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Multifractal analysis of nearly circular Julia set and thermodynamical formalism

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ABSTRACT. – The multifractal description of measures supported by strange sets has been introduced in order to analyse actual experimental data in chaotic systems. A justification of this approach on the basis of the thermodynamical formalism has previously been given for Cantor sets invariant under Markov maps. Here we give the analogous derivation for connected sets invariant under polynomial maps. More precisely, for polynomials close to z^q, we consider the uniform Z_U and the dynamical Z_D partition functions associated to coverings of the Julia set and we show that the corresponding thermodynamical limits F_U and F_D exist, when the size of the pieces of the coverings goes to zero. We then show an explicit relation between F_U and F_D and explain how to expand F_D close to the circle case, corresponding to z^q. Results from the large deviation probability theory are used to show the relation between F_U and the dimension spectrum, which includes Hausdorff dimension as a particular case. The method used here provides a complete description of the multifractal properties of nearly circular Julia sets and an explicit perturbation procedure for the Hausdorff dimension and for the multifractal spectrum.

Key words : Multifractals, Hausdorff dimension, Julia sets, local scaling laws, thermodynamic formalism.

RÉSUMÉ. – L'analyse multifractale des mesures supportées par les ensembles étranges s'est révélée nécessaire à l'étude des résultats expérimentaux obtenus dans les systèmes chaotiques. Dans le cas des ensembles de Cantor laissés invariants par des applications de type markovien, on a pu donner une justification de cette approche, dans le cadre du formalisme thermodynamique. Dans ce travail, nous suivons une procédure analogue pour les ensembles invariants par une transformation polynomiale du plan complexe, dans des cas où ces ensembles sont connexes. Plus précisément, nous considérons les fonctions de partition uniforme Z_U et dynamique Z_D correspondant à des recouvrements de l'ensemble de Julia associés à des polynômes voisins de z^q, et nous montrons que les limites thermodynamiques F_U et F_D existent, quand la taille des pièces des recouvrements tend vers zéro. Nous établissons ensuite une relation explicite entre F_U et F_D et nous montrons comment on peut effectuer un développement de F_D pour le cas des polynômes voisins de z^q, c'est-à-dire quand l'ensemble de Julia est voisin du cercle. Les résultats de la théorie probabiliste des grandes déviations permettent de montrer la relation liant F_U avec le spectre des dimensions, et en particulier la dimension de Hausdorff. Les méthodes que nous utilisons donnent une description complète des propriétés multifractales des ensembles de Julia voisins du cercle, ainsi qu'une procédure perturbative explicite pour calculer la dimension de Hausdorff et le spectre des dimensions multifractales.

Mots clés : Multifractales, dimension de Hausdorff, Ensembles de Julia, lois d'échelles locales, formalisme thermodynamique.

1. INTRODUCTION

In chaotic dynamics, experiments as well as numerical simulations produce unstable trajectories from which only statistical information can be extracted. This is the case when these trajectories belong to a strange attractor, which means that they are highly sensitive to initial conditions. Although it might be easy to display a graphical representation of such an attractor, one needs a specific method to extract numerical parameters associated to it, in order to try a meaningful comparison with theoretical arguments.

A good example is the work of Jensen *et al.* [1] on the convective flow in mercury, where various scaling exponents related to the chaotic attractor are compared with values obtained using universality arguments for critical circle maps. They use a so called thermodynamical approach which has been sketched by various authors ([2]-[14]), and our purpose is to show that an interpretation using large deviation probability theory would clarify many features of this thermodynamical approach for the so called "multifractal" description of strange sets, also considered in the probabilistic framework for the turbulence problem ([8]-[9]).

The relation between large deviation probability theory and themodynamics is well explained in the work of Landford [15], and the same analogy has been applied to multifractals by Collet *et al.* [16], more precisely to invariant measures of expanding Markov maps.

In this paper, we intend to give a simple exposition of this approach, and we will illustrate the results in a somewhat different case, that is a measure invariant under a polynomial transformation on the complex plane. More precisely, we will consider measures supported by Julia sets associated to complex valued polynomials [17]. For the mononomial z^q, with q integer not smaller than 2, the Julia set is the unit circle. We will be particularly interested in small perturbations of the monomial case, and we intend to discuss the variations of the geometrical aspects of the set, as well as the quantitative properties linked to the measure.

One of the outcome of the paper will be the real analyticity properties, as function of the parameters of the polynomial transformation, of the Hausdorff dimension, and of the correlation dimensions of arbitrary order which form the dimension spectrum. This result was already given in Ruelle [18] for the Hausdorff dimension, but is obtained here by a different argument which does not require a systematic analysis of the fixed points. We also show how to extend previous perturbative results on the Hausdorff dimension ([18]-[19]), to the dimension spectrum.

Although our presentation will be centered around the Julia set case, the exposition of the multifractal formalism using large deviation arguments is more general. However the existence of the "thermodynamic limit" needs some specific hypotheses, for instance the invariance of the set and of the measure supported by it, under an expansive transformation. The key point in our case, is that the expansive transformation provides a way of comparing different scales in an uniform way, due to the so-called distortion lemma described in section 2. Then, comparing different scales permits to evaluate the various correlation dimensions, including the Hausdorff dimension of the support.

We consider here a dynamical system in one complex dimension. The invariant set is not an attractor, and the transformation is repulsive on the set, a situation analogous to the cases of Markov maps [16] or cookie cutters [10]. In order to get a strange attractor, one needs to consider a non conformal map in at least two real dimensions, that at least two complex dimensions (for its complex version). The relevance of our analysis to actual physical situations could be questioned, but the usual argument is that what we model is in fact a kind of Poincaré return map associated to a diffeomorphism and the invariant measure we consider is nothing but what we get by considering only the transverse unstable directions [40].

The paper is organised as follows: In section 2, we present the properties of Julia sets relevant to our purpose. In section 3, we give a definition of multifractality using box counting arguments, we then introduce the uniform partition function F_U and recover the multifractal spectrum using large deviations arguments, assuming the existence of the thermodynamic limit. In section 4, we show the relation between the thermodynamic formalism and the various geometrical dimensions, in particular the Hausdorff dimension and the so called dimension spectrum. In section 5, we show the relation between dimension spectrum and local densities of the measure. In section 6, we prove the existence of the thermodynamic limit for the uniform partition function Z_U. In section 7, we introduce the dynamical partition function Z_D and show how the corresponding "free energy" F_D is related to F_U. In section 8, we discuss the analyticity properties of the thermodynamical functions, from which the analyticity properties of the various dimensions result immediately. In section 9, we describe the perturbative expansion for the dimension spectrum and we give a short general comment. The perturbation expansion is developped up to fourth order in the appendix.

This work has already reported in part in two conferences ([20]-[21]), the present version intends to provide all details.

2. NEARLY CIRCULAR JULIA SETS

We consider a polynomial with complex coefficients $T(z)$, of degree $q \geqq 2$, such that:

$$T(z) = z^q + \lambda p(z), \tag{1}$$

where $p(z)$ is a polynomial of degree at most $(q-1)$ and λ a small complex scale parameter. When $|\lambda|$ is sufficiently small, $T(z)$ acts nearly as z^q on the complex plane, more precisely the following proposition can be checked by elementary calculations. For $\rho>1$, we define $\mathscr{A}_\rho$ as the annular region: $z\in\mathscr{A}_\rho$ when and only when $(1/\rho)<|z|<\rho$.

PROPOSITION 2.1. — *Given* $\rho>1$, *there exists* $\varepsilon>0$ *such that* $\forall\lambda$ *with* $|\lambda|<\varepsilon$, *and* $\forall z'\in\mathscr{A}_\rho$, *all preimages of* z', *that is all roots of the equation* $T(z)=z'$, *also satisfy* $z\in\mathscr{A}_\rho$. *In addition, when* z' *describes the circle of radius* r *with* $(1/\rho)<r<\rho$, *each preimage of* z' *describes an arc, and the* q *different arcs joined together form an analytic curve in* $\mathscr{A}_\rho$ *performing one turn around the origin.*

The Julia set J_T associated to T is the boundary of the basin of attraction of infinity under iterations of T. It is also defined as the set of points with forward orbit contained in $\mathscr{A}_\rho$, therefore we have:

$$J_T=\bigcap_{n=1}^{\infty} T^{-n}(\mathscr{A}_p), \tag{2}$$

where $T^{-n}(\mathscr{A})$ denotes the inverse image of $\mathscr{A}$ under the n-th iterate of T. A measure is naturally defined with support on the Julia set, as the asymptotic distribution of predecessors [22]: we first take an arbitrary starting point $z_0\neq 0$, and we consider its q^n preimages, z_i, $i=1, \ldots, q^n$, all being solutions of $T^n(z_i)=z_0$. We then define at order n the discrete measure μ_n as the sum of q^n Dirac measures located at each z_i, affected with equal weights $(1/q^n)$ for normalisation. The sequence of measures μ_n has a limit, in the weak sense, which appears to be the harmonic measure μ on the Julia set associated to T.

From its definition, it is easily seen that the measure μ is invariant under T and balanced [23], which means:

$$\mu(T^{-1}(B))=\mu(B), \tag{3}$$

and:

$$\mu(T_i^{-1}(B))=(1/q)\,\mu(B) \qquad \text{for} \quad i=1, \ldots, q, \tag{4}$$

for any inverse branch T_i^{-1}, and any Borel set B in the complex plane. A consequence of the latter equation will be useful: if the set B is sufficiently small, as well as the set $B'=T(B)$, more precisely if the various preimages of B' are well separated, we have:

$$\mu(T(B))=q\,\mu(B), \tag{5}$$

or for two different sufficiently small Borel sets I and J, with $\mu(J)\neq 0$, we have:

$$(\mu(T(I))/\mu(T(J)))=(\mu(I)/\mu(J)). \tag{6}$$

In the previous equation, we assume $\operatorname{diam}(I)<\delta$, where $\operatorname{diam}(I)$ denotes the diameter of the set I, that is the largest possible distance between any pair of points in I, and δ is somewhat arbitrarily taken to be equal to a half of the distance between two consecutive roots of unity of order q. With such an assumption, the preimages of I and J will be disjoint, at least for ε small enough, as in proposition 2.1.

