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Square integrability of group representations on homogeneous spaces. II. Coherent and quasi-coherent states. The case of the Poincaré group

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ABSTRACT. – The concept of a reproducing triple, developed in the first paper of the series (I), is utilized to give a general definition of a square integrable representation of a group. This definition is applicable to homogeneous spaces of the group, and generalizes earlier attempts at obtaining such notions. Among others, it leads naturally to a notion of equivalence

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among families of coherent states, and also to the concept of quasi-coherent states, or weighted coherent states. The general considerations are applied to the specific example of the Wigner representation of the Poincaré group $\mathscr{P}_+^\uparrow(1, 1)$ in one space and one time dimensions. A whole class of equivalent families of coherent states is derived, each of which corresponds to a continuous frame, in the sense of I.

RÉSUMÉ. — Le concept de triplet reproduisant, étudié dans le premier article de la série (I), permet de donner une définition générale d'une représentation de carré intégrable d'un groupe. Cette définition s'applique au cas d'un espace homogène du groupe et généralise diverses tentatives antérieures d'obtenir de telles notions. Elle mène naturellement, entre autres, à une notion d'équivalence entre familles d'états cohérents, ainsi qu'au concept d'états quasi cohérents ou états cohérents avec poids. La théorie générale est ensuite appliquée au cas spécifique de la représentation de Wigner du groupe de Poincaré $\mathscr{P}_+^\uparrow(1, 1)$ d'un espace-temps de dimension $1+1$. Une classe entière de familles équivalentes d'états cohérents est obtenue, et chacune d'entre elles définit un repère continu, selon la terminologie introduite dans I.

1. INTRODUCTION

In this paper, the second of two [1], we continue the study of generalized coherent states. The aim is to build a theory that is general enough to cover several cases which are physically relevant, but beyond the scope of the standard approach of Perelomov ([2], [3]). The prime examples are the coherent states associated to some representations of the Galilei or the Poincaré group, and more generally to semi-direct products $G = V \wedge S$, where V is a vector group and S a semisimple group of automorphisms of V (this is the case treated, for instance, by DeBièvre [4]).

The starting point, in I, was the consideration of the operator integral

$$\int_X F(x)\, d\nu(x) = I, \tag{1.1}$$

where, in general, X is a homogeneous space G/H of a locally compact group G, ν a G-invariant measure on X, and $\{F(x), x \in X\}$ a family of projection operators in a Hilbert space $\mathscr{H}$ that carries a unitary irreducible representation (UIR) U of G. The convergence of the integral in (1.1) (in the weak sense) is then taken to be the definition of the square-integrability of the representation U. It was argued in I that (1.1) should be generalized in two ways: first, $F(x)$ should be taken to be a *positive-operator valued* function $F : X \to \mathscr{L}(\mathscr{H})^{+}$, and on the r.h.s. of (1.1), the identity operator I must be replaced by a bounded invertible positive operator $A \in \mathscr{L}(\mathscr{H})^{+}$. Thus we arrive at the central notion of a *reproducing triple* $\{\mathscr{H}, F, A\}$, which is in fact quite independent of the group-theoretical context it stems from.

In its most general form, a reproducing triple $\{\mathscr{H}, F, A\}$ consists of a Hilbert space $\mathscr{H}$, a locally compact measure space (X, ν), a ν-measurable, positive-operator valued function $F : X \to \mathscr{L}(\mathscr{H})^{+}$, and a bounded positive, *invertible* operator $A \in \mathscr{L}(\mathscr{H})^{+}$, such that, in the weak sense,

$$\int_X F(x)\, d\nu(x) = A. \tag{1.2}$$

Starting from this, we showed in I how the space $\mathscr{H}$ can be embedded isometrically into a space of vector-valued functions, which is a reproducing kernel Hilbert space, containing an *overcomplete family of states* – precursors of coherent states, so to speak. An order relation may be defined among such families, and it leads to a natural notion of *equivalent* families of states. The whole construction simplifies if the operator $F(x)$ has constant, finite rank equal to n, in which case we use the notation $\{\mathscr{H}, F, A\}_n$. Especially interesting is the particular case where, in addition, the inverse operator A^{-1} is also bounded; then we call $\{\mathscr{H}, F, A\}_n$ a *frame*, since it generalizes to an arbitrary measure space (X, ν) the concept used, in the discrete case, in the theory of nonorthogonal expansions ([5], [6]). Finally we briefly indicated in I how this general setup may be realised in the original group-theoretical problem.

In this second paper we will develop this latter aspect in full detail. In Section 2, the mathematical structure described above is used to obtain a general definition of square integrability of a group representation, not on the group itself, as usually done, but on an *arbitrary* homogeneous space $X = G/H$. A connection is made with K-representations of groups. The main result of this section is a deeper analysis of coherent states (the overcomplete family obtained in I). In particular we show how the dependence of the whole construction on the choice of a section $\sigma : X \to G$ may be circumvented by using the notion of equivalence mentioned above: different sections lead to different, but equivalent sets of coherent states. We discuss also some geometric features of families of coherent states. Section 3 is devoted to a detailed application of the theory to a specific representation of the Poincaré group in one space and one time dimensions, $\mathscr{P}_{+}^{\uparrow}(1, 1)$, namely the Wigner representation of mass $m > 0$. This completes the analysis of this particular representation of $\mathscr{P}_{+}^{\uparrow}(1, 1)$, begun in our previous papers ([7], [8]). We display a class of sections of

$\mathscr{P}_+^\uparrow(1,1)$ which lead to equivalent families of coherent states, and, moreover, each of which defines a nontrivial continuous frame, as described in I. We also construct a general Wigner transform, in the context of a set of generalized orthogonality relations. Finally Section 4 makes contact with the literature and indulges in some speculation on future work.

2. SQUARE INTEGRABLE GROUP REPRESENTATIONS AND COHERENT STATES

In this section we use the mathematical formalism set up in I to give a general definition for the square integrability of a group representation, and to analyze some of its consequences.

2.A. Square integrable group representations

Let G be a locally compact group, $\mathscr{H}$ a Hilbert space (over $\mathbb{C}$) and $g \mapsto \mathrm{U}(g)$ a (strongly) continuous unitary irreducible (UIR) representation of G in $\mathscr{H}$. Let $\mathrm{H} \subset \mathrm{G}$ be a closed subgroup and

$$\mathrm{X}=\mathrm{G}/\mathrm{H} \tag{2.1}$$

the left coset space [9]. We shall denote by x the elements of X, which are cosets, $g\mathrm{H}$, $g \in \mathrm{G}$. We suppose that X carries a (left) *invariant* measure ν [actually it is enough to assume the measure ν to be *quasi-invariant*; this allows the formalism to be extended to certain infinite dimensional groups (*see* Section 4 below)]. Let

$$\sigma: \quad \mathrm{X} \rightarrow \mathrm{G} \tag{2.2}$$

be a (global) measurable (Borel) section, and F a positive operator on $\mathscr{H}$ with finite rank n. Suppose that F has the diagonal representation

$$\left.\begin{aligned} &\mathrm{F}=\sum_{i=1}^{n} \lambda_i |u_i\rangle\langle u_i|, \qquad u_i \in \mathscr{H}, \quad \lambda_i>0, \\ &\langle u_i | u_j\rangle = \delta_{ij}, \qquad i, j=1,2, \ldots, n, \end{aligned}\right\} \tag{2.3}$$

and denote by $\mathbb{P}$, $\mathscr{N}^\perp$ the projection operator and the subspace (of $\mathscr{H}$):

$$\left.\begin{aligned} &\mathbb{P}=\sum_{i=1}^{n} |u_i\rangle\langle u_i|, \\ &\mathscr{N}^\perp = \mathbb{P}\,\mathscr{H} \end{aligned}\right\} \tag{2.4}$$

Using the operator F and the section σ, define the positive operator valued function $\mathrm{F}_\sigma : \mathrm{X} \rightarrow \mathscr{L}(\mathscr{H})^+$:

$$\mathrm{F}_\sigma(x)=\mathrm{U}(\sigma(x))\,\mathrm{F}\,\mathrm{U}(\sigma(x))^*. \tag{2.5}$$

DEFINITION 2.1. — The representation U is said to be *square integrable* mod(H, σ), if there exists a positive operator F, of finite rank n, and a bounded positive invertible operator A_σ on $\mathscr{H}$, such that $\{\mathscr{H}, F_\sigma, A_\sigma\}_n$ is a reproducing triple, that is, one has $\int_X F_\sigma(x)\, d\nu(x) = A_\sigma$, in the sense of weak convergence. In this case we call F a *resolution generator* and the vectors

$$\eta = F^{1/2} u, \qquad u \in \mathscr{N}^\perp, \tag{2.6}$$

admissible vectors mod(H, σ). We also say that the section σ is *admissible* for the representation U.

In particular, we shall be interested in the admissible vectors

$$\eta^i = \lambda_i^{1/2} u_i, \qquad i = 1, 2, \ldots, n, \tag{2.7}$$

with λ_i and u_i as in (2.3).

Note that if U is square integrable mod(H, σ), it is also square integrable mod(H, σ'), where σ' is any other section for which

$$A_{\sigma'} = \int_X F_{\sigma'}(x)\, d\nu(x) = \int_X U(\sigma'(x))\, F U(\sigma'(x))^*\, d\nu(x) \tag{2.8}$$

exists as a bounded positive operator with positive, self-adjoint inverse $A_{\sigma'}^{-1}$. In particular, consider the section σ_g, for any $g \in G$, obtained from the section σ as:

$$\sigma_g(x) = \sigma(x)\, h(g, g^{-1}.x) = g\, \sigma(g^{-1}.x), \tag{2.9}$$

where $g^{-1}.x$ is the translate of the point $x \in X$ (considered as a homogeneous G-space) under $g^{-1} \in G$, and $h : G \times X \to H$ is the cocycle

$$h(g, x) = \sigma(g.x)^{-1}\, g\, \sigma(x). \tag{2.10}$$

Notice [1] that, if g is not the identity element of G, the transformed section σ_g always differs from σ. Now, since

$$\int_X F_\sigma(x)\, d\nu(x) = \int_X U(\sigma(x))\, F U(\sigma(x))^*\, d\nu(x) = A_\sigma, \tag{2.11}$$

[1] This is most easily seen by comparing the present situation with that described in [10]. Indeed (in the notation of that paper), if $G_x = H$ and the homomorphism $\lambda : G_x \to H$ is the identity map, then the principal bundle E_λ reduces to the canonical bundle $G \to G/H$ and the cocycle $h(g, x)$ coincides with the transformation function $\rho^{-1}(g, x)$. Furthermore, by Corollary 1 of [10], the existence of the global section $\sigma : G/H \to G$ implies that λ extends to a smooth map $\Lambda : G \to H$. By Corollary 2, $h(g, x)$ is independent of x iff Λ is a homomorphism, which happens only if G is a direct product $G = K \times H$. Finally, by Corollary 3, $h(g, x) = e$, $\forall g, x$, iff $\lambda(H) = e$, *i.e.* $H = \{e\}$. Thus we always have $\sigma_g \neq \sigma$ if $g \neq e$.

we see that

$$\int_X U(g)\, U(\sigma(x))\, F U(\sigma(x))^*\, U(g)^*\, d\nu(x) = A_{\sigma_g}, \qquad (2.12\,a)$$

where we have defined

$$A_{\sigma_g} = U(g)\, A_\sigma\, U(g)^*. \qquad (2.12\,b)$$

On the other hand,

$$g\,\sigma(x) = \sigma_g(g.x) = \sigma(g.x)\, h(g, x), \qquad (2.13)$$

so that

$$U(g)\, U(\sigma(x)) = U(g\,\sigma(x)) = U(\sigma_g(g.x)). \qquad (2.14)$$

Using (2.14) in (2.12) and the invariance of the measure ν, we get,

$$\int_X F_{\sigma_g}(x)\, d\nu(x) = A_{\sigma_g}, \qquad (2.15\,a)$$

where

$$F_{\sigma_g}(x) = U(\sigma_g(x))\, F U(\sigma_g(x))^*. \qquad (2.15\,b)$$

Thus $\{\mathscr{H}, F_{\sigma_g}, A_{\sigma_g}\}_n$ is a reproducing triple and U is also square integrable mod(H, σ_g), $\forall g \in G$. Moreover, corresponding to any one of the sections σ_g, define the POV-measure $a_{\sigma_g}(\Delta)$ [*see* I, Sec. 2; $\mathscr{B}(X)$ denotes the σ-algebra of Borel sets of X]:

$$\begin{aligned} a_{\sigma_g}(\Delta) &= \int_\Delta F_{\sigma_g}(x)\, d\nu(x), \qquad \Delta \in \mathscr{B}(X), \\ a_{\sigma_g}(X) &= A_{\sigma_g}. \end{aligned} \qquad (2.16)$$

Then, we have the generalized covariance condition (analogue of the imprimitivity relation of Mackey [11]):

$$U(g)\, a_{\sigma_{g'}}(\Delta)\, U(g)^* = a_{\sigma_{gg'}}(g.\Delta), \qquad g, g' \in G, \qquad (2.17)$$

where, again, $g.\Delta$ denotes the translate of the Borel set $\Delta \in \mathscr{B}(X)$ through g.

