ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 55, nº 4 (1991), p. 829-855
<http://www.numdam.org/item?id=AIHPA_1991__55_4_829_0>

Ann. Inst. Henri Poincaré,

Vol. 55, n° 4, 1991, p. 829-855. *Physique théorique*

Square integrability of group representations on homogeneous spaces. I. Reproducing triples and frames

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ABSTRACT. – A connection between a class of positive operator valued measures on a Hilbert space and certain reproducing kernel Hilbert spaces leads to the concept of a reproducing triple. Any such object generates an overcomplete family of vectors, which has most of the attributes of the familiar coherent states. A particular case of such a triple is the notion of frame, which, in a discrete situation, coincides with the structure underlying nonorthogonal expansions. The abstract machinery developed here will be

(1) Work supported in part by the Natural Science and Engineering Research Council (N.S.E.R.C.) of Canada.

(2) Équipe de Recherches 177 du Centre National de la Recherche Scientifique.

used in a second paper to give a general definition of a square integrable representation of a group and the associated coherent states.

RÉSUMÉ. — En faisant le lien entre une classe de mesures à valeurs opérateurs positifs dans un espace de Hilbert et certains espaces de Hilbert à noyau reproduisant, on est mené au concept de triplet reproduisant. Tout objet de ce type engendre une famille surcomplète de vecteurs possédant la plupart des propriétés des états cohérents traditionnels. Un cas particulier d'un tel triplet est la notion de repère (« frame ») qui, dans le cas discret, coïncide avec la structure sous-jacente aux développements en fonctions non orthogonales. Le formalisme abstrait développé ici sera utilisé dans un deuxième article pour définir de façon générale une représentation de carré intégrable d'un groupe et les états cohérents associés.

1. INTRODUCTION

This paper continues the work begun in two previous publications ([1], [2]), in which the problem of square integrability of group representations, and the existence of coherent states, associated to these representations, had been studied. Apart from a few general considerations, the focus of these two papers was the Poincaré group $\mathscr{P}_+^\uparrow(1, 1)$ in one space and one time dimensions. The objective was to arrive at a notion of square integrability for a group representation, which is broad enough to encompass all the classical results on the discrete series representations for locally compact groups, the Perelomov theory of coherent states for Lie groups ([3], [4]) as well as results obtained, in the context of quantization on phase space ([5]-[8]), for the coherent states of the Galilei and the Poincaré groups — the states in question being labelled by points in certain homogeneous spaces of these groups. A related problem is that of obtaining appropriate orthogonality relations for square integrable representations — in this broader sense — which would then also enlarge the notion of the formal dimension of a square integrable representation.

In this work, which consists of two papers, we extend the concept of square integrability and explore some consequences of this generalization. A fairly comprehensive mathematical scheme is presented which, even apart from its relevance to group representations and harmonic analysis, could be of use in other areas of current activity, such as signal analysis, wavelet transform theory and the theory of frames ([9]-[15]). We also return to the Poincaré group, $\mathscr{P}_+^\uparrow(1, 1)$, and explicitly compute how the coherent states associated to a particular representation depend on sections of the group – the latter being considered as a principal bundle over one of its homogeneous spaces.

In all the usual treatments of coherent states for locally compact groups ([3], [4]) the central object is an operator integral

$$\int_X F(x)\, d\nu(x) = I, \qquad (1.1)$$

on a Hilbert space $\mathscr{H}$ which carries a unitary, irreducible represention (UIR) of the group in question. Here X is, in general, a homogeneous space of the group, ν an *invariant* measure on it and I, the identity operator on $\mathscr{H}$. Each $F(x)$, $x \in X$, is a one-dimensional projection operator. These operators $F(x)$ can then be used to map the Hilbert space $\mathscr{H}$ isometrically onto a subspace $\mathscr{H}_K$ of $L^2(X, \nu)$. This subspace is distinguished by the fact that it carries a reproducing kernel, which implies that $\mathscr{H}_K$ is actually a space of (continuous) functions. The operators $F(x)$, together with (1.1), also define a positive operator valued (POV) measure (*see* below) on the Borel sets of X.

To get a more effective tool, in order to analyze a much wider class of coherent states than has been accessible so far (in particular those of the Poincaré group $\mathscr{P}_+^\uparrow(1, 1)$ obtained in [1], [2]), it is necessary to generalize (1.1) in two ways. First, one has to drop the condition that $F(x)$ be a one-dimensional projection operator, and replace it by a more general bounded positive operator; secondly, the identity operator I has also to be replaced, in general, by a positive, bounded operator A admitting an inverse. This also means that the POV-measure mentioned above has now total measure equal to A. In Sections 2 and 3, we shall first study such a POV-measure and show how notions of overcomplete families of vectors and reproducing kernel Hilbert spaces arise very naturally in this abstract context, even when there is no group action on X. The key concept here is that of a *reproducing triple* $\{\mathscr{H}, F, A\}$, where F and A are the operators discussed above. The main result is that, given such a triple, one can construct an overcomplete family of states (generalizing the usual coherent states) and an isometry from $\mathscr{H}$ onto a Hilbert space of vector-valued functions. We also introduce a natural notion of equivalence between reproducing triples.

Then, in Sections 4 and 5, we specialize the discussion to the case where each $F(x)$ is an operator of finite rank, constant for all $x \in X$. An interesting situation arises if, in addition, the inverse operator A^{-1} is required to be bounded – in that case we will call the triple $\{\mathscr{H}, F, A\}$ a *frame*. Indeed, if we assume that X is a discrete space, with ν the counting measure, our structure coincides with that introduced under the same name in the study of nonorthogonal expansions ([10]-[13]). Thus our construction may also be viewed as a unifying one.

In Section 6, we give some brief indications on how this general formalism is used in the case of coherent states. There, X is a homogeneous space G/H of a locally compact group G and $F(x)$ is obtained by transporting a fixed operator F covariantly with respect to a UIR of G. The systematic discussion of this application is postponed to Paper II [16], together with a general treatment of the case of $\mathscr{P}_+^\uparrow(1, 1)$. In this way we will obtain a substantial generalization, and also a better understanding, of our previous results ([1], [2]).

2. POV MEASURES AND REPRODUCING TRIPLES

As stated in the Introduction, we study in this Section the connection between certain POV measures and reproducing triples.

Let $\mathscr{H}$ be an abstract, separable Hilbert space over $\mathbb{C}$, X a locally compact space, $\mathscr{B}(X)$ the σ-algebra of all Borel sets of X and ν a positive, regular Borel measure on $\mathscr{B}(X)$ with support X. Denote by $\mathscr{L}(\mathscr{H})$ the set of all bounded operators on $\mathscr{H}$, and by $\mathscr{L}(\mathscr{H})^+$ the positive cone of $\mathscr{L}(\mathscr{H})$.

A *positive operator valued* (*POV*) *measure on* X (*see* for example [5] or [17]) is a map $a : \mathscr{B}(X) \to \mathscr{L}(\mathscr{H})^+$ satisfying:

(i) $a(\varnothing)=0$, $\varnothing$ = null set, (2.1 *a*)

(ii) $a(X)=A\in\mathscr{L}(\mathscr{H})^+$, (2.1 *b*)

(iii) $a(\bigcup_{i\in J}\Delta_i)=\sum_{i\in J}a(\Delta_i)$, weakly, J = discrete index set,

$\Delta_i\in\mathscr{B}(X)$ and $\Delta_i\cap\Delta_j=\varnothing$, $\forall\, i, j\in J$, s. t. $i\neq j$. (2.1 *c*)

The POV-measure a is said to be *regular* if, $\forall\,\phi\in\mathscr{H}$, the positive Borel measure $\mu_\phi : \mathscr{B}(X)\to\mathbb{R}^+$,

$$\mu_\phi(\Delta)=\langle\phi\,|\,a(\Delta)\,\phi\rangle, \qquad (2.2)$$

is regular. We shall assume that this is the case, that is, we shall consider regular POV-measures only. If in (2.1 *b*), the operator A is the identity, the POV-measure is said to be *normalized*. Suppose now that there exists a weakly measurable positive operator valued function $F : X\to\mathscr{L}(\mathscr{H})^+$ such that, $\forall\,\Delta\in\mathscr{B}(X)$,

$$a(\Delta)=\int_\Delta F(x)\,d\nu(x), \qquad (2.3)$$

the integral being defined weakly. Then a is said to have the *bounded density* F. If in addition we assume that A^{-1} exists as a positive (possibly unbounded), densely defined, self-adjoint operator on $\mathscr{H}$, in that case we call $\{\mathscr{H}, F, A\}$ a *reproducing triple*, over (X, ν), for reasons which become clear shortly.

We proceed to show how overcomplete sets of vectors can be built using $\{\mathscr{H}, F, A\}$. For each $x \in X$, let $\mathscr{N}_x$ be the null space of $F(x)$,

$$\mathscr{N}_x = \{\phi \in \mathscr{H} \mid F(x)\phi = 0\}, \tag{2.4}$$

and $\mathscr{N}_x^\perp$ its orthogonal complement. Let $\mathbb{P}(x) \in \mathscr{L}(\mathscr{H})^+$ be the projection operator onto $\mathscr{N}_x^\perp$:

$$\mathbb{P}(x)\mathscr{H} = \mathscr{N}_x^\perp, \qquad F(x) = \mathbb{P}(x)F(x)\mathbb{P}(x). \tag{2.5}$$

On $\mathscr{N}_x^\perp$ define the new scalar product $(.|.)_x$ as:

$$(\phi_x \mid \psi_x)_x = \langle \phi_x \mid F(x)\psi_x \rangle_{\mathscr{H}}, \qquad \forall \phi_x, \psi_x \in \mathscr{N}_x^\perp. \tag{2.6}$$

Let $\mathscr{K}_x$ be the closure of $\mathscr{N}_x^\perp$ in the corresponding norm. Note, that if the spectrum of $F(x)$, restricted to $\mathscr{N}_x^\perp$, is bounded away from zero, then $(.|.)_x$ gives just the graph norm of $F(x)$ and $\mathscr{K}_x$ is equal to $\mathscr{N}_x^\perp$ as a set. This, for example, would be the case if $F(x)$ were a finite rank operator. In general, however, $\mathscr{N}_x^\perp \subset \mathscr{K}_x$ densely.

