ANNALES DE L'I. H. P., SECTION A

ERIC GOLES

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*Annales de l'I. H. P., section A*, tome 56, nº 1 (1992), p. 75-90
<http://www.numdam.org/item?id=AIHPA_1992__56_1_75_0>

*Ann. Inst. Henri Poincaré,*

Vol. 56, n° 1, 1992, p. 75-90. *Physique théorique*

# Sand pile automata

by

**Eric GOLES (*)**

Departamento de Ingeniería Matemática,
Facultad de Ciencias Físicas y Matemáticas,
Universidad de Chile,
Casilla 170/3, correo 3, Santiago, Chile

ABSTRACT. — We study dynamical aspects of a class of Cellular Automata: its periodic and transient behavior. This class is related to the temporal evolution of some spatially extended physical systems, as for example a sand pile.

RÉSUMÉ. — Nous étudions le comportement périodique et transitoire d'une classe d'automates cellulaires. En physique, ces automates modélisent l'évolution temporel d'une pile de sable.

## 1. INTRODUCTION

Let $G=(V, E)$ be an undirected connected graph without loops with sites in $V=\{1, 2, \ldots, n, \ldots\}$ and let $E \subseteq V \times V$ be the set of links. Each site $i \in V$ is connected to a finite set of neighbors $V_i=\{j \in V/(i, j) \in E\}$, where $d_i=|V_i|$. As a particular case we get the usual lattice $\mathbb{Z}^d$ with nearest interactions links.

(*) Research support in part by Fondecyt 1211-91 and I.S.I.-Turin.

Let us consider natural numbers, called thresholds, $z_i \in \mathbb{N}$, $z_i \geqq d_i$. The automata that we will study throughout this paper is related with the following local transition rule: Let us suppose that, for any $i \in V$, an integer $x_i$ is assigned to site $i$. If $x_i \geqq z_i$, one may change the configuration $(x_k)_{k \in V}$ as follows:

$$\left.\begin{array}{ll} x_i \leftarrow x_i - z_i, & \\ x_j \leftarrow x_j + 1, & \forall j \in V_i \\ x_k \leftarrow x_k, & \text{otherwise} \end{array}\right\} \tag{1.1}$$

A site such that $x_i \geqq z_i$ will be called a firing site. The previous rule can be interpreted as the following game: a legal move (the application of the local rule) consists of selecting a site which has at least as many chips as its threshold $z_i$, and passing one chip to each of its neighboring sites.

For this rule, two dynamics can be defined: the sequential and the parallel update. The sequential one consists to update sites, one at a time, randomly or in a prescribed order.

For the parallel dynamics, all the sites, are updated synchronously. This situation is the usual one in the context of Cellular Automata. The parallel update can be written as follows:

$$x_i(t+1) = x_i(t) - z_i \mathbf{1}(x_i - z_i) + \sum_{j \in V_i} \mathbf{1}(x_j - z_j); \qquad i \in V \tag{1.2}$$

where $\mathbf{1}(u)$ is the Heaveside function, *i.e.* $\mathbf{1}(u) = 1$ iff $u \geqq 0$ and 0 otherwise.

Equation (1.2) is interpreted as follows: a site $i$ loses $z_i$ chips if its number of chips is at least $z_i$ and receives one chip from each firing neighbor.

This model was proposed in [2] to simulate some aspects of sand-piles. The particular case $z_i = d_i = |V_i|$, has been proposed in [1] and some dynamical aspects developped in [4] and [7]. Also, in this case, the number of chips is conserved under the sequential or the parallel dynamics *i.e.* $\sum_i x_i(t) = c$, $\forall t \in \mathbb{N}$.

*Example.* — Let us consider the one dimensional finite lattice, with sites $V = \{0, 1, 2, 3, 4\}$ and links $E = \{(i, j) \in V / |i - j| = 1\}$, *i.e.* the links are defined by the usual nearest interactions. The thresholds, $z_i$, are the degrees of each site: $z_0 = d_0 = 1$, $z_4 = d_4 = 1$, $z_i = 2$, for $i = 1, 2, 3$. Given the initial configuration $x = (3, 0, 0, 0, 0,)$; *i.e.* three chips in site 0, several evolutions may occur depending in the choice of the firing site as well as the update mode. The sequential and the parallel evolution are showed in Figure 1.

In this paper we shall study transient times and periodic behavior of such a class of Automata Networks. First we study the one-dimensional model and its interpretation as a sandpile dynamics. Later we analyze the periodic behavior of high dimensional cases. We prove that, for trees (cellular spaces without circuits, as a finite Cayley lattice), the dynamics

(i)

| | | | | |
|---|---|---|---|---|
| 3* | 0 | 0 | 0 | 0 |
| 2* | 1 | 0 | 0 | 0 |
| 1* | 2 | 0 | 0 | 0 |
| 0 | 3* | 0 | 0 | 0 |
| 1* | 1 | 1 | 0 | 0 |
| 0 | 2* | 1 | 0 | 0 |
| 1 | 0 | 2* | 0 | 0 |
| 1* | 1 | 0 | 1 | 0 |
| 0 | 2* | 0 | 1 | 0 |
| 1* | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |

(ii)

| | | | | |
|---|---|---|---|---|
| 3* | 0 | 0 | 0 | 0 |
| 2* | 1 | 0 | 0 | 0 |
| 1 | 2* | 0 | 0 | 0 |
| 2* | 0 | 1 | 0 | 0 |
| 1* | 1 | 1 | 0 | 0 |
| 0 | 2* | 1 | 0 | 0 |
| 1* | 0 | 2 | 0 | 0 |
| 0 | 1 | 2* | 0 | 0 |
| 0 | 2* | 0 | 1 | 0 |
| 1* | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |

(iii)

| | | | | |
|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 |
| 1 | 2 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 0 | 2 | 1 | 0 | 0 |
| 1 | 0 | 2 | 0 | 0 |
| 0 | 2 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |

FIG. 1. – Sequential and parallel dynamics in a one-dimensional finite lattice. Evolution (i) and (ii) are sequential, (*) denotes the firing site. Evolution (iii) corresponds to the parallel update.

converges to cycles of period 1 or 2. In other cases we present examples with non-bounded periods (in the size of the set of sites, $|V|$).