2.B. Coherent states

Thus, the definition of square integrability adopted here does not depend on the choice of a single section σ, but rather on a whole class of sections, which includes the set $\mathscr{S}_G(\sigma) = \{\sigma_g, g \in G\}$, and which, in a sense to be made precise below, are all equivalent. We observe, at this point, that our definition of square integrability is much more general than usually found in the literature ([2], [12], [13]). Note that if $H \subset G$ is a subgroup for which the quotient space G/H is compact, then U is trivially square integrable mod (H, σ), for any measurable section σ. If G itself is compact, U is of course square integrable in the usual sense, and our construction is not needed. In other words, the interesting cases arise only when X (thus also G) is non-compact.

Let us construct the overcomplete family of states (OFS) associated to the triple $\{\mathscr{H}, F_\sigma, A_\sigma\}$ [*see* I, (2.9)]. From (2.3), (2.5) and (2.7) we get

$$F_\sigma(x) = \sum_{i=1}^{n} \lambda_i \,|\, u_{\sigma,i}(x) \rangle \langle u_{\sigma,i}(x) \,| = \sum_{i=1}^{n} |\, \eta^i_{\sigma(x)} \rangle \langle \eta^i_{\sigma(x)} \,|, \tag{2.18}$$

$$\begin{gathered} u_{\sigma,i}(x) = U(\sigma(x))\, u_i, \\ \eta^i_{\sigma(x)} = U(\sigma(x))\, \eta^i = \lambda_i^{1/2}\, u_{\sigma,i}(x). \end{gathered} \tag{2.19}$$

The support $\mathbb{P}_\sigma(x)$ of $F_\sigma(x)$ and the subspace $\mathscr{N}^\perp_{\sigma(x)}$ corresponding to $\mathbb{P}_\sigma(x)$ are also related to $\mathbb{P}$ and $\mathscr{N}^\perp$ [*see* (2.4)] by:

$$\begin{gathered} \mathbb{P}_\sigma(x) = U(\sigma(x))\, \mathbb{P}\, U(\sigma(x))^*, \\ \mathscr{N}^\perp_{\sigma(x)} = U(\sigma(x))\, \mathscr{N}^\perp. \end{gathered} \tag{2.20}$$

Since, by definition [*see* also (2.11)],

$$\int_X \sum_{i=1}^{n} |\, \eta^i_{\sigma(x)} \rangle \langle \eta^i_{\sigma(x)} \,|\, d\nu(x) = A_\sigma, \tag{2.21}$$

the set

$$\mathfrak{S}_\sigma = \{\, \eta^i_{\sigma(x)} = U(\sigma(x))\, \eta^i \,|\, i = 1, 2, \ldots, n,\ x \in X \,\} \tag{2.22}$$

is an OFS. We call $\mathfrak{S}_\sigma$ a family of *coherent states*, $\eta^i_{\sigma(x)}$. Actually, since for any section σ' in the class $\mathscr{S}_G(\sigma)$ we could construct such a family $\mathfrak{S}_{\sigma'}$ of coherent states, it is useful to define the set

$$\tilde{\mathfrak{S}} = \bigcup_{\sigma' \in \mathscr{S}_G(\sigma)} \mathfrak{S}_{\sigma'} \equiv \bigcup_{i=1}^{n} \{\, U(g)\, \eta^i \,|\, g \in G \,\} \tag{2.23}$$

of *all* coherent states mod (H) for the representation U. Any family $\mathfrak{S}_{\sigma'}$, $\sigma' \in \mathscr{S}_G(\sigma)$ will then be called a *section of coherent states in* $\tilde{\mathfrak{S}}$.

The standard definition of square integrability (*see*, for example, [13]) found in the literature, for discrete series representations, is given in terms of admissible vectors in the following way: a vector $\eta \in \mathscr{H}$ is said to be admissible if the matrix element $\langle U(g)\, \eta \,|\, \eta \rangle$ is square integrable, as a function over G, with respect to the Haar measure μ. If such a vector η exists, and $\eta \neq 0$, one can prove a resolution of the identity of the form

$$N \int_G |\, \eta_g \rangle \langle \eta_g \,|\, d\mu(g) = I, \qquad \eta_g = U(g)\, \eta, \tag{2.24}$$

N being a constant, and the representation is then said to be square integrable. One can, moreover, show in that case that the set of admissible vectors is dense in $\mathscr{H}$ and that, in particular, if η is admissible then $U(g)\eta$ is also admissible, $\forall g \in G$. In the more general situation envisaged in this paper, we could also adopt a similar definition of square integrability if we were to allow the operator A_σ in the reproducing triple $\{\mathscr{H}, F_\sigma, A_\sigma\}_n$ to be unbounded (though still positive, densely defined and having an inverse). In fact we have the following result:

PROPOSITION 2.2. — *Suppose that* $\sigma : X \to G$ *is a continuous section and that there exists a finite set of vectors* $\eta^i \in \mathscr{H}$, $i=1, 2, \ldots, n$, *such that*

(i) *the set* $\eta^i_{\sigma(x)} = U(\sigma(x))\eta^i \,|\, i=1, 2, \ldots, n, x \in X\}$ *is total in* $\mathscr{H}$;

(ii) *for each* $\sigma' \in \mathscr{S}_G(\sigma)$, *the integral*

$$I^k(\sigma') = \sum_{i=1}^{n} \int_X |\langle U(\sigma'(x))\eta^i | \eta^k \rangle|^2 \, d\nu(x) \tag{2.25}$$

is finite, for all $k=1, 2, \ldots, n$.

Then the operator A_σ, *defined through the weak integral*

$$A_\sigma = \int_X \sum_{i=1}^{n} |\eta^i_{\sigma(x)}\rangle \langle \eta^i_{\sigma(x)}| \, d\nu(x), \tag{2.26}$$

is strictly positive and self-adjoint on a dense domain $\mathscr{D}(A_\sigma) \subset \mathscr{H}$, *and has a positive densely defined inverse.* □

The proof is given in the Appendix. We could now use the relation $F_\sigma(x) = \sum_{i=1}^{n} |\eta^i_{\sigma(x)}\rangle \langle \eta^i_{\sigma(x)}|$ (this is not necessarily a rank n operator, since the vectors $\eta^i_{\sigma(x)}$ need not be linearly independent) and define a reproducing triple $\{\mathscr{H}, F_\sigma, A_\sigma\}$. But then the last inclusion in (I-4.8) may no longer be valid. Hence, for the purposes of this paper, we shall continue to define square integrability by Definition 2.1.

Suppose that U is square integrable mod(H, σ) and consider the isometric map (*see* (I-4.9, 4.10)] $W_\sigma : \mathscr{H} \to \mathscr{H}_\sigma$:

$$(W_\sigma \Phi)_i(x) = \langle \eta^i_{\sigma(x)} | \Phi \rangle \tag{2.27}$$

where as a set $\mathscr{H}_\sigma \subset L^2(X, \nu; \mathbb{C}^n)$, and as a Hilbert space it has the scalar product

$$\langle \Phi | \Psi \rangle_\sigma = \sum_{i=1}^{n} \int_X \overline{\Phi_i(x)} (\hat{A}_\sigma^{-1} \Psi)_i(x) \, d\nu(x), \tag{2.28}$$

where

$$\hat{A}_\sigma^{-1} = W_\sigma A_\sigma^{-1} W_\sigma^{-1}. \tag{2.29}$$

On

$$\bigcup_{g' \in G} \mathscr{H}_{\sigma_{g'}} \subset L^2(X, \nu; \mathbb{C}^n), \tag{2.30}$$

σ_g being given by (2.9), let us define the following representation of G:

$$\left.\begin{array}{l}\tilde{U}(g): \bigcup_{g' \in G} \mathcal{H}_{\sigma_{g'}} \to \bigcup_{g' \in G} \mathcal{H}_{\sigma_{g'}} \\ (\tilde{U}(g)\Psi_{\sigma_{g'}})(x) = \Psi_{\sigma_{g^{-1}g'}}(g^{-1}x).\end{array}\right\} \tag{2.31}$$

This is the natural representation of G, generated by U inside $L^2(X, \nu; \mathbb{C}^n)$. It is an example of a *K-representation*. Properties of such representations have been studied in ([14]-[16]). It is sometimes possible to extend $U(g)$ to the whole of $L^2(X, \nu; \mathbb{C}^n)$, but the extension may not be globally unitary ([7], [17], [18]). On the other hand, the image $U_{\sigma_{g'}}$ of U under the unitary map $W_{\sigma_{g'}}$ is clearly unitary. For this we have, for all $\Psi \in \mathcal{H}_{\sigma_{g'}}$ and $\Phi = W_{\sigma_{g'}}^{-1}\Psi$:

$$\left.\begin{array}{l}U_{\sigma_{g'}}(g) = W_{\sigma_{g'}} U(g) W_{\sigma_{g'}}^{-1}, \\ (U_{\sigma_{g'}}(g)\Psi)_i(x) = \langle \eta^i_{\sigma_{g'}(x)} \mid U(g)\Phi \rangle.\end{array}\right\} \tag{2.32}$$

Note, that in (2.32), if the vector $U_{\sigma_{g'}}(g)\Psi$ is considered as lying in $\mathcal{H}_{\sigma_{g^{-1}g'}}$, then we retrieve (2.31). Moreover, denoting by U_l the globally unitary representation on $L^2(X, \nu; \mathbb{C}^n)$:

$$(U_l(g)\Psi(x) = \Psi(g^{-1}x), \qquad \Psi \in L^2(X, \nu; \mathbb{C}^n), \tag{2.33}$$

we easily establish, using (2.31)–(2.33) that:

$$W_{\sigma_{g'}} U(g) = U_l(g) W_{\sigma_{g^{-1}g'}}, \qquad \forall g, g' \in G. \tag{2.34}$$

2. C. Weighted coherent states or quasi-coherent states

Suppose that U is square integrable mod(H, σ) and let $\{\mathcal{H}, F_\sigma, A_\sigma\}_n$ be the corresponding reproducing triple. Let $\{\mathcal{H}, F', A'\}_m$ be another reproducing triple, and denote by $\mathfrak{S}' = \{\eta'^i_x\}$ the corresponding OFS. According to Definition 2.2 of I, $\mathfrak{S}'$ is said to be *weighted* with respect to $\mathfrak{S}_\sigma$ if there exists a (weakly) measurable operator valued function $T: X \to \mathcal{L}(\mathcal{H})$, such that

$$F'(x) = T(x) F_\sigma(x) T(x)^*, \qquad \forall x \in X. \tag{2.35}$$

Then we write, as in I:

$$\mathfrak{S}' \prec \mathfrak{S}_\sigma. \tag{2.36}$$

In that case the vectors of the OFS $\mathfrak{S}' = \{\eta'^i_x\}$ are expressed in terms of the coherent states $\{\eta^j_{\sigma(x)}\}$, after weighting by the operator $T(x)$ and mixing by a certain matrix $t(x)$ [*see* (I-2.21)]. For this reason we shall call the η'^i_x *weighted coherent states* (i.e. weighted w.r.t. the coherent states $\eta^j_{\sigma(x)}$). Note that coherent states such as the $\eta^j_{\sigma(x)}$ are labelled by points $\sigma(x) \in G$ coming from a section, in fact, for fixed i, they constitute the orbit $U(\sigma(x))\eta^i$. Hence, the weighted coherent states η'^i_x are not true coherent states unless there exists a section $\sigma' : X \to G$ for which $\eta'^i_x = U(\sigma'(x))\,\eta'^i$, $\forall\, x \in X$. For that reason, we shall also refer to them as *quasi-coherent states.*

At this point, we would like to mention that in all this construction the group structure of G has not been exploited (except in the definition of the section σ_g and the K-representations). Through the representation U, the group has provided us with a Hilbert space and a nice set of vectors, $\eta^i_{\sigma(x)} = U(\sigma(x))\,\eta^i$. It is more the manifold stucture of G that is important in the construction, and this structure persists even when we consider quasi-coherent states instead of coherent states.

Quasi-coherent states are especially useful when $\{\mathscr{H}, F_\sigma, A_\sigma\}_n$ is a frame, that is, A_σ^{-1} is bounded. Then indeed, using the quasi-coherent states $\eta'^i_x = T(x)\,\eta^i_{\sigma(x)}$, with $T(x) = A_\sigma^{-1/2}\,\mathbb{P}(x)$, we recover the resolution of the identity (tight frame):

$$\left.\begin{aligned} &\int_X F'(x)\,d\nu(x) = I, \\ &F'(x) = \sum_{i=1}^{n} |\eta'^i_x\rangle\langle\eta'^i_x|. \end{aligned}\right\} \tag{2.37}$$

We will perform this construction explicitly in Section 3.B below for the case of the Poincaré group $\mathscr{P}^\uparrow_+(1, 1)$ (thus generalizing results obtained in [8]). Another example may be found in the recent work of Torrésani [19], who has applied the present formalism to the affine Weyl-Heisenberg group and has obtained very interesting generalized wavelets ("wavelet packets").

