ_k of L are nonresonant. But this is precisely the same condition ensuring the solvability of each step of the classical Poincaré method for reducing the system (1) to its linear part [Arn]. Therefore, we can say:

PROPOSITION 1. — *The sufficient conditions on the eigenvalues of the linear part L of (1) which ensure, according to the Poincaré procedure, the reduction of (1) to the linear form by a formal (resp. converging) series, ensure also the existence of a LP symmetry written as a formal (resp. converging) series.* \diamond

The main difference here with respect to Poincaré procedure for reduction to linear or normal forms is the special form of the r. h. s. of all

equation (4). More precisely, the arbitrariness in the matrix Θ [only condition (4.19) is to be satisfied] may allow to solve equation (4), even in the presence of resonances, by choosing Θ , whenever possible, in such a way that also the r. h. s. of all equations in (4) which contain resonant eigenvalues is equal to zero. For instance, we have:

PROPOSITION 2. — *If all nonlinear terms in the system (1) are resonant, there is at least one linear symmetry, which generates the scaling $\tilde{u}_i \rightarrow \tilde{u}_i \exp(\sigma_i \varepsilon)$, $\varepsilon \in \mathbb{R}$.*

Proof. — All equations (4) can be solved by $\psi_i^{(k)} = 0$ for all k, i , choosing

$$\Theta = \text{diag}(\rho_1, \rho_2, \dots, \rho_n)$$

and the numbers ρ_i satisfying all conditions $\rho_k = (\rho, m)$ for all monomials $\tilde{u}^m e_k$ appearing in the nonlinear part of the given system (1). The number of independent solutions ρ_i of all these conditions gives the number of the possible linear symmetries admitted by (1); the hypothesis that all the appearing monomials are resonant ensures that at least the choice $\sigma_i = \rho_i$ is a solution. ●

The above result coincides with one of the results given in [Elp]: indeed, according to the Poincaré-Dulac theorem [Arn], any system (1) may be converted, by a formal or converging series, into a system containing only resonant terms, and the linear symmetry obtained in proposition 2 above is in facts generated by the linear part of the vector field defining the dynamical system.

As well known, an example of this situation is given by the standard Hopf bifurcation problem: at the bifurcation point in facts the eigenvalues of L are resonant and imaginary $\pm i\omega_0$ (we can assume here $n=2$). Once reduced to the normal form, the linear symmetry generated according to Proposition 2 can be written in the form

$$z \rightarrow z \exp(i\omega_0 \varepsilon) \tag{6}$$

having introduced the complex vector $z = u_1 + iu_2$ as usual: then this expresses just the known property that the normal form of this problem exhibits an explicit covariance under the rotation group $S^1 = \text{SO}_2$ ([Sat83], [GS], [GSS]). Reinserting now the control parameter λ , a bifurcating periodic solution, under standard Hopf hypotheses, has the form $z = \hat{z} \exp(i\omega t)$ where \hat{z} , ω depend on λ , and the Hopf problem simply corresponds to a “linearization” [CG86] of the system, where the time translation plays exactly the role of the linear symmetry mentioned in the above proposition. For further details, see [CG90].

6. BIFURCATION POINTS AND SYMMETRY ALGEBRA

We would like to briefly discuss the relation existing between the (LPTI) symmetry algebra \mathcal{G}_Δ of an equation

$$\Delta_\lambda(x, u^{(1)}) \equiv \dot{u} - G(\lambda, u) = 0; \quad u \in U \tag{1}$$

and the existence of a bifurcation point for it.

By the implicit function theorem we have, as already recalled [see section 3, equations (24)-(27)], that in a neighbourhood $A \subset \Lambda \times U$ of (λ_0, u_0) the stationary solution manifold $S_\Delta^{(st)}$ (from now on in this section we will write S_Δ for $S_\Delta^{(st)}$, for ease of notation) can be written as

$$S_\Delta = (S_\Delta^0 \cup S_\Delta^b) \cup S_\Delta^z \tag{2}$$

where \cup denotes disjoint union and S_Δ^z represents big solutions, so that by appropriately reducing the size of A (but keeping it nonzero and finite) we have