## 2. THE ONE-DIMENSIONAL SAND PILE MODEL

The one-dimensional sandpile model was introduced in [2] and [3] to study temporal evolution of some spatially extended physics systems. A one-dimensional sandpile is modelized on the lattice

$$\mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}$$

with nearest interactions. The grains of sand are represented by a non-increasing sequence of non-negative integers $h_1 \geqq h_2 \geqq h_3 \geqq \ldots \geqq h_n \geqq 0$. Each integer, $h_i$, corresponds to the number of sand grains in the $i$-th position (see *Fig.* 2)

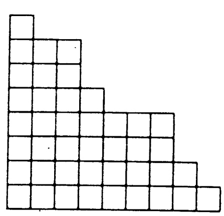

FIG. 2. – The one-dimensionnal sandpile.
(8, 7, 7, 5, 4, 4, 4, 2, 1, 0, ...).

We define the height differences between succesive positions along the sandpile as follows:

$$x_i = h_i - h_{i+1}, \qquad i \geqq 1$$

For instance, given the sandpile $h=(8, 6, 3, 2, 2, 0, \ldots)$ the height differences vector is $x=(2, 3, 1, 0, 2, 0, \ldots)$.

The sand pile dynamics is defined from the introduction of a local rule which takes into account a critical threshold $z \geqq 2$. When the height difference becomes higher than $z$, one grain of sand tumbles to the lower level. The threshold represents the maximal slope permitted without provoking an avalanche. The local rule is written as follows:

If $x_i = h_i - h_{i+1} \geqq z$ then

$$\left.\begin{aligned} h_i &\leftarrow h_i - 1 \\ h_{i+1} &\leftarrow h_{i+1} + 1 \end{aligned}\right\} \tag{2.1}$$

In terms of height differences, (2.1) is equivalent to the following rule:

If $x_i > z$ then

$$\left.\begin{aligned} &\begin{aligned} x_i &\leftarrow x_i - 2 \\ x_{i \pm 1} &\leftarrow x_{i \pm 1} + 1 \end{aligned} \quad \text{for} \quad i \geqq 2 \\ &\text{or} \\ &\begin{aligned} x_1 &\leftarrow x_1 - 2 \\ x_2 &\leftarrow x_2 + 1 \end{aligned} \quad \text{for} \quad i = 1 \end{aligned}\right\} \tag{2.2}$$

For instance, given the sandpile $h=(6, 0, 0, \ldots)$ one gets $x=(6, 0, 0, \ldots)$. Some sequential and parallel evolutions, for $z=2$, are showed in Figure 3.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| (i) | 6* | 6* | (ii) | 6* | 6* | (iii) | 6 | 6 |
| | 5* 1 | 4* 1 | | 5* 1 | 4* 1 | | 5 1 | 4 1 |
| | 4* 2 | 2* 2 | | 4 2* | 2 2* | | 4 2 | 2 2 |
| | 3 3* | 0 3* | | 4* 1 1 | 3* 0 1 | | 3 2 1 | 1 1 1 |
| | 3 2 1 | 1 1 1 | | 3 2 1 | 1 1 1 | | | |
| | sand pile | height diffe-rences | | sand pile | height diffe-rences | | sand pile | heigh diffe-rences |

FIG. 3. – (i) and (ii) are sequential dynamics, (iii) is the parallel dynamics.

It is clear that rule (2.1), on a sand pile, is equivalent to the rule (2.2), for height differences. It suffices to establish the morphism

$$\varphi(h_1, \ldots, h_i, \ldots) = (x_1, x_2, \ldots, x_i, \ldots) \quad \text{where} \quad x_i = h_i - h_{i+1}.$$

This fact is important because the height difference dynamics given by (2.2) (parallel or sequential) is a particular case of the evolutions defined by rule (1.1) of the introduction.

Now we will study the dynamics of a sans pile. That is to say the convergence of a sand pile to a fixed point from the highest sand pile, given by $(n, 0, 0, \ldots)$. Also, we study, for any $n \in \mathbb{N}$, the relaxation time to attain the fixed point and we give a characterisation of it. For that we need some notation and definitions.

Let

$$S_n = \{(h_1, \ldots, h_i, \ldots)/\sum_{i \geqq 1} h_i = n,\ h_i \geqq h_{i+1} \geqq 0,\ h_i \in \mathbb{N}\}$$

be the set of sand pile configurations associated to an integer $n \in \mathbb{N}$. Clearly $S_n$ is the set of non-increasing partitions of the integer $n \in \mathbb{N}$. For instance for $n=6$ one gets

$$S_n = \{(6, 0), (5, 1), (4, 2), (3, 3), (4, 1, 1), (3, 2, 1, 0), (2, 2, 2), (3, 1, 1, 1), (2, 2, 1, 1), (2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}.$$

Given $w, w' \in S_n$, there exists a legal transition from $w$ to $w'$ iff there exists $j \in \mathbb{N}, j \geqq 1$, such that $w_j - w_{j+1} \geqq z$.