Since $\mathscr{N}_x^\perp$ is dense in $\mathscr{K}_x$, we may introduce an orthonormal basis $\{v_i(x), i=1, \ldots, d(x)\}$ in $\mathscr{K}_x$, with each $v_i(x) \in \mathscr{N}_x^\perp$, $d(x)$ being the dimension of $\mathscr{K}_x$. In that case,

$$\begin{gathered}(v_i(x) \mid v_j(x))_x = \langle v_i(x) \mid F(x)v_j(x) \rangle_{\mathscr{H}} = \delta_{ij}, \\ i, j = 1, 2, \ldots, d(x),\end{gathered} \tag{2.7}$$

and the set of vectors $\{u_i(x) \equiv F(x)^{1/2} v_i(x), i=1, \ldots, d(x)\}$ is an orthonormal basis of $\mathscr{N}_x^\perp$ (for the original scalar product of $\mathscr{H}$). Set

$$\eta_x^i = F(x) v_i(x) \tag{2.8}$$

and

$$\mathfrak{S} = \{\eta_x^i \mid i=1, \ldots, d(x); x \in X\} \subset \mathscr{H}. \tag{2.9}$$

Then $\mathfrak{S}$ "looks like" a set of coherent states, for we have the result (proof in Appendix):

PROPOSITION 2.1. – *For each $x \in X$, one has:*

$$F(x) = \sum_{i=1}^{d(x)} |\eta_x^i\rangle\langle\eta_x^i|, \tag{2.10}$$

the sum converging weakly; the set $\mathfrak{S}$ is an overcomplete family of states (OFS) in $\mathscr{H}$, in the sense that

$$\int_X \sum_{i=1}^{d(x)} |\eta_x^i\rangle\langle\eta_x^i| \, d\nu(x) = A, \tag{2.11}$$

the integral converging weakly, and for any $\phi \in \mathscr{H}$, $\langle \eta_x^i \mid \phi \rangle = 0$, $\forall \eta_x^i \in \mathfrak{S}$, implies that $\phi = 0$. □

Thus, in the general situation envisaged here of a reproducing triple $\{\mathscr{H}, F, A\}$, the resolution of the identity in (1.1) is replaced by the "resolution of the positive, bounded, invertible operator" A given by (2.11). There does nevertheless exist an OFS $\mathfrak{S}$ with all the usual properties [4].

For each $x \in \mathrm{X}$ define a map $\mathrm{E}(x) : \mathscr{H} \to \mathscr{K}_x$ by:

$$\mathrm{E}(x)\phi = \mathbb{P}(x)\phi. \tag{2.12}$$

Let $\mathrm{E}(x)^* : \mathscr{K}_x \to \mathscr{H}$ be the adjoint of $\mathrm{E}(x)$. Clearly.

$$\| \mathrm{E}(x) \| = \| \mathrm{E}(x)^* \| = \| \mathrm{F}(x) \|^{1/2}. \tag{2.13}$$

It is straightforward to verify that in terms of the elements of $\mathfrak{S}$, we have:

$$\mathrm{E}(x) = \sum_{i=1}^{d(x)} | v_i(x)) \langle \eta_x^i |, \tag{2.14a}$$

$$\mathrm{E}(x)^* = \sum_{i=1}^{d(x)} | \eta_x^i \rangle (v_i(x) |, \tag{2.14b}$$

where the bra vector $(v_i(x)|$ is the dual of $|v_i(x))$ with respect to the $\mathscr{K}_x$ scalar product $(.|.)_x$. Combining (2.14) with (2.10) and (2.11) we also get

$$\mathrm{E}(x)^* \mathrm{E}(x) = \mathrm{F}(x), \tag{2.15}$$

$$\int_{\mathrm{X}} \mathrm{E}(x)^* \mathrm{E}(x) \, d\nu(x) = \mathrm{A}. \tag{2.16}$$

Suppose now we are given two different reproducing triples $\{\mathscr{H}, \mathrm{F}, \mathrm{A}\}$ and $\{\mathscr{H}, \mathrm{F}', \mathrm{A}'\}$ over the same measure space (X, ν). How do they compare? First, there is a natural notion of subordination, which is given in terms of generalized intertwining operators.

DEFINITION 2.2. — The density function $\mathrm{F}' : \mathrm{X} \to \mathscr{L}(\mathscr{H})^+$ is said to be *weighted* with respect to the density function $\mathrm{F} : \mathrm{X} \to \mathscr{L}(\mathscr{H})^+$ if there exists a (weakly) measurable operator valued function $\mathrm{T} : \mathrm{X} \to \mathscr{L}(\mathscr{H})$, such that

$$\mathrm{F}'(x) = \mathrm{T}(x)\mathrm{F}(x)\mathrm{T}(x)^*, \qquad \forall x \in \mathrm{X}. \tag{2.17}$$

Of course, the operator $\mathrm{T}(x)$ need not be unique.

In order to exploit such a relation, we have to restrict further the density operators $\mathrm{F}(x)$, $\mathrm{F}'(x)$. Let us assume that both F and F' take values in the set $\mathscr{C}(\mathscr{H})^+$ of positive *compact* operators. This applies, in particular, when $\mathrm{F}(x)$ or $\mathrm{F}'(x)$ have *finite* rank for all $x \in \mathrm{X}$. Then, for each $x \in \mathrm{X}$, we may write a spectral decomposition of $\mathrm{F}(x)$, resp. $\mathrm{F}'(x)$:

$$\mathrm{F}(x) = \sum_{i=1}^{d(x)} \lambda_i(x) | u_i(x) \rangle \langle u_i(x) |, \qquad \lambda_i(x) > 0, \tag{2.18a}$$

$$F'(x) = \sum_{i=1}^{d(x)} \lambda'_i(x) \,|\, u'_i(x) \rangle \langle u'_i(x) \,|, \qquad \lambda'_i(x) > 0, \qquad (2.18\,b)$$

where the eigenvectors $\{u_i(x), i=1, \ldots, d(x)\}$ of $F(x)$ constitute, as above, an orthonormal basis of $\mathcal{N}_x^{\perp}$, and similarly for $\{u'_i(x), i=1, \ldots, d'(x)\} \subset \mathcal{N}'^{\perp}_x$, with $\mathcal{N}'_x = \operatorname{Ker} F'(x)$. Then we may re-express (2.17) as:

$$\sum_{i=1}^{d(x)} \lambda'_i(x) \,|\, u'_i(x) \rangle \langle u'_i(x) \,| = \sum_{j=1}^{d(x)} \lambda_j(x) T(x) \,|\, u_j(x) \rangle \langle u_j(x) \,|\, T(x)^*. \quad (2.19)$$

Here and in the sequel, all sums converge weakly, by Proposition 2.1. Thus, writing

$$\eta'^i_x = \lambda'_i(x)^{1/2} u'_i(x), \qquad i=1, 2, \ldots, d'(x), \qquad (2.20)$$

we obtain

$$\eta'^i_x = \sum_{j=1}^{d(x)} \overline{t_{ij}(x)}\, T(x) \,\eta^j_x, \qquad (2.21)$$

where $t_{ij}(x)$ are the $d'(x) \times d(x)$ matrix elements

$$\left. \begin{aligned} & t_{ij}(x) = \left(\frac{\lambda_j(x)}{\lambda'_i(x)}\right)^{1/2} \langle u'_i(x) \,|\, T(x) \,|\, u_j(x) \rangle, \\ & i=1, 2, \ldots, d'(x); \qquad j=1, 2, \ldots, d(x). \end{aligned} \right\} \qquad (2.22)$$

It also follows from (2.18) and (2.22) that

$$\sum_{j=1}^{d(x)} t_{kj}(x) \overline{t_{ij}(x)} = \delta_{ki}, \qquad k, i=1, 2, \ldots, d'(x). \qquad (2.23)$$

From (2.23) it appears that the vectors of the OFS $\mathfrak{S}' = \{\eta'^i_x\}$ of the reproducing triple $\{\mathcal{H}, F', A'\}$ are given in terms of those of the OFS $\mathfrak{S} = \{\eta^j_x\}$ of the reproducing triple $\{\mathcal{H}, F, A\}$, after weighting by the operator $T(x)$ and mixing by the matrix $t(x)$ [with elements $t_{ij}(x)$]. This justifies the terminology: the OFS $\mathfrak{S}'$ is *weighted* w. r. to the OFS $\mathfrak{S}$. We denote this relation by writing

$$\mathfrak{S}' \prec \mathfrak{S}. \qquad (2.24)$$

The relationship $\prec$ is clearly transitive and reflexive, that is, $\prec$ is an order relation. Thus we get a natural notion of equivalence:

DEFINITION 2.3. — Two reproducing triples $\{\mathcal{H}, F, A\}$ and $\{\mathcal{H}, F', A'\}$, with compact density functions $F, F' : X \to \mathcal{C}(\mathcal{H})^+$ and corresponding OFS $\mathfrak{S}$, $\mathfrak{S}'$, are said to be *equivalent* if $\mathfrak{S}' \prec \mathfrak{S}$ and $\mathfrak{S} \prec \mathfrak{S}'$. Then we write $\mathfrak{S} \sim \mathfrak{S}'$.

