

ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 56, n° 1 (1992), p. 1-25

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Hamiltonians with zero-range interactions supported by a Brownian path

by

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ABSTRACT. — Hamiltonians with zero-range interactions supported by some closed set Ω of Lebesgue measure zero are considered. The approach we use to construct such operators is a limiting procedure with ultraviolet cut-off in Fourier-representation. We consider two different cases: without renormalization and with renormalization procedure. Conditions sufficient to the existence of nontrivial hamiltonians with zero-range interactions are given. The obtained results are applied to the case where Ω is a Brownian path in $\mathbb{R}^m$, $m=3, 4$ and 5 .

RÉSUMÉ. — On considère les hamiltoniens avec l'interaction de radius zéro, concentrée sur l'ensemble fermée de la mesure zéro Ω . L'approche qu'on utilise pour construire ces opérateurs c'est une procédure limite avec le couplage ultraviolet dans la représentation de Fourier. On considère deux cas différent : sans renormalisation et avec une procédure de renormalisation. Les conditions suffisantes à l'existence des hamiltoniens non triviaux sont données. Les résultats obtenus sont appliqués au cas où Ω est une trajectoire de la particule brownienne dans $\mathbb{R}^m$, $m=3, 4$, et 5 .

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INTRODUCTION

Study of hamiltonians describing zero-range interactions is the field where research had been doing for a long time both by physicists and by mathematicians. We shall not give here the detailed review of known results, referring to [1].

Let us only note that most of papers are devoted to the case where interactions are concentrated in a discrete (finite or countable) set of points. One can write formally such hamiltonians in the following form:

$$H = -\Delta_x + \sum_j C_j \delta(x - y_j), \quad y_j \in \mathbb{R}^m.$$

At the same time little attention has been devoted to the continuous case:

$$H = -\Delta_x + \int_{\Omega} dm(y) \delta(x - y) F(y) \quad (1)$$

where Ω is some closed set in $\mathbb{R}^m$ of Lebesgue measure 0, dm - some measure on Ω . In particular, Ω can be a line or a surface in $\mathbb{R}^m$, see [2]-[6]. Except of [8] where a nonstandard analysis is applied, the method usually used to construct self-adjoint hamiltonians, corresponding to (1) is based on the theory of extensions of symmetric operators in Hilbert space. The symmetric operator to be extended is given by $-\Delta|C_0^\infty(\mathbb{R}^m \setminus \Omega)$. The known examples of applications of the theory of extensions to this case are valid only for smooth manifolds Ω . At the same time construction of self-adjoint operators in the case where Ω is not smooth is of interest from different points of view, both mathematical and physical. In particular, it is important in the theory of field (see [8]) to construct hamiltonians of type (1), where Ω is a Brownian path in $\mathbb{R}^m$.

We shall describe a quite general scheme which will allow us to construct self-adjoint hamiltonians with zero-range interaction supported by a closed set with Lebesgue measure 0 which can be parametrised. Namely, let $\Omega \equiv \omega(X)$, where $\omega: X \rightarrow \mathbb{R}^m$, X is a space with measure μ . We shall assume that the map ω is measurable, but not necessarily smooth or continuous. The approach we use is a limiting procedure with ultraviolet cut-off in Fourier-representation. This idea was first used by Berezin and Faddeev in [9] in the case where Ω is a single point. Later it was applied by Grossmann, Hoegh-Krohn and Mebkhout [7] to the N-point case. We may consider our method as a continuous version of [9], [7].

The obtained results can be applied to the case where Ω is a Brownian path in $\mathbb{R}^m$, where $m=3, 4, 5$. We get a standard proof of corresponding results of [8]. However, we not only prove the existence of nontrivial self-adjoint operators (1) but also get an explicit formula for their resolvents. This formula can be considered as a concrete realisation of abstract Krein's

formula [10]. It allows us to begin studying of spectral properties of these operators.

1. CONSTRUCTION OF ZERO-RANGE INTERACTIONS WITHOUT RENORMALIZATION

Let Ω be a closed subset of Lebesgue measure 0 in $\mathbb{R}^m$, ω - a Borel measurable map from X to $\mathbb{R}^m$ such that $\Omega = \omega(X)$, X being a space with finite measure μ . Let us try to give a rigorous meaning to the expression (1). We can write formally

$$H = -\Delta_x + \int_X d\mu(t) \delta(x - \omega(t)) F(t),$$

where F is some real function on X . [One can take in particular $F(\omega(t))$, where $F(x)$ is a real function on $\mathbb{R}^m$]. One can construct an approximation of δ -function by a sequence of separable interactions with ultraviolet cut-off:

$$\delta(x - y) \approx \lim_{n \rightarrow \infty} (k_n(y)) k_n(y),$$

where in p -representation

$$k_n(y, p) = (2\pi)^{-m/2} \exp(-i(p, y)) \chi(|p| \leq n), \quad y, p \in \mathbb{R}^m.$$

Let us consider a sequence of operators in $L_2(\mathbb{R}^m)$, given in p -representation by

$$(H_n f)(p) = p^2 f(p) + \int_X d\mu(t) F(t) e_n(t, p) (f, e_n(t)) \equiv p^2 f(p) + (V_n f)(p), \quad (2)$$

where $e_n(t, p) = (2\pi)^{-m/2} \exp(-i(p, \omega(t))) \chi(|p| \leq n)$. We shall assume that F is a real measurable bounded function on X . One can easily see that operators V_n are bounded and self-adjoint in $L_2(\mathbb{R}^m)$, hence H_n are self-adjoint operators with a domain

$$D(H_n) = \left\{ f : \int_{\mathbb{R}^m} dp |f(p)|^2 (p^2 + 1)^2 < +\infty \right\} \quad (3)$$

Note also that

$$\lim_{n \rightarrow \infty} (V_n f, g) = \int_X d\mu(t) F(t) f(\omega(t)) \overline{g(\omega(t))}$$

for all $f, g \in S(\mathbb{R}^m)$. We shall use the following well-known statement [11].

TROTTER-KATO THEOREM. — Let H_n be a sequence of self-adjoint operators in Hilbert space $\mathcal{H}$. Assume that there are two points z_+ and z_- such that $\operatorname{Im} z_+ > 0$, $\operatorname{Im} z_- < 0$ and the strong limits exist:

$$T_{+, -} = s\text{-}\lim_{n \rightarrow \infty} (H_n - z_{+, -})^{-1}.$$

Assume that $\overline{\operatorname{Ran} T_+} = \mathcal{H}$. Then there exists a self-adjoint operator H such that H_n converge to H in the strong resolvent sense.

To use this theorem we have to study resolvents of H_n . Let $\operatorname{Im} z \neq 0$, $g \in L_2(\mathbb{R}^m)$, $f_n = (H_n - z)^{-1} g \equiv R_n(z)g$. We get from (2) that

$$f_n(p) = \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \int_{\mathbb{X}} d\mu(t) e_n(t, p) F(t) (f_n, e_n(t)) \quad (4)$$

Deduce a formula for a function $\rho_n(t) \equiv (f_n, e_n(t))$. By taking the inner product with $e_n(s)$ in $L_2(\mathbb{R}^m)$ we get

$$\rho_n(s) = a_n(s, z) - \int_{\mathbb{X}} d\mu(t) L_n(s, t, z) F(t) \rho_n(t), \quad (5)$$

where

$$a_n(s, z) = \int_{\mathbb{R}^m} dp \frac{g(p) \overline{e_n(s, p)}}{p^2 - z},$$

$$L_n(s, t, z) = \int_{\mathbb{R}^m} dp \frac{\exp(i(p, \omega(s) - \omega(t)))}{p^2 - z} (2\pi)^{-m} \chi(|p| \leq n)$$