In this case we note $w' = T_j(w) = (w_1, \ldots, w_j - 1, w_j + 1, \ldots)$. Clearly, if $z \geqq z'$ and there exist a legal transition for the threshold $z$, then it also exists for $z'$, From previous remark, given a dynamical sequence of sand-piles for $z \geqq 2$, $\{w^t\}_{t \geqq 0}$, such that $w^{t+1} = T_{j_t} w^t$, $j_t \geqq 1$, *i.e.* $w^t_{j_t} - w^t_{j_t+1} \geqq z$, then $w^t_{j_t} - w^t_{j_t+1} \geqq 2$. Hence, since $z \geqq 2$, the same dynamic sequence is produced for the threshold $z=2$. From this fact one concludes that the largest transient sequences appear for the critical threshold $z=2$. Throughout this paragraph we will focus our attention in this case.

A fixed point is a configuration where no legal moves may be done, *i.e.* $h \in S_n$ such that $\forall i \geqq 1$, $h_i - h_{i+1} \leqq 1$.

It is clear that all $n \in \mathbb{N}$ can be written as $n = \frac{k(k+1)}{2} + k'$, $k, k' \in \mathbb{N}$, $0 \leqq k' \leqq k$. For instance:

$$6 = \frac{3.4}{2}; \qquad k=3, \quad k'=0$$

$$7 = \frac{3.4}{2} + 1, \qquad k=3, \quad k'=1$$

$$8 = \frac{3.4}{2} + 3, \qquad k=3, \quad k'=1$$

$$9 = \frac{3.4}{2} + 3, \qquad k=3, \quad k'=3$$

$$10 = \frac{4.5}{2}, \qquad k=4, \quad k'=0.$$

In this context we define the fixed points $s^{(k, k')} \in S_n$ as follows:

$$s^{(k, 0)} = (k, k-1, k-2, \ldots 4, 3, 2, 1, 0 \ldots) \qquad \text{for} \quad k'=0$$

and, for $k' \geqq 1$:

$$s_i^{(k, k')} = \begin{cases} s_i^{(k, 0)} & \text{for} \quad 1 \leqq i \leqq k - k' + 1 \\ s_i^{(k, 0)} + 1, & k - k' + 2 \leqq i \leqq k + 1. \end{cases}$$

It is easy to see that $s^{(k,k')} \in S_n$ are fixed points; *i.e.* $s_i^{(k,k')} - s_{i+1}^{(k,k')} \leqq 1$, $\forall i \geqq 1$. For instance, given $k=4$, $n=\frac{4.5}{2}+0=10$ we get the following fixed points:

$$s^{(4,0)}=(4, 3, 2, 1, 0 \ldots) \in S_{10}$$
$$s^{(4,1)}=(4, 3, 2, 1, 1, 0 \ldots) \in S_{11}$$
$$s^{(4,4)}=(4, 4, 3, 2, 1, 0 \ldots) \in S_{14}$$

The main theorem of this paragraph is the following:

THEOREM 1. – *Given $z=2$ the critical threshold, and*

$$n \in \mathbb{N}, \qquad n=\frac{k(k+1)}{2}+k',$$

*then any sequential trajectory, from the initial sandpile $(n, 0, \ldots) \in S_n$, converges, in exactly $T(n)=\binom{k+1}{3}+kk'-\binom{k'}{2}$ steps, to the fixed point $s^{(k,k')}$.*

Since the parallel dynamics consists in the synchronous update of any legal move, one gets directly the following corollary;

COROLLARY 1. – *For $z=2$ and $n \in \mathbb{N}$, the parallel update applied to $(n, 0, \ldots) \in S_n$ converges to the fixed point $s^{(k,k')}$ and the transient time is bounded by $T(n)$.* ■

In order to prove theorem 1 we need some definitions and lemmas.

Given $h \in S_n$, we define the energy of $h$ as follows:

$$E(h)=\sum_{i \geqq 1} h_i i$$

LEMMA 1. – *Given $w \in S_n$ which accepts a legal move for an index $j \geqq 1$ then:*

$$E(T_j w)-E(w)=1.$$

*Proof.* – We have $T_j w=(w_1, \ldots, w_j-1, w_{j+1}+1, w_{j+2} \ldots)$, hence

$$E(T_j w)-E(w)=\sum_{i \leqq j-1} w_i i+(w_j-1)j$$
$$+(w_j+1)(j+1)+\sum_{i \geqq j+2} w_i i-\sum_{i \geqq 1} w_i i=1. \quad ■$$

COROLLARY 2. – *Any sequential trajectory $\{T_{j_t}(w)\}_{t \geqq 0}$ converges to a fixed point.*

*Proof.* – Let us suppose that there exist a cycle of period $T>1$; *i.e.* a sequence of sandpiles:

$$w^0 \to w^1 \to \ldots \to w^{T-1} \to w^0$$

hence $E(w^0) < E(w') < \ldots < E(w^0)$ which is a contradiction. ■

LEMMA 2. – *Given the initial sandpile* $w \in S_n$, *any sequential trajectory converges to the same fixed point.*

*Proof.* – From corollary 2 we know that any trajectory from $w$, converges to a fixed point. Let us suppose that there exist two fixed points $x$, $y \in S_n$ that are attained from $w$. That is to say we have two trajectories which depend on the updated legal moves:

$$w \to w^1 \to w^2 \to \ldots \to w^q = x$$
$$w \to h^1 \to h^2 \to \ldots \to h^p = y$$

since $x \neq y$, there exists a common sandpile to both trajectories; $w^s = h^m$, such that, $w^{s+1}$ and $h^{m+1}$ do not have a common immediate succesor:

$$\begin{tikzcd} & w^s = h^m \arrow[dl, "j"'] \arrow[dr, "k"] & \\ T_j(w^s) = w^{s+1} & & T_k(h^m) = h^{m+1} \end{tikzcd}$$

where $j$, $k \geqq 1$, $j \neq k$, are indices associated with legal moves.