In that case we have, in addition to (2.17), the relation

$$F(x) = T'(x) F'(x) T'(x)^*, \qquad \forall x \in X, \qquad (2.25)$$

with another measurable function $T' : X \to \mathscr{L}(\mathscr{H})$. From (2.17) and (2.25) we get:

$$\left.\begin{aligned} F(x) &= T'(x)\,T(x)\,F(x)\,(T'(x)\,T(x))^*, \\ F'(x) &= T(x)\,T'(x)\,F'(x)\,(T(x)\,T'(x))^*. \end{aligned}\right\} \tag{2.26}$$

The simplest solution to (2.26) is

$$T'(x)\,T(x) = \mathbb{P}(x), \qquad T(x)\,T'(x) = \mathbb{P}'(x). \tag{2.27}$$

In fact, most OFS will be comparable in the sense of the order relation $\prec$. Take, for instance, $\mathfrak{S}$ and $\mathfrak{S}'$ such that $d'(x) \leq d(x)$, for all $x \in X$: then $\mathfrak{S}' \prec \mathfrak{S}$. Indeed, define

$$T(x) = \sum_{k=1}^{d'(x)} |u'_k(x)\rangle \left(\frac{\lambda'_k(x)}{\lambda_k(x)}\right)^{1/2} \langle u_k(x)|, \tag{2.28}$$

so that

$$t_{ij}(x) = \left\{\begin{array}{lll} \delta_{ij}, & i, j = 1, 2, \ldots, d'(x) & \\ 0, & i = 1, 2, \ldots, d'(x); \quad j = d'(x) + 1, 2, \ldots, d(x). & \end{array}\right\} \tag{2.29}$$

Then the relations (2.17) and (2.21) are immediately verified, so that indeed $\mathfrak{S}' \prec \mathfrak{S}$. Notice that the particular choice (2.28) for the operator $T(x)$ is characterized by the relation $T(x) = \mathbb{P}'(x)\,T(x)\,\mathbb{P}(x)$.

Assume now that $d'(x) = d(x)$, $\forall x \in X$: then $\mathfrak{S}' \sim \mathfrak{S}$. In particular, the choice (2.28) for $T(x)$ and $T'(x)$ leads immediately to the simple relations (2.27). This concept of equivalence of reproducing triples and OFS has an interesting application in the theory of generalized coherent states developed in Paper II [16].

3. THE FUNDAMENTAL ISOMETRY: REPRODUCING KERNEL HILBERT SPACES

The next step in the analysis is to show how a reproducing triple naturally leads to a reproducing kernel Hilbert space. Consider the Cartesian product $\prod_{x \in X} \mathscr{H}_x$ of the spaces $\mathscr{H}_x$. We equip it with its natural structure of vector space and observe next that this vector space contains a subspace $\mathscr{M}$ of ν-measurable vector fields ([18]-[20]). Indeed, let $\{\phi_n\}_{n=1}^{\infty}$ be a countable set of vectors which is total in $\mathscr{H}$, and for each ϕ_n define an element $\Phi_n \in \prod_{x \in X} \mathscr{H}_x$ by

$$\Phi_n(x) = \mathbb{P}(x)\,\phi_n \in \mathscr{H}_x, \qquad \forall x \in X. \tag{3.1}$$

We define also the set:

$$\mathscr{M}=\{\Phi\in\prod_{x\in X}\mathscr{K}_x \mid x\mapsto(\Phi(x)\mid\Phi_n(x))_x \text{ is } \nu\text{-measurable},\quad n=1,2,3,\ldots\} \qquad (3.2)$$

Then we have the following result (proof given in the Appendix):

LEMMA 3.1. – *The sequence* $\{\Phi_n\}_{n=1}^{\infty}\subset\prod_{x\in X}\mathscr{K}_x$ *is a fundamental sequence of* ν*-measurable vector fields, that is:*

(i) $\forall\, m, n$, *the function* $x\mapsto(\Phi_m(x)\mid\Phi_n(x))_x$ *is* ν*-measurable;*

(ii) $\forall\, x\in X$, *the set of vectors* $\{\Phi_n(x)\}_{n=1}^{\infty}$ *is total in* $\mathscr{K}_x$.

Hence, the set $\mathscr{M}$ *is a subspace of* ν*-measurable vector fields, and it is independent of the particular basis* $\{\phi_n\}$ *chosen.* □

Thus, the family of Hilbert spaces $\{\mathscr{K}_x \mid x\in X\}$ together with the set $\mathscr{M}$, defines a measurable field of Hilbert spaces on the measure space $\{X, \nu\}$, and hence [18], we can define the direct integral space,

$$\tilde{\mathscr{H}}=\int_X^{\oplus}\mathscr{K}_x\, d\nu(x). \qquad (3.3)$$

Vectors $\Phi\in\tilde{\mathscr{H}}$ are equivalence classes of elements in $\prod_{x\in X}\mathscr{K}_x$ which are measurable in the sense of (3.2) and satisfy

$$\int_X \|\Phi(x)\|_x^2\, d\nu(x)<\infty.$$

If $d(x)$ is the dimension of $\mathscr{K}_x$, then it follows from the general theory of direct integral Hilbert spaces ([18]-[20]) that the function $x\mapsto d(x)$ is ν-measurable. If $d(x)$ is constant (including ∞), then every $\mathscr{K}_x$ is isomorphic to a fixed Hilbert space $\mathscr{H}_0$, and then $\tilde{\mathscr{H}}\cong L^2(X, \nu; \mathscr{H}_0)$. This will happen in most practical situations, including those described in Sections 4 and 6 below. On the other hand, it is easy to cook up (rather artificial) examples to the contrary, for instance, by taking $d(x)$ to be piecewise constant.

It is the space $\tilde{\mathscr{H}}$ in which we shall identify a subset of vector-valued functions of $x\in X$ (with values in $\mathscr{K}_x$) which will then be made into a Hilbert space, isometrically isomorphic to $\mathscr{H}$. Notice that, as a consequence of Lemma 3.1, while $\tilde{\mathscr{H}}$ could depend on $\mathscr{M}$, it does not depend on the choice of the basis $\{\phi_n\}$.

We first define a linear map $W_K : \mathscr{H}\to\tilde{\mathscr{H}}$ in the obvious way:

$$(W_K\phi)(x)=E(x)\phi, \qquad \phi\in\mathscr{H} \qquad (3.4)$$

The reason for the subscript K will soon become apparent. W_K is bounded, and in fact, since, by the definition of the norm in $\tilde{\mathcal{H}}$,

$$\| W_K \phi \|_{\tilde{\mathcal{H}}}^2 = \int_X (E(x)\phi \mid E(x)\phi)_x \, d\nu(x) = \int_X \langle \phi \mid F(x)\phi \rangle_{\mathcal{H}} \, d\nu(x),$$

by (2.15), (2.6) and (2.5), we see that

$$\| W_K \phi \|_{\tilde{\mathcal{H}}}^2 = \| A^{1/2} \phi \|_{\mathcal{H}}^2 \tag{3.5}$$

in view of (2.16). Thus,

$$\| W_K \| = \| A^{1/2} \|. \tag{3.6}$$

On the range of W_K in $\tilde{\mathcal{H}}$ we can define the inverse map $W_K^{-1} : \mathrm{Ran}(W_K) \to \mathcal{H}$, since by (3.5) and the strict positivity of A, W_K is injective.

It was assumed, for the reproducing triple $\{\mathcal{H}, F, A\}$, that A^{-1} exists as a closed, possibly unbounded, but densely defined positive operator. Since its inverse A is bounded, the spectrum of A^{-1} is bounded away from zero. Let $\mathcal{D}(A^{-1})$ be the domain of A^{-1}, so that $\mathcal{D}(A^{-1}) \subset \mathcal{H}$ densely, and consider its image $W_K[\mathcal{D}(A^{-1})] \subset \mathrm{Ran}(W_K)$ in $\tilde{\mathcal{H}}$. On $W_K[\mathcal{D}(A^{-1})]$ define the operator

$$\mathring{A}_K^{-1} = W_K A^{-1} W_K^{-1} \tag{3.7}$$

and on $\mathrm{Ran}(W_K)$ define

$$\mathring{A}_K = W_K A W_K^{-1}. \tag{3.8}$$

LEMMA 3.2. – *In the Hilbert space* $\overline{\mathrm{Ran}(W_K)}$ (*closure in the* $\tilde{\mathcal{H}}$-*norm*), $\mathring{A}_K$ *is a bounded positive operator and* $\mathring{A}_K^{-1}$ *is a positive essentially self-adjoint operator which is the inverse of* $\mathring{A}_K$ (*on the appropriate domains*). □

(*See* proof in the Appendix.)

We denote by A_K the closure of $\mathring{A}_K$ and by A_K^{-1} the self-adjoint extension of $\mathring{A}_K^{-1}$; its domain $\mathcal{D}(A_K^{-1})$ is a dense subspace of $\overline{\mathrm{Ran}(W_K)}$.