Let us investigate the behaviour of the kernels L_n as $n \rightarrow \infty$. Writing the integral in polar coordinates we get

$$L_n(s, t, z) = \alpha_m \int_0^n dR \frac{R^{m-1}}{R^2 - z} \frac{\sin(R\Delta)}{R\Delta},$$

where $\Delta = \Delta(s, t) = |\omega(s) - \omega(t)|$, α_m is some positive constant, $m \geq 2$ and for $\Delta = 0$ we define $\sin(R\Delta)/(R\Delta) = 1$. Let $\Delta > 0$, $z \in \mathbb{C} \setminus [0, +\infty)$. If $m = 2$ then the limit exists

$$L(s, t, z) \equiv \lim_{n \rightarrow \infty} L_n(s, t, z) = \alpha_2$$

$$\int_0^\infty dR \frac{R}{R^2 - z} \frac{\sin(R\Delta)}{R\Delta} = G_0^{(2)}(\omega(s) - \omega(t), z),$$

where $G_0^{(m)}$ is a Green function of $-\Delta_x$ in $L_2(\mathbb{R}^m)$. From $|\sin(x)/x| \leq C_\delta |x|^{-\delta}$, $0 < \delta < 1$, we get the following estimate:

$$|L_n(s, t, z)| \leq C(\delta, z) \Delta^{-\delta} \quad (6)$$

uniformly on s, t, n . In the case $m=3$

$$L_n(s, t, z) = \frac{\alpha_3}{\Delta} \int_0^{n\Delta} du \frac{\sin u}{u} + \alpha_3 z \int_0^n dR \frac{1}{R^2 - z} \frac{\sin(R\Delta)}{R\Delta},$$

so

$$|L_n(s, t, z)| \leq C_1 \Delta^{-1} + C_2(z), \quad (7)$$

where C_1, C_2 do not depend on s, t, n and

$$L(s, t, z) \equiv \lim_{n \rightarrow \infty} L_n(s, t, z) = G_0^{(3)}(\omega(s) - \omega(t), z) = \frac{\exp(iz^{1/2}\Delta)}{4\pi\Delta}.$$

If $m \geq 4$ then the kernels $L_n(s, t, z)$ do not have pointwise limits as $n \rightarrow \infty$. Therefore we shall assume in the part 1 that $m=2$ or $m=3$. We shall assume also that the following condition is satisfied. If $m=2$ then

(A1) For some $\delta: 0 < \delta < 1/2$

$$\int_{X^2} d\mu(t) d\mu(s) |\omega(s) - \omega(t)|^{-2\delta} < +\infty.$$

If $m=3$ then

(A2) The function

$$N(t, u) = \int_X d\mu(s) (1 + |\omega(s) - \omega(u)|^{-1}) (1 + |\omega(s) - \omega(t)|^{-1})$$

belongs to $L_2(X^2)$.

LEMMA 1. — *Let F be a bounded real function. Consider a sequence of integral operators $B_n(z)$ in $L_2(X)$ with the kernels*

$$B_n(s, t, z) = L_n(s, t, z) F(t), \quad z \in \mathbb{C} \setminus [0, +\infty)$$

The following statements hold:

1. *For all $n \in \mathbb{N}$, $B_n(z)$ are Hilbert-Schmidt operators.*
2. *$B_n(z)$ converge as $n \rightarrow \infty$ in the operator norm to the operators $B(z)$ with the kernels*

$$B(s, t, z) = L(s, t, z) F(t) = G_0^{(m)}(\omega(s) - \omega(t), z) F(t)$$

3. *$B(z)$ are compact in $L_2(X)$.*
4. *$\|B(z)\| \rightarrow 0$ as $\text{Im}(z^{1/2}) \rightarrow +\infty$.*
5. *$B(z)$ is an analytic operator-valued function on $\mathbb{C} \setminus [0, +\infty)$.*

Proof. — The statement 1 follows from boundedness of $B_n(s, t, z)$ and from finiteness of measure μ . Define $\Delta L_n(s, t, z) = L_n(s, t, z) - L(s, t, z)$

and estimate the norm in $L_2(X)$:

$$\begin{aligned} \|B_n(z)h - B(z)h\|^2 &\leq \int_X d\mu(s) \int_X d\mu(t) |h(t)| \|F(t)\| |\Delta L_n(s, t, z)| \\ &\quad \times \int_X d\mu(u) |h(u)| \|F(u)\| |\Delta L_n(s, u, z)| \\ &\leq C \int_{X^2} d\mu(t) d\mu(u) |h(t)| |h(u)| K_n(t, u, z), \quad (8) \end{aligned}$$

where

$$K_n(t, u, z) = \int_X d\mu(s) |\Delta L_n(s, u, z)| |\Delta L_n(s, t, z)|.$$

From (8) we get

$$\|B_n(z) - B(z)\|^2 \leq C \|K_n(z)\|_{\mathcal{Y}} \quad (9)$$

where $\mathcal{Y} \equiv \mathcal{B}(L_2(X^2))$.

Let us show that $\|K_n(z)\|_{\mathcal{Y}} \rightarrow 0$ as $n \rightarrow \infty$.

1. Let $m=2$. Then

$$K_n^2(t, u, z) \leq \int_X d\mu(s) |\Delta L_n(s, t, z)|^2 \int_X d\mu(s) |\Delta L_n(s, u, z)|^2$$

Hence

$$\|K_n(z)\|_{\mathcal{Y}} \leq C \int_{X^2} d\mu(s) d\mu(t) |\Delta L_n(s, t, z)|^2 \quad (10)$$

It follows from condition A1 that $\Delta = |\omega(s) - \omega(t)| > 0$ for a.e. (s, t) . Hence,

$$\lim_{n \rightarrow \infty} |\Delta L_n(s, t, z)|^2 = 0$$

for a.e. $(s, t) \in X^2$. From (6) by dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \|K_n(z)\|_{\mathcal{Y}} = 0.$$

2. Let $m=3$. From A2 we get $N(t, u) < +\infty$ for a.e. (t, u) , and

$$|\omega(s) - \omega(u)| \cdot |\omega(s) - \omega(t)| > 0$$

for a.e. $s \in X$, so

$$\lim_{n \rightarrow \infty} |\Delta L_n(s, t, z)| \cdot |\Delta L_n(s, u, z)| = 0$$

for a.e. $s \in X$. From (7) and A2 by dominated convergence theorem

$$\lim_{n \rightarrow \infty} K_n(t, u, z) = 0 \quad (11)$$

for a. e. (t, u) and

$$|K_n(t, u, z)| \leq C(z) N(t, u) \quad (12)$$

where $C(z)$ does not depend on t, u, n . Applying again the dominated convergence theorem, we obtain from (11), (12) and (A2) that

$$\lim_{n \rightarrow \infty} \|K_n(z)\| = 0.$$

The statement 2 is proved.

The statement 3 follows directly from 1 and 2.

Let us prove 4. If $m=3$ then

$$|B(s, t, z)| \leq C \frac{\exp(-\operatorname{Im}(z^{1/2})\Delta)}{\Delta} \quad (13)$$

From (13) and A2 we get 4. If $m=2$ then

$$|B(s, t, z)| \leq \frac{C(\delta)}{\Delta^\delta} \int_0^\infty dR \frac{R^{1-\delta}}{|R^2 - z|} \quad (14)$$

One can show that the integral in (14) tends to 0 as $\operatorname{Im} z^{1/2} \rightarrow \infty$, hence, 4 is true.

To prove 5 it is sufficient to show that the following limit exists for all $z \in \mathbb{C} \setminus [0, +\infty)$ in operator norm:

$$B'(z) = \lim_{z' \rightarrow z} \frac{B(z) - B(z')}{z - z'}$$

This statement can be proved quite easily, so we shall omit it. The proof of lemma is completed.