$$A \cap S_\Delta = (S_\Delta^0 \cup S_\Delta^b) \cap A \tag{3}$$

Here and in (2), S_Δ^0 corresponds to $u = u_0$, while S_Δ^b to bifurcating solutions. One has therefore

$$S_\Delta^0 \cap S_\Delta^b = (\lambda_0, u_0) \tag{4}$$

$$S_\Delta^0 = \Lambda \times u_0 \tag{5}$$

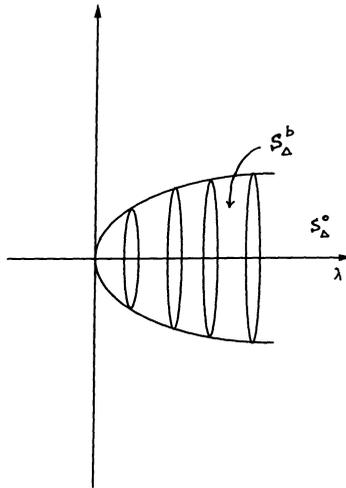


FIG. 4.

The situation is depicted in Figure 4 (notice this applies as well, *mutatis mutandis* to Hopf bifurcation).

Now, if $\eta \in \mathcal{G}_\Delta$, by definition $\eta : S_\Delta \rightarrow TS_\Delta$ (for η LPTI, $\eta : S_\Delta^{(st)} \rightarrow TS_\Delta^{(st)}$); by (4) we have, for $\lambda \neq \lambda_0$,

$$\left. \begin{aligned} \eta : S_\Delta^0 &\rightarrow TS_\Delta^0 \\ \eta : S_\Delta^b &\rightarrow TS_\Delta^b \end{aligned} \right\} \tag{6}$$

On the other end, we have remarked in section 4 that the algebra \mathcal{G}_Δ contains in particular the algebra $\{\partial_t \oplus \eta_G \oplus \mathcal{G}_\Delta^0\}$, where $\eta_G = \partial_t + G(\lambda, u) \partial_u$ and \mathcal{G}_Δ^0 is the algebra of LPTI symmetries. We have also seen that in general we can choose one of the vector fields spanning \mathcal{G}_Δ^0 , say η_1 , to be of the form

$$\eta_\lambda = \partial_\lambda + \xi(\lambda, u) \partial_u \tag{7}$$

while all the others can be chosen in the form

$$\eta_k = \theta_k(\lambda, u) \partial_u \tag{8}$$

Correspondingly, the manifold W of section 3 and the S_Δ^0, S_Δ^b have a natural fibered structure, *i.e.* locally

$$\left. \begin{aligned} W &= \Lambda \times W_\lambda \\ S_\Delta^0 &= \Lambda \times S_\Delta^{0(\lambda)} \equiv \Lambda \times u_0 \\ S_\Delta^b &= \Lambda \times S_\Delta^{b(\lambda)} \end{aligned} \right\} \tag{9}$$

and $W_\lambda, S_\Delta^{j(\lambda)}$ are invariant under η_k ,

$$\left. \begin{aligned} \eta_k : S_\Delta^{0(\lambda)} &\rightarrow TS_\Delta^{0(\lambda)} \\ \eta_k : S_\Delta^{b(\lambda)} &\rightarrow TS_\Delta^{b(\lambda)} \end{aligned} \right\} \tag{10}$$

This means that

$$\theta_k(\lambda, u_0) = 0 \quad (\lambda \neq \lambda_0) \tag{11}$$

and for η_λ , with $\{u_i(s), \lambda = \lambda(s)\}$ a set of bifurcating solutions, *i.e.* $S_\Delta^{b(\lambda)} \in \text{span}\{u_i(s)\}$,

$$\left. \begin{aligned} \xi(\lambda, u_0) &= 0 \quad (\lambda \neq \lambda_0) \\ \xi(\lambda, u_i(s)) &= \frac{\delta u_i(s)}{\delta \lambda(s)} \quad (\lambda \neq \lambda_0) \end{aligned} \right\} \tag{12}$$