On $W_K[\mathcal{D}(A^{-1})]$ let us use the positive operator A_K^{-1} to define the new scalar product $\langle . | . \rangle_K$ and the associated norm $\| . \|_K$ as:

$$\left.\begin{aligned} \langle \Phi \mid \Psi \rangle_K &= \langle \Phi \mid A_K^{-1} \Psi \rangle_{\tilde{\mathcal{H}}}, \\ \| \Phi \|_K^2 &= \langle \Phi \mid \Phi \rangle_K. \end{aligned}\right\} \tag{3.9}$$

Let

$$\mathcal{H}_K = \overline{W_K[\mathcal{D}(A^{-1})]}^K \tag{3.10}$$

be the completion of $W_K[\mathcal{D}(A^{-1})]$ in this norm. Then $\mathcal{H}_K$ is a Hilbert space, and it will turn out that as a set it is contained in $\tilde{\mathcal{H}}$. Indeed let us compute the norm of W_K as a map from $\mathscr{D}(A^{-1})$ into $\mathscr{H}_K$. For any $\phi \in \mathscr{D}(A^{-1})$,

$$\begin{aligned} \| W_K \phi \|_K^2 &= \langle W_K \phi \mid A_K^{-1} W_K \phi \rangle_{\tilde{\mathscr{H}}} \\ &= \langle \phi \mid W_K^* A_K^{-1} W_K \phi \rangle_{\tilde{\mathscr{H}}} \\ &= \| \phi \|_{\mathscr{H}}^2 \end{aligned} \tag{3.11}$$

by (A.8) and (3.7). Thus, as a mapping from $\mathscr{D}(A^{-1}) \subset \mathscr{H}$ to $\mathscr{H}_K$, W_K is an isometry (and hence has norm 1). We can therefore extend W_K by continuity to the whole of $\mathscr{H}$ as a unitary map, $W_K : \mathscr{H} \to \mathscr{H}_K$. This also means that

$$\mathscr{H}_K = \operatorname{Ran}(W_K) \subset \tilde{\mathscr{H}}. \tag{3.12}$$

Thus we have proved:

THEOREM 3.3. – *The range of* W_K *is complete in the* $\|.\|_K$ *norm, hence it is a Hilbert space, denoted* $\mathscr{H}_K$, *and* $W_K : \mathscr{H} \to \mathscr{H}_K$ *is a unitary map.* □

Furthermore, since by Lemma 3.2, A_K^{-1} is a positive self-adjoint operator with a bounded inverse A_K, its spectrum is bounded away from zero. Hence, the norm $\|.\|_K$ is equivalent to the graph norm of $A_K^{-1/2}$, which implies that $\mathscr{D}(A_K^{-1/2}) = \mathscr{H}_K$, and therefore, by (3.12):

$$\mathscr{D}(A_K^{-1/2}) = \mathscr{H}_K = \operatorname{Ran}(W_K) \subset \overline{\operatorname{Ran}(W_K)} \subset \tilde{\mathscr{H}}. \tag{3.13}$$

Note also that now, as a map from $\mathscr{H}_K$ to $\mathscr{H}$, W_K^{-1} is unitary, hence coincides with the adjoint of $W_K : \mathscr{H} \to \mathscr{H}_K$. Moreover, the closures A_K and A_K^{-1} simply become unitary images of A and A^{-1}, so that in particular,

$$\| A_K \|_K = \| A \|_{\mathscr{H}} \tag{3.14}$$

[compare with (A.10)].

Now, the elements in $\operatorname{Ran}(W_K)$, being vectors in $\tilde{\mathscr{H}}$, are equivalence classes (modulo sets of ν-measure zero) of vector valued functions Φ, of $x \in X$, with $\Phi(x) \in \mathscr{K}_x$ for ν-almost all x. However, in each equivalence class [Φ] we can choose a function Φ_K, such that $\Phi_K(x) \in \mathscr{K}_x$, $\forall x \in X$, and this choice, $[\Phi] \to \Phi_K$ can be made linearly for all vectors in $\operatorname{Ran}(W_K)$. Indeed, let us define the *evaluation map* $E_K(x) : \operatorname{Ran}(W_K) \to \mathscr{K}_x$ as follows:

$$E_K(x) = E(x) \circ W_K^{-1}. \tag{3.15}$$

Then, in any equivalence class [Φ], we can choose the vector valued function Φ_K for which

$$\Phi_K(x) = E_K(x)[\Phi]. \tag{3.16}$$

We shall always assume this to have been done and look upon $\operatorname{Ran}(W_K)$ as a space of $\mathscr{K}_x$-valued functions of x. The action of the evaluation map then very simply becomes:

$$E_K(x)\Phi = \Phi_K(x) \tag{3.17}$$

which justifies its name. Note that $E_K(x)$ is linear, but may not be bounded in the $\tilde{\mathscr{H}}$-norm of $\mathrm{Ran}(W_K)$. However, being the composition of the bounded map $E(x)$ with the closed map W_K^{-1}, it is certainly closed.

Thus the crucial point to note in Theorem 3.3 is that it achieves a unitary map of the original Hilbert space $\mathscr{H}$ onto a space $\mathscr{H}_K$ which consists of vector valued *functions*. We now that $\mathscr{H}_K$ is a *reproducing kernel Hilbert space* [21]. First note that the images of $F(x)$ and $a(\Delta)$ [*see* (2.3)] in $\mathscr{H}_K$ under W_K are:

$$F_K(x)=W_K F(x) W_K^{-1}=E_K(x)^* E_K(x), \tag{3.18}$$

$$a_K(\Delta)=W_K a(\Delta) W_K^{-1}=\int_\Delta E_K(x)^* E_K(x)\, d\nu(x), \tag{3.19}$$

where $E_K(x)$ is the evaluation map in (3.15), but now considered as a linear map $E_K(x):\mathscr{H}_K\to\mathscr{K}_x$, and $E_K(x)^*:\mathscr{K}_x\to\mathscr{H}_K$ is its adjoint. Moreover, it follows from (3.18) and (3.15)

$$\|E_K(x)\|=\|E_K(x)^*\|=\|E(x)\| \\ =\|E(x)^*\|=\|F(x)\|^{1/2}=\|F_K(x)\|^{1/2}. \tag{3.20}$$

Equations (3.18) and (3.19) shoulds be compared with (2.15) and (2.16) respectively. Also, (3.19) and the definition of W_K imply that

$$a_K(X)=A_K. \tag{3.21}$$

Thus, $\{\mathscr{H}_K, F_K, A_K\}$ is also a reproducing triple – one that is unitarily equivalent to the original reproducing triple $\{\mathscr{H}, F, A\}$ in the sense that there exists a unitary map $W_K:\mathscr{H}\to\mathscr{H}_K$ for which

$$\left.\begin{aligned} W_K F(x) W_K^{-1}&=F_K(x), \qquad \forall x\in X,\\ W_K A W_K^{-1}&=A_K. \end{aligned}\right\} \tag{3.22}$$

To construct a reproducing kernel on $\mathscr{H}_K$, define a linear operator $K(x, y):\mathscr{K}_y\to\mathscr{K}_x$ by

$$K(x, y)=E_K(x) A_K^{-1} E_K(y)^* \tag{3.23}$$

for each pair $(w, y)\in X\times X$. We first note that $K(x, y)$ is well defined, whenever $\eta_x^i\in\mathscr{D}(A^{-1})$, for all i and all $x\in X$. Otherwise it is defined under integration w.r. to y over elements in $\mathscr{H}_K$. However, even when $\eta_x^i\in\mathscr{D}(A^{-1})$, $K(x, y)$ could be unbounded, but in all the cases that we shall consider, the spaces $\mathscr{K}_x$, $x\in X$, are finite dimensional and then $K(x, y)$ is necessarily bounded. Next we see, from the way in which $E_K(x)$ is defined [*see* (3.15)-(3.17)], that for any measurable vector field $x\mapsto v(x)\in\mathscr{K}_x$ [such as $x\mapsto\psi_K(x)$, for $\psi\in\mathscr{H}_K$], $(v(x)\,|\,K(x, y)\,v(y))_x$ is a measurable function on $X\times X$.

Furthermore, if we set

$$\mathbb{P}_K(x)=W_K\,\mathbb{P}(x)\,W_K^{-1}, \tag{3.24}$$

then the following relation is easily verified:

$$K(x, y)|_{\mathcal{N}_y^\perp} = \mathbb{P}_K(x) A_K^{-1} F_K(y)|_{\mathcal{N}_y^\perp}. \tag{3.25}$$

Summarizing, we have the proposition:

PROPOSITION 3.4. — *Given a reproducing triple* $\{\mathcal{H}, F, A\}$, *one can canonically associate to it a unitarily equivalent reproducing triple* $\{\mathcal{H}_K, F_K, A_K\}$, *where the Hilbert space* $\mathcal{H}_K$ *carries a reproducing kernel* K; *when* $\eta_x^i \in \mathcal{D}(A^{-1})$, $\forall i$, $\forall x \in X$, *the latter is given by linear operators* $K(x, y) : \mathcal{H}_y \to \mathcal{H}_x$, *satisfying:*

(i) $K(x, x) > 0$,

i.e. $\forall x \in X$, $K(x, x)$ *is a strictly positive operator;* (3.26)

(ii) $K(x, y)^* = K(y, x)$,

where $K(x, y)^* : \mathcal{H}_x \to \mathcal{H}_y$ *is the adjoint of* $K(x, y)$; (3.27)

(iii) *for any* $v(x) \in \mathcal{H}_x$ and $v(y) \in \mathcal{H}_y$,

$$\int_X (v(x) | K(x, z) K(z, y) v(y))_x \, d\nu(z) = (v(x) | K(x, y) v(y))_x. \tag{3.28}$$

Condition (iii) is satisfied even when $\eta_x^i \notin \mathcal{D}(A^{-1})$. □

Equation (3.28) is equivalent to the relation:

$$\int_X K(x, y) \Phi_K(y) \, d\nu(y) = \Phi_K(x), \qquad \forall \Phi \in \mathcal{H}_K, \tag{3.29}$$

which is the reproducing property. This also justifies the subscript K on all quantities W_K, $E_K(x)$, $F_K(x)$, etc. related to $\mathcal{H}_K$ that we have been using all along, as well as the term "reproducing triple" for $\{\mathcal{H}, F, A\}$.

Finally, we write for the image of the OFS $\mathfrak{S}$ [*see* (2.9)] in $\mathcal{H}_K$:

$$\mathfrak{S}_K = W_K \mathfrak{S} = \{\xi_x^i = W_K \eta_x^i \,|\, \eta_x^i \in \mathfrak{S}\} \subset \mathcal{H}_K, \tag{3.30}$$

so that $\mathfrak{S}_K$ is an OFS for $\mathcal{H}_K$. Also, by (2.8) and (3.15)-(3.16),

$$\left.\begin{aligned} \xi_x^i &= F_K(x) v_i(x) \\ \xi_x^i(y) &= \mathbb{P}_K(y) F_K(x) v_i(x) \end{aligned}\right\} \tag{3.31}$$

while, by (2.10) and (2.11),

$$F_K(x) = \sum_{i=1}^{d(x)} |\xi_x^i\rangle\langle\xi_x^i|, \tag{3.32a}$$

$$\int_X \sum_{i=1}^{d(x)} |\xi_x^i\rangle\langle\xi_x^i| \, d\nu(x) = A_K, \tag{3.32b}$$

both the sum and the integral being defined weakly in $\mathcal{H}_K$.