LEMMA 2. — *Let F be a real bounded function. Then for all $z: \operatorname{Im} z \neq 0$ the bounded operators $(I + B(z))^{-1}$ exist.*

Proof. — Operators $B(z)$ being compact, it is sufficient to show that there are no nontrivial solutions to the equation

$$f(s) = -(B(z)f)(s), \quad f \in L_2(X). \quad (15)$$

Multiplying (15) by $\overline{f(s)F(s)}$ and integrating over X , we get

$$-(f, Ff) = (B(z)f, Ff)$$

The operators $B_n(z)$ converge to $B(z)$, hence

$$\begin{aligned}
 -(f, Ff) &= \lim_{n \rightarrow \infty} (B_n(z) f, Ff) \\
 &= \lim_{n \rightarrow \infty} \int_{X^2} d\mu(t) d\mu(s) \overline{F(s)} f(s) \overline{F(t)} f(t) (2\pi)^{-m} \\
 &\quad \times \int_{\mathbb{R}^m} dp \frac{\chi(|p| \leq n)}{p^2 - z} \exp(i(p, \omega(s) - \omega(t))) \\
 &\equiv \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{p^2 - z} \chi(|p| \leq n) \equiv \lim_{n \rightarrow \infty} I_n \quad (16)
 \end{aligned}$$

where

$$l(p) = (2\pi)^{-m/2} \int_X d\mu(t) F(t) f(t) \exp(-i(p, \omega(t))), \quad l \in L_\infty(\mathbb{R}^m),$$

and the limit (16) exists.

Therefore,

$$\lim_{n \rightarrow \infty} \operatorname{Im} I_n = \operatorname{Im} z \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2} \quad (17)$$

Since F is real, it follows from (16) and (17) that

$$\int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2} = 0$$

and $l(p) = 0$ for a. e. $p \in \mathbb{R}^m$. Hence,

$$\begin{aligned}
 B(z) f &= L_2(X) - \lim_{n \rightarrow \infty} B_n(z) f \\
 &= L_2(X) - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} dp \frac{\exp(i(p, \omega(s))) l(p)}{p^2 - z} \chi(|p| \leq n) \equiv 0 \quad (18)
 \end{aligned}$$

It follows from (15) and (18) that $f = 0$. The proof is completed.

Remark. — It follows from the proof that for $F(t) \geq 0$ (a repulsive case) the operators $(I + B(z))^{-1}$ exist for all $z = -a$, $a > 0$.

Let us return to the expression for resolvents $R_n(z)$. Fix $z: \operatorname{Im} z \neq 0$. By Lemma 1 and Lemma 2 for $n > N(z)$ operators $(I + B_n(z))^{-1}$ exist and

$$\lim_{n \rightarrow \infty} \|(I + B_n(z))^{-1} - (I + B(z))^{-1}\| = 0 \quad (19)$$

From (4) and (5) we get for all $g \in L_2(\mathbb{R}^m)$

$$\begin{aligned}
 (R_n(z) g)(p) &= \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \\
 &\quad \times \int_X d\mu(t) e_n(t, p) F(t) ((I + B_n(z))^{-1} a_n(z))(t), \quad (20)
 \end{aligned}$$

where

$$a_n(s, z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp \frac{\exp(i(p, \omega(s))) g(p)}{p^2 - z} \chi(|p| \leq n).$$

Obviously,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} dp \frac{|e_n(t, p) - (2\pi)^{-m/2} \exp(-i(p, \omega(t)))|^2}{|p^2 - z|^2} \\ = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} dp \frac{1}{|p^2 - z|^2} \chi(|p| \geq n) (2\pi)^{-m} = 0 \end{aligned} \quad (21)$$

uniformly on X . In a similar manner,

$$\lim_{n \rightarrow \infty} \sup_s |a_n(s, z) - a(s, z)| = 0, \quad (22)$$

where

$$a(s, z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp \frac{g(p) \exp(i(p, \omega(s)))}{p^2 - z}$$

It follows from (20)-(22) that $R_n(z)g$ converge in $L_2(\mathbb{R}^m)$ as $n \rightarrow \infty$, and

$$\begin{aligned} T(z)g \equiv L_2(\mathbb{R}^m) - \lim_{n \rightarrow \infty} R_n(z)g = \frac{g(p)}{p^2 - z} - \frac{(2\pi)^{-m/2}}{p^2 - z} \\ \times \int_X d\mu(t) \exp(-i(p, \omega(t))) F(t) ((I + B(z))^{-1} a(z))(t) \end{aligned} \quad (23)$$

To satisfy to the conditions of Trotter-Kato theorem we should demonstrate that $\text{Ran } T(z)$ is dense in $L_2(\mathbb{R}^m)$. Suppose that it is not true and for some $\varphi \neq 0$, $(\varphi, T(z)g) = 0$ for all $g \in L_2(\mathbb{R}^m)$. Let $f \in C_0^\infty(\mathbb{R}^m \setminus \Omega)$ in x -representation, $g(p) = (p^2 - z)f(p)$. From (23) we get

$$0 = (T(z)g, \varphi) = (f, \varphi) \quad (24)$$

since

$$\begin{aligned} a(s) &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp \frac{g(p) \exp(i(p, \omega(s)))}{p^2 - z} \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp f(p) \exp(i(p, \omega(s))) = f(\omega(s)) \equiv 0. \end{aligned}$$

Since $C_0^\infty(\mathbb{R}^m \setminus \Omega)$ is dense $L_2(\mathbb{R}^m)$, we get from (24) that $\varphi = 0$.

We have proved the main result of part 1.

THEOREM 1. — *Let ω be a measurable map from X to $\mathbb{R}^m$. Assume conditions A1 ($m=2$) or A2 ($m=3$). Let F be a real bounded measurable function on X . Then there exists a self-adjoint operator H in $L_2(\mathbb{R}^m)$ such*

that H_n converge to H in the strong resolvent sense. For all $g \in L_2(\mathbb{R}^m)$ and $z: \operatorname{Im} z \neq 0$ the resolvent of H is given by (23).

Having constructed the self-adjoint operator H , we can study now some of its properties.

Define $H_0 = -\Delta$, $D(H_0) = \{f: f \in H^2(\mathbb{R}^m), f|_{\Omega} = 0\}$. It is clear that H_0 is a closed symmetric operator.

LEMMA 3. — H is a self-adjoint extension of H_0 .

Proof. — Let $f \in D(H_0)$, $g(p) = (p^2 - z)f(p)$, $\operatorname{Im} z \neq 0$. From (23) we have $R(z)g = f$, since $a(s) = f(\omega(s)) = 0$. Hence, $f \in D(H)$ and $(H - z)f = g = (H_0 - z)f$. The proof is completed.

It is important that H be nontrivial extension of H_0 , that is, $H \neq -\Delta|_{H^2(\mathbb{R}^m)}$. We shall denote by $K_R(x_0)$ a ball of radius R centered at x_0 . Define

$$\beta_R(x_0) = \{t: t \in X, \omega(t) \in K_R(x_0)\}$$

LEMMA 4. — Assume that two conditions are satisfied:

1. $\mu(\beta_R(x_0)) > 0$ for some $R > 0$, $x_0 \in \mathbb{R}^m$.
2. For some $M > R$ the function $F(t)$ is strictly positive (or negative) for a. e. $t \in \beta_M(x_0)$.

Then H is a nontrivial extension of H_0 .

Proof. — Suppose that $H = -\Delta$. Let

$$\varphi \in H^2(\mathbb{R}^m), \quad \operatorname{Im} z \neq 0, \quad g(p) = (p^2 - z)\varphi(p), \quad f(p) = (p^2 - \bar{z})\varphi(p).$$

By (23) and $(R(z)g, f) = (R_0(z)g, f)$ we get

$$\int_X d\mu(t) \overline{\varphi(\omega(t))} F(t) (((I + B(z))^{-1} \varphi(\omega(.)))(t)) = 0 \quad (25)$$

Define $h(s) \equiv ((I + B(z))^{-1} \varphi(\omega(.)))(s)$. It follows from (25) that

$$\int_X d\mu(t) F(t) h(t) \overline{((I + B(z))h)(t)} = 0,$$

and

$$\int_X d\mu(t) F(t) |h(t)|^2 = -(Fh, B(z)h) \quad (26)$$

One obtains from (26) and the proof of lemma 2 that $B(z)h = 0$, hence $\varphi(\omega(s)) = ((I + B(z))h)(s) = h(s)$ and by (26)

$$\int_X d\mu(t) F(t) |\varphi(\omega(t))|^2 = 0 \quad (27)$$

Let us choose $\varphi(x)$ such that $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi(x) \equiv 1$ for $x \in K_R(x_0)$ and $\varphi(x) \equiv 0$ for $x \in \mathbb{R}^m \setminus K_M(x_0)$. From (27) and condition 2 of lemma we get

$$\varphi(\omega(t)) = 0$$

for a.e. $t \in \beta_M(x_0)$. On the other hand, $\varphi(\omega(t)) = 1$ for $t \in \beta_R(x_0)$, and by condition 1 $\mu(\beta_R(x_0)) > 0$. This contradiction completes the proof.