2. D. Equivalent coherent states

We end this section by introducing the notion of equivalent sets of coherent states, and a geometric interpretation of such equivalence. Since all reproducing triples considered here have constant, finite rank, any two of them are comparable. Let $\sigma, \sigma' : X \to G$ be two admissible sections for the representation U, that is, there exist operators F, $A_{\sigma'}$, F', $A'_{\sigma'} \in \mathscr{L}(\mathscr{H})^+$, with rank $F = n$, rank $F' = n'$, such that $\{\mathscr{H}, F_{\sigma'}, A_\sigma\}_n$ and $\{\mathscr{H}, F'_{\sigma'}, A'_{\sigma'}\}_{n'}$ are reproducing triples. Let $\mathfrak{S}_\sigma$ and $\mathfrak{S}_{\sigma'}$ be corresponding families of coherent states. Then, according to the discussion in I, Section 2, $\mathfrak{S}_\sigma \prec \mathfrak{S}_{\sigma'}$ if $n \leqq n'$, and $\mathfrak{S}_\sigma$ and $\mathfrak{S}_{\sigma'}$ are equivalent if $n = n'$ (even if $F \neq F'$, provided they have the same rank). This observation allows us to get rid of the annoying dependence of the whole construction on the choice of a section. Indeed, if we consider two coherent state systems $\mathfrak{S}_\sigma$ and $\mathfrak{S}_{\sigma'}$ constructed from the same operator F, with two different admissible sections σ, σ', they are equivalent. For instance, the section of coherent states $\mathfrak{S}_{\sigma_g'}$ coming from the reproducing triples $\{\mathscr{H}, F_{\sigma_g}, A_{\sigma_g}\}_n$, $g \in G$ [*see* (2.15)], are all equivalent.

The equivalence of $\mathfrak{S}_\sigma$ and $\mathfrak{S}_{\sigma'}$ means that, for every $x \in X$, there exist (non-unique) bounded operators $T_{\sigma'\sigma}(x)$, $T_{\sigma\sigma'}(x)$ such that,

$$\left.\begin{aligned} F_{\sigma'}(x) &= T_{\sigma'\sigma}(x)\, F_\sigma(x)\, T_{\sigma'\sigma}(x)^*, \\ F_\sigma(x) &= T_{\sigma\sigma'}(x)\, F_{\sigma'}(x)\, T_{\sigma\sigma'}(x)^*. \end{aligned}\right\} \tag{2.38}$$

More precisely, the transition operators $T_{\sigma'\sigma}$, $T_{\sigma\sigma'}$ define a one-to-one mapping between the basic coherent states; one has indeed, for all $i = 1, \ldots, n$ and $x \in X$:

$$\eta^i_{\sigma'(x)} = T_{\sigma'\sigma}(x)\, \eta^i_{\sigma(x)} \qquad \text{and} \qquad \eta^i_{\sigma(x)} = T_{\sigma\sigma'}(x)\, \eta^i_{\sigma'(x)}. \tag{2.39}$$

The simplest choice for the transition operators is that given by the relation (2.28) in I, namely:

$$T_{\sigma'\sigma}(x) = U(\sigma'(x))\, \mathbb{P}\, U(\sigma(x))^*, \tag{2.40}$$

where $\mathbb{P}$ is the projection on $(\operatorname{Ker} F)^\perp$. For instance, if $\mathbb{P} = |\eta\rangle\langle\eta|$, then we get simply $T_{\sigma'\sigma}(x) = |\eta_{\sigma'(x)}\rangle\langle\eta_{\sigma(x)}|$. In particular,

$$T_{\sigma\sigma}(x) = U(\sigma(x))\, \mathbb{P}\, U(\sigma(x))^* = \mathbb{P}_\sigma(x), \tag{2.41}$$

the projection on $(\operatorname{Ker} F_\sigma(x))^\perp$. Moreover, if σ, σ', σ'' are three admissible sections for U, the corresponding transition operators obey the chain rule:

$$T_{\sigma''\sigma'}(x)\, T_{\sigma'\sigma}(x) = T_{\sigma''\sigma}(x), \qquad \forall x \in X. \tag{2.42}$$

Thus, in this precise sense we may say that the set of coherent states associated to the representation U does not depend on the choice of the section.

If G is a Lie group and σ is a *smooth* admissible section, then the family $\mathfrak{S}_\sigma$ has remarkable geometrical properties. Note, however, that if a global smooth section $\sigma: X \to G$ exists, then the principal bundle (G, X, π), where $\pi: G \to X = G/H$ is the canonical surjection, is trivializable (that is, isomorphic to a product bundle). This is not too surprising: exactly the same situation arises, for instance, in gauge field theory [10], where many physically interesting cases correspond in fact to trivial principal bundles [*see* also Footnote (1)]. If σ is smooth, then $\mathfrak{S}_\sigma$ generates a (trivializable) vector bundle through the frame fields $x \mapsto \eta^i_{\sigma(x)}$, $i = 1, 2, \ldots, n$. Denote this bundle by $B(\mathfrak{S}_\sigma)$. It is a bundle over the base space X, with fibres isomorphic to $\mathbb{C}^n$ [but it is in general not associated to the principal bundle (G, X, π), since there is no natural representation of H on $\mathbb{C}^n$]. Indeed, the fibre over $x \in X$ is spanned by the n basis vectors $\eta^i_{\sigma(x)}$, $i = 1, 2, \ldots, n$. The canonical projection $\pi_\sigma: B(\mathfrak{S}_\sigma) \to X$ has the property,

$$\pi_\sigma^{-1}(x) = E_\sigma(x)^*[\mathbb{C}^n], \qquad \forall x \in X, \tag{2.43}$$

where $E_\sigma(x)^*: \mathbb{C}^n \to \mathscr{H}$, is the map given in (A.13), written out for the particular section σ.

Consider now a second smooth admissible section σ', with resolution generator F', and suppose that $\mathfrak{S}_\sigma \sim \mathfrak{S}_{\sigma'}$. Then by the preceeding discussion, there exists a smooth map $T: X \to \mathscr{L}(\mathscr{H})$, such that [*see* (2.35)]:

$$F'_{\sigma'}(x) = T(x) F_\sigma(x) T(x)^* \tag{2.44}$$

and for which $T: B(\mathfrak{S}_\sigma) \to B(\mathfrak{S}_{\sigma'})$ is a bundle isomorphism:

$$\begin{array}{ccc} B(\mathfrak{S}_\sigma) & \xrightarrow{T} & B(\mathfrak{S}_{\sigma'}) \\ \Big\downarrow{\scriptstyle \pi_\sigma} & & \Big\downarrow{\scriptstyle \pi_{\sigma'}} \\ X & \xrightarrow{\mathrm{Id}_X} & X \end{array} \tag{2.45}$$

In other words, equivalent families of coherent states constitute fields in isomorphic $\mathbb{C}^n$-bundles.

In the Appendix we collect some formulas for reproducing kernels, isometries, etc., related to reproducing triples of the type $\{\mathscr{H}, F_\sigma, A_\sigma\}$. [*See* expressions (A.10) to (A.16)].

3. THE CASE OF THE POINCARÉ GROUP $\mathscr{P}^\uparrow_+(1, 1)$

We undertake in this section a rather comprehensive analysis of the coherent states of the Poincaré group $\mathscr{P}^\uparrow_+(1, 1)$ in one space and one time dimensions, based on the theory developed in the last section. We shall look at a specific representation of this group, corresponding to a particle of mass $m > 0$. The work here is an extension of that begun in [7] and continued in [8], and we shall freely use the notation and concepts introduced there. As mentioned in [7] and [8], and we reiterate here, the restriction to one spatial dimension is a matter of computational and notational neatness alone. Exactly analogous results are obtainable for the four-dimensional Poincaré group $\mathscr{P}^\uparrow_+(1, 3)$ (see, e.g., [18]).

Elements of $\mathscr{P}^\uparrow_+(1, 1)$ are denoted by (a, Λ_p), where $a = (a_0, \mathbf{a}) \in \mathbb{R}^2$ is a space time translation and Λ_p a Lorentz boost. The representation U_w in question is defined on the Hilbert space

$$\mathscr{H}_w = L^2(\mathscr{V}^+_m, d\mathbf{k}/k_0) \tag{3.1}$$

and acts via the unitary operators $U_w(g)$, $g \in G$:

$$\left.\begin{array}{ll} (U_w(g)\phi)(k) = e^{ik \cdot a}\phi(\Lambda_p^{-1} k), & g = (a, \Lambda_p), \\ k.a = k_0 a_0 - \mathbf{k}.\mathbf{a}. & \end{array}\right\} \tag{3.2}$$

3. A. Affine sections and families of coherent states

To construct coherent states, we consider again the homogeneous space $\Gamma = \mathscr{P}_{+}^{\uparrow}(1, 1)/\mathrm{T}$ where T is the subgroup of time translations. Γ has global coordinatization $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2$, and the (left) invariant measure is $d\mathbf{q}\, d\mathbf{p}$. The action of $\mathscr{P}_{+}^{\uparrow}(1, 1)$ on Γ is given by:

$$(a, \Lambda_k) . (\mathbf{q}, \mathbf{p}) = (\mathbf{q}', \mathbf{p}') \tag{3.3a}$$

with

$$p' = \Lambda_k p, \tag{3.3b}$$

$$\mathbf{q}' = \frac{1}{p_0'}(p_0 \mathbf{q} + m \underline{\Lambda}_{p'}^{-1} \underline{a}). \tag{3.3c}$$

Here, as in [8], $\underline{t} \equiv \mathbf{t}$ is another notation for the space component of the 2-vector t.

We fix now the particular section $\sigma_0 : \Gamma \to \mathscr{P}_{+}^{\uparrow}(1, 1)$:

$$\sigma_0(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{q}), \Lambda_p), \qquad p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}) \tag{3.4}$$

(this section was called β in [7]). Any other measurable section then has the form

$$\sigma(\mathbf{q}, \mathbf{p}) = \sigma_0(\mathbf{q}, \mathbf{p})((f(\mathbf{q}, \mathbf{p}), 0), \mathrm{I}) \tag{3.5}$$

where f is a measurable $\mathbb{R}$-valued function. Writing (*Caution:* $\hat{q}$ does *not* denote a unit vector!):

$$\sigma(\mathbf{q}, \mathbf{p}) = (\hat{q}, \Lambda_{\hat{p}}), \tag{3.6}$$

we easily see that

$$\left.\begin{aligned} \hat{q}_0 &= \frac{p_0}{m} f(\mathbf{q}, \mathbf{p}) \\ \hat{\mathbf{q}} &= \mathbf{q} + \frac{\mathbf{p}}{m} f(\mathbf{q}, \mathbf{p}), \\ \hat{p} &= p. \end{aligned}\right\} \tag{3.7}$$

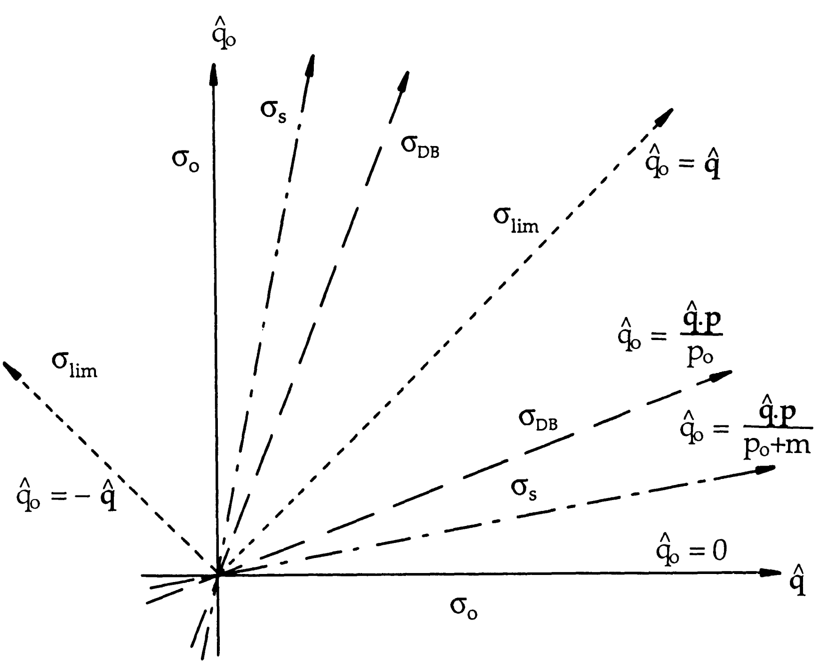
Actually, for reasons which will become clear in a while, we shall consider a class of sections for which f is further restricted. First we impose that $f(\mathbf{q}, \mathbf{p})$ be a continuous affine function of $\mathbf{q}$ (such sections will be called *affine*), *i.e.* take the general form

$$f(\mathbf{q}, \mathbf{p}) = \varphi(\mathbf{p}) + \mathbf{q} . \theta(\mathbf{p}) \tag{3.8}$$

where φ and θ are continuous functions of $\mathbf{p}$ alone. In fact these two function play very different roles. The function $\varphi(\mathbf{p})$ gives an additional freedom in the choice of admissible sections, but otherwise it is completely irrelevant. In particular, it has no influence on the square integrability of the representation $\mathrm{U}_w \bmod (\mathrm{T}, \sigma)$ since, as we shall see below, φ drops out of the calculation. In fact we shall eventually set $\varphi \equiv 0$. If necessary, an arbitrary function φ may always be reinserted by a further multiplication from the right by $((\varphi(\mathbf{p}), 0), I)$ in (3.5) (this is a kind of gauge freedom). With $\varphi = 0$, the formulas (3.7) imply that the section coordinates $\hat{q} = (\hat{q}_0, \hat{\mathbf{q}})$ satisfy the relation:

$$\hat{q}_0 = \frac{p_0 \hat{\mathbf{q}} \cdot \theta(\mathbf{p})}{m + \mathbf{p} \cdot \theta(\mathbf{p})} \tag{3.9}$$

For fixed p, this means that choosing θ amounts to fixing a particular reference frame in $\hat{q}$-space. Several concrete examples are displayed in the figure below in the form of a diagram, inspired by [8].