4. THE CASE OF FINITE, CONSTANT RANK

In this section we specialize the above results to the case where, for every $x \in X$, the operator $F(x)$, in the reproducing triple $\{\mathscr{H}, F, A\}$, has the *finite constant rank n*. This implies that the direct integral $\tilde{\mathscr{H}}$ is independent of the particular set $\mathscr{M}$ in (3.2). Then each $\mathscr{N}_x^{\perp}$ is an n-dimensional subspace of $\mathscr{H}$, and for each $x \in X$, there exists an orthonormal basis $\{u_i(x), i=1, \ldots, n\}$ in $\mathscr{N}_x^{\perp}$ such that

$$\langle u_i(x) \,|\, u_j(x) \rangle_{\mathscr{H}} = \delta_{ij}, \qquad i, j = 1, 2, \ldots, n \tag{4.1}$$

$$\left.\begin{aligned} F(x) &= \sum_{i=1}^{n} \lambda_i(x) \,|\, u_i(x) \rangle \langle u_i(x) \,|, \\ \lambda_i(x) &> 0, \qquad i = 1, 2, \ldots, n. \end{aligned}\right\} \tag{4.2}$$

The corresponding o. n. basis $\{v_i(x), i=1, \ldots, n\}$ for $\mathscr{K}_x$ (which is also clearly n-dimensional) is then [*see* (2.7)]:

$$v_i(x) = \lambda_i(x)^{-1/2} u_i(x), \tag{4.3}$$

and the elements of the OFS $\mathfrak{S}$ are given by [*see* (2.8), (2.9)]

$$\eta_x^i = \lambda_i(x)^{1/2} u_i(x). \tag{4.4}$$

The weak measurability of $x \mapsto F(x)$ then implies that, for each i, $x \mapsto v_i(x)$ is a measurable field of vectors in $\prod_{x \in X} \mathscr{K}_{x'}$. Hence we can define a measurable field of unitary maps $x \mapsto W(x)$, where $W(x) : \mathscr{K}_x \to \mathbb{C}^n$ is given by

$$[W(x)\, v(x)]_i = (v_i(x) \,|\, v(x))_x, \qquad i = 1, 2, \ldots, n, \tag{4.5}$$

$\forall\, v(x) \in \mathscr{K}_x$. Using these, we can map the direct integral Hilbert space $\tilde{\mathscr{H}}$ [*see* (3.3)] unitarily onto $L^2(X, \nu; \mathbb{C}^n)$, the Hilbert space of all $\mathbb{C}^n$-valued measurable functions on X which are square integrable w. r. t. ν. Indeed, denoting this map by $W : \tilde{\mathscr{H}} \to L^2(X, \nu; \mathbb{C}^n)$, we get

$$(W \tilde{\Phi})(x) = W(x) \tilde{\Phi}(x), \tag{4.6}$$

so that, componentwise,

$$\begin{aligned} (W\tilde{\Phi})_i(x) &= (v_i(x) \,|\, \tilde{\Phi}(x))_x \\ &= \langle \eta_x^i \,|\, \tilde{\Phi}(x) \rangle_{\mathscr{H}'} \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{4.7}$$

We small denote the image of $\mathscr{K}_K$ in $L^2(X, \nu; \mathscr{C}^n)$ by $\hat{\mathscr{H}}_K$:

$$W[\mathscr{K}_K] = \hat{\mathscr{H}}_K \subset L^2(X, \nu; \mathbb{C}^n), \tag{4.8}$$

and by $\hat{W}_K$ the corresponding unitary map $\hat{W}_K : \mathscr{H} \to \hat{\mathscr{H}}_K$:

$$\hat{W}_K = W|_{\mathscr{H}_K} \circ W_K. \tag{4.9}$$

Thus, $\forall\, \phi \in \mathscr{H}$,

$$(\hat{W}_K \phi)_i(x) = \langle \eta_x^i \,|\, \phi \rangle. \tag{4.10}$$

Similarly, denoting the images of $F(x)$ and A in $\hat{\mathcal{H}}_K$ by $\hat{F}_K(x)$ and $\hat{A}_K$, respectively, we see that $\{\hat{\mathcal{H}}_K, \hat{F}_K, \hat{A}_K\}$ is a reproducing triple. Moreover $\hat{\mathcal{H}}_K$ is also a reproducing kernel Hilbert space, with kernel

$$\hat{K}(x, y) : \mathbb{C}^n \to \mathbb{C}^n \tag{4.11}$$

given in terms of its matrix elements by

$$\hat{K}_{ij}(x, y) = \langle \eta_x^i | A^{-1} \eta_y^j \rangle_{\mathcal{H}} \tag{4.12}$$

Again $\hat{K}_{ij}(x, y)$ is well defined if we interpret the right-hand side of (4.12) as a bilinear form on $\mathfrak{S} \times \mathfrak{S}$:

$$\langle \eta_x^i | A^{-1} \eta_y^j \rangle_{\mathcal{H}} = (v_i(x) | E_K(x) A_K^{-1} E_K(y)^* v_j(y))_x \tag{4.13}$$

[*see* the discussion following (3.23)].

We then have, $\forall x, y \in X$ and $i, j = 1, 2, \ldots, n$ [*see* (3.26)-(3.29)]:

(i) $\hat{K}_{ii}(x, x) > 0;$ (4.14)

(ii) $\hat{K}_{ij}(x, y) = \overline{\hat{K}_{ji}(y, x)};$ (4.15)

(iii) $\sum_{k=1}^{n} \int_X \hat{K}_{ik}(x, z) \hat{K}_{kj}(z, y) \, d\nu(z) = \hat{K}_{ij}(x, y).$ (4.16)

Also, $\forall \hat{\Psi} \in \hat{\mathcal{H}}_K$,

$$\sum_{j=1}^{n} \int \hat{K}_{ij}(x, y) \hat{\Psi}_{K, j}(y) \, d\nu(y) = \hat{\Psi}_{K, i}(x). \tag{4.17}$$

Here, as in Section 3, we have defined the evaluation map $\hat{E}_K(x) : \hat{\mathcal{H}}_K \to \mathbb{C}^n$:

$$\hat{E}_K(x) = W(x) \circ E_K(x), \tag{4.18}$$

so that

$$(\hat{E}_K(x) \hat{\Psi})_i = \hat{\Psi}_{K, i}(x) \tag{4.19}$$

for all $\hat{\Psi} \in \hat{\mathcal{H}}_K$ and $i = 1, 2, \ldots, n$. Also,

$$\hat{F}_K(x) = \hat{E}_K(x)^* \hat{E}_K(x), \tag{4.20}$$

$$\hat{K}(x, y) = \hat{E}_K(x) \hat{A}_K^{-1} \hat{E}_K(y)^*. \tag{4.21}$$

etc.

We collect together, in the Appendix, some explicit expressions for the various operators $\hat{W}_K$, $\hat{W}_K^{-1}$, $\hat{E}(x)$, etc., as well as displaying the relationships between the different maps, isometries, etc., diagrammatically.

For a reproducing triple with constant, finite rank $(=n)$ density $F(x)$, we shall henceforth adopt the notation $\{\mathcal{H}, F, A\}_n$.

Clearly the notion of *equivalence* introduced in Section 2 applies fully in the present context. Let $\{\mathcal{H}, F, A\}_n$ and $\{\mathcal{H}, F', A'\}_{n'}$ be two such reproducing triples, with corresponding OFS $\mathfrak{S} = \{\eta_x^i, i = 1, \ldots, n\}$, resp. $\mathfrak{S}' = \{\eta_x'^j, i = 1, \ldots, n'\}$. As shown in Section 2, $n' \leq n$ implies that $\mathfrak{S}' \prec \mathfrak{S}$.

We may choose the intertwining operator $T(x)$ as in (2.28), so that $T(x) = \mathbb{P}'(x) T(x) \mathbb{P}(x)$ is of rank n'.

In particular, for $n' = n$ the operator $\mathbb{P}'(x) T(x) \mathbb{P}(x)$ is, for all $x \in X$, an operator of rank n, *invertible* on its range. With the choice (2.28) for $T(x)$ and the corresponding one for $T'(x)$, one gets $T(x) = \mathbb{P}'(x) T(x) \mathbb{P}(x)$, $T'(x) = \mathbb{P}(x) T'(x) \mathbb{P}'(x) = T(x)^{-1}$ on $\mathbb{P}'(x)\mathscr{H}$, an thus the relations (2.17), (2.21) become:

$$\left.\begin{aligned} F'(x) &= T(x) F(x) T(x)^*, \\ \eta_x'^i &= T(x)\,\eta_x^i, \\ \eta_x^i &= T(x)^{-1}\,\eta_x'^i. \end{aligned}\right\} \tag{4.22}$$

Thus we may state:

PROPOSITION 4.1. – *Let* $\{\mathscr{H}, F, A\}_n$ *and* $\{\mathscr{H}, F', A'\}_n$ *be two reproducing triples on* $\mathscr{H}$, (X, ν) *with the same constant, finite rank n. Then the two triples are equivalent, and there exists a measurable family of rank n, invertible operators* $\{T(x)\}$, *that give the equivalence between the corresponding* OFS $\{\eta_x^i, i = 1, \ldots, n\}$, *resp.* $\{\eta_x'^j, i = 1, \ldots, n\}$:

$$\eta_x'^i = T(x)\,\eta_x^i, \qquad \eta_x^i = T(x)^{-1}\,\eta_x'^i \quad (x \in X). \quad \Box$$

5. FRAMES

Especially interesting is the case where $F(x)$ has finite rank n, $\forall x \in X$ and *both* the operators A and A^{-1} are bounded. Then we call the triple $\{\mathscr{H}, F, A\}_n$ a *frame*, and the basic relation (2.11) reads:

$$\int_X \sum_{i=1}^{n} |\eta_x^i\rangle\langle\eta_x^i|\, d\nu(x) = A, \tag{5.1}$$

Actually, in the literature (*see*, for example, [10]-[13]), the term "frame" is only used when the space X is discrete. If we take $X = \{1, 2, \ldots\}$ and ν the counting measure, Equation (5.1) becomes

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n} |\eta_k^i\rangle\langle\eta_k^i| = A, \tag{5.2}$$

and this is indeed the very statement that $\{\eta_k^i, i = 1, \ldots, n, k = 1, 2, \ldots\}$ is a frame in the usual sense. Thus we have here a genuine generalization of the familiar concept of a frame which plays such an important role in wavelet analysis ([11]-[14]).

