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Complex entropy for dynamical systems

by

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ABSTRACT. — This paper introduces a notion of complex entropy in a quite intrinsic way. This number can be attached to a family of linear symplectic transforms but also to a large class of dynamical systems.

RÉSUMÉ. — Cet article introduit une notion d'entropie complexe d'une façon intrinsèque. Ce nombre peut être attaché à une famille de transformations linéaires symplectiques mais aussi à une grande classe de systèmes dynamiques.

1. INTRODUCTION

This paper contains a new effective approach to the rotation number of a family of linear symplectic transforms. The rotation number appears to be, in a natural way, the imaginary part of a complex number that we call *complex entropy* since its real part is related to the usual entropy. This approach is quite intrinsic, and we extend it to the general framework of second order differential equations attached to variational problems on a manifold.

A rotation number can be attached to a continuous family of symplectic transforms. In order to do that one has to deal with symplectic geometry.

Before going into more details, let us say that before us several authors gave a good definition of the rotation number [7] but in a slightly different way. They use the fact that the fundamental group of the linear symplectic group $\mathrm{Sl}(2n, \mathrm{R})$ is $\mathbb{Z}$ and that any symplectic transform S has a good polar decompose.

To be more intrinsic than is generally done (to avoid trivial fibrations in the case of manifolds), we observe that in addition to a symplectic 2-form we suppose that we are given a positive complex structure (not necessarily integrable in general). Let us notice that this additional structure exists for a large class of dynamical systems including geodesic flows of Riemannian metrics, and more generally for all systems whose evolution is governed by a convex variational problem (in particular, Hamiltonain systems admitting a Legendre transform).

Our approach deals with the set Σ of Lagrangian subspaces. This set has been extensively studied, and is known to be the homogeneous space $\mathrm{U}(n)/\mathrm{O}(n)$. The famous Maslov class shows that its fundamental group is also $\mathbb{Z}$.

In part 2, we study the linear symplectic transport of a Lagrangian space. We first show that for any Lagrangian subspace L and a reference Lagrangian space V being given we can build a unitary symmetric operator. This provides a nice representation of $\mathrm{U}(n)/\mathrm{O}(n)$, leaving aside the problem of transversality encountered in the definition of the Maslov index. Then, to a family S_t, and to L, we associate a complex family Z_t of operators. Using this operator, we define what we call the *complex entropy* of a Lagrangian space L. In the linear case, this complex entropy is given when it exists is defined by:

$$\gamma(\mathrm{L}) = \lim_{t \to \infty} 1/2t \int_{-t}^{t} d/dt \log(\det(\mathrm{Z}_t^{\mathrm{T}} \mathrm{Z}_t))\, dt.$$

In proposition 1, we show that the imaginary part α of γ does not depend on L. We call α the rotation number of the family S_t. Proposition 2 explains why this number is the same as the one defined by D. Ruelle, and in fact more generally shows that there is only one possible rotation number. At this stage we observe that for explicit computations, the use of Lagrangian spaces induces simpler calculations. In a recent paper [1], using this approach we have been able to obtain estimates on the complex entropy for quadratic Hamiltonians in the adiabatic limit. For a probability space the ergodic subadditive theorem shows the almost sure existence of α. Then, we observe that the real part h of γ is related [5], [6] through ergodic theory to the sum of the Lyapunov exponents, hence our terminology.

In part 3, we deal with manifolds. Using the geometric properties of second order differential equation, especially the one, coming from a convex variationnal problem, we show that we can extend the notion of complex entropy to a large class of dynamical systems. For compact manifolds, we relate the real part of the complex entropy to the classic metric entropy of a flow, and this justifies our terminology. The relevant dynamical systems include in particular the geodesic flows on Riemannian manifolds. Theorem 3 shows that, in this context, the rotation number (divided by π) is the average number of conjugate points counted with their multiplicities. This provides the extension of Sturm-Liouville theory.