Reference frames in $\hat{q}$-space (section coordinates) corresponding to the various sections $\sigma : \Gamma \to \mathscr{P}^{\uparrow}_{+}(1, 1)$ described in the text.

The second restriction we will impose is that the part of the translation 2-vector $\hat{q}$ which is linear in $\mathbf{q}$ (*i. e.* $\hat{q}|_{\varphi} = 0$) be *space-like*. A straightforward calculation shows this condition to be equivalent to any one of the following equivalent inequalities, valid for all $p \in \mathscr{V}_m^+$:

$$-\frac{p_0 - \mathbf{p}}{m} < \theta(\mathbf{p}) < \frac{p_0 + \mathbf{p}}{\mathbf{m}} \tag{3.10a}$$

$$\left| \theta(\mathbf{p}) - \frac{\mathbf{p}}{m} \right| < \frac{p_0}{m} \tag{3.10b}$$

$$|\theta(\mathbf{p})\, p_0| < |m + \theta(\mathbf{p}) . \mathbf{p}| \tag{3.10c}$$

Actually the function $\theta(\mathbf{p})$ may be further restricted. In the full (1+3)-dimensional case, $\mathbf{q}$, $\mathbf{p}$, θ are 3-vectors; since $\hat{q}_0$ must be rotation invariant, it follows that $\theta(\mathbf{p})$ must be proportional to $\mathbf{p}$, *i.e.*

$$\theta(\mathbf{p}) = \frac{\mathbf{p}}{m} \lambda(|\mathbf{p}|), \tag{3.11}$$

with λ a dimensionless scalar function of $|\mathbf{p}|$. Alternatively we may rewrite (3.9) as

$$\hat{q}_0 = \frac{\hat{\mathbf{q}} \cdot \mathbf{p}}{p_0} \omega(|\mathbf{p}|), \tag{3.12}$$

in terms of another scalar function of $|\mathbf{p}|$. The relation between λ and ω is given by

$$\omega = \frac{p_0^2 \lambda}{m^2 + \lambda \mathbf{p}^2}, \qquad \lambda = \frac{m^2 \omega}{p_0^2 - \omega \mathbf{p}^2}. \tag{3.13}$$

In terms of these, the inequalities (3.10) read simply:

$$\begin{aligned} |\omega(|\mathbf{p}|)| &< \frac{p_0}{|\mathbf{p}|}, \\ |\lambda(|\mathbf{p}|) - 1| &< \frac{p_0}{|\mathbf{p}|}. \end{aligned} \tag{3.14}$$

When conditions (3.10) are satisfied, one gets $1 + \frac{\mathbf{p}}{m} . \theta(\mathbf{p}) \neq 0$ (the l. h. s. is in fact positive, *see* the proof of Theorem 3.2 below), *i.e.* the map $\sigma : \Gamma \to \mathscr{P}_+^\uparrow(1, 1)$ is injective, and then $\mathbf{q}$ and $\mathbf{p}$ can be solved in terms of the section coordinates $\hat{q}$, $\hat{p}(=p)$. In terms of these, the (left) invariant measure reads:

$$\begin{aligned} d\sigma &= \left[1 + \frac{\mathbf{p}}{m} . \theta(\mathbf{p}) \right]^{-1} d\hat{\mathbf{q}}\, d\hat{\mathbf{p}} \\ &= \left[1 - \frac{\mathbf{p}^2}{p_0^2} \omega(|\mathbf{p}|) \right] d\hat{\mathbf{q}}\, d\hat{\mathbf{p}}. \end{aligned} \tag{3.15}$$

An arbitrary element $(q, \Lambda_p) \in \mathscr{P}^{\uparrow}_{+}(1, 1)$ has the coset decomposition [according to $\mathscr{P}^{\uparrow}_{+}(1, 1)/T$]:

$$(q, \Lambda_p) = \left(\left(0, \mathbf{q} - \frac{q_0}{p_0}\mathbf{p}, \Lambda_p\right)\right)\left(\left(\frac{mq_0}{p_0}, \mathbf{0}\right), \mathrm{I}\right). \tag{3.16}$$

On the other hand, with $g = (a, \Lambda_k)$ and writing

$$g\,\sigma(\mathbf{q}, \mathbf{p}) = (q', \Lambda_{p'}), \tag{3.17}$$

we get:

$$\left.\begin{aligned} q'_0 &= a_0 + \frac{1}{m} f(\mathbf{q}, \mathbf{p})(\Lambda_k p)_0, \\ \mathbf{q}' &= \mathbf{a} + \underline{\Lambda}_k \underline{q} + \frac{1}{m} f(\mathbf{q}, \mathbf{p}) \underline{\Lambda}_k \underline{p}, \\ \mathrm{p}' &= \Lambda_k p, \end{aligned}\right\} \tag{3.18}$$

where $q = (0, \mathbf{q})$. Combining (3.10), (3.13) and (3.18) a straightforward but tedious computation shows that if σ satisfies (3.10) the σ_g satisfies it also.

It was shown in [7], [8] that the representation U_w is square integrable mod(T, σ) when $\sigma = \sigma_0$ or $\sigma = \sigma_s$ (the section obtained in [8] by contraction from the de Sitter group and denoted there by β_s). We shall show here that U_w is square integrable mod(T, σ) for any $\sigma \in \mathscr{S}_{\mathrm{A}}$, when $\mathscr{S}_{\mathrm{A}}$ denotes the class of all sections σ obeying the two restrictions stated above: σ is *affine*, *i.e.* f is continuous and of the form (3.8), with an arbitrary φ, and the part of $\hat{q}$ linear $\mathbf{q}$ is space-like, *i.e.* θ satisfies the conditions (3.10) (obviously, both σ_0 and σ_s belong to the class $\mathscr{S}_{\mathrm{A}}$). As mentioned above, the class $\mathscr{S}_{\mathrm{A}}$ is stable under the action $(a, \Lambda_k) : \sigma \mapsto \sigma_{(a, \Lambda_k)}$ of $\mathscr{P}^{\uparrow}_{+}(1, 1)$.

For convenience we first particularize to the case of $\mathscr{P}^{\uparrow}_{+}(1, 1)$ the general Definition 2.1 of admissible vectors.

DEFINITION 3.1. – A vector $\eta \in \mathscr{H}_w$ is said to be *admissible* mod(T, σ), $\sigma \in \mathscr{S}_{\mathrm{A}}$, if there exists a positive bounded operator A_σ, admitting a positive, self-adjoint, densely defined inverse A_σ^{-1}, and such that

$$\int_{\mathbb{R}^2} \mathrm{F}_\sigma(\mathbf{q}, \mathbf{p})\, d\mathbf{q}\, d\mathbf{p} = \mathrm{A}_\sigma, \tag{3.19}$$

where

$$\mathrm{F}_\sigma(\mathbf{q}, \mathbf{p}) = \mathrm{U}_w(\sigma(\mathbf{q}, \mathbf{p})) \,|\, \eta \rangle \langle \eta \,|\, \mathrm{U}_w(\sigma(\mathbf{q}, \mathbf{p}))^*. \tag{3.20}$$

To find vectors which are admissible mod(Γ, σ), it is necessary to look at integrals of the type:

$$I_\sigma^\eta(\phi, \psi) \equiv \langle \phi \mid A_\sigma \psi \rangle = \int_{\mathbb{R}^2} \langle \phi \mid F_\sigma(\mathbf{q}, \mathbf{p}) \psi \rangle \, d\mathbf{q} \, d\mathbf{p}, \quad \phi, \psi \in \mathscr{H} \qquad (3.21)$$

Actually, it is convenient to start with the more general integral

$$I_\sigma(\eta_1, \eta_2; \phi, \psi) = \int_{\mathbb{R}^2} \overline{\langle U_w(\sigma(\mathbf{q}, \mathbf{p})) \eta_1 \mid \phi \rangle} \times \langle U_w(\sigma(\mathbf{q}, \mathbf{p})) \eta_2 \mid \psi \rangle \, d\mathbf{q} \, d\mathbf{q}, \qquad (3.22)$$

from which $I_\sigma^\eta(\phi, \psi)$ may be obtained by setting $\eta_1 = \eta_2 = \eta$. Now,

$$I_\sigma(\eta_1, \eta_2; \phi, \psi) = \int_{\mathbb{R}^2} d\mathbf{q} \, d\mathbf{p} \left[\int_{\mathscr{V}_m^+} e^{ik \cdot \hat{q}} \eta_1 (\Lambda_p^{-1} k) \overline{\phi(k)} \frac{d\mathbf{k}}{k_0} \times \int_{\mathscr{V}_m^+} e^{-ik' \cdot \hat{q}} \overline{\eta_2 (\Lambda_p^{-1} k')} \psi(k') \frac{d\mathbf{k}'}{k'_0} \right],$$

which after a rearrangement, and use of Fubini's theorem, can be brought into the form:

$$I_\sigma(\eta_1, \eta_2; \phi, \psi) = \int_{\mathbb{R}^2 \times \mathscr{V}_m^+ \times \mathscr{V}_m^+} d\mathbf{q} \, d\mathbf{p} \frac{d\mathbf{k}}{k_0} \frac{d\mathbf{k}'}{k'_0} \exp\left[i(k-k') \cdot \frac{p}{m} \varphi(\mathbf{p}) + i \left\{ (k_0 - k'_0) \frac{p_0}{m} \theta(\mathbf{p}) - (\mathbf{k} - \mathbf{k}') \left(1 + \frac{\mathbf{p}}{m} \cdot \theta(\mathbf{p}) \right) \right\} \cdot \mathbf{q} \right] \times \eta_1 (\Lambda_p^{-1} k) \overline{\eta_2 (\Lambda_p^{-1} k')} \, \overline{\phi(k)} \psi(k') \qquad (3.23)$$

In arriving at (3.23), use has been made of (3.7) and (3.8). At this point, let us introduce a change of variables, $\mathbf{k} \mapsto \mathbf{X}$:

$$\mathbf{X}(\mathbf{k}) = -\frac{k_0 p_0}{m} \theta(\mathbf{p}) + \mathbf{k} \left(1 + \frac{\mathbf{p}}{\mathbf{m}} \cdot \theta(\mathbf{p}) \right). \qquad (3.24)$$

For this to be a one-to-one map, the condition

$$\frac{d\mathbf{X}}{d\mathbf{k}} = \frac{1}{k_0} [k_0 - \theta(\mathbf{p}) \cdot (\underline{\Lambda}_p^{-1} \underline{k})] \neq 0 \quad \text{for any } \mathbf{k}, \mathbf{p} \qquad (3.25)$$

has to be satisfied. But then we have the result (proof in the Appendix):

THEOREM 3.2. – *Let σ be an affine section, with a continuous function $\theta(\mathbf{p})$. Then the following three conditions are equivalent:*

(i) $\dfrac{d\mathbf{X}}{d\mathbf{k}} > 0$ *for any* $k, p \in \mathscr{V}_m^+$;

(ii) $\left| \theta(\mathbf{p}) - \dfrac{\mathbf{p}}{m} \right| < \dfrac{p_0}{m}$;

(iii) *the part linear in q of the 2-vector q is space-like.* □

Theorem 3.2 justifies the introduction of the class $\mathcal{S}_A$: these are precisely the sections for which square integrability of U_w may be proved explicitly, by the same calculation as the one performed in [8] for the section σ_s.