Example. — If $F > 0$ (or $F < 0$) on X , then the statement of lemma holds.

LEMMA 5. — *The operator H is bounded from below.*

Proof. — By lemma 1 we have $\|B(z)\| \rightarrow 0$ as $\text{Im}(z^{1/2}) \rightarrow +\infty$. Hence, for some $A > 0$ the inverse operators $(I + B(z))^{-1}$ exist and

$$\|(I + B(z))^{-1}\| \leq 2 \quad (28)$$

for all $z \in V_A = \{z: \text{Re } z \leq -A, 0 < |\text{Im } z| \leq 1\}$. One can easily show from (23) and (28) that $\|R(z)g\| \leq C$ uniformly on V_A and

$$R(-a + i0) = R(-a - i0)$$

for all $a \geq A$. This implies $(-\infty, -A) \in \rho(H)$. The proof is completed.

LEMMA 6. — *The spectrum of H on $(-\infty, 0)$ is discrete with only possible accumulation point $z = 0$.*

The proof of this lemma is identical to that in lemma 12 of part 2, so we shall omit it.

LEMMA 7. — *Assume that $F(t) \geq 0$ for a.e. $t \in X$. Then $(-\infty, 0) \in \rho(H)$, that is, $H \geq 0$ and there are no bound states with negative energy.*

The result follows from the remark to lemma 2.

An important property of H is locality [1].

LEMMA 8. — *Let $f \in D(H)$ and for some ball $K_R f(x) = 0$ for a.e. $x \in K_R$. Then $(Hf)(x) = 0$ a.e. on K_R .*

The result follows from lemma C 2 of appendix C in [1].

Applications

1. Ω is a part of line in $\mathbb{R}^2$. For instance, $X = [0, 1]$, $\omega(t) = (t, 0)$. The condition A1 is equivalent to

$$\int_0^1 \int_0^1 dt ds |t - s|^{-2\delta} < +\infty,$$

that is true. One can apply our results also in the case where Ω is a part of a smooth curve.

2. Ω is a part of plane in $\mathbb{R}^3$. The fact to verify is whether a function

$$N(s, t) = \int_{\mathbb{R}^2} du \frac{\chi(|u| \leq C)}{|s-u| \cdot |u-t|}, \quad s, t \in \mathbb{R}^2,$$

belongs to $L_{2, \text{loc}}(\mathbb{R}^4)$. One can easily show that N has a logarithmic singularity at $s=t$, hence the condition A2 is satisfied.

3. Ω is a sphere in $\mathbb{R}^3$. Our results can be applied to construct hamiltonians with zero-range interactions which are not spherically symmetric.

4. Ω is a Brownian path in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $X=[0, 1]$, $\omega(t)$ be a Brownian path. For $m=2$ we have to show that

$$\Phi(\omega(.)) = \int_0^1 \int_0^1 dt ds |\omega(t) - \omega(s)|^{-2\delta} < +\infty \quad (29)$$

Let μ_{Wien} be a Wiener measure on $C([0, 1]; \mathbb{R}^m)$. One can easily show (see [8]) that

$$\int d\mu_{\text{Wien}} \Phi(\omega(.)) < +\infty$$

for $\delta < 1$. Hence, (29) is satisfied for a. e. $\omega(.) \in C([0, 1]; \mathbb{R}^m)$.

Let $m=3$. To verify condition A2, it is sufficient to show that

$$I = \int d\mu_{\text{Wien}} \Psi(\omega(.)) < +\infty \quad (30)$$

where

$$\begin{aligned} \Psi(\omega(.)) &= \int_0^1 \int_0^1 dt ds \left(\int_0^1 du |\omega(s) - \omega(u)|^{-1} |\omega(u) - \omega(t)|^{-1} \right)^2 \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 dt ds du dv \Delta(s, u)^{-1} \Delta(s, v)^{-1} \Delta(t, u)^{-1} \Delta(t, v)^{-1}, \end{aligned}$$

and $\Delta(s, t) \equiv |\omega(s) - \omega(t)|$. We shall demonstrate that (30) holds for all $m \geq 3$. One should consider all possible orderings of points s, t, u, v . We may assume that $s < t$ and $u < v$ since points s, t and u, v are equivalent. There are 6 cases:

1. $s < t < u < v$, 2. $u < v < s < t$, 3. $s < u < t < v$, 4. $u < s < v < t$, 5. $s < u < v < t$, 6. $u < s < t < v$.

The cases 1 and 2, 3 and 4, 5 and 6 are similar (we have to do a substitution $s \rightarrow u, t \rightarrow v$). Finally we have 3 essentially different cases: 1, 3 and 5. Denote by I_1, I_3, I_5 corresponding contributions in I .

Case 1:

$$I_1 = \int_{\mathbb{P}} ds dt du dv \chi(s < t < u < v) J_1,$$

where $P = [0, 1]^4$,

$$J_1(s, t, u, v) = C_m \int_{(\mathbb{R}^m)^4} dx_1 dx_2 dx_3 dx_4 |x_1 - x_3|^{-1} \\ \times |x_2 - x_3|^{-1} |x_1 - x_4|^{-1} |x_2 - x_4|^{-1} \\ \times \frac{\exp(-(x_1^2/s + (x_2 - x_1)^2/(t-s) + (x_3 - x_2)^2/(u-t) + (x_4 - x_3)^2/(v-u)))}{s^{m/2} (t-s)^{m/2} (u-t)^{m/2} (v-u)^{m/2}},$$

C_m is some constant. Obviously,

$$\int_0^1 ds \frac{\exp(-x^2/s)}{s^{m/2}} = C |x|^{2-m} \\ \times \int_0^{1/x^2} d\tau \tau^{-m/2} \exp(-1/\tau) \leq C_m |x|^{2-m} \exp(-x^2/2), \quad m \geq 3.$$

Consequently,

$$I_1 \leq C_m \int_{(\mathbb{R}^m)^4} dx_1 dx_2 dx_3 dx_4 \\ \times \exp(-1/2(x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + (x_4 - x_3)^2)) \\ |x_1|^{2-m} |x_2 - x_1|^{2-m} |x_3 - x_2|^{2-m} |x_4 - x_3|^{2-m} \\ |x_1 - x_3|^{-1} |x_2 - x_3|^{-1} |x_1 - x_4|^{-1} |x_2 - x_4|^{-1} \quad (31)$$

Let us change the variables in (31): $u = x_4 - x_3$, $v = x_3 - x_2$, $w = x_2 - x_1$, $x = x_1$. Then

$$I_1 \leq \int_{\mathbb{R}^m} dv |v|^{1-m} \exp(-v^2/2) L(v), \quad (32)$$

where

$$L(v) = \int_{\mathbb{R}^m} dw |u|^{2-m} |u+v|^{-1} \exp(-u^2/2) \\ \times \int_{\mathbb{R}^m} dw |w|^{2-m} |w+v|^{-1} |u+v+w|^{-1} \exp(-w^2/2) \\ \equiv \int_{\mathbb{R}^m} du |u|^{2-m} |u+v|^{-1} \exp(-u^2/2) M(u, v) \quad (33)$$

Obviously,

$$M(u, v) \leq \int_{\mathbb{R}^m} dw |w|^{2-m} \exp(-w^2/2) (|w+v|^{-2} + |u+v+w|^{-2}) \quad (34)$$

The integral

$$D(y) = \int_{\mathbb{R}^m} dw |w|^{2-m} |w+y|^{-2} \exp(-w^2/2)$$

can be easily estimated:

$$D(y) \leq C(1 + \chi(|y| \leq 1) |\ln(|y|)|) \quad (35)$$

It follows from (33)-(35) that

$$\begin{aligned} L(v) \leq C \int_{\mathbb{R}^m} du |u|^{2-m} |u+v|^{-1} \exp(-u^2/2) \\ (1 + \chi(|v| \leq 1) |\ln(|v|)| + \chi(|u+v| \leq 1) \\ |\ln(|u+v|)|) \leq C(1 + \chi(|v| \leq 1) |\ln(|v|)|) \quad (36) \end{aligned}$$