Another useful property is related to the fact that the parameter ρ can be chosen in such a way that: $|T'(z)| > \eta > 1$ for some η, $\forall z \in \mathscr{A}_\rho$. In other words T is expansive on $\mathscr{A}_\rho$. A consequence is the so called distorsion lemma [24]:

PROPOSITION 2.2 (Distorsion Lemma). — *Under the same conditions as proposition* 1.1, *consider two orbits of length n, x_i and y_i, for $i=1, \ldots, n$, defined for $i \geqq 1$ by: $x_{i+1} = T(x_i)$ and $y_{i+1} = T(y_i)$ such that $\forall i$, $|x_i - y_i| < C < \delta$, with δ as above, and such that $\forall i$, $x_i \in \mathscr{A}_\rho$ and $y_i \in \mathscr{A}_\rho$. Then we have: $|x_i - y_i| < C\eta^{i-n}$, for all $i=1, \ldots, n$. Moreover there exists $\gamma > 1$ such that:*

$$\forall j=1, \ldots, n \quad (1/\gamma) < \left| \frac{T^{j'}(x_1)}{T^{j'}(y_1)} \right| < \gamma. \tag{7}$$

Finally, for any sets I *and* J *such that $I_n \subset \mathscr{A}_\rho$ and $J_n \subset \mathscr{A}_\rho$ where $I_n = T^n(I)$ and $J_n = T^n(J)$, with* diam(I_n) *and* diam(J_n) *both smaller than δ, and such that* I *and* J *are obtained from* $T^n(I)$ *and* $T^n(J)$ *respectively, by application of the same inverse branch of* T^n, *we have:*

$$(1/\gamma) < \left(\frac{\text{diam}(I)}{\text{diam}(J)} \right) \Big/ \left(\frac{\text{diam}(T^j(I))}{\text{diam}(T^j(J))} \right) < \gamma, \qquad \forall j=1, \ldots, n. \tag{8}$$

The important fact is that the constant γ in equations (7) and (8) does not depend on n. The last property we need about Julia sets is the following statement [22]:

PROPOSITION 2.3. — *Given any open set $\mathcal{O}$ such that $\mathcal{O} \cap J_T \neq 0$, there exists a finite* N *such that $T^N(\mathcal{O}) \supset J_T$. In fact one can also find another integer* R *such that $T^R(\mathcal{O}) \supset \mathscr{A}_\rho$.*

Indeed we will only use the existence when $\mathcal{O}$ is small, of a finite N such that $T^N(\mathcal{O})$ has a diameter of the order of δ, and therefore a measure of the order of $(1/q)$, that is a finite number. We have formulated the propositions in the present section only for the case of Julia sets close to unit circle, because they can receive in this case elementary justifications, and we will not need here to consider the more general case. However it is worth mentioning that corresponding statements can be made for conformal transformations in the hyperbolic case, the difficulty occuring mainly in the case where critical points of T belong to the Julia set J_T. Such a situation does not occur for Julia sets associated to polynomials close to z^q, since the derivative of the polynomial in the vicinity of the unit circle has a modulus close to q.

3. BOX COUNTING AND LARGE DEVIATIONS

We consider a probability measure μ with support contained in a bounded set $\mathcal{S}$ in the complex plane – or equivalenty in the two dimensional real plane – and we assume normalisation as:

$$\int_{\mathcal{S}} d\mu = 1. \tag{9}$$

We draw in the plane a square lattice, each individual square-called a box-with size 2^{-n}. The number of boxes needed to cover $\mathcal{S}$ is bounded by $A2^{2n}$, for n large, where A is the area of some bounded set including $\mathcal{S}$. Now select boxes b such that for $t>0$:

$$t < \frac{(-1)}{n} \log_2 (\mu(b)) < t + dt, \tag{10}$$

and call $N_n(t)\,dt$ their number. The "box counting" assumption is:

$$\lim_{n \to \infty} \frac{1}{n} \log_2 (N_n(t)) = f(t). \tag{11}$$

In order to introduce the thermodynamical formalism, it is now convenient to introduce the uniform partition function Z_U defined by:

$$Z_U^{(n)}(\beta) = \sum_b (\mu(b))^\beta. \tag{12}$$

The previous sum extends over all boxes b needed to cover $\mathcal{S}$, such that $\mu(b) \neq 0$, therefore negative values of β are allowed. The box counting assumption allows to evaluate the behaviour for large n of $Z_U^{(n)}(\beta)$. Indeed we have:

$$Z_U^{(n)}(\beta) \simeq \int_0^\infty N_n(t)\, 2^{-n\beta t}\, dt. \tag{13}$$

Since $N_n(t) \sim 2^{nf(t)}$, we can estimate the partition function as:

$$Z_U^{(n)}(\beta) \simeq \int_0^\infty 2^{n(f(t) - \beta t)}\, dt. \tag{14}$$

Therefore $Z_U^{(n)} \sim 2^{-nF_U(\beta)}$, that is we have the "thermodynamic limit":

$$\lim_{n \to \infty} \frac{(-1)}{n} \log_2 (Z_U^{(n)}(\beta)) = F_U(\beta), \tag{15}$$

where $F_U(\beta)$, the uniform "free energy" associated to the measure μ, is defined by the following condition:

$$F_U(\beta) = \inf_{t \geqq 0} (\beta t - f(t)). \tag{16}$$

The relation between $f(t)$ and $F_U(\beta)$ is the usual Legendre transform, similar to the relation between entropy and free energy in classical thermodynamics. We shall give more details in the next section.

In fact the previous arguments receive a rigourous treatment in the frame of the large deviation probability theory ([15], [25]-[26]). Following Collet *et al.* [16], we shall see that the existence of the thermodynamical limit ensures some kind of box counting statement. More precisely, we define $N_n^<(t)$ (resp. $N_n^>(t)$), as the number of boxes b with size 2^{-n}, such that $\mu(b)>2^{-nt}$ (resp. $\mu(b)<2^{-nt}$). The opposite choice of the direction of the latter inequality is intented in order to get the following equations (18) and (19). Then we have the following proposition:

PROPOSITION 3.1. – *Given the measure* μ, *we assume that the uniform partition function defined in equation* (12), *fulfils the thermodynamical limit* [*equation* (15)]. *We also assume that the resulting* $F_U(\beta)$ *is continuously derivable. We then define the function* $f(t)$ *as:*

$$f(t)=\inf_{\beta}(t\beta-F_U(\beta)). \tag{17}$$

The function $f(t)$ *is thus a convex function, so there is a value* t_m *such that for* $t<t_m$, *the function* $f(t)$ *is non-decreasing, and for* $t>t_m$, *the function* $f(t)$ *is non-increasing. Then we have:*

$$\textit{for:}\quad t<t_m,\qquad f(t)=\lim_{n\to\infty}\frac{1}{n}\log_2(N_n^<(t)), \tag{18}$$

and:

$$\textit{for:}\quad t>t_m,\qquad f(t)=\lim_{n\to\infty}\frac{1}{n}\log_2(N_n^>(t)). \tag{19}$$

The large deviation property have been derived in many areas of probability theory and this analysis goes back to Cramér and Chernoff (*see* Ellis [25]). An easy proof of the previous statement adapted to our purpose is given by Plachky *et al.* ([27]-[28]). For $\bar{t}$ larger than t_m, we note that in the result (19), the condition $t>\bar{t}$ can be replaced, in some loose sense, by $t=\bar{t}$ because the number of boxes decreases exponentially with n, with a rate which becomes faster when t increases. A similar but reversed statement holds for t smaller than t_m. Exceptionally, $f(t)$ may be constant on some interval and therefore t_m is not unique, in which case the above statements (18) and (19) remain true for any t_m in this interval.

In the case where there is no constant plateau, and therefore a unique value t_m, we see that the number of boxes corresponding to a value $t\neq t_m$, is negligible in the large n limit, in comparison to the number of boxes corresponding to $t=t_m$. Nevertheless the proposition 3.1 gives a precise evaluation of their probability of occurence. This is the meaning of large deviation arguments, which tell precisely how large deviations amount to small probabilities.

As it has been explicitely stated, the proposition 3.1 requires not only the existence of the thermodynamic limit, equation (15), but also the derivability of the limit at the point β considered. Some statements can be made when no such regularity assumption are made on this limit [28], but they are more cumbersome and not necessary here. Nevertheless such considerations are needed in the case where discontinuities occur in the derivative of the free energy, which reflects the presence of a phase transition ([29]-[30]). We shall see in the following sections how the existence of the limit can be proven to exist with the suitable regularity assumptions in the case of the Julia sets close to unit circle.

4. HAUSDORFF DIMENSION AND DIMENSION SPECTRUM

The purpose of this section is to relate the functions $f(t)$ and $F_U(\beta)$ to geometrical dimensions related to the measure μ and to its support. We will recall first some general properties of the Legendre transform which connects $F_U(\beta)$ and $f(t)$, given by equation (16). When $f(t)$ is continuous but not convex, the function $F_U(\beta)$ given by (16) is nevertheless convex. Then the new f, say $\bar{f}$, obtained from F by (17) is nothing else than the convex envelope of the initial f, *i.e.* the smallest possible convex function greater than f. When the starting function f is convex continuous, applying the Legendre transform twice gives back the same function f. In order to avoid a special treatment of some particular cases, it is convenient to include the value $-\infty$ as a permitted value for F or f, with the convention that a convex continuous function must be infinitely negative for either all values of the variable greater than α, or for all values smaller than α, as long as it is infinitely negative for α. Discontinuities are permitted only at such a value, but must in fact be interpreted as the presence of vertical lines in the graph of the function, and may therefore be ignored.

We recall the classical formulas when the functions are differentiable. From (16) by derivation we get:

$$f(t)=t\,\beta-F_U(\beta), \quad \text{where } \beta \text{ is given by: } \frac{\partial F_U(\beta)}{\partial \beta}=t. \tag{20}$$

$$F_U(\beta)=\beta\,t-f(t), \quad \text{where } t \text{ is given by: } \frac{\partial f(t)}{\partial t}=\beta. \tag{21}$$

The geometrical interpretation of the Legendre transformation is then as follows: the tangent line with slope β to the graph of the function $f(t)$

intersects the vertical axis at a point with ordinate $-F_U(\beta)$. The analogy with classical thermodynamics is now clear: if F plays the role of a free energy, f will in turn play the role of an entropy. Special values of β give rise to useful interpretations:

(1) $\beta=1$. From the normalisation condition (9), and the definitions given in relations (12) and (15), we get $F_U(1)=0$. The tangent line to the curve $f(t)$ with slope 1 goes through the origin.