2. COMPLEX ENTROPY FOR CONTINUOUS PATHS IN $\mathrm{Sp}(2n, \mathrm{R})$

2.1. Symplectic vector spaces with compatible positive complex structures

Let (E, ω) be a real symplectic vector space with $\dim \mathrm{E} = 2n$. In addition, we suppose we have chosen a complex structure J, and a linear automorphism of E satisfying:

$$\mathrm{J}^2 = -\mathrm{I}_d. \tag{1}$$

This complex structure is assumed to be compatible (or symplectic) and positive that is for every ζ and η in E:

$$\omega(\zeta, \eta) = \omega(\mathrm{J}(\zeta), \mathrm{J}(\eta)), \tag{2}$$

and the scalar product g defined by:

$$g(\zeta, \eta) = \omega(\mathrm{J}(\zeta), \eta) \tag{3}$$

is positive definite (Riemannian). Such a structure is called *positive compatible complex*. The possibility of such a choice on a vector space is insured by the existence of symplectic bases.

2.2. Lagrangian spaces and complexifications

A Lagrangian space L is a maximal isotropic vector space for ω, *i.e.* $\omega(\xi, \eta) = 0$ for every ξ and η in L. These spaces play a particular rule because if V is Lagrangian then:

$$\mathrm{E} = \mathrm{V} \oplus \mathrm{JV} \tag{4}$$

with the corresponding projectors.

$$\mathrm{I}_d = \mathrm{P}_v + \mathrm{Q}_v \tag{5}$$

One easily checks that the projectors satisfy:

$$\mathrm{P}_v \mathrm{J} = \mathrm{J} \mathrm{Q}_v \tag{6}$$

$$\mathrm{Q}_v \mathrm{J} = \mathrm{J} \mathrm{P}_v \tag{7}$$

This gives a complexification of E that we will write V^c: any vector ζ in E can be split into:

$$\zeta = v_1 + J\, v_2. \tag{8}$$

Thus, ζ can be represented as an element z of V^c by:

$$z = v_1 + i v_2. \tag{9}$$

Let us remark that there is a natural Hilbertian inner product g^c on V^c:

$$g^c(v_1 + i v_2,\, u_1 + i u_2) = g(v_1 - i v_2,\, u_1 + i u_2).$$

Furthermore, if $\zeta = v_1 + J\, v_2$ and $\eta = u_1 + J\, u_2$, we can also express g^c as:

$$g^c(v_1 + i v_2,\, u_1 + i u_2) = g(\zeta, \eta) + i\,\omega(\zeta, \eta)$$

which is the standard complexification of Lagrangian spaces. Two Lagrangian spaces, L and V, give rise to two possible complexifications of E and so induce an isomorphism $U : L^c \to V^c$. Using the two possible decompositions $\zeta = l_1 + J\, l_2 = v_1 + J\, v_2$ and (6), (7) we see that:

$$v_1 = P_v(l_1) + JQ_v(l_2)$$
$$v_2 = P_v(l_2) - JQ_v(l_1),$$

and thus $v_1 + i v_2 = (P_v - i\, JQ_v)(l_1 + i l_2)$. Notice that we use only the restriction of Q_v and P_v to L. So, $U : L^c \to V^c$ is provided by:

$$U = P_v/L - i\, JQ_v/L \tag{10}$$

For the sake of brevity we will denote $P_v/L \to V$ by P and $Q_v/L : L \to JV$ by Q.

LEMMA 1. – *The operator* $U : L^c \to V^c$ *defined above is unitary, that is* $U^* U = I_d(L^c)$.

Proof. – $U^* = P^T + i(JQ)^T = P^T - i Q^T J$ where the transposed operator is defined with respect to the metric g given by (3). Hence,

$$U^* U = P^T P + Q^T Q - i(Q^T JP + P^T JQ).$$

The real part is clearly the identity, and the imaginary part is the wellknown Wronskian which vanishes because L is Lagrangian.

Remark. – The map $U^T U : L^c \to L^c$ is a unitary symmetric operator for which the complex determinant is well-defined. This number belongs to the unit circle, and we call its argument the "*angle*" between L and V.