From (32) and (36) we get

$$I_1 \leq C \int_{\mathbb{R}^m} dv |v|^{1-m} \exp(-v^2/2) (1 + \chi(|v| \leq 1) |\ln(|v|)|) < +\infty$$

Case 3. — In an analogous manner we have to estimate the integral

$$\begin{aligned} I_3 \leq C \int_{(\mathbb{R}^m)^3} du dv dw \exp(-1/2(u^2 + v^2 + w^2)) \\ \times |u|^{2-m} |v|^{2-m} |w|^{2-m} |u|^{-1} |v|^{-1} |w|^{-1} |u+v+w|^{-1} \\ \leq C \int_{(\mathbb{R}^m)^2} dv dw \exp(-1/2(v^2 + w^2)) |v|^{1-m} |w|^{1-m} \\ \times \int_{\mathbb{R}^m} du |u|^{1-m} |u+v+w|^{-1} \exp(-u^2/2) \\ \leq C \int_{(\mathbb{R}^m)^2} dv dw \exp(-1/2(v^2 + w^2)) \\ |v|^{1-m} |w|^{1-m} (1 + \chi(|v+w| \leq 1) |\ln(|v+w|)|) < +\infty \end{aligned}$$

Case 5:

$$\begin{aligned} I_5 \leq C \int_{\mathbb{R}^m} du |u|^{1-m} \exp(-u^2/2) \int_{\mathbb{R}^m} dv |v|^{2-m} |u+v|^{-1} \exp(-v^2/2) \\ \times \int_{\mathbb{R}^m} dw |w|^{1-m} |u+v+w|^{-1} \exp(-w^2/2) \leq C \int_{\mathbb{R}^m} du |u|^{1-m} \exp(-u^2/2) \\ \times \int_{\mathbb{R}^m} dv |v|^{2-m} |u+v|^{-1} \exp(-v^2/2) (1 + \chi(|u+v| \leq 1) |\ln(|u+v|)|) \\ \leq C \int_{\mathbb{R}^m} du |u|^{1-m} \exp(-u^2/2) < +\infty. \end{aligned}$$

We have just proved that for $m \geq 3$

$$\psi(\omega(.)) < +\infty$$

for a. e. $\omega(.) \in C([0, 1]; \mathbb{R}^m)$. Therefore we have constructed corresponding semibounded self-adjoint hamiltonians in $L_2(\mathbb{R}^2)$ and $L_2(\mathbb{R}^3)$. This can

be considered as a standard proof of corresponding statements in [8] for $m \leq 3$, obtained there by means of nonstandard analysis.

2. HAMILTONIANS OBTAINED BY A RENORMALIZATION PROCEDURE

The considerations of the part 1 show that in certain cases the construction procedure without renormalisation fails. If $m \geq 4$ then the kernels $B_n(s, t, z)$ have not pointwise limit as $n \rightarrow \infty$. If $m=2$ and Ω is discrete or $m=3$ and Ω is discrete or a smooth curve, operators $B_n(z)$ have not a bounded limit as $n \rightarrow \infty$. Hence, we need another procedure. It is well known in the case where Ω consists of one or several separated points in $\mathbb{R}^m$, $m=2, 3$. The corresponding method was elaborated in [9], [7]. What we shall give below for $m=3, 4, 5$ is a continuous version of it.

The main idea is again to construct a sequence of operators H_n given by

$$(H_n f, g) = \int_{\mathbb{R}^m} dp p^2 f(p) \overline{g(p)} + \int_X d\mu(t) (F_n \varphi_n)(t) \overline{\psi_n(t)}, \quad (37)$$

where $\varphi_n(t) = (f, e_n(t))$, $\psi_n(t) = (g, e_n(t))$. The difference from part 1 is that F_n should be now some operators in $L_2(X)$ depending on n , $\omega(\cdot)$.

Assume that F_n are bounded self-adjoint operators in $L_2(X)$. One can see that H_n are self-adjoint in $L_2(\mathbb{R}^m)$ with the same domain (3). By the same consideration as in the part 1 we get

$$f_n(p) = \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \int_X d\mu(t) e_n(t, p) (F_n \rho_n)(t), \quad (38)$$

and

$$\rho_n(s) = a_n(s, z) - \int_X d\mu(t) L_n(s, t, z) (F_n \rho_n)(t), \quad (39)$$

where $f_n = (H_n - z)^{-1} g$, $\text{Im } z \neq 0$; ρ_n , a_n and L_n are given by the same formulae. We shall assume for a while that $g \in S(\mathbb{R}^m)$.

Assume that operators F_n have bounded inverses. Then

$$(F_n^{-1} h_n)(s) = a_n(s, z) - \int_X d\mu(t) L_n(s, t, z) h_n(t), \quad (40)$$

where $h_n = F_n \rho_n$. As we just have mentioned, the operators $L_n(z)$ may have not bounded limits as $n \rightarrow \infty$. Therefore we should separate the most singular part in $L_n(z)$ as $n \rightarrow \infty$:

$$L_n(z) = L_n^{\text{sing}} + L_n^{\text{reg}}(z)$$

Then we shall choose F_n^{-1} such that L_n^{sing} be compensated. Let

$$F_n^{-1} = A - L_n^{\text{sing}} \quad (41)$$

where A is some bounded self-adjoint operator in $L_2(X)$ which has a bounded inverse. From (40)-(41) we get formally

$$h_n = (A + L_n^{\text{reg}}(z))^{-1} a_n(z)$$

If $L_n^{\text{reg}}(z)$ converge in the operator norm to some operators $L^{\text{reg}}(z)$ as $n \rightarrow \infty$ and $(A + L^{\text{reg}}(z))^{-1}$ exist, then h_n and f_n have limits as $n \rightarrow \infty$.

Let us elaborate this strategy in detail. We begin by writing $L_n(s, t, z)$ in the form

$$\begin{aligned} L_n(s, t, z) = & \alpha_m \int_0^n dR R^{m-3} \frac{\sin(R\Delta)}{R\Delta} \\ & + \alpha_m z \int_0^n dR R^{m-3} \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \equiv L_n^{\text{sing}}(s, t) + L_n^{\text{reg}}(s, t, z) \end{aligned} \quad (42)$$

where $z \in \mathbb{C} \setminus [0, +\infty)$; $m = 3, 4, 5$; $\Delta \equiv \Delta(s, t) = |\omega(s) - \omega(t)|$. We need that operators in the right-hand side of (41) be invertible. Fix the operator A and consider a family of operators

$$D_n(\lambda) = \lambda A - L_n^{\text{sing}} = A(\lambda - A^{-1} L_n^{\text{sing}}), \quad \lambda \in \mathbb{R}.$$

The operators L_n^{sing} are compact self-adjoint in $L_2(X)$ for all n , hence, $A^{-1} L_n^{\text{sing}}$ are also compact. Let E_n be the spectrum of $A^{-1} L_n^{\text{sing}}$, this is a countable set. Hence $E = \bigcup_n E_n$ is also countable. For all $\lambda \in \mathbb{R} \setminus E$ the

inverse operators $D_n^{-1}(\lambda)$ exist for all n . We shall choose $\lambda \in \mathbb{R} \setminus E$ and calculate the strong limit of $R_n(z)$. Then one can take another limit as $\lambda \rightarrow 1$, $\lambda \notin E$.

Thus, $F_n = (\lambda A - L_n^{\text{sing}})^{-1}$ exist for all n and are bounded self-adjoint operators in $L_2(X)$. Let us study the behaviour of the kernels $L_n^{\text{reg}}(s, t, z)$ as $n \rightarrow \infty$.

If $m = 3$, then for all $s, t \in X, z \in \mathbb{C} \setminus [0, +\infty)$

$$L^{\text{reg}}(s, t, z) \equiv \lim_{n \rightarrow \infty} L_n^{\text{reg}}(s, t, z) = \alpha_m z \int_0^\infty dR \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \quad (43)$$

The estimate holds

$$|L_n^{\text{reg}}(s, t, z)| \leq C |z| \int_0^\infty dR |R^2 - z|^{-1} \leq C(z), \quad (44)$$

where $C(z)$ does not depend on s, t, n .