Example. – Let us consider the equation

$$\dot{u} = (\lambda - u^2)u, \quad u \in \mathbb{R}^2 = \{x, y\}$$

Then (12₁) implies $\xi(\lambda, 0) = 0$, and on the bifurcating manifold

$$S_\Delta^b = \{(\lambda, u)/u^2 = \lambda\} \in \text{span}\{u_1 = \sqrt{\lambda} \cdot (1, 0); u_2 = \sqrt{\lambda} \cdot (0, 1)\}$$

we get $\xi = x/2\lambda$; equivalently we choose

$$\eta_\lambda = 2\lambda \partial_\lambda + u \partial_u$$

As for the η_k , there is only one such vector field, corresponding to $SO(2)$:

$$\eta_2 = x \partial_y - y \partial_x$$

Notice that if λ is considered as a *given*, not varying, parameter [so that all LPTI vector fields are of the form (8)] our condition (10) reads $\eta = \eta_2$ or $\theta = \theta[u(\lambda - u^2)]$ for $\lambda \neq \lambda_0$, while θ is arbitrary for $\lambda = \lambda_0$. (Notice that $\theta = \theta[u(\lambda - u^2)]$ corresponds to a trivial symmetry.)

This corresponds to the fact that in $\lambda = \lambda_0$, $TS_{\Delta}^{b(\lambda)} \cap (\lambda_0 \times \mathbb{R}^2) = \mathbb{R}^2$, see Figure 4. Notice also that $\eta_2|_{S_{\Delta}} \equiv 0$. \diamond

A little thinking shows that the situation met at the end of the above example is actually general. We formalize this as follows:

PROPOSITION. — For equation (1), let $\mathcal{G}_{\Delta}^{0(\lambda)}$ be the algebra of LPTI vector fields leaving $S_{\Delta}^{(\lambda)}$ invariant, where

$$S_{\Delta}^{(\mu)} = S_{\Delta} \cap (\mu \times U) \subset \Lambda \times U$$

$$\mathcal{G}_{\Delta}^{0(\mu)} = \{ \theta(\lambda, u) \partial_u / \theta(\mu, u) : S_{\Delta}^{(\mu)} \rightarrow TS_{\Delta}^{(\mu)} \}$$

Then if (1) admits a bifurcation point at $\lambda = \lambda_0$, for μ a neighbourhood of λ_0 in Λ and A a neighbourhood of (λ_0, u_0) in $\Lambda \times U$, the algebrae $\mathcal{G}_{\Delta}^{0(\mu)}$ are isomorphical among them for different μ 's for $\mu \neq \lambda_0$, while $\mathcal{G}_{\Delta}^{0(\lambda_0)}$ is made of arbitrary vector fields $\theta \partial_u$ with θ satisfying only $\theta(\lambda_0, 0) = 0$. \diamond

The above discussion furnishes as well the setting to make precise the requirement that “the degeneration in the critical eigenvalues is entirely due to the symmetry of the problem”: this is expressed, in the above notation, by the requirement that \mathcal{G}_{Δ}^{0} acts transitively on $S_{\Delta}^{(\lambda)}$ for any $\lambda \neq \lambda_0$.

We would like to thank professor Tanizhmani for raising the question discussed in this section.

7. EXTENSIONS

Up to now we have been discussing autonomous ODE, and actually only their stationary solutions. In this “final remark” section, we would like to comment on the possibility to extend our procedure and results to more general cases.

First of all (as recalled before), the standard bifurcation of periodic solutions (Hopf bifurcation) can be reconducted to the setting of the stationary case and Lyapounov-Schmidt (L/S) reduction by considering the operator $\tilde{A}(\lambda, \cdot)$; $\tilde{A}(\lambda, u) \equiv \frac{\partial u}{\partial t} - G(\lambda, u)$ and introducing the auxiliary parameter $\omega = 2\pi/T$, where T is the period of the solutions, and ω varies with λ ([IJ], [Sat79]), or more precisely one has families $u = u(s)$, $\lambda = \lambda(s)$,

$\omega = \omega(s)$, so there is no difficulty in extending our method and results to Hopf bifurcation.

We also remark that by this (Sattinger-Joseph) approach, the “temporal” $SO(2)$ symmetry intrinsic to Hopf bifurcation, and corresponding to time shifts along the periodic solutions is mapped into an explicit (“spatial”) $SO(2)$ symmetry in solution space ([GS], [GSS]).