(2) $\beta=0$. Then in (12), we just count how many boxes are needed to cover the support $\mathscr{S}$ of μ. So we expect a relation between $F_U(0)$ and the Hausdorff dimension [31] $\dim(\mathscr{S})$ of the set $\mathscr{S}$. Indeed we have: $(-F_U(0)) \geqq \dim(\mathscr{S})$. Equality requires some additional assumptions which are true in the case we consider here (Julia sets close to unit circle), as we shall see in proposition 5.1, and more generally for invariant sets under expansive maps.

(3) derivative at $\beta=1$. We have:

$$\left.\frac{\partial Z_U^{(n)}}{\partial \beta}\right|_{\beta=1} = \sum_b \mu(b) \log(\mu(b)), \tag{22}$$

therefore derivative of $F_U(\beta)$ at $\beta=1$ is nothing else than the so called information dimension ([5]-[6]) σ, under similar hypotheses as for the Hausdorff dimension case obtained for $\beta=0$. The information dimension is usually defined as: $\sigma = \lim_{n\to\infty} (-1/n)\sum_b \mu(b)\log(\mu(b))$. In fact we have $\sigma = D_1$, where D_1 is defined in equation (23) below.

(4) β integer and positive. One easily sees that the value of $F_U(\beta)$ for β positive integer bigger than one, is equal to the generalized correlation exponent [5], and is related to the generalized correlation dimension (or Rényi dimension) D_q by:

$$D_q = \frac{1}{(q-1)} F_U(q). \tag{23}$$

This formula extends to the case $q=0$ (*see* item 1 above), and allows to recover the Hausdorff dimension of the support $\mathscr{S}$: $D_0 = \dim(\mathscr{S})$. For $q=1$ one recovers the information dimension D_1 (*see* item 2 above). In itself, this formula defines correlation dimensions for non integer order q.

(5) It is interesting to evaluate the total measure of the reunion of boxes b satisfying (10) for a given t. Using (11) the result is that this total measure vanishes for all but one value t_1 of t, such that $t-f(t)$ is minimum, which correspond to $\beta=1$ in equation (16). This particular value t_1 is in general different from the value t_m for which $f(t)$ is maximum and equal to the Hausdorff dimension of the support $\mathscr{S}$. If we discard some particular, but very interesting, cases corresponding to strictly self similar fractals (as the original Cantor set), we see that in fact almost all

the measure is contained is a set of Hausdorff dimension t_1 strictly smaller than the dimension of the support. As discussed above (*see* item 3), t_1 is nothing else than the information dimension $\sigma = D_1$. One has to be therefore very careful in discussing Hausdorff dimensions through measure arguments. The number $t_1 = D_1$ is in fact the so-called Hausdorff dimension of the measure, that is the smallest possible dimension of sets with full measure. So by removing from the support $\mathscr{S}$, itself of dimension t_m, sets of measure zero, one can get a resulting set of dimension t_1.

5. DIMENSION SPECTRUM AND LOCAL DENSITY EXPONENTS

The precise relation between the function $f(t)$ and the geometrical dimensions associated to the measure μ will be now expressed in terms of the local density exponent $\theta(x)$, defined as:

$$\theta(x) = \limsup_{r \to 0} \left[\frac{\log_2(\mu(\mathscr{B}_r(x)))}{\log_2(r)} \right], \tag{24}$$

where $\mathscr{B}_r(x)$ is a ball of radius r centered at point x in the complex plane. We will now sketch the proof of the following:

PROPOSITION 5.1. — *We assume the thermodynamic limit,* i.e. *equation* (15), *not only for a covering of the full set* $\mathscr{S}$, *but also that the same limit is obtained when we restrict in* (12) *the covering to the intersection of* $\mathscr{S}$ *with any small open set. Then defining* f *by equation* (17), *and assuming that property* (8) *in the description of the distortion lemma* 2.2 *is verified, we have, when* $f(t) \geqq 0$:

$$f(t) = \dim(\mathrm{B}(t)), \tag{25}$$

where $\mathrm{B}(t)$ *is the set of points* x *such that the local exponent* $\theta(x) = t$. *Moreover, the same result* (25) *holds if we replace in* (24) *the superior limit by an inferior limit.*

We first remark that the inequality $f(t) \geqq \dim(\mathrm{B}(t))$ results immediatly from the existence of the covering of the set $\mathscr{S}$ by boxes as we did in section 3. In order to prove equality, we need to prove also the reverse inequality, in other words, we must find a lower bound to the Hausdorff dimension $\dim(\mathrm{B}(t))$. Such a lower bound is given by an argument due to Frostman ([32]-[33]):

LEMMA 5.2. — *Suppose we know the existence of a measure* ν, *and a set* $\mathscr{D}$, *such that for some* κ *and some positive constant* c *we have:*

$$(1)\ \nu(\mathscr{D}) = 1, \tag{26}$$

$$(2)\ \ \nu(\mathscr{B}_\varepsilon) \leqq c\,\varepsilon^{\varkappa}, \tag{27}$$

for any ball $\mathscr{B}_\varepsilon$ of small radius ε, then we get the inequality:

$$\dim(\mathscr{D}) \geqq \kappa. \tag{28}$$

It is not necessary to assume that $\mathscr{D} \cap \mathscr{B}_\varepsilon \neq \varnothing$, since (27) would be automatically true otherwise. For the proof, we first recall that the κ-measure of the set $\mathscr{D}$ is given by:

$$\operatorname{mes}_{\varkappa}(\mathscr{D}) = \sup_{\varepsilon}\ \inf_{\mathscr{R}(\varepsilon)} \sum_{\mathscr{B} \in \mathscr{R}(\varepsilon)} (\operatorname{diam}(\mathscr{B}))^{\varkappa}, \tag{29}$$

where $\mathscr{R}(\varepsilon)$ is an arbitrary covering of $\mathscr{D}$ by balls of radius smaller than ε. Then using equation (27), we get $(\operatorname{diam}(\mathscr{B}))^{\varkappa} \geqq (1/c)\,\nu(\mathscr{B})$, therefore for the special covering $\mathscr{R}_i$ for which the infimum in (29) is reached, we have:

$$\operatorname{mes}_{\varkappa}(\mathscr{D}) \geqq (1/c) \sup_{\varepsilon} \sum_{\mathscr{B} \in \mathscr{R}_i} \nu(\mathscr{B}) \geqq (1/c)\,\nu(\mathscr{D}) = (1/c), \tag{30}$$

where the last inequality results from the fact that $\mathscr{R}_i$ is a covering of $\mathscr{D}$. If the infimum in (29) is not reached for a particular $\mathscr{R}_i$, there exists a covering which reaches this infimum value up to an arbitrarily small amount. For this covering, (30) is fulfilled up to an arbitrarily small error. As a consequence, $\operatorname{mes}_{\varkappa}(\mathscr{D})$ is strictly positive, and therefore we get (28), hence the lemma.

The strategy is therefore as follows [16]: we shall in the remainder of this section give an explicit construction of a measure with support on $\mathrm{B}(t)$, satisfying the conditions of lemma 4.2, with $\kappa = f(t)$. Under these conditions, the proof of proposition 4.1 will be complete.

We start from the same assumptions as in proposition 3.1 and we will assume for simplicity that $t < t_m$, obvious changes allowing a similar argument when $t > t_m$. We shall construct a measure on $\mathrm{B}(t)$ in successive steps corresponding to a sequence of increasing integers: $n_1 < n_2 < \ldots < n_k < \ldots$ At the first step, we select among boxes I with size 2^{-n_1} covering the set $\mathscr{S}$, the boxes such that $\mu(\mathrm{I}) > 2^{-n_1 t}$. Let us call $\mathrm{B}_1(t)$ the union of all such boxes. A convenient way of writing these conditions is to define the set of boxes $\mathrm{A}_1(t)$ as:

$$\mathrm{A}_1(t) = \{\mathrm{I},\ |\mathrm{I}| = 2^{-n_1},\ \mu(\mathrm{I}) > 2^{-n_1 t}\}, \tag{31}$$

then using the large deviations property:

$$\#\,(\mathrm{A}_1(t)) \sim 2^{n_1 f(t)}, \tag{32}$$

and:

$$\mathrm{B}_1(t) = \bigcup_{\mathrm{I} \in \mathrm{A}_1(t)} \mathrm{I}. \tag{33}$$

In the above equations $|\mathrm{I}|$ denotes the size of I, and $\#\,(\mathrm{A})$ the number of elements of the set A.

We will now define the set function $\nu(\mathrm{I})$ by giving first its value for boxes I of size 2^{-n_1} as following:

$$\nu(\mathrm{I})=0 \quad \text{when } \mathrm{I}\notin \mathrm{A}_1(t), \tag{34}$$

and:

$$\nu(\mathrm{I})=\left(\frac{1}{\#(\mathrm{A}_1(t))}\right) \quad \text{when } \mathrm{I}\in \mathrm{A}_1(t). \tag{35}$$

We can now give a recursive definition of $\mathrm{A}_k(t)$:

$$\mathrm{A}_k(t)=\{\mathrm{I},\ |\mathrm{I}|=2^{-n_k},\ \mu(\mathrm{I})>2^{-n_k t},\ \mathrm{I}\subset \mathrm{B}_{k-1}(t)\}, \tag{36}$$

then:

$$\mathrm{B}_k(t)=\bigcup_{\mathrm{I}\in \mathrm{A}_k(t)} \mathrm{I}. \tag{37}$$

Once again, we refine the definition of the set function $\nu(\mathrm{I})$ by giving now its value for boxes of size 2^{-n_k} as following:

$$\nu(\mathrm{I})=0 \quad \text{when } \mathrm{I}\notin \mathrm{A}_k(t), \tag{38}$$

and:

$$\nu(\mathrm{I})=\left(\frac{1}{\#(\mathrm{A}_k(t))}\right) \quad \text{when } \mathrm{I}\in \mathrm{A}_k(t). \tag{39}$$

The problem is now to evaluate $\#(\mathrm{A}_k)$, and to check that it does not vanish. For that purpose, take any box in $\mathrm{A}_k(t)$, call it I, and notice that from definition (36), there exists a box J in $\mathrm{A}_{k-1}(t)$, such that $\mathrm{I}\subset\mathrm{J}$. Then proposition 2.3 asserts that there exists an integer n such that $\mu(\mathrm{T}^n(\mathrm{J}))$ is of order one, in fact of the order $(1/q)$, and $|\mathrm{T}^n(\mathrm{J})|$, is also of order one, in fact of order δ, as discussed in section 2.