If $m = 4$ then for $\Delta(s, t) > 0$

$$L^{\text{reg}}(s, t, z) \equiv \lim_{n \rightarrow \infty} L_n^{\text{reg}}(s, t, z) = \alpha_m \int_0^n dR R \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \quad (45)$$

The uniform estimate holds:

$$|L_n^{\text{reg}}(s, t, z)| \leq C(\delta, z) \Delta^{-\delta} \alpha_m \times \int_0^n dR \frac{R^{1-\delta}}{|R^2 - z|} \leq C(\delta, z) \Delta^{-\delta}, \quad 0 < \delta < 1 \quad (46)$$

If $m=5$ then for $\Delta > 0$

$$L_n^{\text{reg}}(s, t, z) = \alpha_m z \Delta^{-1} \int_0^{n\Delta} du \frac{\sin u}{u} + \alpha_m z^2 \int_0^n dR \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \quad (47)$$

Hence,

$$L^{\text{reg}}(s, t, z) \equiv \lim_{n \rightarrow \infty} L_n^{\text{reg}}(s, t, z) = \alpha_m z \Delta^{-1} \pi/2 + \alpha_m z^2 \int_0^n dR \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \quad (48)$$

It follows from (47) that

$$|L_n^{\text{reg}}(s, t, z)| \leq C(z)(1 + \Delta(s, t)^{-1}), \quad (49)$$

where $C(z)$ does not depend on s, t, n .

Assume that for $m=4$ the map ω satisfies to the condition A1 of part 1 and for $m=5$ ω satisfies to A2. As to the case $m=3$, we shall not assume anything about ω except of measurability.

Let $L_n^{\text{reg}}(z)$, $L^{\text{reg}}(z)$ be integral operators with the kernels $L_n^{\text{reg}}(s, t, z)$, $L^{\text{reg}}(s, t, z)$; $z \in \mathbb{C} \setminus [0, +\infty)$.

LEMMA 9. — *The following statements hold:*

1. $L_n^{\text{reg}}(z)$ are Hilbert-Schmidt operators in $L_2(X)$ for all n .
2. $L_n^{\text{reg}}(z)$ converge in the operator norm in $L_2(X)$ to $L^{\text{reg}}(z)$ as $n \rightarrow \infty$.
3. $L^{\text{reg}}(z)$ are compact in $L_2(X)$.
4. $L^{\text{reg}}(z)$ is an analytic operator-valued function on $\mathbb{C} \setminus [0, +\infty)$.

The proof is identical to that in lemma 1.

LEMMA 10. — *For all z : $\text{Im } z \neq 0$ the inverse operators $(A + L^{\text{reg}}(z))^{-1}$ exist.*

Proof. — The operator A is invertible, hence

$$A + L^{\text{reg}}(z) = A(I + A^{-1}L^{\text{reg}}(z)),$$

where by lemma 9 the operator $A^{-1}L^{\text{reg}}(z)$ is compact. Therefore it is sufficient to show that the equation

$$A^{-1}L^{\text{reg}}(z)f = -f \quad (50)$$

has no nonzero solutions in $L_2(X)$. From (50) we get

$$-(Af, f) = \lim_{n \rightarrow \infty} ((L_n(z)f, f) - (L_n^{\text{sing}}f, f)) \quad (51)$$

The proof of lemma 2 implies that

$$(\mathbf{L}_n^{\text{sing}} f, f) = \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{p^2} \chi(|p| < n) \geq 0, \quad (52)$$

$$(\mathbf{L}_n(z) f, f) = \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{p^2 - z} \chi(|p| \leq n), \quad (53)$$

where $l(p) = (2\pi)^{-m/2} \int_X d\mu(t) f(t) \exp(-i(p, \omega(t)))$. From (51)-(53) we get

$$\begin{aligned} 0 = -\text{Im}(\mathbf{A} f, f) &= \text{Im } z \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2} \chi(|p| \leq n) \\ &= \text{Im } z \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2}, \end{aligned} \quad (54)$$

where the integral over $\mathbb{R}^m$ converges. Since $\text{Im } z \neq 0$, (54) implies $l(p) = 0$ a. e. By (52)-(53) we have $\mathbf{L}^{\text{reg}}(z) f = 0$, hence $f = 0$. The proof is completed.

Remark. — It follows from the proof that $\mathbf{L}_n^{\text{sing}} \geq 0$ and $\mathbf{L}_n^{\text{reg}}(-a) \leq 0$, $\mathbf{L}^{\text{reg}}(-a) \leq 0$, $a > 0$.

Thus the inverse operators $(\lambda \mathbf{A} + \mathbf{L}^{\text{reg}}(z))^{-1}$ exist for all z : $\text{Im } z \neq 0$, $\lambda \in \mathbb{R}$. By the statement 2 of lemma 9, the operators $(\lambda \mathbf{A} + \mathbf{L}_n^{\text{reg}}(z))^{-1}$ exist for $n > N(z)$ and the resolvents $\mathbf{R}_n(z)$ are given by

$$\begin{aligned} (\mathbf{R}_n(z) g)(p) &= \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \\ &\quad \times \int_X d\mu(t) e_n(t, p) ((\mathbf{A} + \mathbf{L}_n^{\text{reg}}(z))^{-1} a_n(z))(t) \end{aligned} \quad (55)$$

Define

$$v_n(p) = \frac{1}{p^2 - z} \int_X d\mu(t) h_n(t) (e_n(t, p) - (2\pi)^{-m/2} \exp(-i(p, \omega(t))))$$

where $h_n = (\lambda \mathbf{A} + \mathbf{L}_n^{\text{reg}}(z))^{-1} a_n(z)$, $g \in \mathcal{S}(\mathbb{R}^m)$. It is obvious that

$$\lim_{n \rightarrow \infty} \sup_s |a_n(s, z) - a(s, z)| = 0,$$

where

$$\begin{aligned} a(s, z) &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp g(p) (p^2 - z)^{-1} \exp(i(p, \omega(s))), \\ a(z) &\in \mathbf{L}_\infty(X) \in \mathbf{L}_2(X) \equiv \mathcal{H}. \end{aligned}$$

The operators $(\mathbf{A} + \mathbf{L}_n^{\text{reg}}(z))^{-1}$ converge to $(\mathbf{A} + \mathbf{L}^{\text{reg}}(z))^{-1}$, hence

$$\lim_{n \rightarrow \infty} \|h_n - h\|_{\mathbf{L}_2(X)} = 0, \quad (56)$$

where $h = (\lambda A + L^{\text{reg}}(z))^{-1} a(z)$. It follows from (56), in particular, that

$$\|h_n\|_{\mathcal{H}} \leq C \quad (57)$$

uniformly on $n \in \mathbb{N}$. Let us show that

$$\lim_{n \rightarrow \infty} \|v_n\|_{\mathcal{M}} = 0, \quad \mathcal{M} \equiv L_2(\mathbb{R}^m).$$

Consider the following integrals, where $d \geq n$:

$$\begin{aligned} J(d, n) &\equiv \int_{\mathbb{R}^m} dp |v_n(p)|^2 \chi(|p| \leq d) = \int_{X^2} d\mu(t) d\mu(s) h_n(t) \overline{h_n(s)} (2\pi)^{-m} \\ &\int_{\mathbb{R}^m} dp \frac{\chi(n \leq |p| \leq d)}{|p^2 - z|^2} \exp(i(p, \omega(s) - \omega(t))) = \alpha_m \int_{X^2} d\mu(t) d\mu(s) h_n(t) \overline{h_n(s)} \\ &\int_n^d dR \frac{R^{m-1}}{|R^2 - z|^2} \frac{\sin(R\Delta)}{R\Delta}, \\ &\Delta \equiv \Delta(s, t) = |\omega(s) - \omega(t)|. \end{aligned} \quad (58)$$