2.3. Complex entropy

Let us consider a given continuous family S_t of linear symplectic transforms (with respect to ω) and a fixed Lagrangian space V. Through S_t, a Lagrangian space L is mapped onto a Lagrangian space L_t. Let

$(S_t/L)^c$ be the complexified of the restriction $S_t/L : L \to L_t$. Now, let us define the operator $Z_t : L^c \to V^c$ by:

$$Z_t = U(L_t) \circ (S_t/L)^c \tag{11}$$

where $U(L_t) : L_t^c \to V^c$ is the unitary operator associated to the pair (L_t, V) as above. The family of operators Z_t can be written as

$$Z_t = X_t + i Y_t$$

where

$$X_t = (P_t \circ S_t/L)^c, \qquad Y_t = -(JQ_t \circ S_t/L)^c \tag{12}$$

(the superscript c denotes the natural complexification), P_t, Q_t are the projectors of L_t onto $V \oplus JV$.

Definition. — Consider a continuous family S_t of linear symplectics transforms of a space E endowed with a compatible positive complex structure J. Let V be a fixed Lagrangian space. To a given Lagrangian space L, we associate the number $\gamma(L)$ given by:

$$\gamma(L) = \lim_{t \to \infty} 1/2t \int_{-t}^{t} d/dt \log(\det(Z_t^T Z_t))\, dt \tag{13}$$

when it exists the number will be called the *complex entropy* of L.

Remark. — Only the determinant of $Z_t^T Z_t$ is relevant since it is an endomorphism of L^c (in fact an automorphism).

Proposition 1. — *When it exists, the number* $\alpha = \operatorname{Im}(\gamma(L))$ *does not depend on the chosen Lagrangian spaces* L *and* V. *This number will be called the* rotation number *of the family* S_t.

Proof of proposition 1. — Since $Z_t = U(L_t) \circ (S_t/L)^c$, we have:

$$\log(\det(Z_t^T Z_t)) = \log(\det([S_t/L)^c]^T (S_t/L)^c) + \log(\det(U(L_t)^T U(L_t)), \tag{14}$$

and the first term is real. Thus, we have to prove that, when it exists, the number:

$$\alpha = \lim_{t \to \infty} 1/2t \operatorname{Im} \int_{-t}^{t} d \log(\det[U(L_t)^T U(L_t)]) \tag{15}$$

does not depend on L. We shall study the relative "angle" between $V_t = S_t(V)$ and $L_t = S_t(L)$. Thus, we use the unitary operator $U_t : L_t^c \to V_t^c$ defined as in (10). The following diagram is obviously commutative:

$$\begin{tikzcd} L_t^c \arrow[r, "U_t"] \arrow[dr, "U(L_t)"'] & V_t^c \arrow[d, "U(V_t)"] \\ & V^c \end{tikzcd}$$

Thus we have:

$$\log(\det[U^T(L_t)U(L_t)]) - \log(\det[U^T(V_t)U(V_t)]) = \log(\det[U_t^T U_t]). \quad (16)$$

Now, let us suppose that the argument of $\det[U_t^T U_t]$ varies by more than $2n\pi$. Then, necessarily one of the eigenvalues of the symmetric unitary operator $U_t^T U_t$ (they can be followed continuously) should have crossed 1 at a certain time, say τ. So, there exists ζ in L_τ^c such that $U_\tau^T U_\tau(\zeta) = \zeta$. Writing the congugated equation and noticing that $(U_\tau^T U_\tau)^* = (U_\tau^T U_\tau)^{-1}$ we see that ζ^* also satisfies the same relation, thus $U_\tau^T U_\tau$ admits a real eigenvector ξ with eigenvalue 1. Let us remark that, by construction, we have:

$$U_\tau^T U_\tau = (p_\tau + iJq_\tau)^{-1}(p_\tau - iJq_\tau) \quad (17)$$

where p_τ and q_τ are defined as in (10) with the pair (L_τ, V_τ). (17) yields:

$$(p_\tau - iJq_\tau)(\xi) = (p_\tau + iJq_\tau)(\xi) \quad (18)$$

which implies that $q_\tau(\xi) = 0$. Thus ξ lies in $L_\tau \cap V_\tau$. The intersection is invariant by S_t. Thus $S_\tau^{-1}(\xi)$ was in $L \cap V$, and so still a eigenvector associated with the eigenvalue 1. Consequently, the eigenvalue 1 and its multiplicity are invariant. This shows that either an eigenvalue is different from 1, and then it can never cross 1 or is equal to 1 and then remains 1. Thus, the argument of $\det[U_t^T U_t]$ cannot vary in time by more than $2n\pi$. Therefore, the difference in (16) is bounded by $2\pi n$ which ends the proof of Proposition 1.