If $k \geqq j+2$ one gets:

$$w^s = (\ldots, w_j^s, w_{j+1}^s, \ldots, w_k^s, w_{k+1}^s, \ldots)$$

and

$$T_j(w^s) = (\ldots, w_j^s - 1, w_{j+1}^s + 1, \ldots, w_k^s, w_{k+1}^s \ldots) = w^{s+1}$$

so, we can apply the legal move $T_k$ to $w^{s+1}$:

$$T_k(w^{s+1}) = T_k T_j(w^s) = (\ldots, w_j^s - 1, w_{j+1}^s + 1, \ldots, w_k^s - 1, w_{k+1}^s + 1, \ldots)$$

In a similar way, for $h^m = w^s$:

$$T_k(h^m) = T_k(w^s) = (\ldots, w_j^s, w_{j+1}^s, \ldots, w_k^s - 1, w_{k+1}^s + 1, \ldots) = h^{m+1}$$

and

$$\begin{aligned} T_j(h^{m+1}) &= T_j T_k(h^m) \\ &= (\ldots, w_j^s - 1, w_{j+1}^s + 1, \ldots, w_k^s - 1, w_{k+1}^s + 1, \ldots) = T_k T_j(w^s). \end{aligned}$$

That is to say, the sandpile $T_j T_k(w^s)$ is a common succesor of $h^{m+1}$ and $w^{s+1}$, which is a contradiction.

The other case is $k = j+1$; *i.e.* $T_j$ and $T_{j+1}$ are legal moves. So:

$$T_j(w^s) = (\ldots, w_j^s - 1, w_{j+1}^s + 1, w_{j+2}, \ldots) = w^{s+1}$$
$$T_{j+1}(w^s) = (\ldots, w_j^s, w_{j+1}^s - 1, w_{j+2}^s + 1, \ldots) = h^{m+1}$$

since $T_j$, $T_{j+1}$ are legal moves: $w_j^s - w_{j+1}^s \geqq 2$ and $w_{j+1}^s - w_{j+1\ 2}^s \geqq 2$ hence $h_j^{m+1} - h_{j+1}^{m+1} = w_j^s - w_{j+1}^s + 1 \geqq 3$ and $w_{j+1}^{s+1} - w_{j+2}^{s+1} = w_{j+1}^s + 1 - w_{j+2}^s \geqq 3$. So one can applied the legal moves $T_{j+1}$, $T_j$ to $w^{s+1}$, $h^{m+1}$ respectively:

$$v=T_{j+1}T_j(w^s)=(\ldots, w_j-1, w_{j+1}, w_{j+2}+1, \ldots)$$

which is a common successor of $w^{s+1}$ and $h^{m+1}$, which is a contradiction. ■

*Remark.* – We can also say that legal moves commute: $T_j T_k = T_k T_j$.

LEMMA 3. – *Given* $n=\dfrac{k(k+1)}{2}$, *there exists a sequential trajectory which converges to the fixed point* $s^{(k,0)}=(k, k-1, \ldots, 4, 3, 2, 1, 0, \ldots)$.

*Proof.* – The idea consist to transport each grain of sand from the rightmost active sandpile of $(k, k-1, \ldots 3, 2, 1, 0 \ldots)$ to the first one until obtaining $(n, 0, \ldots)$. First, one translates the unique grain of $k$-th pile as follows:

$$\begin{aligned}&(k, k-1, \ldots, 3, 2, 1, 0\ldots)\\ &\quad\leftarrow(k, k-1, \ldots 4, 3, 3, 0, 0\ldots)\leftarrow(k, k-1, \ldots 4, 4, 2, 0, \ldots)\leftarrow\ldots\\ &\qquad\leftarrow(k, k-1, \ldots, l+1, l, l, l-2, l-3, \ldots, 4, 3, 2, 0, 0, \ldots)\\ &\qquad\qquad\leftarrow(k+1, k-1, \ldots, 3, 2, 0\ldots)\end{aligned}$$

After that, one translates the two grains of pile $k-1$, and in general, when one gets the situation corresponding to translate piles $k, k-1, \ldots, k-s+2$:

$$\left(k+\frac{s(s-1)}{2}, k-1, k-2, \ldots, s+2, s+1, s, 0, \ldots\right),$$

one translates the $s$ grains of position $k-s+1$ as follows:

For $j\in\{1, \ldots, s\}$: we construct the sequence:

$$\begin{aligned}&\left(k+\frac{s(s-1)}{2}+j-1, k-1, \ldots, s+1, s-j+1, 0, \ldots\right)\\ &\quad\leftarrow\left(k+\frac{s(s-1)}{2}+j-1, \ldots, s+2, s+2, s-j, 0, \ldots\right)\leftarrow\ldots\\ &\qquad\leftarrow\left(k+\frac{s(s-1)}{2}+j-1, k-1, \ldots, l+1,\right.\\ &\qquad\qquad\left. l, l, l-2, \ldots, s+2, s+1, s-j, 0, \ldots\right)\leftarrow\ldots\\ &\qquad\leftarrow\left(k+\frac{s(s-1)}{2}+j, k-1, k-2, \ldots, s+2, s+1, s-j, 0, \ldots\right)\end{aligned}$$

Any transition in the sequence corresponds to a legal move. In fact, given two consecutive configurations $h'\leftarrow h$:

$$h'=\left(k+\frac{s(s+1)}{2}+j-1, \ldots, l+1, l, l, l-2, l-3, \ldots, s+1, s-j, 0\ldots\right)$$

$$h=\left(k+\frac{s(s+1)}{2}+j-1, \ldots, l+2, l+1, l+1, l-1, l-2, \ldots, s+1, s-j, 0 \ldots\right)$$

we obtain $h_{k-l+1}-h_{k-l+2}=(l+1)-(l-1)=2$ hence $h'=\mathrm{T}_{k-l-1}(h)$. So we have generated a sequential trajectory from $(n, 0, \ldots)$ to $s^{(k, 0)}$. ■

In a similar way we have:

LEMMA 4. – *Given* $n=\dfrac{k(k+1)}{2}+k'$, $1 \leqq k'<k$, *there exists a sequential trajectory which converges to the fixed point* $s^{(k, k')}$.