Let us estimate the integrals

$$I(\Delta, d, n) = \int_n^d dR \frac{R^{m-1}}{|R^2 - z|^2} \frac{\sin(R\Delta)}{R\Delta}, \quad \Delta > 0.$$

It is obvious that for $m = 3, 4$

$$|I(\Delta, d, n)| \leq C(\delta, z) \Delta^{-\delta} \gamma_n(z), \quad (59)$$

where $\delta = 0$ for $m = 3$, $0 < \delta < 1$ for $m = 4$, $\gamma_n(z)$ does not depend on d, Δ and

$$\lim_{n \rightarrow \infty} \gamma_n(z) = 0 \quad (60)$$

If $m = 5$ then

$$\begin{aligned} I(\Delta, d, n) &= \Delta^{-1} \int_n^d dR \frac{\sin(R\Delta)}{R} + \Delta^{-1} \int_n^d dR \frac{\sin(R\Delta)}{R} \left(\frac{R^4}{|R^2 - z|^2} - 1 \right) \\ &\equiv I_1(\Delta, d, n) + I_2(\Delta, d, n) \end{aligned} \quad (61)$$

where

$$I_1(\Delta, d, n) = \Delta^{-1} (Si(d\Delta) - Si(n\Delta)); \quad Si(x) = \int_0^x du \frac{\sin u}{u}$$

It is easy to show that

$$|I_2(\Delta, d, n)| \leq C \Delta^{-1} \gamma_n(z), \quad (62)$$

where $\gamma_n(z)$ does not depend on Δ, d and

$$\lim_{n \rightarrow \infty} \gamma_n(z) = 0 \quad (63)$$

Now we can estimate the norm of v_n in $L_2(\mathbb{R}^m)$. From (58)-(59) we get for $m=3, 4$

$$\|v_n\|^2 = \sup_{d>n} J(d, n) \leq C(\delta, z) \gamma_n(z) \times \int_{X^2} d\mu(t) d\mu(s) |h_n(t)| |h_n(s)| |\omega(s) - \omega(t)|^{-\delta} \quad (64)$$

If $m=3$ than $\delta=0$ and from (57) we get

$$\|v_n\|^2 \leq C(\delta, z) \gamma_n(z) \rightarrow 0$$

as $n \rightarrow \infty$. If $m=4$, then by condition A1 the operator with the kernel $|\omega(t) - \omega(s)|^{-\delta}$ is bounded in $L_2(X)$, hence by (58), (59) and (64) we get $\|v_n\| \rightarrow 0$.

Let us consider the case $m=5$. By (58), (62) and condition A2

$$\lim_{n \rightarrow \infty} \sup_{d>n} |J_2(d, n)| = 0, \quad (65)$$

where J_2 is a corresponding contribution in J . As to J_1 , it can be written as follows:

$$J_1(d, n) = \alpha_m \int_{X^2} d\mu(t) d\mu(s) h_n(t) \overline{h_n(s)} \Delta^{-1}(Si(d\Delta) - Si(n\Delta)), \quad (66)$$

$$\Delta = |\omega(s) - \omega(t)|$$

Let $U(d, n)$ be the operator with the kernel $\Delta^{-1}(Si(d\Delta) - Si(n\Delta))$, $U(n)$ be the operator with the kernel $\Delta^{-1}(\pi/2 - Si(n\Delta))$. By condition A2 and the dominated convergence theorem

$$\lim_{d \rightarrow \infty} \|U(d, n) - U(n)\| = 0, \quad (67)$$

$$\lim_{n \rightarrow \infty} \|U(n)\| = 0. \quad (68)$$

From (65)-(67) we get

$$\|v_n\|^2 = \overline{\lim}_{d \rightarrow \infty} (J_1(d, n) + J_2(d, n)) \leq \sup_{d>n} |J_1(d, n)| + \lim_{d \rightarrow \infty} J_2(d, n) = \sup_{d>n} |J_1(d, n)| + J_2(n), \quad (69)$$

where $J_2(n) = \alpha_m (U(n) h_n, h_n)$. It follows from (65), (57), (68) and (69) that $\|v_n\| \rightarrow 0$.

To calculate the strong limit of resolvents we should consider now the following sequence in $L_2(\mathbb{R}^m)$:

$$Q_n(p) = \frac{(2\pi)^{-m/2}}{p^2 - z} \int_X d\mu(t) \exp(-i(p, \omega(t))) h_n(t)$$

It follows from considerations we have made above that $Q_n \in L_2(\mathbb{R}^m)$ for all n . Define

$$Q(q) = \frac{(2\pi)^{-m/2}}{p^2 - z} \int_X d\mu(t) \exp(-i(p, \omega(t))) h(t)$$

As we made it for $\|v_n\|^2$, we write

$$\begin{aligned} V(d) &\equiv \int_{\mathbb{R}^m} dp |Q_n(p) - Q(p)|^2 \chi(|p| \leq d) \\ &= \alpha_m \int_{X^2} d\mu(t) d\mu(s) (h_n(t) - h(t)) \overline{(h_n(s) - h(s))} \overline{I(\Delta, d, 0)} \end{aligned}$$

and it can be shown that

$$\|Q_n - Q\|^2 = \lim_{d \rightarrow \infty} V(d) = (U(h_n - h), h_n - h), \quad (70)$$

where U is an operator with the kernel

$$U(s, t) = \lim_{d \rightarrow \infty} \alpha_m \int_0^d dR \frac{R^{m-1}}{|R^2 - z|^2} \frac{\sin(R\Delta)}{R\Delta}$$

One can show from (59)-(62) that U is a bounded operator in $L_2(X)$. Hence from (56) and (70) we have

$$\lim_{n \rightarrow \infty} \|Q_n - Q\| = 0,$$

in particular, $Q \in L_2(\mathbb{R}^m)$.

We have just shown that for all $z: \operatorname{Im} z \neq 0$, $g \in S(\mathbb{R}^m)$ the sequence $R_n(z)g$ converge in $L_2(\mathbb{R}^m)$ as $n \rightarrow \infty$ and

$$\begin{aligned} T(z)g &\equiv \lim_{n \rightarrow \infty} R_n(z)g = \frac{g(p)}{p^2 - z} - \frac{(2\pi)^{-m/2}}{p^2 - z} \\ &\times \int_X d\mu(t) \exp(-i(p, \omega(t))) ((\lambda A + L^{\operatorname{reg}}(z))^{-1} a(z))(t), \quad (71) \end{aligned}$$

where

$$a(s, z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp \frac{g(p) \exp(i(p, \omega(s)))}{p^2 - z} = (R_0(z)g)(\omega(s))$$

in space-representation. The operators $R_n(z)$ are resolvents of self-adjoint operators in $L_2(\mathbb{R}^m)$, hence for all n

$$\|R_n(z)\| \leq |\operatorname{Im} z|^{-1}$$

Therefore $R_n(z)g$ converge in $L_2(\mathbb{R}^m)$ for all $g \in L_2(\mathbb{R}^m)$ and

$$\|T(z)\| \leq |\operatorname{Im} z|^{-1}$$

However the formula for $a(z)$ holds for any g only for $m=3$.

Let us show that $\text{Ran } T(z)$ is dense in $L_2(\mathbb{R}^m)$. Suppose that $(\varphi, T(z)g) = 0$ for all $g \in S(\mathbb{R}^m)$. For $f \in C_0^\infty(\mathbb{R}^m \setminus \Omega)$ in space-representation define $g(p) = (p^2 - z)f(p) \in S(\mathbb{R}^m)$. By (71) we get

$$T(z)g = R_0(z)g = f,$$

since

$$a(s) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp f(p) \exp(i(p, \omega(s))) = f(\omega(s)) = 0.$$

Hence

$$0 = (\varphi, T(z)g) = (\varphi, f)$$

for all $f \in C_0^\infty(\mathbb{R}^m \setminus \Omega)$. This set is dense in $L_2(\mathbb{R}^m)$, so $\varphi = 0$.

We have proved the following result.