Let us denoted by $\sigma(A)$ the spectrum of A, and by $m(A)$, $M(A)$ the bounds of $\sigma(A)$:

$$m(A) = \inf_{\|\phi\|=1} \langle\phi|A\phi\rangle, \qquad M(A) = \sup_{\|\phi\|=1} \langle\phi|A\phi\rangle, \tag{5.3}$$

so that

$$\sigma(A) \subset [m(A), M(A)] \tag{5.4}$$

and both $m(A)$ and $M(A)$ belong to $\sigma(A)$. Similarly,

$$\sigma(A^{-1}) \subset [M(A)^{-1}, m(A)^{-1}] \tag{5.5}$$

and $M(A)^{-1}, m(A)^{-1} \in \sigma(A^{-1})$.

With these notations, the relation (5.1) reads now, $\forall \phi \in \mathscr{H}$:

$$m(A) \|\phi\|^2 \leq \sum_{i=1}^{n} \int_X |\langle \eta_x^i | \phi \rangle|^2 \, d\nu(x) \leq M(A) \|\phi\|^2, \tag{5.6}$$

i. e. $m(A)$ and $M(A)$ are the familiar *frame bounds* ([10], [11]).

Clearly, the frame is characterized by the family of measurable maps $\eta^i : x \mapsto \eta_x^i \in \mathscr{H}$ $(i=1, 2, \ldots, n)$; hence we shall often denote it by $\mathscr{F}\{\eta^i, A, n\}$.

Given a frame $\mathscr{F}\{\eta^i, A, n\}$, the *dual frame* is $\mathscr{F}'\{\eta'^i, A', n\}$, where $\eta'^i = A^{-1} \eta^i$ $(i=1, 2, \ldots, n)$ and $A' = A^{-1}$. In other words, the dual frame is the reproducing triple $\{\mathscr{H}, F', A^{-1}\}_n$, where

$$F'(x) = A^{-1} F(x) A^{-1}, \qquad x \in X, \tag{5.7}$$

is also a positive bounded operator of rank n. Clearly, the frame bounds of $\mathscr{F}'$ are $M(A)^{-1}$ and $m(A)^{-1}$. In the discrete case, the dual frame is a crucial ingredient for various *reconstruction formulae* ([10]-[12]); in the present language, this means expanding elements of $\mathscr{H}$ in terms of those of $\mathscr{H}_K$, by a suitable approximation (truncated power expansion) of the operator W_K^{-1}.

The *width* of the frame $\mathscr{F}\{\eta^i, A, n\}$ is the number (called *snugness* in [12])

$$w(\mathscr{F}) = \frac{M(A) - m(A)}{M(A) + m(A)}. \tag{5.8}$$

Clearly, $0 \leq w(\mathscr{F}) < 1$, and $w(\mathscr{F})$ measures the spectral width of the operator A. The frame $\mathscr{F}$ is called *tight* if $w(\mathscr{F}) = 0$. In this case, $A = \lambda I$, with $\lambda > 0$. Notice also that a frame and its dual frame have the same width:

$$w(\mathscr{F}) = w(\mathscr{F}'). \tag{5.9}$$

The interest of this parameter is that, in the discrete case ([10], [11]), the reconstruction formula is an expansion in powers of $w(\mathscr{F})$. When the latter is small, convergence is very fast and the reconstruction series can be truncated after a small number of terms, often a single one. In concrete applications, this is the main ingredient of the efficiency of wavelet analysis for the resynthesis of signals (sounds, images). We will see in Paper II [16] that, in an explicit, non-discrete situation (Poincaré coherent states), the width has a physical meaning, being related to the nonrelativistic approximation.

In the case of a frame, the whole construction simplifies. Consider first the general discussion preceding Theorem 3.3. Since A and A^{-1} are bounded, $\mathscr{D}(A^{-1})=\mathscr{H}$, the K-norm $\|\cdot\|_K$ is equivalent to the norm induced by $\tilde{\mathscr{H}}$ on $\mathrm{Ran}(W_K)$. Therefore, by (3.13), $\mathscr{H}_K=\mathrm{Ran}(W_K)=\overline{\mathrm{Ran}(W_K)}$, that is, $\mathrm{Ran}(W_K)$ is a closed subspace of $\tilde{\mathscr{H}}$. This is the situation familar in the usual case, $A=I$ (corresponding to a *normalized* POV-measure [5], [17]). Next we go isometrically into $L^2(X, \nu; \mathbb{C}^n)$, via the map W, as in Section 4, and consider the bounded linear map $W_{\mathscr{F}}\equiv\hat{W}_K:\mathscr{H}\to L^2(X, \nu; \mathbb{C}^n)$ defined in (4.10):

$$(W_{\mathscr{F}}\phi)_i(x)=\langle \eta^i_x \mid \phi\rangle, \qquad \phi\in\mathscr{H} \tag{5.10}$$

On $W_{\mathscr{F}}[\mathscr{H}]$, the frame $\mathscr{F}$ defines the inner product

$$\langle \hat{\Phi}_K \mid \hat{\Psi}_K\rangle_{\mathscr{F}}=\langle W_{\mathscr{F}}^{-1}\hat{\Phi}_K \mid A^{-1}W_{\mathscr{F}}^{-1}\hat{\Psi}_K\rangle_{\mathscr{H}} \tag{5.11}$$

where $W_{\mathscr{F}}^{-1}$ is the inverse of $W_{\mathscr{F}}$ considered as a map from $\mathscr{H}$ to $W_{\mathscr{F}}[\mathscr{H}]$, and $\hat{\Phi}_K, \hat{\Psi}_K\in W_{\mathscr{F}}[\mathscr{H}]$. Since A^{-1} is positive and bounded away from zero, the norm $\|\,.\,\|_{\mathscr{F}}$ on $W_{\mathscr{F}}[\mathscr{H}]$ (which is the image of the K-norm on $\mathscr{H}_K$) is equivalent to the norm inherited from $L^2(X, \nu; \mathbb{C}^n)$. Hence $W_{\mathscr{F}}[\mathscr{H}]\equiv\hat{\mathscr{H}}_K$ is closed in both norms and is a reproducing kernel Hilbert space, with projection $\mathbb{P}_K$:

$$\mathbb{P}_K L^2(X, \nu; \mathbb{C}^n)=\hat{\mathscr{H}}_K \tag{5.12}$$

and reproducing kernel given as in (4.12):

$$\hat{K}_{ij}(x, y)=\langle \eta^i_x \mid A^{-1}\eta^j_y\rangle. \tag{5.13}$$

Thus we have, for each $\Psi\in L^2(X, \nu; \mathbb{C}^n)$:

$$(\mathbb{P}_K\Psi)_i(x)=\sum_{j=1}^{n}\int \hat{K}_{ij}(x, y)\,\Psi_j(y)\,d\nu(y), \tag{5.14}$$

and for every $\hat{\Psi}_K\in\hat{\mathscr{H}}_K$:

$$\hat{\Psi}_{K,i}(x)=\sum_{j=1}^{n}\int \hat{K}_{ij}(x, y)\,\hat{\Psi}_{K,j}(y)\,d\nu(y). \tag{5.15}$$

The reproducing kernel $\hat{K}$ in (5.13) will be called the *frame kernel* for the frame $\mathscr{F}$. Two frames $\mathscr{F}$ and $\bar{\mathscr{F}}$ will be said to be *kernel equivalent* if they have the same frame kernel $\hat{K}$. Denote by $[\mathscr{F}]_K$ this equivalence class. Clearly, if $\mathscr{F}$ and $\bar{\mathscr{F}}$ are kernel equivalent, then $W_{\mathscr{F}}[\mathscr{H}]=W_{\bar{\mathscr{F}}}[\mathscr{H}]$. The properties of this equivalence relation are summarized in the following theorem, the easy proof of which is left to the reader (*see* also [22]).

THEOREM 5.1. – *Two frames $\mathscr{F}\{\eta^i, A, n\}$ and $\bar{\mathscr{F}}\{\bar{\eta}^i, \bar{A}, n\}$ are kernel equivalent iff there exists a bounded operator $T\in\mathscr{L}(\mathscr{H})$, with bounded* *inverse* $T^{-1} \in \mathscr{L}(\mathscr{H})$, *for which*

$$\bar{\eta}_x^i = T\,\eta_x^i, \qquad \eta_x^i = T^{-1}\,\bar{\eta}_x^i \quad (i=1,2,\ldots,n, x \in X). \tag{5.16}$$

In particular, a frame $\mathscr{F}$ *and its dual frame* $\mathscr{F}'$ *are kernel equivalent. In each equivalence class* $[\mathscr{F}]_K$, *there exists, up to unitary equivalence, a unique tight frame* $\hat{\mathscr{F}}\{\hat{\eta}^i, \hat{A}, n\}$ *which is self-dual,* i. e. $\hat{\mathscr{F}}' = \hat{\mathscr{F}}$, *so that* $\hat{A} = \hat{A}^{-1} = I$. □

Comparing this result with Proposition 4.1, we see that kernel equivalence is stronger than equivalence: two frames of the same rank are always equivalent, with a measurable family of intertwining operators $\{T(x)\}$, but they are kernel equivalent iff the function $T: X \to \mathscr{L}(\mathscr{H})$ is constant.