*Proof.* – By using the same procedure of the previous proof we buid a sequential trajectory, $\mathscr{C}_1$ between $(n, 0, \ldots)$ and the sand pile $w=(k+k', k-1, k-2, \ldots, 4, 3, 2, 1, 0 \ldots)$, where $w_j=0$, $\forall j \geqq k+1$.

From $w$ to the fixed point $s^{(k, k')}$ we build the following sequential trajectory, $\mathscr{C}_2$:

$$\begin{aligned}
w &\rightarrow (k+k'-1, k, k-2, \ldots, 3, 2, 1, 0 \ldots)\\
&\rightarrow (k+k'-1, k-1, k-1, k-3, \ldots, 3, 2, 1, 0) \rightarrow \ldots\\
&\rightarrow (k+k'-1, k-1, k-2, \ldots, l+1, l, l, l-2, \ldots, 3, 2, 1, 0 \ldots)\\
&\rightarrow (k+k'-1, k-1, k-2, \ldots, l+1,\\
&\qquad l, l-1, l-1, l-3, \ldots, 3, 2, 1, 0 \ldots) \rightarrow \ldots\\
&\rightarrow (k+k'-1, k-1, k-2, \ldots, 3, 2, 2, 0 \ldots)\\
&\rightarrow (k+k'-1, k-1, \ldots, 3, 2, 1, 1, 0 \ldots)
\end{aligned}$$

After, we generate, for $s \in \{1, \ldots, k'-1\}$:

$$\begin{aligned}
&\rightarrow (k+k'-s, k-1, k-2, \ldots, s+1\\
&\qquad s, s, s-1, s-2, \ldots, 3, 2, 1, 1, 0, \ldots)\\
&\rightarrow (k+k'-s-1, k, k-2, \ldots, 3, 2, 1, 1, 0 \ldots)\\
&\rightarrow (k+k'-s-1, k-1, k-2, \ldots, l+1, l, l, l-2, \ldots,\\
&\qquad s+2, s+1, s, s, s-1, \ldots, 3, 2, 1, 1, 0 \ldots) \rightarrow \ldots\\
&\rightarrow (k+k'-s-1, k-1, k-2, \ldots, \ldots,\\
&\qquad s+2, s+1, s+1, s, s-1, \ldots, 3, 2, 1, 1, 0 \ldots)
\end{aligned}$$

So the trajectory $\mathscr{C}_1 \cup \mathscr{C}_2$ goes from $(n, 0 \ldots)$ to $s^{(k, k')}$. ■

From the uniqueness of the fixed point (lemma 2) and the previous lemmas 3 and 4 we conclude:

COROLLARY 3. – *Any sequential trajectory from* $(n, 0, \ldots, 0) \in \mathrm{S}_n$ *converges to the fixed point* $s^{(k, k')}$, *where* $n=\dfrac{k(k+1)}{2}+k'$; $0 \leqq k'<k$. ■

Also we may calculate the sequential transient time:

LEMMA 5. – *Given* $(n, 0, \ldots, 0) \in \mathrm{S}_n$, *the sequential transient time to get the fixed point is:*

$$\mathrm{T}(n)=\binom{k+1}{3}+kk'-\binom{k'}{2}, \qquad \forall k \in \mathbb{N}, \quad 0 \leqq k' \leqq k.$$

*Proof.* – Directly from lemma 1 and lemma 4, since the energy increases exactly one unit step by step, the transient time is given by:

$$T(n)=E(s^{(k,k')})-E((n,0,\ldots))=\binom{n+1}{3}+kk'-\binom{k'}{2} \quad \blacksquare$$

*Remarks.*

1. The proof of Theorem 1 follows directly from previous results.

2. In all the cases the relaxation transient time is $O(n\sqrt{n})$, *i.e.* it is a fast dynamics.

3. For any other critical treshold $z>2$ the convergence, from the initial configuration $(n, 0, \ldots)$ to a fixed point is faster, and the same analysis can be done.

Taking into account the morphism between one dimensional sand-piles and height differences we have, directly from theorem-1, the following result:

THEOREM 2. – *The sequential trajectories of the automaton defined by rule* (2.2):

$$\text{If } x_i \geqq 2;\ x_i \leftarrow x_i-2, \qquad x_{i\pm 1} \leftarrow x_{i\pm 1}+1 \qquad \text{for} \quad i \geqq 2$$
$$\text{or } x_1 \leftarrow x_1-2, \qquad x_2 \leftarrow x_2+1 \qquad \text{for} \quad i=1.$$

*converges in exactly* $T(n)$ *steps to the fixed point* $(1, 1, \ldots, 1)$ *for* $n=\dfrac{k(k+1)}{2}$ *or* $(1 \ldots 101 \ldots 1)$ *if* $n=\dfrac{k(k+1)}{2}+k'$, $1 \leqq k' < k$. *In the second case, the component zero appears in position* $k-k'+1$.

*Proof.* – Direct from theore 1 and the morphism:

$$\varphi(w_1, w_2, \ldots, w_i \ldots)=(w_1-w_2, w_2-w_3, \ldots, w_i-w_{i+1}, \ldots) \quad \blacksquare$$

In Figure 4 we give, for $n=18$, an example of a sequential and the parallel dynamics of a sand pile and the height differences.

## 3. HIGH DIMENSIONAL CASES

Here we shall study the asymptotic behavior of the parallel iteration of the automaton given by rule (1.2); *i.e.*:

$$x_i(t+1)=x_i(t)-z_i\mathbf{1}(x_i(t)-z_i)+\sum_{j\in V_i}\mathbf{1}(x_j-z_j),$$
$$z_i \geqq d_i, \qquad i \in V$$

where V is the set of sites and $|V|<\infty$. Recall that $d_i=|V_i|$ is the degree of site $i$.