THEOREM 2. — *Let $m=3, 4$ or 5 and ω satisfies to condition A1 for $m=4$ and to A2 for $m=5$. Assume that A is a bounded invertible self-adjoint operator in $L_2(X)$. Then there exists a countable set $E \in \mathbb{R}$ such that for all $\lambda \in \mathbb{R} \setminus E$, $n \in \mathbb{N}$ the operators $F_n = (\lambda A - L_n^{\text{sing}})^{-1}$ exist. If $\lambda \in \mathbb{R} \setminus E$ then the self-adjoint operators H_n given by (37), converge in the strong resolvent sense to some self-adjoint operator H . Its resolvent is given by (71) for all $z: \text{Im } z \neq 0$, $g \in S(\mathbb{R}^m)$.*

One can consider now the limit of (71) as $\lambda \rightarrow 1$. Applying again the Trotter-Kato theorem we get that the right-hand side of (71) is for $\lambda = 1$ also a resolvent of some self-adjoint operator in $L_2(\mathbb{R}^m)$. Therefore we can put $\lambda = 1$.

We may study now some properties of H . The fact that H is a self-adjoint extension of $H_0 = -\Delta$ with the domain $C_0^\infty(\mathbb{R}^m \setminus \Omega)$ can be proved by the same way as in lemma 3.

LEMMA 11. — *Assume that $(A\varphi, \varphi) = 0$ iff $\varphi = 0$. Then H is nontrivial.*

Proof. — Suppose that H is trivial. Then for all $z: \text{Im } z \neq 0$, $g \in S(\mathbb{R}^m)$

$$\begin{aligned} 0 &= ((R(z) - R_0(z))(p^2 - z)g(p), (p^2 - \bar{z})g(p)) \\ &= - \int_X d\mu(t) \overline{\rho(t)} ((A + L^{\text{reg}}(z))^{-1} \rho)(t) \end{aligned} \quad (72)$$

where

$$\rho(s) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp g(p) \exp(i(p, \omega(s))) = g(\omega(s)),$$

g taken in space-representation. Let $\varphi = (A + L^{\text{reg}}(z))^{-1} \rho$. From (72) we get

$$(\varphi, A\varphi + L^{\text{reg}}(z)\varphi) = 0 \quad (73)$$

for all $z: \operatorname{Im} z \neq 0$. From (73) and (52)-(54) we have $L^{\operatorname{reg}}(z)\varphi=0$, hence, $(\varphi, A\varphi)=0$, and by condition of lemma $\varphi=0$. Therefore

$$g(\omega(s)) = \rho(s) = (A + L^{\operatorname{reg}}(z))\varphi = 0$$

This is not true all $g \in S(\mathbb{R}^m)$. The contradiction proves the lemma.

Remark. — The condition of lemma is satisfied, in particular, if $A > 0$ or $A < 0$.

The locality of H (see lemma 8) follows again from [1].

The semiboundedness of H is a difficult problem. The reason is that operators $L^{\operatorname{reg}}(-a)$ may have not a bounded limit as $a \rightarrow \infty$. Let us discuss this question in detail. By lemma 9, lemma 10 and analytical Fredholm theorem [11] the operator-valued function $(A + L^{\operatorname{reg}}(z))^{-1}$ is meromorphic on $\mathbb{C} \setminus [0, +\infty)$ and is analytic on $\mathbb{C} \setminus [0, +\infty) \setminus S$, where S is a discrete set on $(-\infty, 0)$, having as possible accumulation points 0 and $-\infty$.

LEMMA 12. — *Let $z \in (-\infty, 0) \setminus S$. Then z belongs to resolvent set of H .*

This result follows directly from (71).

LEMMA 13. — *Assume that $(A\varphi, \varphi) \leq -\lambda(\varphi, \varphi)$ for some $\lambda > 0$ for all $\varphi \in L_2(X)$. Then $(-\infty, 0) \in \rho(H)$, that is, $H \geq 0$ and it has no negative eigenvalues.*

This result follows from lemma 12 and the remark to lemma 10.

Applications

1. Ω is a single point in $\mathbb{R}^3$. One can take $X = \{0\}$, $\mu(0)=1$, $\omega(0)=y \in \mathbb{R}^3$. In such case the operator F_n is in fact a real number $(A - 4\pi n)^{-1}$. This renormalization scheme has been considered in [9].

2. Ω is a finite set in $\mathbb{R}^3$. In this case our consideration is similar to [7]. One can take $X = \{1, 2, \dots, N\}$, $\mu(j)=1$ for all $j \in X$, $\omega(j)=y_j \in \mathbb{R}^3$. We assume that $y_j \neq y_1$ for $j \neq 1$. The operator A is a matrix such that $A_{ik} = \bar{A}_{ki}$, the operator F_n^{-1} also being a matrix:

$$(F_n^{-1})_{ik} = A_{ik} - d_{ik}^{(n)},$$

where $d_{ii}^{(n)} = 4\pi n$; $d_{ik}^{(n)} = 4\pi \int_0^n dR \frac{\sin(R\Delta)}{R\Delta}$, $i \neq k$, $\Delta = |y_i - y_k|$. It is clear that for $i \neq k$ there exist finite limits

$$d_{ik} = \lim_{n \rightarrow \infty} d_{ik}^{(n)}$$

Therefore one can choose A such that operators F_n be asymptotically diagonal as $n \rightarrow \infty$. Let

$$A_{ii} = \alpha_i; \quad A_{ik} = d_{ik}, \quad i \neq k.$$

Then

$$(F_n^{-1})_{ik} = \delta_{ik} \left(-\frac{1}{4\pi n} + \frac{\alpha_k}{(4\pi n)^2} \right) + o(n^{-2})$$

as $n \rightarrow \infty$. In this concrete case we could take L_n^{sing} diagonal for all n , $(L_n(z))_{ik}$ having finite limits for $i \neq k$. The operators F_n^{-1} and F_n would be diagonal for all n . However, if Ω is not a discrete set, we cannot do it and we should consider the asymptotic diagonality of F_n .

Let us explain the importance of diagonality of F_n . As it has been mentioned in [1], it can be interpreted as independence of zero-range interactions in different points. One can see it from the definition of H_n . It seems quite natural that the condition of independence should be satisfied, at least asymptotically as $n \rightarrow \infty$. In [1] it makes it possible to choose the N -parameter family of self-adjoint extensions, the whole family of extensions being N^2 -parameter.

In the part I this condition is obviously satisfied since F is a function. Certainly, one could apply the renormalization procedure to the case considered in the part I. However, the operators L_n^{sing} having a bounded limit L^{sing} as $n \rightarrow \infty$, the renormalization is equivalent to the consideration of hamiltonians (2), where F is a bounded self-adjoint operator in $L_2(X)$, $F = (A - L^{\text{sing}})^{-1}$. Only for $A = F^{-1} + L^{\text{sing}}$, where F is diagonal, we have self-adjoint extensions H interesting from physical point of view. Other extensions are less attractive.

The situation where renormalization is really necessary occurs when operators $(I + L_n(z)F)^{-1}$ have not nonzero bounded limits as $n \rightarrow \infty$. In this case L_n^{sing} have not bounded limit, and the main problem is to choose operators A such that $F_n = (A - L_n^{\text{sing}})^{-1}$ be asymptotically diagonal.

3. Ω is a Brownian path in $\mathbb{R}^4$ or $\mathbb{R}^5$.

The conditions A1 (for $m=4$) or A2 (for $m=5$) are satisfied, as we have proved in the part I. Therefore for a.e. ω we have self-adjoint operators with expected properties, whose resolvents on a dense set are given by (71). If $A \leq -\lambda I$, where $\lambda > 0$, then these operators are nontrivial by lemma 11 and nonnegative (hence, semibounded) by lemma 13. Since $L_n^{\text{sing}} \geq 0$ and most probably have not a bounded limit, we can expect that in some sense F_n are "negative infinitesimal" operators. This is in agreement with corresponding results of [8] for $m=4, 5$, obtained there by nonstandard methods.

The question we cannot answer at present is for what A operators F_n are asymptotically diagonal.

ACKNOWLEDGEMENTS

I wish to thank Dr. J. F. Brasche who put my attention to this problem. I am very grateful to K. A. Makarov for stimulating discussions.

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(Manuscript received June 8, 1990.)