Then the distortion lemma tells that:

$$\frac{|\mathrm{T}^n(\mathrm{I})|}{|\mathrm{T}^n(\mathrm{J})|}\simeq\frac{|\mathrm{I}|}{|\mathrm{J}|}, \tag{40}$$

where once again $|\mathrm{I}|$ denotes either the size of the box I or the diameter of the set I. The possible $\sqrt{2}$ ambiguity factor is inessential in the present evaluations. Using (6), one also gets:

$$\frac{\mu(\mathrm{T}^n(\mathrm{I}))}{\mu(\mathrm{T}^n(\mathrm{J}))}=\frac{\mu(\mathrm{I})}{\mu(\mathrm{J})}. \tag{41}$$

From (40) and (41) one gets:

$$|\mathrm{T}^n(\mathrm{J})|\sim 2^{-(n_k-n_{k-1})}, \tag{42}$$

and:

$$\mu(\mathrm{T}^n(\mathrm{J}))>2^{-(n_k-n_{k-1})t}. \tag{43}$$

But $\#(A_k)$ is nothing else than the number of boxes fulfilling (42) and (43) and this number is once more given by the large deviation argument, proposition 3.1. Here we use in fact the so-called conditionned large deviation property, that is the hypothesis that the thermodynamical limit reaches the same value $F_U(\beta)$ if we restrict the covering to J (which always contains a small open set) instead of $\mathcal{S}$. So we get:

$$\#(A_k) \sim 2^{(n_k - n_{k-1}) f(t)}. \tag{44}$$

Of course there remains many details to check in the above argument, such as the control of the deformation of J under the action of T^n, which allows to still compare the result to a box of size given in (42). One also has to be somewhat more careful to get the right inequality in (43), but an exact treatment would only bring corrections with growth smaller than exponential as n goes large. Finally the use of the large deviation argument in (44) requires that also the difference $(n_k - n_{k-1})$ be large enough, which we can always decide since no previous choice for the n_k was made.

Now with definitions (35) and (39), one easily checks that ν can be extended toward a σ-additive function of sets, that is a measure with:

$$\nu(B_k(t)) = 1, \tag{45}$$

and moreover that:

$$\nu(K(t)) = 1 \qquad \text{where} \quad K(t) = \bigcap_k B_k(t). \tag{46}$$

The last thing we need to check is property (27), but that is easy: first find the integer p such that:

$$2^{-n_{p+1}} \leqq \varepsilon < 2^{-n_p}, \tag{47}$$

Then observe that for any ball $\mathcal{B}_\varepsilon$ such that $\mu(\mathcal{B}_\varepsilon) \neq 0$, there exists $I \in A_p(t)$ and $J \in A_{p+1}(t)$ such that $J \subset \mathcal{B}_\varepsilon \subset I$. This in fact requires that we choose a sequence n_p which does not grow too fast. Then $\nu(\mathcal{B}_\varepsilon) \leqq \nu(I)$, and $\nu(I) \sim \varepsilon^{f(t)}$ in view of (47) and the definition of A_p.

Finally we observe that any point in $B(t)$ is contained in some $A_k(t)$ since it is the center of a ball with some small radius ε the μ-measure of it being of the order ε^t. Therefore we have $B(t) \supset K(t)$ and therefore $\nu(B(t)) = 1$.

So we have completed the conditions to apply lemma 5.2 to $B(t)$, which completes the proof of proposition 5.1, therefore the relation between the function $f(t)$ and the dimension of the set with local exponent t for the measure μ is established. The inferior limit case is treated in a similar way, and in fact a careful proof would lead to the equality between $f(t)$ and the dimension of $B(t)$ up to an arbitrarily small amount, which gives the expected result.

Let us remark that we used the distortion lemma as a way to compare small scales to normal order one scales, and condition (6) to control the corresponding scaling properties of the measure μ. In fact what was really needed, was a kind of homogeneity property for the set and the measure supported by it. More explicitly any set I at level k, thas is belonging to $A_k(t)$, must contain roughly the same number of sets J at level $(k+1)$. This homogeneity is obtained through a scale comparison procedure, which is the main reason for which the expansivity properties under T has been used, besides proposition 2.3, which reflects a kind of mixing property for the measure μ.

6. THE THERMODYNAMIC LIMIT FOR THE UNIFORM PARTITION FUNCTION

In this section we will show the existence of the limit, when n goes to infinity in equation (15), for the uniform partition function $Z_U^{(n)}(\beta)$ which has been defined in (12). In this definition, the set $\mathcal{S}$ has been covered by squares of size 2^{-n} forming a partition for any n over a bounded set in the complex plane. We will call $\mathcal{P}_n$ such a partition.

We first observe that if (15) is valid with $b \in \mathcal{P}_n$ in (12), the same limit in (15) is obtained if instead we assume b in (12) to belong to a covering $\mathcal{R}_n$ – made with sets not necessarily square shaped and not necessarily forming a partition – such that the following properties holds: (1) there exists an integer h such that for any n and any b in $\mathcal{P}_n$, there exists sets $b_>$ and $b_<$ with $b_> \in \mathcal{R}_{n-h}$ and $b_< \in \mathcal{R}_{n+h}$ such that $b_< \subset b \subset b_>$. (2) The same property holds if we exchange the roles of the partitions $\mathcal{R}_n$ and $\mathcal{P}_n$. The proof is immediate and left to the reader, it makes only use of the fact that each piece of the covering is included in a finite number of pieces of the square shaped partition of order shifted by h, and conversely. In particular, any set in $\mathcal{R}_n$ has a non void intersection with a number of sets in $\mathcal{P}_n$, which is not only finite, but bounded independently of n. This remark shows that the restriction to coverings by squares is inessential: coverings by balls would yield the same result.

In order to prove the existence of the limit we will use the classical argument given below, which derives the result from a subadditivity assumption on the logarithm of the functions.

LEMMA 6.1. – *Suppose that the sequence of positive numbers* Z_n *fulfils the inequality:*

$$Z_{n+m} \leqq C Z_n Z_m, \tag{48}$$

where the constant C *is independant of* n, *then* $(1/n)\log_2(Z_n)$ *has a limit when* n *goes to infinity. Without additional assumption, this limit can be*

$-\infty$. *If we also assume a reversed inequality such as:* $Z_{n+m} \geqq C' Z_n Z_m$, *where the constant* $C'>0$ *is also independant on n, then the above mentioned limit is finite.*

The proof is simple and deserves to be shortly reproduced. Let $a_n = \log_2(Z_n)$ and $c = \log_2(C)$. Then from (48), we get the subadditivity inequality:

$$a_{n+m} \leqq a_n + a_m + c. \tag{49}$$

Let u be a fixed integer, and write the result of the euclidean division of n by u as: $n = bu + r$. Then we get, by repeated use of (49):

$$\frac{a_n}{n} = \frac{a_{bu+r}}{bu+r} \leqq \frac{ba_u + a_r + bc}{bu+r}. \tag{50}$$

By taking the superior limit for n going to infinity, with u arbitrary but fixed, we get, since b also goes to infinity, r remaining bounded:

$$\limsup_{n \to \infty} (a_n/n) \leqq ((a_u + c)/u). \tag{51}$$

Now the inferior limit of the last expression gives:

$$\limsup_{n \to \infty} (a_n/n) \leqq \liminf_{u \to \infty} (a_u/u), \tag{52}$$

which states the existence of $\lim_{n \to \infty} (a_n/n)$, but with the restriction that this limit can be $-\infty$. Now if an inequality as (48) holds in the reversed order, we apply the previous argument to the sequence Z_n^{-1}, and we deduce that $(-1/n)\log_2(Z_n)$ does not go to $+\infty$, hence the lemma.

Now we will prove the existence of the thermodynamic limit for the uniform partition function.

PROPOSITION 6.2. — *For the uniform partition function* $Z_U^{(n)}(\beta)$ *which has been defined in* (12), *the expression:* $(-1/n)\log_2(Z_U^{(n)}(\beta))$ *has a finite limit when n goes to infinity.*

For simplicity, we will write in the following proof Z_n for $Z_U^{(n)}(\beta)$, and we recall the notation $\mathscr{P}_n$ for the partition made of squares with size 2^{-n}. We then have:

$$Z_{n+m} = \sum_{b \in \mathscr{P}_{n+m}} (\mu(b))^\beta \leqq \sum_{d \in \mathscr{P}_n} (\mu(d))^\beta \sum_{b \in \mathscr{P}_{n+m},\, b \cap d \neq \varnothing} C\left(\frac{\mu(b)}{\mu(d)}\right)^\beta. \tag{53}$$

The sum over boxes d has to be restricted to the condition $\mu(d) \neq 0$, and the constant C does not depend on n and m. More precisely, when summing over d and then over all b intersecting d, there is a possibility of counting some of the boxes b more than one time, so (53) holds in fact with $C=1$. On the other hand, obvious geometrical considerations show that the number of possible overcounting is bounded by some number K independant on n, so the inequality derived from (53) by reversing the inequality sign also holds provided we also replace $C=1$ by $C=(1/K)$. In the previous equation, using (6), one can replace under the last summation, both b and d by $T^N(b)$ and $T^N(d)$. We choose N in such a way that both $\mu(T^N(d))$ and $|T^N(d)|$ be of order one, in fact of order $(1/q)$ and δ respectively, following proposition 2.3, as we did in the previous section. We now replace the sum over b by the sum over $T^N(b)$, and get with a suitable rescaling of the constant C:

$$Z_{n+m} \leqq \sum_{d \in \mathscr{P}_n} (\mu(d))^\beta \sum_{T^N(b)} C\,\mu(T^N(b))^\beta, \tag{54}$$

where now the sum over $T^N(b)$ is taken over all b such that $T^N(b)$ intersects $T^N(d)$ which is a set with size of order one. If we knew that under these conditions, the set $T^N(b)$ intersects only a finite number of boxes in $\mathscr{P}_m$, we could, up to some multiplicative constant, replace now the last sum by a sum of boxes in $\mathscr{P}_m$, and we would get the required inequality: $Z_{n+m} \leqq C Z_n Z_m$, where C has been suitably rescaled. But the distortion lemma 2.2 asserts that:

$$\frac{|T^N(b)|}{|T^N(d)|} \sim \frac{|b|}{|d|} = 2^{-m}. \tag{55}$$

Therefore $|T^N(b)| \sim 2^{-m}$, from which we deduce that $|T^N(b)|$ intersects only a finite number of pieces in $\mathscr{P}_m$, which was the missing part of the proof.