6. APPLICATION TO COHERENT STATES AND OUTLOOK

The results obtained in Sections 2 and 3 apply potentially to many fields. Most promising, in our opinion, is the generalized notion of frame discussed in Section 4. However, the main domain of application, and actually the motivation of this work, is to the construction of coherent states. This theory will be developed at length in the accompanying paper [16], yet it is instructive to give here already its main features.

Let G be a locally compact group (quite often a Lie group) and U a strongly continuous unitary irreducible representation in a Hilbert space $\mathscr{H}$. As is well-known ([3], [4]), the properties of coherent states associated to U are linked to its square-integrability. In order to be general enough, we will take for X an arbitrary coset space G/H of X, assuming it carries a G-left invariant measure ν (actually it is enough to assume the measure ν to be *quasi-invariant*; this allows the formalism to be extended to certain infinite dimensional groups [26]). Let $\sigma: X \to G$ be a measurable section of G and F a positive operator on $\mathscr{H}$ with finite rank n. Then we define a positive operator valued function $F_\sigma: X \to \mathscr{L}(\mathscr{H})^+$, and the corresponding POV measure, by transporting the operator F covariantly under U, that is:

$$F_\sigma(x) = U(\sigma(x))\, F\, U(\sigma(x))^*. \tag{6.1}$$

We say that the representation U is *square integrable* mod (H, σ) if together with F, there exists a positive, bounded, invertible operator A_σ such that $\{\mathscr{H}, F_\sigma, A_\sigma\}_n$ is a reproducing triple. In particular we get

$$\int_X F_\sigma(x)\, d\nu(x) = A_\sigma. \tag{6.2}$$

To get coherent states, we simply diagonalize F:

$$F=\sum_{i=1}^{n} \lambda_i | u_i \rangle \langle u_i |, \qquad u_i \in \mathcal{H}, \quad \lambda_i > 0,$$
$$\langle u_i | u_j \rangle = \delta_{ij}, \qquad i,j=1,2,\ldots,n, \tag{6.3}$$

and define the vectors

$$\eta^i_{\sigma(x)} = \lambda_i^{1/2}\, U(\sigma(x))\, u_i. \tag{6.4a}$$
$$= U(\sigma(x))\, \eta^i, \tag{6.4b}$$

where we have defined the particular *admissible* vectors:

$$\eta^i = \lambda_i^{1/2} u_i. \tag{6.5}$$

Then the *family of coherent states*

$$\mathfrak{S}_\sigma = \{ \eta^i_{\sigma(x)} \,|\, i=1,2,\ldots,n, x \in X \} \tag{6.6}$$

is an overcomplete set of vectors, which has all the expected properties.

In the simplest case, X is G itself, ν the left invariant Haar measure and $F = |\eta\rangle\langle\eta|$ the projection on a single vector η. Then η is *admissible* or U is *square integrable* (in the usual sense) if

$$\int_G |\eta_g\rangle\langle\eta_g|\, dg = A, \tag{6.7}$$

where $\eta_g = U(g)\eta$ and A is a bounded positive operator (usually $A=I$), in other words, if the integral

$$I_\eta = \int_G |\langle U(g)\eta \,|\, \phi\rangle|^2\, dg \tag{6.8}$$

converges for every $\phi \in \mathcal{H}$. This is for instance the case for the representation of the affine group that leads to the theory of wavelets ([14], [23]). Notice that, if η_1 and η_2 are *both* admissible, then the more general integral

$$I_{\eta_1,\eta_2} = \int_G \langle \phi \,|\, U(g)\eta_1\rangle\langle U(g)\eta_2 \,|\, \psi\rangle\, dg \tag{6.9}$$

also converges, for all ϕ, $\psi \in \mathcal{H}$ (by the Schwarz inequality). In operator terms, this amounts replacing the positive operator F in (6.1) by the non-self-adjoint, rank one, operator $F_{12} = |\eta_1\rangle\langle\eta_2|$. Such operators are routinely used in signal analysis [24], where η_2 is the analyzing wavelet, η_1 the reconstruction wavelet, and the two need not coincide. In our context, integrals of the type (6.9) lead to generalized orthogonality relations, that will be discussed in [16] for the case of the Poincaré group.

The next case is that of Perelomov [3]. Given $\eta \in \mathcal{H}$, H is taken to be the subgroup of G that leaves η invariant up to a phase:

$$U(h)\eta = e^{i\alpha(h)}\eta. \tag{6.10}$$

The square integrability condition then reads

$$\int_{G/H} |\langle U(g)\eta | \phi \rangle|^2 \, d\nu(x) < \infty, \qquad \forall \phi \in \mathscr{H}, \tag{6.11}$$

where the integrand depends indeed only on the coset gH of g, by (6.10). The coherent states are again the vectors in the orbit of η under U:

$$\eta_g = U(g)\eta. \tag{6.12}$$

This is the situation described at length in the monograph of Perelomov [3], e. g. for the Weyl-Heisenberg group (canonical coherent states), the compact simple Lie groups [e. g. SU (2)] or the discrete series representations of noncompact simple Lie groups [e. g. SU (1.1)]. In the latter case, representations of the continuous series may also be used, but these are not square integrable and then the corresponding coherent states, still defined by (6.12), lack many of the useful properties of the previous cases.

The interesting aspect of the general theory developed here is that it applies to more general situations, such as that of the Poincaré group, or more generally semi-direct products $S \wedge V$ of a vector group V by a semisimple group S of automorphisms of V, as described by DeBièvre [25]. This general construction will be developed at length in the next paper [16], and then applied explicitly to the Poincaré group $\mathscr{P}_+^\uparrow(1, 1)$ in one space and one time dimensions. Thus we recover and put into perspective our earlier results ([1], [2]). The main point is that the whole construction now depends on the choice of the section σ. However, we shall get rid of this dependence by using the equivalence relation defined in Section 2 above: replacing one suitable section by another one will amount to going from one set of coherent states to an equivalent one. Furthermore, we will exhibit, in the Poincaré case, a class of sections that lead to non-trivial, *i. e.* non-tight, frames.

APPENDIX

We collect in this Appendix proofs of some results, theorems, etc. mentioned in the body of the text, as well as some related results.

A.1. Proof of Proposition 2.1

To establish (2.10) in the weak sense, let ϕ, $\psi \in \mathscr{H}$. Then

$$\sum_{i=1}^{d(x)} \langle \phi \mid \eta_x^i \rangle \langle \eta_x^i \mid \psi \rangle = \sum_{i=1}^{d(x)} \langle \phi \mid F(x) v_i(x) \rangle \langle F(x) v_i(x) \psi \rangle, \quad \text{by (2.11)},$$

$$= \sum_{i=1}^{d(x)} \langle \phi \mid F(x)^{1/2} [\mid F(x)^{1/2} v_i(x) \rangle \langle F(x)^{1/2} v_i(x) \mid] F(x)^{1/2} \psi \rangle, \quad \text{(A.1)}$$

since $F(x)$, being a positive operator, has a well-defined square root. Next, since $\{ v_i(x),\ i=1, \ldots, d(x) \}$ is an orthonormal basis for $\mathscr{K}_x$, by (2.7) it follows that $\{ F(x)^{1/2} v_i(x),\ i=1, \ldots, d(x) \}$ is an orthonormal basis for $\mathscr{N}_x^{\perp}$. Thus,

$$\sum_{i=1}^{d(x)} \mid F(x)^{1/2} v_i(x) \rangle \langle F(x)^{1/2} v_i(x) \mid = \mathbb{P}(x). \quad \text{(A.2)}$$

Inserting (A.2) into (A.1) and using (2.6),

$$\sum_{i=1}^{d(x)} \langle \phi \mid \eta_x^i \rangle \langle \eta_x^i \mid \psi \rangle$$

$$= \langle \phi \mid F(x)^{1/2} \mathbb{P}(x) F(x)^{1/2} \psi \rangle = \langle \phi \mid F(x) \psi \rangle, \quad \text{(A.3)}$$

which proves (2.10). The relation (2.11), in the weak sense, then follows from (A.3) and (2.1 *b*).

Finally, let $\phi \in \mathscr{H}$ be such that $\langle \eta_x^i \mid \phi \rangle = 0$, $\forall\, \eta_x^i \in \mathfrak{S}$. Then, by (2.11), for any $\psi \in \mathscr{H}$, $\langle \psi \mid A \phi \rangle = 0$. But since A is strictly positive, this implies that $\phi = 0$. □

A.2. Proof of Lemma 3.1

(i) The measurability of $x \mapsto (\Phi_m(x) \mid \Phi_n(x))_x$ follows from the weak measurability of $x \mapsto F(x)$. Indeed,

$$(\Phi_m(x) \mid \Phi_n(x))_x = \langle \phi_n \mid F(x) \phi_m \rangle_{\mathscr{H}} \quad \text{(A.4)}$$

by (3.1), (2.6) and (2.5), and the assertion is obvious.

(ii) Since the set $\{ \phi_n \}_{n=1}^{\infty}$ is total in $\mathscr{H}$, it is total in every subspace of $\mathscr{H}$. In particular, $\{ \mathbb{P}(x) \phi_n \}_{n=1}^{\infty} = \{ \Phi_n(x) \}_{n=1}^{\infty}$ is total in $\mathscr{N}_x^{\perp}$. But $\mathscr{N}_x^{\perp} \subset \mathscr{K}_x$ densely, hence we can find an orthonormal basis, $\{ v_i(x) \}_{i=1}^{d(x)}$, for $\mathscr{K}_x$ [having dimension $d(x)$], with each vector $v_i(x)$ being a finite linear combination of vectors from the set $\{ \Phi_n(x) \}_{n=1}^{\infty}$ (e.g., by a Gram-Schmidt orthogonalization process). Thus, for any $v_x \in \mathscr{K}_x$, $(v_x \mid \Phi_n(x))_x = 0$, $\forall\, n$, implies $(v_x \mid v_i(x))_x = 0$, $\forall\, i$ and therefore $v_x = 0$. Thus $\{ \Phi_n(x) \}_{n=1}^{\infty}$ is total in $\mathscr{K}_x$.