Indeed, for $\sigma \in \mathcal{S}_A$, the change of variables (3.24) is one-to-one. Also note that in this case,

$$\mathbf{X}(\mathbf{k}) = \mathbf{X}(\mathbf{k}') \quad \text{implies } \mathbf{k} = \mathbf{k}' \text{ and } k_0 = k'_0. \tag{3.26}$$

Hence, for $\sigma \in \mathcal{S}_A$,

$$\begin{aligned} I_\sigma(\eta_1, \eta_2; \phi, \psi) = 2\pi \int_{\mathbb{R}^2 \times \mathcal{V}_m^+} d\mathbf{p} \frac{d\mathbf{k}'}{k'_0} \int_{\mathbb{R}} d\mathbf{X} \\ \times \exp\left[i(k - k') \cdot \frac{p}{m} \varphi(\mathbf{p}) \right] \frac{1}{k_0 - \theta(\mathbf{p}) \cdot (\underline{\Lambda}_p^{-1} \underline{k})} \delta(\mathbf{X}(\mathbf{k}) - \mathbf{X}(\mathbf{k}')) \\ \times \eta_1(\Lambda_p^{-1} k) \overline{\eta_2(\Lambda_p^{-1} k')}\; \overline{\phi(k)} \psi(k') \end{aligned} \tag{3.27}$$

Using (3.26), we see that whenever $I_\sigma(\eta_1, \eta_2; \phi, \psi)$ converges as an integral, we have

$$\begin{aligned} I_\sigma(\eta_1, \eta_2; \phi, \psi) = \int_{\mathcal{V}_m^+ \times \mathcal{V}_m^+} \frac{d\mathbf{p}}{p_0} \frac{d\mathbf{k}}{k_0} \frac{2\pi p_0}{k_0 - \theta(\mathbf{p}) \cdot (\underline{\Lambda}_p^{-1} \underline{k})} \\ \times \eta_1(\Lambda_p^{-1} k) \overline{\eta_2(\Lambda_p^{-1} k)}\; \overline{\phi(k)} \psi(k). \end{aligned} \tag{3.28}$$

Changing $\mathbf{p}$ into $-\mathbf{p}$, using the fact that $\Lambda_p k = \Lambda_k p$ and the invariance of the measures, we obtain:

$$I_\sigma(\eta_1, \eta_2; \phi, \psi) = \int_{\mathcal{V}_m^+ \times \mathcal{V}_m^+} \frac{d\mathbf{p}}{p_0} \frac{d\mathbf{k}}{k_0} \mathcal{A}_\sigma(k, p) \eta_1(p) \overline{\eta_2(p)}\; \overline{\phi(k)} \psi(k) \tag{3.29}$$

where the kernel $\mathcal{A}_\sigma(k, p)$ is given by

$$\mathcal{A}_\sigma(k, p) = 2\pi \frac{(\Lambda_p^{-1} p)_0}{k_0 - \theta(\underline{\Lambda}_p^{-1} \underline{k}) \cdot \mathbf{p}}.$$

This kernel is strictly positive for all $k, p \in \mathcal{V}_m^+$ and any θ obeying (3.10). This fact results from Theorem 3.2, since

$$0 < \frac{d\mathbf{X}}{d\mathbf{k}} = 2\pi \frac{p_0}{k_0} [\mathcal{A}_\sigma(k, \Lambda_k \mathcal{I} p)]^{-1}, \tag{3.31}$$

where $\mathcal{I}$ is the space inversion operator $\mathcal{I} : p = (p_0, \mathbf{p}) \mapsto \mathcal{I} p = (p_0, -\mathbf{p})$. Notice the identities:

$$\mathcal{I} \Lambda_k \mathcal{I} = \Lambda_k^{-1} = \Lambda_{\mathcal{I}k}, \qquad \underline{\Lambda}_k \underline{\mathcal{I} p} = -\underline{\Lambda}_k^{-1} \underline{p} = \underline{\Lambda}_p^{-1} \underline{k}. \tag{3.32}$$

More than that, the kernel $\mathscr{A}_\sigma$ satisfies the following inequalities, which result from (3.10 *a*):

$$\frac{2\pi}{m}(p_0 - |\mathbf{p}|) < \mathscr{A}_\sigma(k, p) < \frac{2\pi}{m}(p_0 + |\mathbf{p}|), \qquad (3.33)$$

for all $k, p \in \mathscr{V}_m^+$ and any θ obeying (3.10), in particular, for any section $\sigma \in \mathscr{S}_A$. This relation will be crucial in the sequel.

First it implies the next lemma, which settles the question of convergence of (3.29). On $\mathscr{H}_w$, define the self-adjoint (unbounded) operators $P_0 > 0$ and $\mathbf{P}$:

$$P_0 \phi(k) = k_0 \phi(k), \qquad (3.34\,a)$$
$$\mathbf{P} \phi(k) = \mathbf{k} \phi(k), \qquad (3.34\,b)$$

for any state ϕ in their respective (dense) domains. Note that the states in $\mathscr{D}(P_0^{1/2})$ are precisely those with finite energy.

LEMMA 3.3. — *The integral* $I_\sigma(\eta_1, \eta_2; \phi, \psi)$ *converges, for all* $\phi, \psi \in \mathscr{H}_w$, *iff* $\eta_1, \eta_2 \in \mathscr{D}(P_0^{1/2})$. *More precisely we have the estimate:*

$$|I_\sigma(\eta_1, \eta_2; \phi, \psi)| < \frac{2\pi}{m} |\langle \phi | \psi \rangle| . |\langle \eta_2 | (P_0 + |\mathbf{P}|) \eta_1 \rangle|. \quad \square \qquad (3.35)$$

Using this lemma, we can identify the admissible vectors as the finite energy states $\eta \in \mathscr{D}(P_0^{1/2})$. Hence the representation U_w is square integrable mod(T, σ) for all $\sigma \in \mathscr{S}_A$. Actually we can go further and obtain a complete characterization of the reproducing triple $\{\mathscr{H}_w, F_\sigma, A_\sigma^\eta\}$, where, according to (3.21), $A_\sigma \equiv A_\sigma^\eta$ yields the decomposition over coherent states (the integral converging weakly):

$$A_\sigma^\eta = \int_{\mathbb{R}^2} U_w(\sigma(\mathbf{q}, \mathbf{p})) |\eta\rangle\langle\eta| U_w(\sigma(\mathbf{q}, \mathbf{p}))^* \, d\mathbf{q}\, d\mathbf{p}, \qquad (3.36)$$

THEOREM 3.4. — *A vector* $\eta \in \mathscr{H}_w$ *is admissible* mod(T, σ), *for any* $\sigma \in \mathscr{S}_A$, *iff* $\eta \in \mathscr{D}(P_0^{1/2})$. *The operator* $A_\sigma \equiv A_\sigma^\eta$ *is a multiplication operator given by:*

$$(A_\sigma^\eta \psi)(k) = A_\sigma^\eta(k) \psi(k), \qquad (3.37)$$

$$A_\sigma^\eta(k) = \int_{\mathscr{V}_m^+} \mathscr{A}_\sigma(k, p) |\eta(p)|^2 \frac{d\mathbf{p}}{p_0}; \qquad (3.38)$$

it is bounded and positive, and has a bounded inverse. Hence the reproducing triple $\{\mathscr{H}_w, F_\sigma, A_\sigma^\eta\}$ *is a frame, with constant rank* $n = 1$. *The frame bounds* $m(A_\sigma^\eta)$, $M(A_\sigma^\eta)$ *obey the following estimates, independent of* $\sigma \in \mathscr{S}_A$:

$$m(A_\sigma^\eta) \geqq m(\eta), \qquad M(A_\sigma^\eta) \leqq M(\eta), \qquad (3.39)$$

where

$$m(\eta)\equiv \inf_{\sigma\in\mathscr{S}_A} m(A_\sigma^\eta)=2\pi\left\langle \eta\left|\frac{P_0-|\mathbf{P}|}{m}\eta\right.\right\rangle \equiv 2\pi\left\langle\frac{P_0-|\mathbf{P}|}{m}\right\rangle_\eta, \tag{3.40a}$$

$$M(\eta)\equiv \sup_{\sigma\in\mathscr{S}_A} M(A_\sigma^\eta)=2\pi\left\langle\frac{P_0+|\mathbf{P}|}{m}\right\rangle_\eta. \quad\square \tag{3.40b}$$

Note that $A_\sigma^\eta(k)$ in (3.38) can also be seen as a mean value

$$A_\sigma^\eta(k)\equiv\langle \mathscr{A}_\sigma(k, P)\rangle_\eta. \tag{3.41}$$

The proof of Theorem 3.4 is immediate. First, putting $\eta_1=\eta_2=\eta$ in (3.29), we may rewrite the integral $I_\sigma^\eta(\phi, \psi)$ in (3.21) as:

$$I_\sigma^\eta(\phi, \psi)=\int_{\mathscr{V}_m^+}\overline{\phi(k)}(A_\sigma^\eta\psi)(k)\,\psi(k)\frac{d\mathbf{k}}{k_0} \tag{3.42}$$

$$=\int_{\mathscr{V}_m^+\times\mathscr{V}_m^+}\frac{d\mathbf{p}}{p_0}\,\frac{d\mathbf{k}}{k_0}\mathscr{A}_\sigma(k, p)\,|\eta(p)|^2\,\overline{\phi(k)}\,\psi(k), \tag{3.43}$$

which proves (3.37) and (3.38). The rest follows from the inequalities (3.33), which imply:

$$m(\eta)\,\|\phi\|^2\langle\langle\phi\,|\,A_\sigma^\eta\phi\rangle\langle M(\eta)\,\|\phi\|^2. \tag{3.44}$$

We emphasize that the quantities $m(\eta)$, $M(\eta)$ give estimates which are valid for all sections in $\mathscr{S}_A$. For any given σ, the actual frame bounds may be sharper, as will be seen in the examples described below. The width of the frame class (*see* I, Section 5) is given by:

$$w(\eta)=\frac{M(\eta)-m(\eta)}{M(\eta)+m(\eta)}=\frac{\langle|\mathbf{P}|\rangle_\eta}{\langle P_0\rangle_\eta}. \tag{3.45}$$

An interesting open question is whether there exists a vector η which minimizes this width.

In fact it is possible to find couples (σ, η) such that the corresponding frame is tight, *i.e.* such that $A_\sigma^\eta=I$. As a first example (*see* Table A.1 in the Appendix), consider the section σ_{DB} corresponding to $\theta(\mathbf{p})=\dfrac{\mathbf{p}}{m}$. For this choice, the kernel $\mathscr{A}_\sigma(k, p)$ does *not* depend on k, so that the operator A_σ^η is a multiple of the identity, and the frame is tight for *any* admissible η.

More generally, start from (3.42). Since

$$\left.\begin{aligned}(\Lambda_k^{-1}p)_0&=\frac{1}{m}(k_0p_0-\mathbf{k}.\mathbf{p}),\\ \underline{\Lambda}_k^{-1}\underline{p}&=\frac{1}{m}(k_0\,\mathbf{p}-\mathbf{k}\,p_0),\end{aligned}\right\} \tag{3.46}$$

the kermel $\mathscr{A}_\sigma(k, p)$ may be rewritten as

$$\mathscr{A}_\sigma(k, p)=\frac{2\pi}{m}p_0\left[1-\frac{\mathbf{k}/p_0-\theta(\underline{\Lambda}_p^{-1}\underline{k})}{k_0/\mathbf{p}-\theta(\underline{\Lambda}_p^{-1}\underline{k})}\right]. \tag{3.47}$$

Inserting (3.47) into the integral (3.42), we obtain the expression:

$$\mathrm{I}_\sigma(\phi, \psi)=\frac{2\pi}{m}\langle \mathrm{P}_0\rangle_\eta\langle\phi\,|\,\psi\rangle - \int_{\mathbb{R}\times\mathscr{V}_m^+} d\mathbf{p}\frac{d\mathbf{k}}{k_0}\left[\frac{\mathbf{k}/p_0-\theta(\underline{\Lambda}_p^{-1}\underline{k})}{k_0/\mathbf{p}-\theta(\underline{\Lambda}_p^{-1}\underline{k})}\right]|\eta(p)|^2\,\overline{\phi(k)}\,\psi(k). \tag{3.48}$$

If η can be chosen in a way which makes the integral on the r.h.s. of (3.48) vanish, for all $\phi, \psi\in\mathscr{H}_w$, then using the normalization

$$\frac{2\pi}{m}\langle \mathrm{P}_0\rangle_\eta=1, \tag{3.49}$$

we see from (3.37) and (3.48) that

$$\mathrm{A}_\sigma^\eta=\mathrm{I}. \tag{3.50}$$

This precisely was the case envisaged in [7]. However, for arbitrary $\sigma\in\mathscr{S}_\mathrm{A}$, there may not exist any nontrivial $\eta\in\mathscr{D}(\mathrm{P}_0^{1/2})$ for which the integral on the r.h.s. of (3.48) vanishes. An example is the section σ_s considered in [8]. In that case, in fact, the spectrum $\sigma(\mathrm{A}_\sigma^\eta)$ of A_σ^η is always continuous, whatever state η is used (*see* also the Appendix). We summarize these results in a proposition.

PROPOSITION 3.5. – *Let $\mathscr{S}_\mathrm{A}$ denote the class of all affine space-like sections $\sigma:\Gamma\to\mathscr{P}_+^\uparrow(1, 1)$. Then:*

(1) *$\mathscr{S}_\mathrm{A}$ is invariant under the action of $\mathscr{P}_+^\uparrow(1, 1)$;*

(2) *$\mathscr{S}_\mathrm{A}$ contains at least one section of each of the following types:*

(i) *for any admissible vector η, one has $\mathrm{A}_\sigma^\eta=\lambda\,\mathrm{I}$, thus the frame is tight;*

(ii) *one can find an admissible vector η such that $\mathrm{A}_\sigma^\eta=\lambda\,\mathrm{I}$, but not all admissible vectors lead to tight frames;*

(iii) *for any admissible vector η, the spectrum of A_σ^η is purely continuous; in that case, the frame is never tight.* □

An interesting open question is whether $\mathscr{S}_\mathrm{A}$ contains in fact a *unique* section of any of those types.

Sections of coherent states can now be constructed for any $\sigma\in\mathscr{S}_\mathrm{A}$ and any $\eta\in\mathscr{D}(\mathrm{P}_0^{1/2})$. One obtains:

$$\mathfrak{S}_\sigma=\{\eta_{\sigma(\mathbf{q},\mathbf{p})}=\mathrm{U}_w(\sigma(\mathbf{q},\mathbf{p}))\,\eta\,|\,(\mathbf{q},\mathbf{p})\in\Gamma\}, \tag{3.51}$$

$$\int_\Gamma |\eta_{\sigma(\mathbf{q},\mathbf{p})}\rangle\langle\eta_{\sigma(\mathbf{q},\mathbf{p})}|\,d\mathbf{q}\,d\mathbf{p}=\mathrm{A}_\sigma^\eta. \tag{3.52}$$

From the discussion in Section 2.C, it is clear that the sections of coherent states $\mathfrak{S}_\sigma$, $\sigma\in\mathscr{S}_\mathrm{A}$, are all equivalent.