It remains to check that $(-1/n)\log_2(Z_n)$ does not go to $-\infty$, but similar counting arguments show that inequality (54) can be reversed in the same way as we already noticed for inequality (53). This leads to a super-additivity property for $\log_2(Z_n)$, that is subadditivity for its opposite, which shows, as mentioned at the end of the lemma 6.1, that the limit remains bounded. If we restrict now the covering to a small open box $\mathscr{O}$ instead of $\mathscr{S}$, we only need to iterate the covering sufficiently many times to get a covering of $\mathscr{S}$ and check that the distortion lemma allows to compare the final covering of $\mathscr{S}$ to the initial covering of $\mathscr{O}$. The comparison yields to the same thermodynamical limit $F_U(\beta)$.

It is interesting to notice once more that the only properties needed in the proof are the same as in the previous section. We need the scale comparison procedure provided by the distortion lemma and the "mixing" property—any small open set expands to cover the whole set under iterations of T—which results from the expanding character of the map T.

7. THE DYNAMICAL PARTITION FUNCTION

In this section we will consider the dynamical partition function defined by the following property:

$$Z_{\mathrm{D}}^{(n)}(\alpha)=\sum_{b\in\mathscr{D}_n}(|b|)^{-\alpha}, \tag{56}$$

where the sum has to be taken over the sets b in the partition $\mathscr{D}_n$, defined recursively as follows: the pieces of the partition $\mathscr{D}_n$ are the preimages under T of the pieces of the partition $\mathscr{D}_{(n-1)}$. Once more $|b|$ denotes the diameter of the set b, and one can write: $b\in\mathscr{D}_n$ if an only if b is one of the q^n preimages of an open set including the Julia set $\mathrm{J_T}$, with an arbitrary, but unambiguous, choice of the q different inverse branches of T.

It is convenient to introduce a more general mixed kind of partition functions:

$$Z_{\mathscr{C}}^{(n)}(\alpha,\beta)=\sum_{b\in\mathscr{C}_n}(|b|)^{\alpha}(\mu(b))^{\beta}, \tag{57}$$

where the sum has to be taken over the pieces b of a partition $\mathscr{C}_n$, with sizes $|b|$ decreasing approximately exponentially-like in n. Obvious arguments show that when we take for $\mathscr{C}_n$ the uniform partitions $\mathscr{P}_n$ defined in the previous section 6, we have, up to a multiplicative constant independant on n:

$$\text{If:}\quad \mathscr{C}_n=\mathscr{P}_n\quad\text{then}\quad Z_{\mathscr{P}}^{(n)}(\alpha,\beta)=2^{-n\alpha}Z_{\mathrm{U}}^{(n)}(\beta). \tag{58}$$

On the other hand, when we take for $\mathscr{C}_n$ the dynamical partitions $\mathscr{D}_n$ defined above, we have in view of (4) for the invariant and balanced measure μ:

$$\text{If:}\quad \mathscr{C}_n=\mathscr{D}_n\quad\text{then}\quad Z_{\mathscr{D}}^{(n)}(\alpha,\beta)=q^{-n\beta}Z_{\mathrm{D}}^{(n)}(-\alpha). \tag{59}$$

Now consider the partition function $Z_{\mathscr{P}}$ associated to the uniform partition function given in (58). By comparing it to the dynamical partition function, we shall show the following result:

PROPOSITION 7.1. – *The dynamical partition function* $Z_{\mathrm{D}}^{(n)}(\alpha)$, *has a thermodynamic limit:*

$$\lim_{n\to\infty}(1/n)\log_q(Z_{\mathrm{D}}^{(n)}(\alpha))=\mathrm{F_D}(\alpha), \tag{60}$$

which satisfies:

$$\mathrm{F_D}(\mathrm{F_U}(\beta))=\beta. \tag{61}$$

In fact we repeat an argument very close to the one given for proposition 6.2. We start from equation (58), which defines the mixed partition function associated to $\mathscr{P}_n$, and choosing n and m large, but with $n\gg m$ in such a way that each piece of $\mathscr{D}_m$ contains a large number of pieces of $\mathscr{P}_n$, we get:

$$Z_{\mathscr{P}}^{(n)}(\alpha, \beta) = \sum_{b \in \mathscr{P}_n} (|b|)^\alpha (\mu(b))^\beta$$

$$\sim \sum_{d \in \mathscr{D}_m} (\mu(d))^\beta \sum_{b \in \mathscr{P}_n,\, b \cap d \neq \varnothing} (|b|)^\alpha \left(\frac{\mu(b)}{\mu(d)}\right)^\beta. \quad (62)$$

Now once more, given δ, we choose N in such a way that both $\mu(T^N(d))$ and $|T^N(d)|$ be of order one, in fact of order $(1/q)$ and δ respectively, following proposition 2.3, as we did in the previous section. We now replace the sum over b by the sum over $T^N(b)$, and get:

$$Z_{\mathscr{P}}^{(n)}(\alpha, \beta) \sim 2^{-n\alpha} \sum_{d \in \mathscr{D}_m} (\mu(d))^\beta \sum_{T^N(b)} (\mu(T^N(b)))^\beta, \quad (63)$$

where now the sum over $T^N(b)$ is taken over all b such that $T^N(b)$ intersects $T^N(d)$ which is a set with size of order one. Now we use again the distortion lemma 2.2 in order to evaluate the size of $T^N(b)$. We get:

$$|T^N(b)| \sim \frac{|T^N(b)|}{|T^N(d)|} \sim \frac{|b|}{|d|} = 2^{-(n+\log_2(|d|))}. \quad (64)$$

So we control the size of $T^N(b)$, and one can consider the covering by $T^N(b)$ as a uniform covering. Therefore, using in (15) the size 2^{-n} of the pieces of the uniform partition, and rewriting the definition of $F_U(\beta)$ as:

$$\sum_b (\mu(b))^\beta \sim (|b|)^{F_U(\beta)}, \quad (65)$$

we can write:

$$Z_{\mathscr{P}}^{(n)}(\alpha, \beta) \sim 2^{-n\alpha} \sum_{d \in \mathscr{D}_m} (\mu(d))^\beta 2^{-(n+\log_2(|d|))F_U(\beta)}, \quad (66)$$

which reads:

$$Z_{\mathscr{P}}^{(n)}(\alpha, \beta) \sim 2^{-n\alpha} 2^{-nF_U(\beta)} \sum_{d \in \mathscr{D}_m} (\mu(d))^\beta (|d|)^{-F_U(\beta)}. \quad (67)$$

The resulting relation is:

$$Z_{\mathscr{P}}^{(n)}(\alpha, \beta) \sim 2^{-n\alpha} 2^{-nF_U(\beta)} Z_{\mathscr{D}}^{(m)}(-F_U(\beta), \beta). \quad (68)$$

The previous relation is in fact a way to show the existence of the thermodynamic limit, that is the limit of $(1/m)\log_q(Z_D^{(m)}(\alpha))$, when m goes to infinity, which is a well known result in the thermodynamic formalism [36]. As in (60), let us denote $F_D(\alpha)$ this limit. Using (58) and (59) one gets:

$$2^{-n\alpha} 2^{-nF_U(\beta)} \sim 2^{-n\alpha} 2^{-nF_U(\beta)} q^{-m\beta} q^{mF_D(F_U(\beta))},$$

and finally:

$$F_D(F_U(\beta)) = \beta. \quad (70)$$

from which the proposition results. Notice that F_U is defined using base-2 logarithms, whereas F_D uses base-q logarithms, but this is just a matter of normalisation. The choice of signs in the definitions (15) and (60), and in (12) and (56) for the exponents, are made in order to get the simplest form for the relation (61). The relation between F_U and F_D is surprisingly simple and seemed unnoticed before the work by Collet *et al.* [16].

8. ANALYTICITY PROPERTIES IN THE THERMODYNAMICAL FORMALISM

In the previous section, a simple connection has been established between the uniform and dynamical partition functions. Here we will use this relation in order to deduce analyticity properties for F_U from analyticity properties of F_D.

We first state some alternate expressions for the dynamical partition function $Z_D(\beta)$ defined in (56). The dynamical partition function admits the following representations:

(1) The original definition is:

$$Z_D^{(n)}(\beta) = \sum_{b \in \mathcal{P}_n} (|b|)^{-\beta} = \sum_i^{q^n} (|T_i^{-n}(\mathcal{A})|)^{-\beta}, \tag{71}$$

where the second sum runs over the q^n inverse branches of the function $T^{(n)}$ and $\mathcal{A}$ is a somewhat arbitrary bounded open set including J_T.

(2) For T as in (1), with $|\lambda|$ small enough, there exists a repulsive fixed point ξ close to one, that is such that $\xi = 1 + O(|\lambda|)$ and $T(\xi) = \xi$. Then, through an argument which uses the distortion lemma, we have:

$$Z_D^{(n)}(\beta) \sim \sum_{x,\, T^n(x) = \xi} |T^{n'}(x)|^{\beta}, \tag{72}$$

where the sum runs over the preimages of ξ.