(i)

| | sand pile | height differences |
|---|---|---|
| | 1 8 | 1 8 0 |
| | 1 7 1 | 1 6 1 |
| | 1 6 2 | 1 4 2 |
| | 1 5 3 | 1 2 3 |
| | 1 4 4 | 1 0 4 |
| | 1 3 5 | 8 5 |
| | 1 2 6 | 6 6 |
| | 1 1 7 | 4 7 |
| | 1 0 8 | 2 8 |
| | 9 9 | 0 9 |
| | 9 8 1 | 1 7 1 |
| | 9 7 2 | 2 5 2 |
| | 8 8 2 | 0 6 2 |
| | 8 7 3 | 1 4 3 |
| | 8 6 4 | 2 2 4 |
| | 8 5 5 | 3 0 5 |
| | 8 5 4 1 | 3 1 3 1 |
| | 8 5 3 2 | 3 2 1 2 |
| | 8 4 4 2 | 4 0 2 2 |
| | 8 4 3 3 | 4 1 0 3 |
| $w=$ | 8 4 3 2 1 | 4 1 1 1 1 |
| | 7 5 3 2 1 | 2 2 1 1 1 |
| | 7 4 4 2 1 | 3 0 2 1 1 |
| | 7 4 3 3 1 | 3 1 0 2 1 |
| | 7 4 3 2 2 | 3 1 1 0 2 |
| | 7 4 3 2 1 1 | 3 1 1 1 0 1 |
| | 6 5 3 2 1 1 | 1 2 1 1 0 1 |
| | 6 4 4 2 1 1 | 2 0 2 1 0 1 |
| | 6 4 3 3 1 1 | 2 1 0 2 0 1 |
| | 6 4 3 2 2 1 | 2 1 1 0 1 1 |
| | 5 5 3 2 2 1 | 0 2 1 0 1 1 |
| | 5 4 4 2 2 1 | 1 0 2 0 1 1 |
| fixed point | 5 4 3 3 2 1 | 1 1 0 1 1 1 |

(ii)

| sand pile | height differences |
|---|---|
| 1 8 | 1 8 |
| 1 7 1 | 1 6 1 |
| 1 6 2 | 1 4 2 |
| 1 5 2 1 | 1 3 1 1 |
| 1 4 3 1 | 1 1 2 1 |
| 1 3 3 2 | 1 0 1 2 |
| 1 2 4 1 1 | 8 3 0 1 |
| 1 1 4 2 1 | 7 2 1 1 |
| 1 0 4 3 1 | 6 1 2 1 |
| 9 5 2 2 | 4 3 0 2 |
| 8 5 3 1 1 | 3 2 2 0 1 |
| 7 5 3 2 1 | 2 2 1 1 1 |
| 6 5 4 2 1 | 1 1 2 1 1 |
| 6 5 3 3 1 | 1 2 0 2 1 |
| 6 4 4 2 2 | 2 0 2 0 2 |
| 5 5 3 3 1 1 | 0 2 0 2 0 1 |
| 5 4 4 2 2 1 | 1 0 2 0 1 1 |
| 5 4 3 3 2 1 | 1 1 0 1 1 1 |

FIG. 4. — Sand pile dynamics for $n=18=\frac{5.6}{2}+3$, $k=5$, $k'=3$. (i) A sequential update with $n=18$ and T $(n)=32$. (ii) The parallel update.

Let $S(t)=\sum_{j\in V_i} x_j(t)$. Clearly if there exists a firing site $i$ [*i.e.* $x_i(t)\geqq z_i$] with $z_i>d_i$, hence this site receives at most $d_i$ chips ad loses $z_i$, that it to say $S(t+1)<S(t)$. From this remark it is easy to see that the steady state only admits fixed points. Hence the critical case, which permits periodic behavior, is given by the situation $z_i=d_i$ for any $i\in V$. In this case $S(t)=c$, $\forall t\geqq 0$, *i.e.* the network does not lost chips.

Clearly, given an initial condition $x(0)\in\mathbb{N}$, the dynamics occurs in the finite set $Q=\{s\in\mathbb{N}/s\leqq\sum_i x_i(0)\}$. We shall say that a finite connect graph without loops $G=(V, E)$ is a tree iff it does not contain circuits. In this context our main result is the following:

THEOREM 3. — 1. *Given the automaton with rule* (1.2) *on a tree. Then, in the steady state, the limit cycles have period* 1 *or* 2.

2. *If* G *is not a tree, cycles which period depends of* $n=|V|$ *may appear.*

*Proof.* — To prove 2 it suffices to take a one-dimensional torus $G=(V, E)$, where $V=\{0, 1, \ldots, n-1\}$ and the set of links

$$V=\{(i, j)/|(i-j) \bmod n|=1\}.$$

In this context, the configuration $(0, 2, 1, \ldots, 1) \in \{0, 1, 2\}^n$ belongs to a cycle of period $n$. Two-dimensional examples may be seen in [4]. ■

To prove (1) we need some definitions and lemmas.

Let $(x(0), \ldots, x(t-1))$ a limit cycle with period T. We define local cycles as follows:

$$\forall i \in V, \quad x_i=(x_i(0), \ldots, x_i(T-1)) \in Q^T$$

and the local traces:

$$\bar{x}_i=(\bar{x}_i(0), \ldots, \bar{x}_i(T-1)) \in \{0, 1\}^T$$

where $\bar{x}_i(t)=\mathbf{1}(x_i(t)-d_i)$.