3. B. Weighted coherent states or quasi-coherent states for $\mathscr{P}_+^\uparrow(1, 1)$

Now, given any section of coherent states $\mathfrak{S}_\sigma$, $\sigma \in \mathscr{S}_A$, it is possible to find weighted coherent states, as defined in Section 2:

$$\eta'_{\mathbf{q}, \mathbf{p}} = \mathrm{T}(\mathbf{q}, \mathbf{p})\, \eta_{\sigma(\mathbf{q}, \mathbf{p})} \tag{3.53}$$

with $(\mathbf{q}, \mathbf{p}) \mapsto \mathrm{T}(\mathbf{q}, \mathbf{p}) \in \mathscr{L}(\mathscr{H}_w)$ a measurable operator valued function, such that the resolution of the identity holds:

$$\int_\Gamma |\eta'_{\mathbf{q}, \mathbf{p}}\rangle \langle \eta'_{\mathbf{q}, \mathbf{p}}|\, d\mathbf{q}\, d\mathbf{p} = \mathrm{I}. \tag{3.54}$$

To see this, let us return to the integral $\mathrm{I}_\sigma(\eta_1, \eta_2; \phi, \psi)$, $\eta_1, \eta_2 \in \mathscr{D}(\mathrm{P}_0^{1/2})$, $\phi, \psi \in \mathscr{H}_w$, defined in (3.22). Taking $\eta_1 = \eta_2 = \eta$ in (3.28), this integral becomes

$$\mathrm{I}_\sigma(\phi, \psi) = \int_{\mathscr{V}_m^+ \times \mathscr{V}_m^+} \frac{d\mathbf{p}}{p_0} \frac{d\mathbf{k}}{k_0} c_\sigma(k, p)\, |\eta(\Lambda_p^{-1} k)|^2\, \overline{\phi(k)}\, \psi(k). \tag{3.55}$$

where

$$c_\sigma(k, p) \equiv \frac{2\pi p_0}{k_0 - \theta(\mathbf{p}) . (\underline{\Lambda}_p^{-1} \underline{k})} = \mathscr{A}_\sigma(k, \Lambda_k \mathscr{I} p). \tag{3.56}$$

To go from (3.55) to (3.54), clearly we have to "divide" by the factor $c_\sigma(k, p)$ [this makes sense, since $c_\sigma(k, p) > 0$]. The precise result needed here is the following (proof in the Appendix):

LEMMA 3.6. – *For any $\sigma \in \mathscr{S}_A$ and for each $p \in \mathscr{V}_m^+$, the operator $\mathrm{C}_\sigma(p)^{-1/2} \equiv c_\sigma(\mathrm{P}, p)^{-1/2}$ defined by the relation:*

$$(\mathrm{C}_\sigma(p)^{-1/2} \phi)(k) = (c_\sigma(k, p))^{-1/2} \phi(k), \tag{3.57}$$

is positive, self-adjoint and defined for all $\phi \in \mathscr{D}(\mathrm{P}_0^{1/2})$. □

Now we are ready to construct weighting operators $\mathrm{T}(\mathbf{q}, \mathbf{p})$, as discussed in Section 2.6 above. Let $\mathbb{P}_{\sigma(\mathbf{q}, \mathbf{p})} = \|\eta\|^{-2} (|\eta_{\sigma(\mathbf{q}, \mathbf{p})}\rangle \langle \eta_{\sigma(\mathbf{q}, \mathbf{p})}|)$ denote the one dimensional projection operator (in $\mathscr{H}_w$) corresponding to the vector $\eta_{\sigma(\mathbf{q}, \mathbf{p})}$. Since $\mathscr{D}(\mathrm{P}_0^{1/2})$ is stable under $\mathrm{U}_w(g)$, $\forall g \in \mathscr{P}_+^\uparrow(1, 1)$, we see that

$$\mathrm{T}(\mathbf{q}, \mathbf{p}) = \|\eta\|^{-1}\, \mathrm{C}_\sigma(p)^{-1/2}\, \mathbb{P}_{\sigma(\mathbf{q}, \mathbf{p})} \tag{3.58}$$

is a bounded operator on $\mathscr{H}_w$, as a consequence of Lemma 3.6. Thus defining $\eta'_{\mathbf{q}, \mathbf{p}}$ as in (3.53), it is easy to derive (3.54), following essentially the steps leading from (3.21) to (3.28), but using the weighted density function

$$\mathrm{F}'_\sigma(\mathbf{q}, \mathbf{p}) = \mathrm{T}(\mathbf{q}, \mathbf{p})\, \mathrm{F}_\sigma(\mathbf{q}, \mathbf{p})\, \mathrm{T}(\mathbf{q}, \mathbf{p})^* \tag{3.59}$$

in place of F_σ [*see* (2.35) and (3.20)].

This completes the construction of coherent states for the UIR U_w of $\mathscr{P}_+^\uparrow(1, 1)$. Comparing with the general theory of Section 2, the reproducing triples in the present case are of the type $\{\mathscr{H}_w, F_\sigma, A_\sigma\}_{n=1}$, $\sigma \in \mathscr{S}_A$. Moreover, using the weighted coherent states $\eta'_{\mathbf{q},\mathbf{p}}$, we obtain the reproducing triple $\{\mathscr{H}_w, F'_\sigma, I\}_{n=1}$. In this case, for any Borel set Δ of the phase space Γ, the operators

$$a(\Delta) = \int_\Delta F'_\sigma(\mathbf{q}, \mathbf{p})\, d\mathbf{q}\, d\mathbf{p}$$

are quantum mechanical localization operators with $\langle \phi \mid a(\Delta)\phi \rangle$ giving the probability of localization of the system in the set Δ when in the state ϕ.

We note in passing that if the four-dimensional Poincaré group had been used, reproducing triples with $n=2$ (or, more generally, $n=2s+1$) could have been obtained.

3. C. Orthogonality relations and the Wigner transform

Finally we obtain some general orthogonality relations for the sections $\sigma \in \mathscr{S}_A$. Let $\mathscr{B}_2(\mathscr{H}_w)$ be the Hilbert space of all Hilbert-Schmidt operators on $\mathscr{H}_w$, with scalar product:

$$\langle \rho_1 \mid \rho_2 \rangle_{\mathscr{B}_2(\mathscr{H}_w)} = \operatorname{tr}[\rho_1^* \rho_2]. \tag{3.60}$$

Consider the linear span of the set of all vectors in $\mathscr{B}_2(\mathscr{H}_w)$, which are of the form

$$\rho_{\phi,\eta} = |\phi\rangle\langle\eta|, \qquad \phi \in \mathscr{H}, \quad \eta \in \mathscr{D}(P_0^{1/2}), \tag{3.61}$$

that is, Hilbert-Schmidt operators with separable kernel

$$\rho_{\phi,\eta}(k, p) = \phi(k)\overline{\eta(p)}. \tag{3.62}$$

This set, which we denote by $\mathscr{H}_w \otimes \overline{\mathscr{D}(P_0^{1/2})}$ (the overbar meaning a complex conjugation), is clearly dense in $\mathscr{B}_2(\mathscr{H}_w)$. On this domain define the operator $\mathscr{A}_\sigma$:

$$(\mathscr{A}_\sigma \rho_{\phi,\eta})(k, p) = \mathscr{A}_\sigma(k, p)\phi(k)\overline{\eta(p)}, \qquad \forall k, p \in \mathscr{V}_m^+, \tag{3.63}$$

with $\mathscr{A}_\sigma(k, p)$ as in (3.30). Then, combining (3.22) and (3.29) we obtain

$$\int_\Gamma \overline{\operatorname{tr}[U_w(\sigma(\mathbf{q}, \mathbf{p}))^* \rho_{\phi,\eta_1}]}\operatorname{tr}[U_w(\sigma(\mathbf{q}, \mathbf{p}))^* \rho_{\psi,\eta_2}]\, d\mathbf{q}\, d\mathbf{p} = \langle \rho_{\phi,\eta_1} \mid \mathscr{A}_\sigma \rho_{\psi,\eta_2} \rangle_{\mathscr{B}_2(\mathscr{H}_w)}. \tag{3.64}$$

Since $\mathscr{A}_\sigma(k, p) > 0$, $\forall k, p$, $[\mathscr{A}_\sigma(k, p)]^{-1}$ exists, and one has [*see* (3.33)]:

$$\frac{1}{2\pi m}(p_0 - |\mathbf{p}|) < [\mathscr{A}_\sigma(k, p)]^{-1} < \frac{1}{2\pi m}(p_0 + |\mathbf{p}|), \tag{3.65}$$

Using this fact we define the positive operator $\mathscr{A}_\sigma^{-1}$ on $\mathscr{B}_2(\mathscr{H}_w)$ by giving its action on all vectors of the type $\rho_{\phi,\eta} \in \mathscr{H}_w \otimes \overline{\mathscr{D}(P_0^{1/2})}$:

$$(\mathscr{A}_\sigma^{-1/2} \rho_{\varphi,\eta})(k,p) = [\mathscr{A}_\sigma(k,p)]^{-1/2} \phi(k) \overline{\eta(p)} = \left[\frac{k_0 - \theta(\underline{\Lambda_p^{-1} k}) \cdot \mathbf{p}}{2\pi(\Lambda_p^{-1} k)_0}\right]^{-1/2} \phi(k) \overline{\eta(p)}. \quad (3.66)$$

Let us now define the *Wigner transform* map $\mathscr{W}: \mathscr{H}_w \otimes \overline{\mathscr{D}(P_0^{1/2})} \to L^2(\Gamma, d\mathbf{q}\, d\mathbf{p})$:

$$(\mathscr{W} \rho)(\mathbf{q}, \mathbf{p}) = \operatorname{tr}[U_w(\sigma(\mathbf{q}, \mathbf{p}))^* \mathscr{A}_\sigma^{-1/2} \rho]. \quad (3.67)$$

Then from (3.64) it immediately follows that

(i) $\mathscr{W}$ is a linear isometry, and hence can be extended to $\mathscr{B}_2(\mathscr{H}_w)$:

$$\langle \mathscr{W} \rho_1 | \mathscr{W} \rho_2 \rangle_{L^2(\Gamma)} = \langle \rho_1 | \rho_2 \rangle_{\mathscr{B}_2(\mathscr{H}_w)}, \quad (3.68)$$

for all $\rho_1, \rho_2, \in \mathscr{B}_2(\mathscr{H}_w)$, and

(ii) the following *orthogonality relation* holds for all $\rho_1, \rho_2 \in \mathscr{H}_w \otimes \overline{\mathscr{D}(P_0^{1/2})}$:

$$\int_\Gamma \overline{\operatorname{tr}[U_w(\sigma(\mathbf{q}, \mathbf{p}))^* \rho_1]} \operatorname{tr}[U_w(\sigma(\mathbf{q}, \mathbf{p}))^* \rho_2]\, d\mathbf{q}\, d\mathbf{p} = \langle \mathscr{A}_\sigma^{1/2} \rho_1 | \mathscr{A}_\sigma^{1/2} \rho_2 \rangle_{\mathscr{B}_2(\mathscr{H}_w)}. \quad (3.69)$$

It is clear that (3.69) includes, as special cases, the orthogonality relations studied in [7] and [8]. Furthermore, these relations may be used to construct a relativistic Weyl transform, for any section $\sigma \in \mathscr{S}_A$, as was done in [7] for the particular section σ_0. We leave the details to the reader.

4. CONCLUSION

Relativistic coherent states have been treated before in the literature, from various points of view. The type of coherent states we have analyzed here, and also in [7], were first introduced by Prugovecki [20], in the study of massive spin 0 particles in a phase space setting, generalizing the corresponding nonrelativistic states obtained earlier [21]. Among other things, these coherent states allowed one to formulate a relativistic quantum mechanics on phase space with a positive, covariant conserved current. However in [20], only the section σ_0 is (implicitly) used and the group representation problem is not considered.

A different approach was taken by Kaiser in [22]. There, the starting point is the observation that positive energy solutions of the Klein-Gordon equation [in $(n+1)$ space-time dimensions] are boundary values on the Minkowski space $\mathbb{R}^{n+1}$ of functions holomorphic in the forward tube $\mathbb{T} = \mathbb{R}^{n+1} + iC$, where C is the forward lightcone in $\mathbb{R}^{n+1}$. Thus one obtains a family of states $\mathfrak{S}_{\mathbb{T}}=\{e_z,\ z\in\mathbb{T}\}$, that transform covariantly under the Poincaré group $\mathscr{P}_+^{\uparrow}(1, n)$:

$$(a, \Lambda)\, e_z = e_{\Lambda z + a}, \qquad \forall (a, \Lambda)\in\mathscr{P}_+^{\uparrow}(1, n), \quad \forall z\in\mathbb{T}, \tag{4.1}$$

and have a number of interesting properties. In fact, specializing to $n=1$ and identifying $z\equiv\hat{q}-i\hat{p}/m^2\in\mathbb{T}$, with $(\hat{q}, \Lambda_{\hat{p}})=\sigma(\mathbf{q}, \mathbf{p})$, one sees easily that e_z coincides with the vector $\eta_{\sigma(\mathbf{q}, \mathbf{p})}=\mathrm{U}_w(\sigma(\mathbf{q}, \mathbf{p}))\,\eta_0$, where η_0 is the particular vector $\eta_0(\mathbf{k})=\exp(-k_0/m)$. It is interesting to notice, from the explicit transformation law (3.2), that this vector η_0 is the *only* one in $\mathrm{L}^2(\mathscr{V}_m^+, d\mathbf{k}/k_0)$ that leads to an (anti)holomorphic function, namely $e_z(k)=\exp(i\bar{z}k)$.