For generic bifurcating solutions, our picture still holds if we consider only *asymptotic solutions*. More precisely, we consider the center-unstable manifold W_{cu} ([GH], [HPS], [Rue89]) for the system

$$\begin{aligned} \dot{u} &= G(\lambda, u) \\ \dot{\lambda} &= 0 \end{aligned}$$

[Then solutions starting in a neighbourhood of W_{cu} (which is in general only a *local* manifold) are exponentially attracted towards W_{cu} , and in particular for $t \rightarrow \infty$ all the solutions in a neighbourhood of $(u, \lambda) = (u_0, \lambda_0)$ belong to W_{cu} .]

We can then repeat our treatment of section 3, with the role of W played by W_{cu} . Anyway, now the solutions on the manifold W_{cu} have necessarily nearby solutions not contained in W_{cu} (e.g. those converging to solutions in W_{cu}), so that Lemma I of section 3 has no reason to hold; our discussion extends therefore, if the bifurcation equation is obtained via a center manifold reduction, only to those symmetries $\eta \in \mathcal{G}_\Delta$ that, once restricted to W_{cu} , satisfy $\eta: W_{cu} \rightarrow TW_{cu}$.

Notice that $\dot{\lambda} = 0$ ensures that W_{cu} is fibered as $\Lambda \times W_{cu}^\lambda$ by invariant manifolds W_{cu}^λ which are tangent to \mathcal{N} in u_0 and depend smoothly on λ .

Our treatment does therefore essentially extend to the case of generic bifurcation for (autonomous) ODE.

Let us now consider the case of (autonomous) time-evolution PDE, referring again for simplicity to the bifurcation of stationary solutions, *i.e.* to the L/S case (the method of Sattinger and Joseph applies anyway to PDE as well, so this also covers the periodic bifurcation case again).

The L/S reduction has essentially the role of selecting a finite-dimensional function space $\mathcal{N} \subset \mathcal{B}$, where now the equation would be

$$\dot{u} = G(\lambda, u); \quad u \in \mathcal{B}, \quad G: \Lambda \times \mathcal{B} \rightarrow T\mathcal{B}$$

\mathcal{B} a Banach (at least) space. In this case, W would still be a *finite dimensional* manifold, but it is naturally a submanifold of an infinite-dimensional (function) space.

Now, if we cast the L/S reduction in jet space, as we have done in section 3, there is a problem: to select a function space, or functions, means giving conditions on the derivatives of all orders, as only by knowing derivatives of all orders one can reconstruct (C^∞) functions. Therefore, in order to see W as a submanifold of jet space, we are forced

to consider the (formal) infinite-order jet space $M^{(\infty)}$, and infinite order prolongations of functions [Olv].

By the way, we notice that also imposing boundary conditions does select a function space, and can therefore be interpreted geometrically as selecting a (generically, infinite-dimensional) submanifold of $M^{(\infty)}$. This means that when we deal with PDE we are naturally lead, unless we want to consider only a formal equation, to the infinite order jet space.

At this point, if one is ready to work in $M^{(\infty)}$, there is no reason to consider only Lie-point symmetries rather than generalized ones.

On the other end, our discussion still applies, since we considered only operations (restriction to a finite-dimensional submanifold W and projections from this to the tangent space $\Lambda \times \mathcal{N}$) which are still defined. Therefore, our method and results do formally apply to PDE as well, and the correspondence theorem of section 3 should actually hold for generalized symmetries of the original equation as well.

We would like to point out that dealing with $M^{(\infty)}$ is customary in considering generalized symmetries [Olv] and that our method would seem to be naturally cast, in the PDE case, in the language of *diffeities* developed by Vinogradov *et al.* ([Vin84], [Vin89]). We hope to be able to treat these topics in a future work.

It should be remarked, anyway, that recently it was developed a method to transform e. g. a PDE with Neumann boundary condition on a square into a PDE on a torus, the solution of the original problem corresponding to the solution of the problem on the torus which satisfy some symmetry conditions; we refer to [Cra] for details and applications of this method (which can clearly be extended to other geometries). Here we just want to stress that for certain kinds of boundary conditions a PDE problem can be set naturally in terms of finite order jets, so that in this case our method can be extended to PDE bypassing the problem of consideration of boundary conditions.

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