Traces are coding steps with firing sites. In this way, for any $i \in V$ we define:

$$\operatorname{supp}(\bar{x}_i)=\{t \in [0, T-1]/\bar{x}_i(t)=1\}$$

Clearly $\operatorname{supp}(\bar{x}_i)=\bigcup_{k=1}^{p_i} C_k^i$ where $C_k^i$ are maximal sets of $[0, T-1]$ codding blocks of 1's in the $i$-th trace, *i.e.*:

$$C_k^i=\{l, l+1, \ldots, l+s\} \quad \text{for some } l, s \in [0, T-1]$$

such that $C_k^i \subset \operatorname{supp}(\bar{x}_i)$ and $\bar{x}_i(l-1)=\bar{x}_i(l+s+1)=0$.

For the particular case $\bar{x}_i=\vec{1}$ or $\vec{0}$ (*i.e.* the local cycle is always or never fired) one gets $C_1^i=[0, T-1]$ and $C_k^i=\varnothing$, $\forall k \in [0, T-1]$, respectively.

The same definition can be done for maximal set coding blocks of 0's. If $(\operatorname{supp} \bar{x}_i)^c$ is the complementary set of the $i$-th support, we define, for any $i \in V$:

$$(\operatorname{supp}(\bar{x}_i))^c=\bigcup_{k=1}^{q_i} D_k^i$$

where $D_k^i$ are maximal sets coding blocks of 0's.

From that we define the maximum number of consecutive 1's, M, and 0's, N, in the network, as follows.

$$M=\max_{i \in V} \max_{1 \leqq k \leqq p_i} |C_k^i|$$

and

$$N = \max_{i \in V} \max_{1 \leqq k \leqq q_i} |D_k^i|$$

For instance, given $V = \{1, 2, 3, 4\}$ and the local traces $\bar{x}_1 = (000111110)$, $\bar{x}_2 = (001111110)$, $\bar{x}_3 = (010101000)$, $\bar{x}_4 = (111111000)$, one gets $M = 6$, $N = 4$.

Clearly, $1 \leqq M$, $N \leqq T$. In the extreme cases; *i.e.*, M or N equal 1 or T, the steady state is a fixed point. So, the interesting case to analyze is $2 \leqq M$, $N \leqq T - 1$.

LEMMA 6. – *For any finite connected graph* $G = (V, E)$, *a limit cycle of rule* (1.2) *verifies:*

1. *If there exist a fixed trace,* $\bar{x}_k = \vec{0}$ *or* $\vec{1}$, *then* $T = 1$; *i.e. the limit cycle is a fixed point.*

2. *Let* $[s-k, s] \subseteq \operatorname{supp}(\bar{x}_i)$ *be a maximal set, then there exists a site* $j \in V_i$; *such that* $[s-k-1, s-1] \subseteq \operatorname{supp}(\bar{x}_j)$.

3. Let $[s-k, s] \subseteq \operatorname{supp}(\bar{x}_i))^c$, then there exists $j \in V_i$ such that $[s-k-1, s-1] \subseteq (\operatorname{supp}(\bar{x}_i))^c$.

*Proof.* – 1. We prove the case $\bar{x}_k = \vec{0}$, the other one is analogous. Clearly it suffices to prove that for any $j \in V_k$ (*i.e.* the neighbors of site $k$) $\bar{x}_j = \vec{0}$. From the definition of local rule (1.2), we have:

$$x_k(t+1) = x_k(t) + \sum_{j \in V_k} \bar{x}_j(t) \geqq x_k(t)$$

hence $x_k(0) \geqq x_k(1) \geqq \ldots \geqq x_k(T-1) \geqq x_k(0)$, so $x_k(t) = a \in \{0, 1\}$ for any $t \in [0, T-1]$, *i.e.* the site $k$ is a local fixed point. Now, let us suppose that there exists $j \in V_k$ and $t^* \in [0, T-1]$ such that $\bar{x}_j(t^*) = 1$. So:

$$x_k(t^*+1) \geqq x_k(t^*) + 1 > x_k(t^*)$$

which is a contradiction because $x_k$ is a fixed point.

2. By inductive application of rule (1.2) one gets:

$$x_i(s) = x_i(s-k) - k\,d_i + \sum_{j \in V_i} \sum_{t=1}^{k} \bar{x}_j(s-t)$$

since $[s-k, s]$ is maximal $\bar{x}_i(s-k-1) = 0$, so

$$x_i(s) = x_i(s-k-1) - k\,di + \sum_{j \in V_i} \sum_{t=1}^{k+1} \bar{x}_j(s-t)$$

Let us suppose that the property does not hold, *i.e.* $\forall j \in V_i$, $\exists t_j \in [s-k-1, s-1]$ with $\bar{x}_j(t_j)=0$. Hence:

$$\sum_{j \in V_i} \sum_{t=1}^{k+1} \bar{x}_j(s-t) \leqq (k+1)\,d_i - d_i = k\,d_i$$

so $x_i(s) \leqq d_i - 1 - k\,d_i + k\,d_i = d_i - 1$, then $\bar{x}_i(s)=0$ which is a contradiction because $s \in \operatorname{supp}(\bar{x}_i)$.

3. This property is proved in a similar way to the previous one. ■

*Remark.* – The previous lemma holds for arbitrary graphs.

LEMMA 7. – *Let* $G=(V, E)$ *be a tree then:*

$$0<M<T \;\Rightarrow\; M=1$$
$$0<N<T \;\Rightarrow\; N=1$$

i.e. *when the limit cycle is not a fixed point* (*cases* $M, N \neq 0$ *or* $T$), *the maximal blocks of* 1*'s and* 0*'s are of size* 1.