(3) We define for the preimages of ξ the following labelling procedure: let us denote the q^n preimages of ξ as: $x_{i_1, i_2, \ldots, i_n}$ such that:

$$T(x_{i_1, i_2, \ldots, i_n}) = x_{i_1, i_2, \ldots, i_{n-1}}, \tag{73}$$

where i_k for $k = 1, 2, \ldots, n$ takes one of the values $1, 2, \ldots, q$, according to which inverse branch $T_{i_n}^{-1}$ of T goes back from $x_{i_1, i_2, \ldots, i_{n-1}}$ to $x_{i_1, i_2, \ldots, i_n}$. Of course x_{i_1} denotes one of the q preimages of ξ, that is $T(x_{i_1}) = \xi$. Then we have:

$$Z_D^{(n)}(\beta) \sim \sum_{\{i\}} \prod_{k=0}^{(n-1)} (|T'(T^k(x_{\{i\}}))|)^{\beta}, \tag{74}$$

where the sum has to be taken over all possible values of the multi-indices $\{i\}$ which run for $i_1, i_2, \ldots, i_n$, and T^n denotes the n-th iterate of T, such that $T^n(z) = T(T^{n-1}(z))$ and $T^0(z) = z$. So we get:

$$Z_D^{(n)}(\beta) \sim \sum_{\{i\}} \prod_{k=1}^{n} (|T'(x_{i_1, i_2, \ldots, i_k})|)^\beta, \tag{75}$$

and:

$$Z_D^{(n)}(\beta) \sim \sum_{\{i\}} \exp(-\beta H(x_{\{i\}})), \tag{76}$$

where:

$$(H(x_{\{i\}})) = -\sum_{k=1}^{n} \ln(|T'(x_{i_1, i_2, \ldots, i_k})|). \tag{77}$$

(4) In the previous formulas (72-77) a special role (starting point for the computation of preimages) was given to the repulsive fixed point ξ which goes to one when c goes to zero, because ξ and its preimages belong to the Julia set. In fact, using the distortion lemma, one can show that similar formulas can be written by replacing the starting point ξ by any point in (or close to) J_T, and we will see in the appendix that it is convenient to choose the starting point $(1+c)$ in order to simplify the perturbative calculations. Moreover it is convenient to write (74) for an arbitrary starting point and then take the mean value using the invariant and balanced measure μ introduced in section 2, equations (3) and (4). Then we get:

$$Z_D^{(n)}(\beta) \sim q_n \int \prod_{k=0}^{(n-1)} (|T'(T^k(z))|)^\beta \, d\mu(z), \tag{78}$$

and:

$$Z_D^{(n)}(\beta) \sim q^n \int \exp(-\beta H(z)) \, d\mu(z), \tag{79}$$

where:

$$H(z) = -\sum_{k=0}^{n-1} \ln(|T'(T^k(z))|). \tag{80}$$

Going from (1) to (2) requires the distortion lemma 2.2, the same lemma is used to obtain the independance of (74) on the starting point ξ of the backward iterations. The invariance (3) and the balance property (4) of the measure μ allows to replace $x_{\{i\}}$ by z in (78), and lead to the factor q^n. The details of the arguments are left to the reader.

We then can state the following:

PROPOSITION 8.1. – *The limit* $F_D(\beta)$ *of* $(1/n)\log_q(Z_D^{(n)}(\beta))$ *when n goes to infinity, as well as the limit* $F_U(\beta)$ *of* $(-1/n)\log_2(Z_U^{(n)}(\beta))$, *exists, is analytic in* β *and real analytic in the coefficients occurring in the polynomial* T, *for* $|\lambda|$ *sufficiently small in equation* (1).

We recall the definition used here for the real analyticity property: a function $f(z)$ is said real analytic if it is an analytic function separately of the real and the imaginary part of z, and takes real values as long as both variables remain real. For the proof, we observe that the expression (75) for Z_D is just the same as for a partition function for a one dimensional Ising like system, with q possible states at each site, and an interaction between all sites, but decreasing sufficiently fast with the distance for the application of the usual results. The proposition 8.1 for F_D is a consequence of the existence of the limit, and of the uniform bound provided by Dobrushin [34]. Various regularity results in β, are available in earlier litterature ([35-[38]), including real analyticity properties for the Hausdorff dimension [18]. We feel that the approach permitted by Dobrushin may give a more direct look for our present purposes. The analyticity properties in proposition 8.1 for F_U result from the implicit function theorem which can be applied to the equation (61) relating F_U and F_D. No problems arise from the functional inversion in (61) for polynomials T close to z^q, since for the polynomial z^q we have the following expressions which are easy to get:

$$F_U(\beta)=(\beta-1), \tag{81}$$
$$F_D(\beta)=(\beta+1). \tag{82}$$

In view of (23), one can deduce from proposition 8.1 the analyticity properties of the Hausdorff dimension and of the various higher order correlation dimensions building the dimension spectrum D_q in (23) (where q is the order of the correlation and not merely the degree of T). So we have now completed our program which consisted in considering the analyticity properties of the multifractal properties of Julia sets close to unit circle. Of course if we consider polynomials far from z^q, one expects interesting singularities in the thermodynamical functions which remain to be analysed. Such a situation seems to occur for the polynomial $z^2+\frac{1}{4}$, as shown by numerical calculations [29].

9. A SKETCH FOR PERTURBATIVE ARGUMENTS AND CONCLUDING REMARKS

Once F_D is expressed in form of a classical statistical mechanics model, the usual perturbative methods can be used here, and will permit a perturbative expansion for both F_D and F_U, and therefore for the dimension spectrum D_q, defined in equation (23). We will only sketch here the arguments, and refer to the appendix for details.

In order to simplify the calculations, we will restrict ourselves in the following discussion to the case:

$$T(z) = z^q + c. \tag{83}$$

We now decribe the procedure which might be followed in order to perform perturbation expansions, that is expansions on powers of the real part and the imaginary part of c, for the various thermodynamical functions associated with the Julia set J_T and its invariant and balanced measure μ. In fact we have to go through the following steps:

(1) compute the expansion of F_D on powers of c;

(2) with (70) compute the expansion of F_U on powers of c;

(3) use (23) to expand the correlation dimensions D_q on powers of c;

(4) use (20) to expand the function $f(t)$ on powers of c.

There is no special comments about steps (2) and (3), but there is a specific difficulty in step (4), that is the fact that the unperturbed function $f(t)$ corresponding to the unit circle with the Lebesgue measure, is singular and takes ontly one value different from $-\infty$, that is $f(1)=1$. But there is no difficulty to compute the perturbed inverse function $t(f)$, the unperturbed one being taken as the constant function with value equal to one.

So the main problem is to compute the expansion of F_D. There are two possibilities, to start either from equations (75) or from equation (78). Starting from (75) leads to the usual expansions occurring in statistical mechanics, in a case where the interactions are not limited to nearest neighbours. Some tricks used in these calculations are in fact very similar to those used for the statistic of fixed points [18]. In the appendix, we give explicit calculations up to fourth order in c. But although the computations are lengthy, they do not reveal any special difficulties in mastering the thermodynamical limit. The remaining problem is the relevance of this expansion for actual evaluations of effective quantities like D_q or $f(t)$, with only a truncated series limited to a small number of terms.

One the other hand, starting from (78) requires in addition an effective way of computing the perturbation expansion for mean values taken on the perturbed measure μ, for instane for its moments. Such an expansion is possible using a technique analogous to the computations by Widom *et al.* [19]. We expect that such an approach would allow some resummations in the expansion, by procedures using the invariance properties of μ under the transformation T.

We think that our approach deserves the following general comment: it allows a direct evaluation of the thermodynamical quantities, in a somewhat different spirit than the approach based on classical thermodynamic formalism ([35]-[36]). In some sense, this direct approach recalls the observation made long ago by Billingsley [39], who pointed out the analogy between the dimension and the entropy of the transformation which leaves invariant some usual Cantor sets. The large deviation argument may clarify the relation between box counting, that is geometrical measurements, and true dimensions as Hausdorff dimension, as it is shown in earlier papers ([16], [10], [40]), and recently announced works [41]. These considerations apply to many systems, in particular to those for which a distortion lemma applies. A difficulty remains when the thermodynamical functions display discontinuities. An example of this situation occurs when the box counting procedure gives a non convex function $f(t)$. In this case we have to look for a more detailed way of computing partition functions, which will provide without any doubt an other interpretation in the large deviation formalism. The last remark is that we have restricted our discussion to the apparently academic problem of polynomial iterations, but we think this example as a non trivial model problem, in a domain which has been growing very fast in the recent years [38]. Clearly the analysis of the geometrical properties of chaotic dynamics will still receive a large developpment in the close future.

APPENDIX: THE PERTURBATION EXPANSION UP TO FOURTH ORDER

We shall calculate the perturbation expansion around $c=0$ for the correlation dimensions associated to the invariant and balanced measure on the Julia set defined by the polynomial:

$$\mathrm{T}(z)=z^q+c. \tag{84}$$

For arbitrary integer q, $(q \geqq 2)$, we will give the expansion up to fourth order on powers of c. We recall the recursive definition of preimages:

$$x_{i_1, i_2, \ldots, i_n}=\mathrm{T}_{i_n}^{-1}(x_{i_1, i_2, \ldots, i_{n-1}}) \qquad \text{with} \quad x_{i_1}=\mathrm{T}_{i_1}^{-1}(x_0) \tag{85}$$

The starting point x_0 is in fact arbitrary, for instance the repulsive fixed point ξ [*see* (72)]. We consider the dynamical partition function $\mathrm{Z}_{\mathrm{D}}^{(n)}(\beta)$ defined in equation (75). Whenever it makes no ambiguity we will use the abbreviated notation x_n for $x_{i_1, i_2, \ldots, i_n}$, and $\{i\}_n$ for the set of values of the multi-index $i_1, i_2, \ldots, i_n$. We also define the following notations:

$$\left.\begin{array}{c} \gamma=(1/q), \\ \tau_k=\exp(2 i\pi\gamma i_k) \quad (i_k \text{ is the last index in } i_1, i_2, \ldots, i_k) \\ k_i=1-(c/x_i), \\ \text{so that} \quad x_i^q=x_{i-1}-c \text{ gives } x_i=\tau_i x_{i-1}^{\gamma}\kappa_{i-1}^{\gamma} \\ b_j=\dfrac{\beta}{2}(1-\gamma^j), \qquad p_k^{(i)}=-c(x_k\kappa_k)^{-\gamma^i} \end{array}\right\} \tag{86}$$