Let us now discuss the existence of the limit (15) in the framework of probability theory. Let (Ω, P, T^t) be a probability space with a continuous P-preserving shift T^t. And let $S(t, \omega)$ be a continuous (in t) and measurable (in ω) family of linear symplectic transforms satisfying:

$$S(t+t', \omega) = S(t', T^t\omega).S(t, \omega) \quad (19)$$

Then, it is well-known that for any Lagrangian space L, the real part of $\gamma_\omega(L)$ exists with probability one. Now, if we consider any absolutely continuous measure ν on the set of Lagrangian spaces, then ν-a. s. this limit is equal to the sum of the positive Lyapunov exponents. Furthermore, this limit $\chi(\omega)$ is invariant under the shift and thus a. s. constant in the ergodic case. The situation is simpler for the rotation number since it does not depend on L: if we set:

$$\alpha(t, L, \omega) = \operatorname{Im}\int_0^t d\log(\det[U_\omega(L_t)^T U_\omega(L_t)]), \quad (20)$$

we have:

$$\begin{aligned}\alpha(t+t', L, \omega) &= \alpha(t, L, \omega) + \alpha(t', L_t, T^t\omega)\\ &\leqq \alpha(t, L, \omega) + \alpha(t', L, T^t\omega) + 2n\pi \quad (21)\end{aligned}$$

since (16) was shown to be bounded by $2\pi n$. And the subadditive ergodic theorem ensures that α_ω exists with probability one and does not depend on L by Proposition 1. As previously α_ω is invariant under the shift and thus is a. s. constant in the ergodic case.

2.4. Properties of the rotation number

PROPOSITION 2. – *Let* $S_1(t)$ *and* $S_2(t)$ *be two continuous family of linear symplectic transforms satisfying* $S_1(0)=S_2(0)=\mathrm{Id}$. *We suppose that the rotation numbers* α_1 *and* α_2 *exist. Then, the rotation number* α *of the family* $S_2(t)\circ S_1(t)$ *exists and is equal to* $\alpha_1+\alpha_2$.

Proof. – Let L be a Lagrangian space and $L_t=S_1(t)L$. Let $U_1(t): L_t^c \to L^c$ and $U_2(t): (S_2(t)L_t)^c \to L_t^c$ be the unitary operators defined as in (10). Then, setting $U=U_2\circ U_1$ we have:

$$\alpha = \lim_{t\to\infty} 1/2t\,\mathrm{Im}\int_{-t}^{t} d\,\mathrm{Log}(\det(U^T U) \tag{22}$$

$$= \lim_{t\to\infty} 1/2t\,\mathrm{Im}\int_{-t}^{t} d\,\mathrm{Log}(\det(U_2^T U_2)$$

$$+ \lim_{t\to\infty} 1/2t\,\mathrm{Im}\int_{-t}^{t} d\,\mathrm{Log}(\det(U_1^T U_1) \tag{23}$$

By definition, the second term in (23) is α_1. Thus we have to prove that the first term is α_2. The difficulty is that the initial Lagrangian space L_t depend on t so that this limit is not, *a priori*, the rotation number of $S_2(t)$. The following lemma ends the proof of Proposition 2.

LEMMA 2. – *Let* S_t *be a continuous family of linear symplectic transforms [satisfying* $S(0)=\mathrm{Id}$] *and* L_t *be a continuous family of Lagrangian spaces. Then the rotation number* α *of* S_t *is equal to the relative rotation number of* $S_t(L_t)$ *with respect to* L_t.

Proof. – Let us study the motion on the interval [0, T]. Let $U(t): (S_t L_t)^c \to L_t^c$ be defined as in (10), then the winding number $w(T)=\int d\,\mathrm{Arg}(\det(U^T U)$ up to time T modulo 2π is the angle beetween $S_T L_T$ and L_T, that is the argument of det $(U^T(T)U(T))$. Let us consider now the continuous family $L_{t,\lambda}=L_{\lambda t+(1-\lambda)T}$ (which verifies $L_{t,1}=L_t$, $L_{t,0}=L_T$ and $L_{T,\lambda}=L_T$). The winding $w_\lambda(T)$ does not depend on λ modulo $2\pi\,(w_\lambda(T)=\mathrm{Arg}(\det(U^T(T)U(T))$ modulo $2\pi)$ and is a continuous function function of λ, hence is constant with respect to λ. In particular, the winding number is the same for the family L_t and for the Lagrangian space L_T. The limit $T\to\infty$ ends the proof of the Lemma 2.