However, the family $\mathfrak{S}_{\mathbb{T}}$ is much too big ($\mathbb{T}$ is therefore called an *extended* phase space) and a restriction has to be made to a subfamily $\mathfrak{S}_{\Sigma}=\{e_z,\ z\in\Sigma\}$, where Σ is a submanifold of $\mathbb{T}$ of real codimension 2; one takes for Σ either the set $\mathrm{P}_\lambda=\{z=x-iy\in\mathbb{T} : x^0=0,\ y\in\mathscr{V}_\lambda^+,\ \lambda>0\}$ or, more generally, $\Sigma=\mathrm{S}-i\mathscr{V}_\lambda^+$, with S a hypersurface in $\mathbb{R}^{n+1}$ of space- or light-like type. Clearly this amounts exactly to a restriction to the section σ_0 in the first case and to a certain section σ, essentially of affine type, in the second one. Thus, quite naturally, each family $\mathfrak{S}_{\Sigma}$ of coherent states generates a resolution of the identity. Thus (for $n=1$) the coherent states in [22] are a special case of ours, corresponding to the particular vector η_0, but this choice seems to have a special physical relevance. The group-theoretical underpinnings of the problem are also not discussed in [22].

More recently Unterberger ([23], [24]) has rediscovered the coherent states of [22] and used them as a basic tool for his comprehensive relativistic generalization of Weyl's operational calculus (under the name of Fuchs and Klein-Gordon calculus). In particular, he obtains appropriate resolutions of the identity, by selecting suitable submanifolds of $\mathbb{T}$. An interesting aspect of Unterberger's work is it geometrical content. Indeed the tube domain $\mathbb{T}$ is a homogeneous space for the conformal group $\mathrm{SO}_0(2, n)$, $\mathbb{T}\cong\mathrm{SO}_0(2, n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$, and a Riemannian globally symmetric space [25]. However, $\mathbb{T}$ is not a homogeneous space for $\mathscr{P}_+^{\uparrow}(1, n)$, and the restriction to $\mathscr{P}_+^{\uparrow}(1, n)$ of the relevant (discrete series) representation of $\mathrm{SO}_0(2, n)$ involves a direct integral decomposition ([24], [26]); this is a major difficulty in the derivation of the Klein-Gordon calculus.

Thus Unterberger's approach suggests a class of spaces X for which the present analysis applies, namely the Riemannian globally symmetric spaces of noncompact type ([2], [25]). Indeed, if $\mathrm{X}=\mathrm{G}/\mathrm{H}$ is such a space, in other words, a noncompact classical domain, then there exists a smooth global section $\sigma : \mathrm{X}\to\mathrm{G}$ [25], Theorem VI.1.1, and hence the principal bundle $\mathrm{G}\to\mathrm{X}=\mathrm{G}/\mathrm{H}$ is trivializable (*see* Section 2). Typical examples, besides $\mathbb{T}$ itself, are $\mathrm{SU}(1,1)/\mathrm{U}(1)$ (already discussed in [8]), or $\mathrm{SU}(2,2)/$

$S(U(2) \times U(2))$. In the general case, G is a connected noncompact semisimple Lie group and H is the maximal compact subgroup of G. As pointed out already in [7], there is a considerable literature about the realization of the discrete series (*i.e.* square integrable) representations of G on the corresponding symmetric space G/H (*see* for instance [27]-[29], and the references contained in those papers). On the other hand, in the case of a Riemannian globally symmetric space of *compact* type, $X = G/H$, no smooth global section $X \rightarrow G$ exists (although a global Borel section always does), but our construction is of course not needed, as pointed out already in Section 2.

Another class consists of semi-direct products $G = S \wedge V$, with V a vector space and S a semisimple subgroup of GL(V), as considered by DeBièvre [4]. Such groups usually arise by contraction from those of the preceding class, which do have a discrete series (the simplest case is the contraction $SO_0(1, 2) \rightarrow \mathscr{P}_+^\uparrow(1, 1)$ studied in [8]). This explains why they have representations square integrable on a coset space G/H.

In fact there is a close relationship between our approach and that of [4], and we feel it interesting to discuss it in some detail [30]. The key point is that the family of affine sections σ introduced in Section 3 is in one-to-one correspondence with the so-called parallel bundles of [4]. Furthermore sections with spacelike $\hat{q}$ correspond to the admissible parallel bundles of [4], Definition 2.2, as follows. We recall that, in the case of a general group $G = S \wedge V$, the coordinate transformation corresponding to (3.24) is nonsingular only on a piece of the orbit X, namely a neighborhood U_{x_0} of a base point $x_0 \in X$. Then one proves easily the following statement:

LEMMA 4.1. – *Let $\sigma \in \mathscr{S}_A$ be an affine, spacelike section. Then the corresponding parallel bundle is admissible and $U_p = \mathscr{V}_m^+$, for all $p \in \mathscr{V}_m^+$.* □

One should notice that such a regular situation results from the restriction that $\hat{q}$ be spacelike; if one drops it, a nontrivial neighborhood U_{x_0} must be introduced, and the admissible vector η must have its support restricted to U_{x_0}. The same holds true for other groups, such as the Euclidean group E(2) analyzed in [4], no matter what section is chosen. Finally one should notice that the calculation of the integral $I_\sigma(\eta_1, \eta_2; \phi, \psi)$ in (3.22) coincides with the calculation made in the proof of Theorem 3.2 of [4] – and the similar calculation made in [7].

We should also mention the work of Bohnké [31], in which discrete frames associated to the Poincaré group (in any dimension) are constructed, and that of Klauder and Streater [32], who circumvent the lack of square integrability of the Wigner representation by using a *reducible* representation, obtained by taking a direct integral over the mass m.

Where do we go now? As indicated in the introduction, the mathematical structure developed here and in the preceding paper [7] is broad enough to encompass an array of applications. In a succeeding paper we intend to analyze spin coherent states and Dirac coherent states, within the framework of the full Poincaré group $\mathscr{P}_+^\uparrow(1, 3)$. In these cases the density functions $F_\sigma(\mathbf{q}, \mathbf{p})$ involve rank-2 and rank-4 operators F, respectively. Further work is also underway on the construction, within the present theory, of general relativistic frames [33], that is, frames corresponding to the Galilei and the anti-de Sitter groups, in addition to the Poincaré case; a description of squeezed states of light using sections in the semi-direct product of the Weyl-Heisenberg group with the metaplectic group [34]; and the construction of coherent states for certain infinite dimensional groups (diffeomorphism groups of manifolds) [35]. Another problem is the application of the present analysis of the massless case; the $1+3$ dimensional case is straightforward, but in $1+1$ dimensions new difficulties appear, because 2-D massless fields require an indefinite metric, and the method has to be generalized further. Massless Poincaré coherent states have been obtained recently by Moschella and one of us [36] and the extension to a conformal setup is under study. Another open problem is a general analysis of the K-representation (2.31) and its possible connection with general orthogonality relations of the type (3.66)-(3.67).

Finally, a characterization of group representations which are square integrable in the sense of Definition 2.1 would be highly desirable. In particular it would be useful to obtain a criterion for determining when a given representation is square integrable. Such results could have bearing on problems of quantization ([35]-[40]).

APPENDIX

A.1. Proof of Proposition 2.2

On $\mathscr{H}$, consider the formal operator

$$A_\sigma = \int_X \sum_{i=1}^{n} |\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}|\, d\nu(x). \tag{A.1}$$

Define vectors $\eta^i_g \in \mathscr{H}$

$$\eta^i_g = U(g)\,\eta^i, \qquad g \in G, \quad i = 1, 2, \ldots, n. \tag{A.2}$$

Since U is irreducible, this set of vectors is dense in $\mathscr{H}$ (assuming, of course, there is at least one $\eta^i \neq 0$). Then,

$$\langle \eta_g^j | A_\sigma \eta_g^j \rangle = \sum_{i=1}^{n} \int_X |\langle U(g)\eta^j | \eta^i_{\sigma(x)} \rangle|^2 \, d\nu(x)$$

$$= \sum_{i=1}^{n} \int_X |\langle \eta^i_{g^{-1}\sigma(x)} | \eta^j \rangle|^2 \, d\nu(x)$$

$$= \sum_{i=1}^{n} \int_X |\langle \eta^i_{\sigma_{g^{-1}}(x)} | \eta^j \rangle|^2 \, d\nu(x),$$

by (2.9) and the left invariance of ν. Hence, from (2.25),

$$\langle \eta_g^j | A_\sigma \eta_g^j \rangle = I^j(\sigma_{g^{-1}}) < \infty, \tag{A.3}$$

and thus A_σ is densely defined. Clearly A_σ is also positive (since, as we show next, it is self-adjoint). Let $\mathscr{D}(A_\sigma) \subset \mathscr{H}$ be the largest set on which A_σ is defined. For any $\phi \in \mathscr{D}(A_\sigma)$, we have

$$\langle \phi | A_\sigma \phi \rangle = \sum_{i=1}^{n} \int_X |\langle \eta^i_{\sigma(x)} | \phi \rangle|^2 \, d\nu(x). \tag{A.4}$$

Hence, if $\langle \phi | A_\sigma \phi \rangle = 0$, then $\langle \eta^i_{\sigma(x)} | \phi \rangle = 0$, for each $i = 1, 2, \ldots, n$ and for ($\nu-$) almost all $x \in X$. But since $\sigma : X \to G$ is continuous, the function $x \mapsto \langle \eta^i_{\sigma(x)} | \phi \rangle$ is continuous. Also the measure ν is invariant. Hence $\langle \eta^i_{\sigma(x)} | \phi \rangle = 0$, for *all* $x \in X$, and $i = 1, 2, \ldots, n$. Since by condition (i), the set $\{ \eta^i_{\sigma(x)} \}$ is total, this means that $\phi = 0$. Thus A_σ is strictly positive on $\mathscr{D}(A_\sigma)$. We prove that it is closed.

Indeed, consider the map $W_\sigma : \mathscr{D}(A_\sigma) \to L^2(X, \nu; \mathbb{C}^n)$ given by

$$(W_\sigma \phi)_i(x) = \langle \eta^i_{\sigma(x)} | \phi \rangle. \tag{A.5}$$

Let $\{ \phi_n \}_{n=1}^{\infty}$ be a sequence of vectors in $\mathscr{D}(A_\sigma)$, converging strongly to $\phi \in \mathscr{H}$ and suppose that $\{ \psi_n = W_\sigma \phi_n \}_{n=1}^{\infty}$ converges strongly to $\psi \in L^2(X, \nu; \mathbb{C}^n)$ in the norm of $L^2(X, \nu; \mathbb{C}^n)$. Then, ψ_n also converges weakly in $L^2(X, \nu; \mathbb{C}^n)$ to ψ and the sequence of norms $\| \psi_n \|_{L^2(X, \nu; \mathbb{C}^n)}$ is bounded. Also, by the continuity of the scalar product in $\mathscr{H}$, the sequence of complex numbers $\{ \langle \eta^i_{\sigma(x)} | \phi_n \rangle \}_{n=1}^{\infty}$ converges to $\langle \eta^i_{\sigma(x)} | \phi \rangle$. Hence,

$$\psi_{n,i}(x) \to \langle \eta^i_{\sigma(x)} | \phi \rangle \tag{A.6}$$

$\forall x \in X$ and $i = 1, 2, \ldots, n$. Thus,

$$\psi_i(x) = \langle \eta^i_{\sigma(x)} | \phi \rangle \tag{A.7}$$

and since $\psi \in L^2(X, \nu; \mathbb{C}^n)$, this means that $\phi \in \mathscr{D}(A_\sigma)$. Hence,

$$W_\sigma \phi = \psi \tag{A.8}$$

and, therefore, W_σ is a closed map. Also, from (A.4) and (A.5),

$$\| W_\sigma \phi \|^2 = \langle \phi | A_\sigma \phi \rangle, \tag{A.9}$$

$\forall \phi \in \mathscr{D}(A_\sigma)$, which implies that A_σ is closed on $\mathscr{D}(A_\sigma)$, hence self-adjoint.

Finally, since A_σ is strictly positive and self-adjoint on $\mathscr{D}(A_\sigma)$, it has a positive self-adjoint inverse A_σ^{-1}, which is also densely defined. □

A.2. Some explicit relations for $\{\mathscr{H}, F_\sigma, A_\sigma\}_n$

In the special case of a reproducing triple $\{\mathscr{H}, F_\sigma, A_\sigma\}_n$, arising from a representation U of G, which is square integrable mod (H, σ), the various operators entering in the general construction developed in I [4] assume special forms. We list here, for reference purposes, several relevant formulas, corresponding to I-(A.18), (21)-(23).