With these notations one gets:

$$\kappa_n=1+\bar{\tau}_n p_{n-1}^{(1)} \qquad \text{and} \qquad p_k^{(i)}=p_{k-1}^{(i+1)}(\bar{\tau}_k\kappa_k^{-1})^{\gamma^i} \tag{87}$$

A strainghtforward recursive argument gives:

$$|x_k|=|x_0|^{\gamma^k}\prod_{j=1}^{k}|k_{j-1}|^{\gamma^{k+1-j}}, \tag{88}$$

$$\begin{aligned} Z_D^{(n)}(\beta)=\sum_{\{i\}}\prod_{k=1}^{n}(|T'(x_{i_1, i_2, \ldots, i_k})|)^{\beta}&=q^{n\beta}|x_0|^{\beta(1-\gamma^n)}\sum_{\{i\}_n}\prod_{j=0}^{n-1}|\kappa_j|^{\beta(1-\gamma^{n-j})} \\ &=q^{n\beta}|x_0|^{\beta(1-\gamma^n)}(\kappa_0\bar{\kappa}_0)^{\beta/2(1-\gamma^n)}q^n S_{(n-1)}, \end{aligned} \tag{89}$$

where $S_{(n)}$ is defined as:

$$q^n S_{(n)}=\sum_{\{i\}_n}\prod_{j=1}^{n}(k_j\bar{\kappa}_j)^{b_{n+1-j}}. \tag{90}$$

The "free energy" is then given by:

$$F_D(\beta)=\lim_{n\to\infty}\frac{1}{n}\log_q(Z_D^{(n)}(\beta))=\beta+1+\lim_{n\to\infty}\frac{1}{n}\log_q(S_{(n)}). \tag{91}$$

The explicit dependence on x_0 and κ_0 disappears in $F_D(\beta)$ in the limit $n\to\infty$. We therefore have to evaluate the sums:

$$S_{(n)}=\frac{1}{q}\sum_{i_1}(\kappa_1\bar{\kappa}_1)^{b_n}\frac{1}{q}\sum_{i_2}(\kappa_2\bar{\kappa}_2)^{b_{n-1}}\ldots\frac{1}{q}\sum_{i_n}(\kappa_n\bar{\kappa}_n)^{b_1}. \tag{92}$$

In fact $S_{(n)}$ can be written this way since κ_k does not depend on $i_{k+1}, \ldots, i_n$. Therefore the sums over the indices i_k will be performed step by step, using (87), the related binomial expansion, and then the sum over roots of unity:

$$\frac{1}{q}\sum_{i_k=0}^{q-1}\tau_k^m=\sum_{\lambda=-\infty}^{+\infty}\delta_{m, q\lambda}. \tag{93}$$

In the binomial expansion we introduce as usual $\binom{b}{k}$ for any real b, as the polynomial in b: $\binom{b}{k} = \frac{1}{k!} x(x-1)\dots(x-k+1)$ for $k \geqq 0$ (it vanishes identically for negative k). The last sum $\Sigma_n = \frac{1}{q} \sum_{i_n} (\kappa_n \bar{\kappa}_n)^{b_1}$ in (92) gives:

$$\Sigma_n = \frac{1}{q} \sum_{i_n} (1 + \bar{\tau}_n p_{n-1}^{(1)})^{b_1} (1 + \tau_n \bar{p}_{n-1}^{(1)})^{b_1}$$

$$= \sum_{\lambda, j, k} \delta_{k-j, q\lambda} \binom{b_1}{j} \binom{b_1}{k} (p_{n-1}^{(1)})^j (\bar{p}_{n-1}^{(1)})^k \quad (94)$$

Due to the relation $k-j=q\lambda$, there is no phase ambiguity in expressing x_n and $p_{n-1}^{(1)}$ in terms of x_{n-1} as in (86). Now we can perform recursively the computation of the last k sums in (92):

$$\Sigma_{n-k+1} = \frac{1}{q} \sum_{i_{n-k+1}} (\kappa_{n-k+1} \bar{\kappa}_{n-k+1})^{b_k} \dots \frac{1}{q} \sum_{i_n} (\kappa_n \bar{\kappa}_n)^{b_1}$$

$$= \sum_{\{\lambda, j, k\}_k} \delta_{k_1 - j_1, q\lambda_1} \delta_{k_2 - j_2, q\lambda_2 - \lambda_1} \times \dots \times \delta_{k_k - j_k, q\lambda_k - \lambda_{k-1}}$$

$$\times \binom{b_1}{j_1} \binom{b_1}{k_1} \binom{b_2 - \gamma j_1}{j_2} \binom{b_2 - \gamma k_1}{k_2} \times \dots$$

$$\times \binom{b_k - \gamma j_{k-1} - \dots - \gamma^{k-1} j_i}{j_k} \binom{b_k - \gamma k_{k-1} - \dots - \gamma^{k-1} k_1}{k_k}$$

$$\times (p_{n-k}^{(k)})^{j_1} (\bar{p}_{n-k}^{(k)})^{k_1} (p_{n-k}^{(k-1)})^{j_2} (\bar{p}_{n-k}^{(k-1)})^{k_2} \times \dots \times (p_{n-k}^{(1)})^{j_k} (\bar{p}_{n-k}^{(1)})^{k_k}. \quad (95)$$

This leads to the following expression for $S_{(n)}$:

$$S_{(n)} = \sum_{\{\lambda, j, k\}_n} \prod_{l=1}^{n} \left\{ \binom{J_l}{j_l} \binom{K_l}{k_l} \delta_{k_l - j_l, q\lambda_l - \lambda_{l-1}} (p_0^{(n+1-l)})^{j_l} (\bar{p}_0^{(n+1-l)})^{k_l} \right\}, \quad (96)$$

where we have set $\lambda_0 = 0$ and:

$$\left.\begin{array}{llll} J_n = b_n - \gamma j_{n-1} - \dots - \gamma^{n-1} j_1, & & K_n = b_n - \gamma k_{n-1} - \dots - \gamma^{n-1} k_1 & \\ J_1 = b_1, & J_2 = b_2 - \gamma j_1, \dots, & K_1 = b_1, & J_2 = b_2 - \gamma k_1, \dots \end{array}\right\} \quad (97)$$

A straightforward calculation shows that:

$$J_n - K_n = \lambda_{n-1}. \quad (98)$$

One might wonder what we have got through these heavy manipulations: in fact equation (96) is now easily ordered in increasing powers of c and c, using the definition (86) of the $p_k^{(i)}$, namely $p_0^{(i)} = -c(x_0-c)^{-\gamma^i}$. One gets:

$$S_{(n)} = \sum_{\{\lambda, j, k\}_n} \prod_{l=1}^{n} \left\{ \binom{J_l}{j_l} \binom{K_l}{k_l} \right.$$
$$\left. \times \delta_{k_l - j_l,\, q\lambda_l - \lambda_{l-1}} (-c)^{j_l} (-\bar{c})^{k_l} (x_0 - c)^{-j_l \gamma^{n+1-l}} (\bar{x}_0 - \bar{c})^{-k_l \gamma^{n+1-l}} \right\}. \quad (99)$$

The remarkable fact is that the free energy (91) does not depend on x_0, and choosing $x_0 = c+1$ makes the computation tractable, since it results that the powers of $x_0 - c$ and $\bar{x}_0 - \bar{c}$ disappear. So we consider:

$$\tilde{S}_{(n)} = S_{(n)} |_{x_0 = c+1} = \sum_{\{\lambda, j, k\}_n} \prod_{l=1}^{n} \left\{ \binom{J_l}{j_l} \binom{K_l}{k_l} \right.$$
$$\left. \times \delta_{k_l - j_l,\, q\lambda_l - \lambda_{l-1}} (-c)^{j_l} (-\bar{c})^{k_l} \right\}. \quad (100)$$

We now evaluate the coefficients of the expansion:

$$\tilde{S}_{(n)} = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{s+t} A_{s,t}^{(n)} c^s \bar{c}^t, \qquad A_{s,t}^{(n)} = A_{t,s}^{(n)}. \quad (101)$$

The coefficients $A_{s,t}^{(n)}$ are real. Since in equation (100) the indices j_l and k_l are non negative, the computation of the coefficient $A_{s,t}^{(n)}$ requires only a finite number of terms in the sum over $\{\lambda, j, k\}_n$. For instance, one gets in the $q=2$ case:

$$A_{0,0}^{(n)} = 1, \qquad A_{1,0}^{(n)} = 0, \qquad A_{2,0}^{(n)} = \binom{b_n}{2}, \qquad A_{1,1}^{(n)} = \sum_{p=1}^{n} b_p^2, \quad (102)$$

$$A_{3,0}^{(n)} = \binom{b_{n-1}}{2}(b_n - 1), \qquad A_{2,1}^{(n)} = \sum_{p=1}^{n-1} \binom{b_p}{2} b_{p+1}, \quad (103)$$

$$A_{4,0}^{(n)} = \binom{b_n}{4} + \binom{b_{n-1}}{4} + (b_n - 1)(b_{n-1} - 1)\binom{b_{n-2}}{2}, \quad (104)$$

$$A_{3,1}^{(n)} = b_n \binom{b_n}{3} + \sum_{l=1}^{n-1} b_l^2 \binom{b_n - \gamma^{n-l}}{2} + \sum_{l=1}^{n-2} \binom{b_l}{2} (b_{l+1} - 1) b_{l+2}, \quad (105)$$

$$A_{2,2}^{(n)} = \sum_{l=1}^{n} \binom{b_l}{2}^2 + \sum_{m=2}^{n} \sum_{l=1}^{m-1} b_l^2 (b_m - \gamma^{m-l})^2. \quad (106)$$