Proposition 2 shows that the rotation number defined by D. Ruelle is the same than ours. Let us recall that his rotation number for a family S_t is the rotation number associated with the unitary transform U_t of the polar decomposition in $S_t = |S_t| U_t$. Thus we know that the rotation number of S_t is the sum of the rotation numbers of the two families $|S_t|$ and U_t. We need the following lemma:

LEMMA 3. — *The rotation number of a continuous family* S_t *of symmetric positive symplectic transforms is zero.*

Proof. — If S_t is symmetric positive definite for every ζ in V we have $g(\zeta, S_t(\zeta)) > 0$. Thus $S_t(\zeta)$ never lies in JV and so $P_t(\zeta)$ never vanishes. Then formula (17) shows that -1 can never be an eigenvalue of the unitary operator associated with the pair $(S_t(V), V)$. Thus the winding number is bounded by πn. This ends the proof.

Consequently the rotation number of a continuous family of symplectic transforms is equal to the rotation number of the unitary family of operators associated to the polar decomposition (left or right). For explicit calculations, our approach is easier since it avoids the difficult problem of effective polar decomposition of symplectic transforms. Moreover, in the case where the family S_t is given by a differential equation, it is simpler to follow the evolution of a Lagrangian space (only for an n-dimensional space) rather than solving the problem for all the operator S_t.

3. COMPLEX ENTROPY FOR SECOND ORDER DIFFERENTIAL EQUATIONS

3.1. Geometric properties of second order differential equations

Let M be an n-dimensional manifold and $F : TM \to \mathbb{R}$ a Lagrangian (F is positively homogeneous of degree 1 on the fibres). This function defines a variational problem when we want to extremize the action:

$$I(c) = \int F(c')\, dt \tag{19}$$

It is well-known that any Lagrangian can be mapped onto an homogeneous variational problem. The geometrical structure of these problem has been tackled in [2] and we just recall there some basic results without proofs. The convenient space to study the variational problem is not the tangent bundle TM but rather the homogeneous fibre bundle HM (the fibre of the tangent half-lines). The Euler-Lagrange solution is a vector field X on HM. This vector field is of particular type and is called a second order differential equation. This second order differential equation indues a splitting of THM into the 3 supplementary bundles:

$$\mathrm{THM} = \mathbb{R}\,\mathrm{X} \oplus \mathrm{VHM} \oplus h_x\,\mathrm{HM} \tag{20}$$

with the associated projectors,

$$\mathrm{I}_d(\mathrm{THM}) = p_{\mathrm{X}} + \mathrm{P} + \mathrm{Q}. \tag{21}$$

The bundle VHM is the *vertical bundle i.e.*, made of spaces tangent to the fibres of HM (it does not depends on X). The bundle h_{X} HM is called the *horizontal bundle* associated with X. There exists an almost-complex structure I^{X} on $\mathrm{VHM} \oplus h_{\mathrm{X}}\,\mathrm{HM}$. If we assume that the Lagrangian is convex on the fibres (then one speaks of a Finsler structure and, this includes Riemannian geometry), then there exists a 1-form A on HM which is a contact form, that is $d\mu_{\mathrm{L}} = \mathrm{A} \wedge (d\mathrm{A})^{n-1}$ is a volume form. The volume form is invariant under X and dA induces a symplectic structure on $\mathrm{VHM} \oplus h_{\mathrm{X}}\,\mathrm{HM}$. Furthermore, the complex structure I^{X} is positive compatible [the metric defined as in (3) is positive definite], and VHM, h_{X} HM are Lagrangian and related by $h_{\mathrm{X}}\,\mathrm{HM} = \mathrm{I}^{\mathrm{X}}(\mathrm{VHM})$.