First the basic isometry now reads $W_\sigma : \mathscr{H} \to \mathscr{H}_\sigma \subset L^2(X, \nu; \mathbb{C}^n)$, with

$$(W_\sigma \phi)_i(x) = \lambda_i^{1/2} \langle u_i | U(\sigma(x))^* \phi \rangle, \qquad (A.10)$$

with u_i and λ_i, $i = 1, 2, \ldots, n$, given by (2.3). The expression for the inverse map is now:

$$W_\sigma^{-1} \Phi = \sum_{j=1}^{n} \lambda_j^{1/2} \int_X \Phi_j(x) A_\sigma^{-1} U(\sigma(x)) | u_j \rangle \, d\nu(x), \qquad (A.11)$$

$\forall \Phi \in \mathscr{H}_\sigma$. Similarly, the evaluation map $\hat{E}(x)$ (compare I, Fig. A.2) and its adjoint become $E_\sigma(x) : \mathscr{H} \to \mathbb{C}^n$:

$$E_\sigma(x) = \sum_{i=1}^{n} \lambda_i^{1/2} | e_i \rangle \langle u_i | U(\sigma(x))^*, \qquad (A.12)$$

$$E_\sigma(x)^* = \sum_{i=1}^{n} \lambda_i^{1/2} U(\sigma(x)) | u_i \rangle \langle e_i |, \qquad (A.13)$$

$$F_\sigma(x) = E_\sigma(x)^* E_\sigma(x). \qquad (A.14)$$

Finally, the reproducing kernel K^σ for $\mathscr{H}_\sigma$ is [*see* I, (3.23) and (4.12)]:

$$K^\sigma(x, y) = E_\sigma(x) A_\sigma^{-1} E_\sigma(y)^*, \qquad (A.15)$$

$$K_{ij}^\sigma(x, y) = \langle e_i | K^\sigma(x, y) e_j \rangle_{\mathbb{C}^n} = (\lambda_i \lambda_j)^{1/2} \langle u_i | U(\sigma(x))^* A_\sigma^{-1} U(\sigma(y)) u_j \rangle. \qquad (A.16)$$

A.3. Proof of Theorem 3.2

The equivalence between (ii) and (iii), under the three variants (3.10), results from a straightforward calculation (*see* Section 3).

We show (ii) $\Rightarrow$ (i). Inserting the relation

$$\underline{\Lambda}_p^{-1} \underline{k} = \frac{1}{m} (p_0 \mathbf{k} - k_0 \mathbf{p})$$

into (3.25), we see that (i) is equivalent to the inequality:

$$\frac{p_0}{m}\theta(\mathbf{p}).\mathbf{k}<k_0\left(1+\frac{\mathbf{p}}{m}.\theta(\mathbf{p})\right) \tag{A.17}$$

Assume (ii). Since $\mathbf{k}<k_0$ for any $k\in\mathscr{V}_m^+$, we may write

$$\frac{p_0}{m}\theta(\mathbf{p}).\mathbf{k}<\frac{p_0}{m}|\theta(\mathbf{p})|k_0<\left|1+\frac{\mathbf{p}}{m}.\theta(\mathbf{p})\right|k_0, \quad \text{by (3.10}\,c\text{)}.$$

Now the quantity $1+\frac{\mathbf{p}}{m}.\theta(\mathbf{p})$ does not vanish, again by (3.10 c), hence it is always positive (since it equals 1 for $\mathbf{p}=0$ and θ is continuous). Thus it equals its absolute value and (A.17) is proved.

Conversely, assume (i) to be true, *i.e.* (A.17) is satisfied, for all k, $p\in\mathscr{V}_m^+$:

$$1+\frac{\mathbf{p}}{m}.\theta(\mathbf{p}) > \frac{p_0}{m}\theta(\mathbf{p}).\frac{\mathbf{k}}{k_0}$$

This holds, in particular, in the limits $\mathbf{k}\to\pm\infty$, where $\frac{\mathbf{k}}{k_0}\to\pm 1$. For $\mathbf{k}\to+\infty$, we get:

$$\theta(\mathbf{p})<\frac{m}{p_0-\mathbf{p}}=\frac{p_0+\mathbf{p}}{m},$$

while the limit $\mathbf{k}\to-\infty$ yields:

$$\theta(\mathbf{p})>-\frac{m}{p_0+\mathbf{p}}=-\frac{p_0-\mathbf{p}}{m}.$$

By (3.10 a), this is precisely the condition (ii). □

A.4. Proof of Lemma 3.5

As shown in Section 3, one has $c_\sigma(k,p)>0$ for all k, $p\in\mathscr{V}_m^+$, thus $c_\sigma(k,p)^{-1/2}$ is well-defined. Next, for fixed p, consider the function $\alpha:\mathscr{V}_m^+\to\mathbb{R}^+$ given by

$$\alpha(k)=\mathrm{k}_0-\theta(\mathbf{p}).(\underline{\Lambda}_p^{-1}\underline{k}). \tag{A.18}$$

Clearly, $\alpha(k)\to\infty$, either as $\mathbf{k}\to\infty$ or as $\mathbf{k}\to-\infty$. On the other hand, a standard computation shows that $\alpha(k)$ attains its minimum at

$$\left.\begin{aligned} k_0&=\frac{m+\theta(\mathbf{p}).\mathbf{p}}{[1+2\theta(\mathbf{p}).\mathbf{p}/m-\theta(\mathbf{p})^2]^{1/2}}\\ k/k_0&=\frac{\theta(\mathbf{p})p_0}{m+\theta(\mathbf{p}).\mathbf{p}}\end{aligned}\right\} \tag{A.19}$$

The minimum is

$$\alpha_{\min}=m[1+2\theta(\mathbf{p}).\mathbf{p}/m-\theta(\mathbf{p})^2]^{1/2}. \tag{A.20}$$

Thus, as a positive operator, $C_\sigma(p)^{-1/2}$ has spectrum $[(\alpha_{\min}/2\pi p_0)^{1/2}, \infty)$. Moreover, writing $c_\sigma(k, p)^{-1/2}$ as:

$$c_\sigma(k, p)^{-1/2}=k_0^{1/2}\left[\frac{1}{2\pi p_0}-\frac{\theta(\mathbf{p})}{2\pi p_0}\cdot\frac{(\underline{\Lambda}_p^{-1}\underline{k})}{k_0}\right]^{1/2}, \tag{A.21}$$

and noting that, for fixed p, $(\underline{\Lambda}_p^{-1}\underline{k})/k_0$ is bounded in k, we immediately see that $\mathscr{D}(C_\sigma(p)^{-1/2})=\mathscr{D}(P_0^{1/2})$, and that on this domain $C_\sigma(p)^{-1/2}$ is self-adjoint. □

A.5. Some explicit examples of affine sections

For the sake of completeness, we list in Table A.1 below the parameters corresponding to be concrete sections that have been used in earlier work, namely:

– σ_0, the section of reference, introduced in [7] and also used implicitly by Unterberger ([23], [24]);

– σ_s, the so-called deSitterian section derived in [8];

– σ_{DB}, the section used by DeBièvre [4];

– $\sigma_{\lim}$, the section underlying the work of Bertrand and Bertrand [41].

The last section is a limiting case, it does *not* belong to $\mathscr{S}_A$; indeed, for this choice, the inequalities (3.10*b*), (3.10*c*), (3.14) become equalities, corresponding to the fact that the vector $\hat{q}_{\lim}$ is now light-like (see the Figure in Section 3.A).

By (3.38), the sixth column in Table A.1 yields the function $A_\sigma^\eta(k)$ for the various sections, and this in turn specifies the operator A_σ^η. The results, which are summarized in the last column of the table, are the following (a more detailed discussion will be given elsewhere [33]):

(i) *for the section* σ_0: $\frac{m}{2\pi}A_\sigma^\eta(k)=\langle P_0\rangle_\eta-\frac{\mathbf{k}}{k_0}\langle\mathbf{P}\rangle_\eta$; thus for a general admissible vector η, one gets a continuous spectrum

$$\sigma\left(\frac{m}{2\pi}A_\sigma^\eta\right)\doteq[\langle P_0\rangle_\eta-|\langle\mathbf{P}\rangle_\eta|, \langle P_0\rangle_\eta+|\langle\mathbf{P}\rangle_\eta|]; \tag{A.22}$$

however, if η is chosen such that $\langle\mathbf{P}\rangle_\eta=0$ (typically, if $\eta(\mathbf{p})$ is an even function (*see* [7]), or, mimicking 3-dimensional terminology, a "rotation"-invariant function), then the spectrum collapses to a single point and the frame becomes tight. An example is the vector $\eta_0(\mathbf{p})=\exp(-p_0/m)$ used in [22] (*see* Section 4);

(ii) *for the section* σ_s: an explicit calculation shows that the range of the function $A_\sigma^\eta(k)$ is always a full interval, which however depends

TABLE A.1. – Values of some characteristic parameters for the various sections $\sigma : \Gamma \rightarrow \mathscr{P}_+^{\uparrow}(1, 1)$ described in the text.

σ	$\theta(\mathbf{p})$	$\lambda(\lvert\mathbf{p}\rvert)$	$\omega(\lvert\mathbf{p}\rvert)$	$\hat{q}$	$\mathscr{A}_\sigma(k, p)/2\pi$	$2\pi c_\sigma(k, p)^{-1}$	$\alpha_{\min}$	$\sigma\left(\frac{m}{2\pi} \mathrm{A}_\sigma^\eta\right)$
σ_0	$\mathbf{0}$	0	0	$\hat{q}_0 = 0$	$\frac{(k.p)}{mk_0}$	$\frac{k_0}{p_0}$	m	– continuous if $\langle \mathbf{P} \rangle_\eta \neq 0$: $[\langle \mathrm{P}_0 \rangle_\eta - \lvert\langle \mathbf{P} \rangle_\eta\rvert, \langle \mathrm{P}_0 \rangle_\eta + \lvert\langle \mathbf{P} \rangle_\eta\rvert]$ – point if $\langle \mathbf{P} \rangle_\eta = 0$: $\langle \mathrm{P}_0 \rangle_\eta$
σ_s	$\frac{\mathbf{p}}{p_0 + m}$	$\frac{m}{p_0 + m}$	$\frac{p_0}{p_0 + m}$	$\hat{q}_0 = \frac{\hat{\mathbf{q}} \cdot \mathbf{p}}{p_0 + m}$	$\frac{(k.p) + m^2}{m(k_0 + p_0)}$	$\frac{(k.p) + mk_0}{m(p_0 + m)}$	$p_0 \left[\frac{2m}{p_0 + m}\right]^{1/2}$	Always continuous: – if $\langle \mathbf{P} \rangle_\eta = 0$: $[m\lVert\eta\rVert^2, \langle \mathrm{P}_0 \rangle_\eta]$ – if $\langle \mathbf{P} \rangle_\eta \neq 0$: more complicated
σ_{DB}	$\frac{\mathbf{p}}{m}$	1	1	$\hat{q}_0 = \frac{\hat{\mathbf{q}} \cdot \mathbf{p}}{p_0}$	$\frac{m}{p_0}$	$\frac{(k.p)}{m^2}$	p_0	Point: $m^2 \langle \mathrm{P}_0^{-1} \rangle_\eta$
σ_{lim}	$\frac{\mathbf{p}}{\lvert\mathbf{p}\rvert}\left(\frac{p_0 + \lvert\mathbf{p}\rvert}{m}\right)$	$1 + \frac{p_0}{\lvert\mathbf{p}\rvert}$	$\frac{p_0}{\lvert\mathbf{p}\rvert}$	$\hat{q}_0 = \frac{\hat{\mathbf{q}} \cdot \mathbf{p}}{\lvert\mathbf{p}\rvert}$	$\frac{p_0}{m} + \frac{(\mathbf{k}\text{-}\mathbf{p})\,\mathbf{p}}{\lvert\mathbf{k}\text{-}\mathbf{p}\rvert m}$	$\frac{\lvert\mathbf{p}\rvert(k.p) - \mathbf{p}(k_0 \mathbf{p} - p_0 \mathbf{k})}{m^2 \lvert\mathbf{p}\rvert}$	0	Continuous: $[\langle \mathrm{P}_0 - \lvert\mathbf{P}\rvert \rangle_\eta, \langle \mathrm{P}_0 \rangle_\eta + \lvert\langle \mathbf{P} \rangle_\eta\rvert]$

on the state η, as in the previous case. If $\langle \mathbf{P} \rangle_\eta = 0$, $\sigma\left(\frac{m}{2\pi} A_\sigma^\eta\right) = [m \|\eta\|^2, \langle P_0 \rangle_\eta]$. If $\langle \mathbf{P} \rangle_\eta \neq 0$, the situation is more complicated, several cases must be distinguished, but, for any admissible state η, the spectrum of A_σ^η is purely continuous and the frame is never tight;

(iii) *for the section* σ_{DB}: the kernel $\mathscr{A}_\sigma(k, p)$ does not depend on k, hence the function $A_\sigma^\eta(k)$ is constant; thus A_σ^η is a multiple of the identity and the frame is tight;

(iv) *for the section* $\sigma_{\lim}$: the spectrum of A_σ^η is purely continuous, namely:

$$\sigma\left(\frac{m}{2\pi} A_\sigma^\eta\right) = [\langle P_0 - |\mathbf{P}| \rangle_\eta, \langle P_0 \rangle_\eta + |\langle \mathbf{P} \rangle_\eta|]$$

As a final comment, we may remark, by inspection of Table A.1, that the section σ_{DB} yields distinctly simpler results than the others. The reason is that it is the only one which is Lorentz invariant. Indeed σ_{DB} is specified by the invariant condition $(\hat{q}.p) = 0$, and this is actually the way in which it was introduced in the paper of DeBièvre [4].

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