We now compute the logarithm of $\tilde{S}_{(n)}$ by using the cumulants of the expansion (101) rewritten in increasing powers of $|c|$. The limit $n \to \infty$

can then be taken, and one gets for $q=2$:

$$F_D(\beta)=\beta+1+\frac{1}{\ln(2)}\left\{\frac{1}{4}\beta^2|c|^2+\frac{1}{16}\beta^2(2-\beta)(c+\bar{c})|c|^2\right.$$
$$+\frac{1}{32}\beta^2(2-\beta)^2(c^2+\bar{c}^2)|c|^2$$
$$\left.+\left(\frac{7}{48}\beta^2-\frac{5}{16}\beta^3-\frac{1}{64}\beta^4\right)|c|^4\right\}+O(|c|^5). \quad (107)$$

Similarly, one gets for $q=3$:

$$F_D(\beta)=\beta+1+\frac{1}{\ln(3)}\left\{\frac{1}{4}\beta^2|c|^2+\frac{1}{96}\beta^2(2-\beta)(4-\beta)(c^2+\bar{c}^2)|c|^2\right.$$
$$\left.+\left(\frac{3}{32}\beta^2-\frac{3}{16}\beta^3-\frac{1}{64}\beta^4\right)|c|^4\right\}+O(|c|^5), \quad (108)$$

and for $q\geqq 4$:

$$F_D(\beta)=\beta+1+\frac{1}{\ln(q)}\left\{\frac{1}{4}\beta^2|c|^2\right.$$
$$\left.+\left(\frac{3+q^2}{16(q^2-1)}\beta^2-\frac{3+q}{16(q-1)}\beta^3-\frac{1}{64}\beta^4\right)|c|^4\right\}+O(|c|^5). \quad (109)$$

We remark here that the case $q=2$ is the most complicated. On the contrary if q is large (here $q>4$), all terms $A_{s,t}^{(n)}$ in (101) vanish for $s+t\leqq 4$ when $s\neq t$, and for $q=4$ the only non-vanishing "off diagonal" terms $A_{4,0}$ and $A_{0,4}$ give no contribution in the limit $n\to\infty$. As a result, the coefficient of $|c|^4$ in (109) is the same as in equations (107) and (108), up to replacement of q by its actual value.

We now derive the perturbation expansion of $F_U(\beta)$ up to the same order, using (70), *i.e.* $F_U(F_D(\beta))=\beta$. We rewrite equations (107-109) in replacing c by λc with real λ, and we get:

$$F_D(\beta)=1+\beta+\lambda^2\Phi_2(\beta)+\lambda^3\Phi_3(\beta)+\lambda^4\Phi_4(\beta)+O(\lambda^5). \quad (110)$$

Then one easily writes:

$$F_U(\beta)=-1+\beta-\lambda^2\varphi_2(\beta)-\lambda^3\varphi_3(\beta)-\lambda^4\varphi_4(\beta)+O(\lambda^5), \quad (111)$$

where:

$$\varphi_2(\beta+1)=\Phi_2(\beta), \qquad \varphi_3(\beta+1)=\Phi_3(\beta), \quad (112)$$

and

$$\varphi_4(\beta+1)=\Phi_4(\beta)-\Phi_2(\beta)\frac{d}{d\beta}\Phi_2(\beta). \quad (113)$$

Setting $\lambda=1$, one gets for q integer, $q \geqq 2$:

$$\mathrm{F_U}(\beta)=\beta-1-\frac{(\beta-1)^2}{\ln(q)}\left\{\frac{1}{4}|c|^2+\frac{\delta_{q,2}}{16}(3-\beta)(c+\bar{c})|c|^2\right.$$
$$+\frac{3(3-\beta)\delta_{q,2}+(5-\beta)\delta_{q,3}}{96}(3-\beta)(c^2+\bar{c}^2)|c|^2$$
$$+\left(\frac{3+q^2}{16(q^2-1)}-\left(\frac{3+q}{16(q-1)}+\frac{1}{8\ln(q)}\right)(\beta-1)\right.$$
$$\left.\left.-\frac{1}{64}(\beta-1)^2\right)|c|^4\right\}+O(|c|^5). \quad (114)$$

We now recall equation (23) and evaluate the generalized dimensions

$$\mathrm{D}_n=\frac{1}{(n-1)}\mathrm{F_U}(n). \quad (115)$$

For any q, we observe that $\mathrm{D}_1=1$. This is due to the property that at any order in $|c|$, there is at least a coefficient $(\beta-1)^2$ in the expansion of $\mathrm{F_U}(\beta)$. This property is easily derived from the expansion (100). Now the Hausdorff dimension of the support (*i.e.* the Julia set) is D_0. We get for q integer, $q \geqq 2$:

$$\mathrm{D}_0=1+\frac{1}{\ln(q)}\left\{\frac{|c|^2}{4}+\frac{3\delta_{q,2}}{16}(c+\bar{c})|c|^2+\frac{9\delta_{q,2}+5\delta_{q,3}}{32}(c^2+\bar{c}^2)|c|^2\right.$$
$$\left.+\left(\frac{25+16q+7q^2}{64(q^2-1)}+\frac{1}{8\ln(q)}\right)|c|^4\right\}+O(|c|^5). \quad (116)$$

These values coincide with previous results ([18]-[19]) on order 2 and 3. The correlation dimension is obtained for $n=2$, so that for q integer, $q \geqq 2$:

$$\mathrm{D}_2=1-\frac{1}{\ln(q)}\left\{\frac{|c|^2}{4}+\frac{\delta_{q,2}}{16}(c+\bar{c})|c|^2+\frac{\delta_{q,2}+\delta_{q,3}}{32}(c^2+\bar{c}^2)|c|^2\right.$$
$$\left.+\left(\frac{1-16q-q^2}{64(q^2-1)}-\frac{1}{8\ln(q)}\right)|c|^4\right\}+O(|c|^5). \quad (117)$$

For $c=0$, the function $f(t)$ defined in equation (20) is singular: it takes values $-\infty$ for all values of $t \neq 1$, and $f(1)=1$. Therefore it is not possible to expand $f(t)$ on powers of c. If we truncate the expansion (114) at second order in c, so that:

$$\mathrm{F_U}(\beta)=\beta-1-\frac{(\beta-1)^2}{4\ln(q)}|c|^2. \quad (118)$$

The corresponding $f(t)$ reads:

$$f(t)=t-\frac{\ln(q)}{|c|^2}(t-1)^2. \quad (119)$$

In order to get a good limit when c goes to zero, one should invert the function $f(t)$, writing now t as a function of f. One gets:

$$t(f)=1+\frac{|c|^2}{2\ln(q)}\pm\frac{|c|}{\sqrt{\ln(q)}}\sqrt{1+\frac{|c|^2}{4\ln(q)}-f}. \tag{120}$$

There are two branches in the above equation, since $f(t)$ has a maximum for $t=t_m$. In the general case, call t_m the value of t for which $f'(t)=0$. Then $f(t_m)$ is the Hausdorff dimension of the support of the measure. From the relation $t=\frac{d}{d\beta}\mathrm{F_U}(\beta)$ one gets:

$$t_m=\left.\frac{d\mathrm{F_U}}{d\beta}\right|_{\beta=0}, \tag{121}$$

and:

$$t-t_m=\sum_{k=1}^{\infty}\frac{1}{k!}\left.\frac{d^{k+1}\mathrm{F_U}(\beta)}{d\beta^{k+1}}\right|_{\beta=0}\beta^k. \tag{122}$$

From $f(t)=t\beta-\mathrm{F_U}(\beta)$, one gets:

$$f(t_m)-f(t)=\sum_{k=2}^{\infty}\left(\frac{1}{k!}-\frac{1}{(k-1)!}\right)\left.\frac{d^k\mathrm{F_U}(\beta)}{d\beta^k}\right|_{\beta=0}\beta^k. \tag{123}$$

So one can express both $t-t_m$ and $\sqrt{f(t_m)-f(t)}$ in power series of β with non-vanishing lower term of power one. Eliminating β between the two equations allows to express $t-t_m$ in powers series of $\sqrt{f(t_m)-f(t)}$. In fact a consistent perturbative procedure (in powers of $|c|$) requires to express $t-t_m$ in terms of the variable ψ, where:

$$\psi=\sqrt{\frac{f(t_m)-f(t)}{|c|^2}}. \tag{124}$$

We must then keep for t_m and $f(t_m)$ only terms up to a given order in the parameter c, and then express from (122) and (123) the quantities $t-t_m$ and $\psi^2=|c|^{-2}(f(t_m)-f(t))$ up to the same order. Doing that, one observes that only a finite number of powers of β survive. Elimination of β (assuming ψ to be constant) between the two resulting relations leads to the requested expansion. Equation (120) is in fact equivalent to the lowest order case (up to order $|c|^2$ included). To illustrate the procedure we give the result obtained when keeping terms up to order 3 in $|c|$:

$$t-t_m=\pm\frac{|c|^2}{\sqrt{\ln(q)}}\left(1+\frac{5}{8}(c+\bar{c})\delta_{q,2}\right)\psi \\ +\frac{1}{4}\delta_{q,2}(c+\bar{c})|c|^2\psi^2+O(|c|^4), \tag{125}$$

where:

$$t_m = 1 + \frac{|c|^2}{2\ln(q)} + \frac{7|c|^2}{16\ln(q)}\,\delta_{q,\,2}(c+\bar{c})$$

$$f(t_m) = 1 + \frac{|c|^2}{4\ln(q)} + \frac{3|c|^2}{16\ln(q)}\,\delta_{q,\,2}(c+\bar{c}). \tag{126}$$

We want to emphasize that expressing t as a function of f brings no new information compared to the expansion (114).

We have not analyzed the rate of convergence of expansion (114). Numerical tests show that the number of terms needed in the expansion of $F_U(\beta)$ on powers of $|c|$, as in (114), increases with the size of the domain in β in which a given accuracy is asked. One of the problems comes from the fact that truncating the expansion as in (114) does not lead in general to a convex function for all real β, but only on some limited interval, and the accuracy is certainly destroyed outside the convexity interval. Accuracy is always better around $\beta=1$ than for large $|\beta|$. Some explicit bounds could be derived from Dobrushin's technique [34]. However we remark that the value $\beta=0$ needed for computing the Hausdorff dimension is not far from $\beta=1$. The same observation holds for $\beta=2$ which leads to the correlation dimension.

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