3.2. Complex entropy

Let φ_t the one-parameter pseudo-group associated with X and $\mathrm{T}\varphi_t$ its linearisation. The bundle $\mathrm{VHM} \oplus h_{\mathrm{X}}\,\mathrm{HM}$ is invariant under $\mathrm{T}\varphi_t$, and furthermore the action of $\mathrm{T}\varphi_t$ on it is symplectic. In order to apply the results of the first part, we need a parallel transport to identify the spaces at different points. It is proven in [2] that along any orbit Γ of X on HM, there exists such a parallel transport τ_t leaving VHM and h_{X} HM invariant. Now, being given a point z in HM and a Lagrangian space L_z in $\mathrm{VHM}_z \oplus h_{\mathrm{X}}\,\mathrm{HM}_z$, we define the operator $\mathrm{Z}_t : \mathrm{L}_z^c \to \tau_t(\mathrm{V}_z)^c$ by:

$$\mathrm{Z}_t = \mathrm{U}(\mathrm{T}\varphi_t(\mathrm{L}_z)) \circ (\mathrm{T}\varphi_t/\mathrm{L}_z)^c. \tag{22}$$

We thus define the complex entropy for any Lagrangian space L at the point z as in (13). Let us remark that, by construction, this entropy is invariant, that is:

$$\gamma(\mathrm{L}, z) = \gamma(\mathrm{T}\varphi_t(\mathrm{L}), \varphi_t(z))$$

for every t in $\mathbb{R}$ and z in HM. Now, everything works as in part 1. In particular, Proposition 1 is in force and, following the previous remark, the rotation number is a dynamical invariant. By (22), the real part of the entropy $h(\mathrm{L}, z)$ is determined by:

$$h(\mathrm{L}, z) = \lim_{t \to \infty} 1/2t \,\mathrm{Log}(\det[(\mathrm{T}\varphi_t/\mathrm{L}_z)^{\mathrm{T}}\, \mathrm{T}\varphi_t/\mathrm{L}_z]) \tag{23}$$

Now, we suppose that M is compact without boundary. A convex Lagrangian is given, and we normalize the diffuse measure μ. Following

Oseledec, for a C^1 flow, $\varphi : \mathbb{R} \times \mathrm{HM} \to \mathrm{HM}$, there exists a subset Ω in HM of full measure, invariant by φ, and such that, for any z in Ω, there exists a unique splitting of $T_z \mathrm{HM} = E_z^0 \oplus E_z + \oplus E_z^-$ satisfying:

$$\lim_{t \to \pm\infty} \frac{1}{t} \operatorname{Log} \| T\varphi_t \xi \| = 0 \qquad \text{for} \quad \xi \text{ in } E_z^0$$

$$\lim_{t \to \pm\infty} \frac{1}{t} \operatorname{Log} \| T\varphi_t \xi \| < 0 \qquad \text{for} \quad \xi \text{ in } E_z^-$$

$$\lim_{t \to \pm\infty} \frac{1}{t} \operatorname{Log} \| T\varphi_t \xi \| > 0 \qquad \text{for} \quad \xi \text{ in } E_z^+$$

Furthermore, the limit $h(L_z, z)$ in (22) exists for any subspace L_z satisfying $E_z^0 \oplus E_z^+ \supset L_z \supset E_z^+$, does not depend on L_z, and is the sum $\chi(z)$ of the Lyapunov exponents. D. Ruelle [6] has proven that, if μ is an invariant probability measure, the metric entropy $h_\mu(\varphi)$ satisfies:

$$h_\mu(\varphi) \leqq \int_{\mathrm{HM}} \chi(z)\, d\mu$$

if φ is C^2 and μ absolutely continuous. Then, following Pesin [5]:

$$h_\mu(\varphi) = \int_{\mathrm{HM}} \chi(z)\, d\mu \tag{24}$$

This justifiės the name "complex entropy".

As in the linear case, the ergodic subadditive Theorem provides a simple proof of the existence with probability one of the rotation number $\alpha(z)$.