*Proof.* – We prove for M, (for N the proof is similar). Let us suppose $M \geqq 2$. let $i_0 \in V$ be a site where a maximum set of 1's occurs, *i.e.*: $\operatorname{supp}(\bar{x}_{i_0}) \supseteqq C^0 = [t, t+M-1]$, where $C^0$ is a maximal set. From previous lemma 6.2 there exists $i_1 \in V_{i_0}$ such that $\operatorname{supp}(\bar{x}_{i_1}) \supseteqq C^1 = [t-1, t+M-2]$. Furthermore, since M codes the maximum length of a set of 1's, $C^1$ is maximal. Also, from lemma 6.2 there exists $i_2 \in V_{i_1}$ such that $\operatorname{supp}(\bar{x}_{i_2}) \supseteqq C^2 = [t-2, t+M-3]$ a maximal set and $i_0 \neq i_2$, because, since $t-1 \in \operatorname{supp}(\bar{x}_2)$, $i_0 = i_2$ implies $t-1 \in \operatorname{supp}(\bar{x}_{i_0})$, which is a contradiction. By recursive application of Lemma 6.2 and since G is a finite tree, one determines a finite sequence of different sites $\{i_0, i_1, \ldots, i_s\}$ such that $i_s$ is a leaf (*i.e.* $d_{i_s}=1, V_{i_s}=\{i_{s-1}\}$) and:

$$C^0=[t, t+M-1], \qquad C^1=[t-1, t+M-2], \qquad \ldots,$$
$$C^s=[t-s, t+M-(s+1)]$$

Now, Lemma 6.2 applied to the maximal set $C^s$ implies that a maximal set $[t-s-1, t+M-(s+2)]$ must be contained in $\operatorname{supp}(\bar{x}_{s-1})$ hence $\bar{x}_{i_{s-1}}(t-s)=1$ which is a contradiction, because $i_{s-1} \notin C^{s-1}$. ■

*Remark.* – Let us suppose that there exists a cycle $(x(0), \ldots, x(T-1))$ with period $T>1$. We have to prove that $T=2$. From lemma 6 we get $M=N=1$, *i.e.* the period of the traces is 1 or 2; that is to say, $\forall i \in V$, $\bar{x}_i(t+2)=\bar{x}_j(t)=1$.

*Proof of Theorem* 3. – Directly from lemma 6, if $(x(0), \ldots, x(T-1))$ is not a fixed point, hence $M=N=1$, *i.e.* the period of the traces is 1 or 2; that is to say, $\forall i \in V$, $\bar{x}_i(t+2)=\bar{x}_i(t)$.

Now, let $i \in V$, $t \in [0, T-1]$ such that $\bar{x}_i(t)=1$, with exactly $m$ firing neighbors, $\bar{x}_j(t)=1$. From the two periodic property of traces one gets:

$$x_i(t+1)=x_i(t)-d_i+m$$
$$x_i(t+2)=x_i(t+1)+d_i-m=x_i(t)$$

Similarly if $\bar{x}_i(t)=0$ we obtain $x_i(t+2)=x_i(t)$, then $\forall t \in [0, T-1]$, $x(t+2)=x(t)$. ■

Examples of two-periodic behavior are given in Figure 5.

(i)

1-1-3-0-2-0
↓↑
0-3-1-2-0-1

(ii)

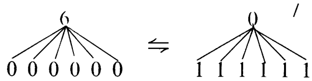

FIG. 5. — (i) Two cycle in a finite one dimensional lattice.
(ii) Two-cycle in a star-tree.

## 4. CONCLUSIONS

We have studied some aspects of the dynamical behavior of local rules modeling temporal evolution of spatially extended physics systems. In the one-dimensional case we studied both the transient and the steady state behavior and exact transient times were calculated. Furthermore, our theoretical framework can be easily adapted to study finite sandpiles with boundary conditions introduced in [2], [3]; *i.e.* the evolution in the lattice $\{1, \ldots, m\}$ with nearest interactions and the particular update rule, for the rightmost site, $m$, accordingly to equations (3.2) or (3.3).

If $x_m \geqq z_m$

$$x_m \leftarrow x_m - 2, \qquad x_{m-1} \leftarrow x_{m-1}+1 \tag{3.2}$$

or If $x_m \geqq z_m$

$$x_m \leftarrow x_m - 1, \qquad x_{m-1} \leftarrow x_{m-1}+1 \tag{3.3}$$

Equation (3.2) represents a non-conservative model. Each update of site $m$ diminishes the number of grain in the system. On the other hand, (3.3) is conservative. In this last situation, as a particular case of a finite tree, the parallel update converges to fixed points or two cycles.

For the two-dimensional model we have proved that the dynamical behavior of the parallel update in trees is independent of the lattice size: periods one or two. In more general cases, *i.e.* graph with circuits, there exist counter-examples where periods depend on the size $n=|V|$ of the lattice. In any case, computer experiments and the property 2 of lemma 6 seem to indicate that for any finite lattice the period is bounded by $n$ ([4], [6]).

It is important to remark that for the particular tree $G=(V, E)$, $V=\{1, \ldots, n\}$ $E=\{(i, j)/(|i-j|=1\}$ (*i.e.* a finite one-dimensional lattice with nearest interactions), the parallel dynamics of the traces is driven by the Lyapunov functional:

$$E(\bar{x}(t)) = -\sum_{i=1}^{n} \bar{x}_i(t) \sum_{j \in V_i} \bar{x}_j(t-1) + \sum_{i=1}^{n} (\bar{x}_i(t) + \bar{x}_i(t-1)) \qquad (3.1)$$

that is to say, $\Delta_t E = E(\bar{x}(t)) - E(\bar{x}(t-1)) \leqq 0$. The proof is direct from the definition of the local rule and the fact that graph G is undirected with degrees $\leqq 2$ [4].

We point out that expression (3.1) has also been determined for the parallel update of Ising models and symmetric Neural Networks [5]. In this context, the parallel dynamics of finite one dimensional sand piles is not to far of Ising models. Nevertheless, this analogy is no longer true for more general trees because expression (3.1) is not a Lyapunov operator.

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(*Manuscript received August* 28, 1990;
*revised version received November* 22, 1990.)