3.3. Conjugate points: generalized Sturm-Liouville theory

The conjugate points naturally appear in the framework of geodesic flows on Riemannian manifolds and this notion can be easily extended [3] to the case of Lagrangian systems. Let z be a point in HM and Γ the orbit of z under the flow defined by the Euler-Lagrange equation. The points conjugate to z in HM are the points $\varphi_t(z)$ such that:

$$T\varphi_t(V_z \mathrm{HM}) \cap V_{\varphi_t(z)} \mathrm{HM} \neq 0. \tag{25}$$

One can evaluate the number of z-conjugate points for $t > 0$ taking into account their multiplicities (the dimension of the intersection). Their time average number is linked to the rotation number through the following theorem:

THEOREM 3. — *For the Euler-Lagrange equation* X *of a convex variational problem, the average number of conjugate points (counted with their multiplicities) to a given point z in* HM *is equal to α/π where α is the rotation number of z (whenever it exists).*

Remark. — According to the precedent paragraph, when M is compact the limit of Proposition 3, exists on a set of full measure.

Proof of the theorem. — Before starting the proof we need to recall (without proofs), some facts [2]relative to the special case of second order differential equations. To a second order differential equation on HM is attached a a first order differential operator γ_X called dynamical derivation satisfying for every vector field ζ over HM an any differentiable function f the relation

(i) $\gamma_X(f\zeta)=L_X f.\gamma_X(\zeta)$; (26)

(ii) $\gamma_X(X)=0$;

(iii) γ_X leaves the splitting $\mathbb{R}X\oplus VHM\oplus h_X HM$ invariant;

(iv) the dynamical derivation γ_X commutes with the almost complex structure I^X on $VHM\oplus h_X HM$. The parallel displacement is done relatively to γ_X.

If X is the Euler Lagrange of a convex variationnal problem then γ_X is compatible with the metric g, that is for two vector fields ζ, η on HM

$$L_X(g(\zeta,\eta))=g(\gamma_X(\zeta),\eta)+g((\zeta,\gamma_X(\eta)) \tag{27}$$

Let z be a point in HM, ζ any vector in $T_z HM$ with decomposition $\zeta=aX+Y+h$ relative to the splitting $\mathbb{R}X\oplus VHM\oplus h_X HM$, for the vector field $\zeta_t=T\varphi_t(\zeta)$ the following relations [2] (known as the *Jacobi equation* in the Riemannain context) are satisfied

$$\gamma_X(I^X(h_t))+Y_t=0 \tag{28}$$
$$\gamma_X(Y_t)-\theta_X(I^X(h_t))=0 \tag{29}$$

Relation (28) and (29) are essential to understand the particular properties of second order differential equations. We will not discuss here these equations, let us just say that θ_X is a field of linear endomorphisms of VHM, and is called the *Jacobi endomorphism of the second order differential equation.*

Proof of theorem 3. — Choose a point z in HM, a Lagrangian space L over z and suppose that there exists Y in $V_z HM\cap L$. Then using (10) and observing that $Q(Y)=0$, $P(Y)=Y$, we see that Y is an eigenvector with eigenvalue 1. Thus, the result follows the fact that for second order differential equations, any eigenvalue 1 necessarily moves in the positive sense on the unit circle.

Indeed, let ε be a small time, on the eigenspace associated with the eigenvalue 1 the evolution (after parallel displacement) is provided [using (28), (29) up to order ε^2] by $T=Id-\varepsilon I^X$. Noticing that for any vertical vector Y the 2-space generated by Y and $I^X(Y)$ is invariant by P and Q (this is also true for any restriction of this two operators to any Lagrangian space) we get:

$$U_\varepsilon^T U_\varepsilon (Y - \varepsilon I^X(Y)) = e^{2i\varepsilon}(Y - \varepsilon I^X(Y)) + O(\varepsilon^2)$$

for Y in the previous eigenspace. Thus the eigenvalues 1 behave for small time ε like $e^{2i\varepsilon}$.

By continuity we can follow the set G of the eigenvalues of $U_t^T U_t$ on the universal covering and build n continuous function f_i, $n \geqq i \geqq 1$ by labeling the set G with the convention $f_j \geqq f_i$ for $j \geqq i$. Our precedent argument shows that $f_i(t)$ can never decrease of more than 2π so there is a monotonous increasing integer function $k_i(t)$ such that $2\pi k_i \leqq f_i \leqq 2\pi(k_i+1)$ and k_i is the number of times f_i has crossed 1. Thus, on the interval [0, T] the winding number (15) satisfies,

$$\sum 2\pi k_i \leqq 2T.\alpha(T) \leqq \sum 2\pi(k_i+1). \tag{30}$$

This ends the proof.

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