It then follows from (i) and (ii), and Lemma IV.8.10 in [18], that the set $\mathscr{M}$ defined in (3.2) is a subspace of ν-measurable vector fields. Suppose now that $\{\psi_n\}$ is another total set. Since each ψ_n is a linear combination of the ϕ_m's, and measurability of $\Phi_n(x)$ depends only on $\mathbb{P}(x)$, it follows that the functions $x \mapsto (\Phi(x) \mid \mathbb{P}(x)\psi_n)_x$ are also ν-measurable; thus $\{\psi_n\}$ generates the same set $\mathscr{M}$. □

A.3. Proof of Lemma 3.2

For any $\Phi \in \mathrm{Ran}(W_K)$,

$$\|\mathring{A}_K \Phi\|^2_{\tilde{\mathscr{H}}} = \int_X ((\mathring{A}_K \Phi)(x) \mid (\mathring{A}_K \Phi)(x))_x \, d\nu(x)$$

$$= \int_X (E(x) A W_K^{-1} \Phi \mid E(x) A W_K^{-1} \Phi)_x \, d\nu(x),$$

by (3.4) and (3.8),

$$= \int_X \langle A W_K^{-1} \Phi \mid F(x) A W_K^{-1} \Phi \rangle_{\mathscr{H}} \, d\nu(x)$$

by (2.15) and (2.6).
Thus

$$\|\mathring{A}_K \Phi\|^2_{\tilde{\mathscr{H}}} = \langle W_K^{-1} \Phi \mid A^3 W_K^{-1} \Phi \rangle_{\mathscr{H}}, \tag{A.5}$$

by (2.1 *b*). Now consider the adjoint $W_K^* : \tilde{\mathscr{H}} \to \mathscr{H}$ of W_K. Then $\|W_K^*\| = \|W_K\|$ and by (3.6),

$$\|W_K^*\| = \|A^{1/2}\|. \tag{A.6}$$

Moreover, $\forall \Phi \in \mathrm{Ran}(W_K)$ and $\psi \in \mathscr{H}$,

$$\langle W_K^* \Phi \mid \psi \rangle_{\mathscr{H}} = \langle \Phi \mid W_K \psi \rangle_{\tilde{\mathscr{H}}}$$

$$= \int_X (\Phi(x) \mid (W_K \psi)(x))_x \, d\nu(x)$$

$$= \int_X \langle \Phi(x) \mid F(x) \psi \rangle_{\mathscr{H}} \, d\nu(x), \quad \text{by (2.15), (2.12) and (3.4).}$$

Thus,

$$W_K^* \Phi = \int_X F(x) \Phi(x) \, d\nu(x). \tag{A.7}$$

Furthermore, $\Phi = W_K \phi$, for some $\phi \in \mathscr{H}$. Thus, writing $W_K \phi$ for Φ in (A.7) and using (3.4), (2.12) and (2.1 *b*) we find

$$W_K^* W_K = A, \tag{A.8}$$

and thus

$$W_K^* |_{\text{Ran } W_K} = AW_K^{-1}. \tag{A.9}$$

Putting (A.9) into (A.5), we get, $\forall \Phi \in \text{Ran}(W_K)$,

$$\begin{aligned} \| \mathring{A}_K \Phi \|_{\tilde{\mathscr{H}}}^2 &= \langle W_K^* \Phi | AW_K^* \Phi \rangle_{\mathscr{H}} \\ &\leqq \| A \| . \| W_K^* \Phi \|_{\mathscr{H}}^2 \\ &\leqq \| A \|^2 \| \Phi \|_{\tilde{\mathscr{H}}}^2, \quad \text{by (A.6)}. \end{aligned} \tag{A.10}$$

Hence $\mathring{A}_K$ is bounded on $\text{Ran}(W_K)$ and can thus be extended by continuity to a bounded self-adjoint operator A_K on the Hilbert space $\overline{\text{Ran}(W_K)}$ (closure in the $\tilde{\mathscr{H}}$-norm). We next show that A_K is a positive operator. Indeed, $\forall \Phi \in \text{Ran}(W_K)$,

$$\begin{aligned} \langle \Phi | A_K \Phi \rangle_{\tilde{\mathscr{H}}} &= \langle \Phi | W_K A W_K^{-1} \Phi \rangle_{\tilde{\mathscr{H}}}, \quad \text{by (3.8)}, \\ &= \langle \Phi | W_K W_K^* \Phi \rangle_{\tilde{\mathscr{H}}}, \quad \text{by (A.9)}, \\ &= \| W_K^* \Phi \|_{\mathscr{H}}^2 \geqq 0. \end{aligned} \tag{A.11}$$

Since $\text{Ran}(W_K)$ is dense in $\overline{\text{Ran}(W_K)}$, this implies that $A_K \geqq 0$. Incidentally, (A.9) also implies that

$$\mathring{A}_K = W_K W_K^* |_{\text{Ran}(W_K)}. \tag{A.12}$$

Next, $\mathring{A}_K^{-1}$ is defined on $W_K[\mathscr{D}(A^{-1})]$. Since $\mathscr{D}(A^{-1}) \subset \mathscr{H}$ densely and W_K is continuous [*see* (3.6)], it follows that $W_K[\mathscr{D}(A^{-1})] \subset \overline{\text{Ran}(W_K)}$ densely. Since W_K^{-1} is closed, its restriction to $W_K[\mathscr{D}(A^{-1})]$ is closable; W_K and A^{-1} being closed, it follows from (3.7) that $\mathring{A}_K^{-1}$ is also closable. The fact that A_K and $\mathring{A}_K^{-1}$ are inverses on appropriate domains follows from (3.7) and (3.8). Finally, the positivity of $\mathring{A}_K^{-1}$ is also clear since, as shown below, it is essentially self-adjoint, and since, for any $\Phi \in W_K[\mathscr{D}(A^{-1})] \subset \text{Ran}(W_K)$, one has

$$\begin{aligned} \langle \Phi | \mathring{A}_K^{-1} \Phi \rangle_{\tilde{\mathscr{H}}} &= \langle \Phi | W_K A^{-1} W_K^{-1} \Phi \rangle_{\tilde{\mathscr{H}}} \\ &= \int_X (E(x) W_K^{-1} \Phi | E(x) A^{-1} W_K^{-1} \Phi)_x \, d\nu(x), \quad \text{by (3.4)}, \\ &= \int_X \langle W_K^{-1} \Phi | F(x) A^{-1} W_K^{-1} \Phi \rangle_{\mathscr{H}} \, d\nu(x), \quad \text{by (2.15)}, \\ &= \| W_K^{-1} \Phi \|^2 \geqq 0, \quad \text{by (2.1}\,b). \end{aligned} \tag{A.13}$$

To show that $\mathring{A}_K^{-1}$ is an essentially self-adjoint operator, it is enough to note that it has deficiency indices (0, 0). Indeed, let $z \in \mathbb{C}$ with $\text{Im}(z) \neq 0$ and suppose there exists a $\Phi \in \text{Ran}(W_K)$ such that

$$\langle \Phi | (\mathring{A}_K^{-1} - z) \Psi \rangle_{\tilde{\mathscr{H}}} = 0, \tag{A.14}$$

for arbitrary $\Psi \in W_K[\mathscr{D}(A^{-1})] \subset \text{Ran}(W_K)$. Then if $\Phi = W_K \phi$, $\Psi = W_K \psi$, for $\phi, \psi \in \mathscr{H}$, using (3.7) and (A.8) we find that (A.14) implies

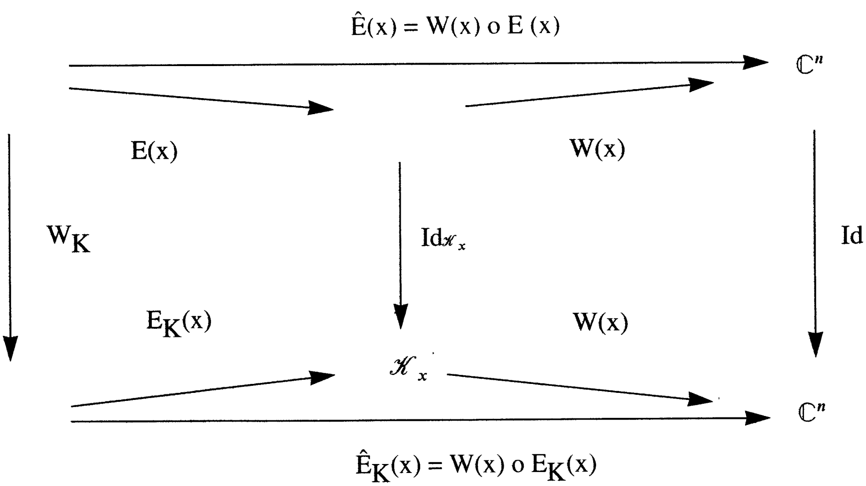
$$\langle \phi | \psi \rangle = z \langle \phi | A \psi \rangle_{\mathscr{H}} \qquad \forall \phi \in \mathscr{H}. \tag{A.15}$$

Thus, by the strict positivity of A, we must have $\phi=0$, and therefore $\Phi_K=0$, by (3.5). Since $\mathrm{Ran}(W_K)$ is dense in $\overline{\mathrm{Ran}(W_K)}$, this implies that $\mathring{A}_K^{-1}$ has deficiency indices (0, 0), and thus is essentially self-adjoint. □

A.4. Some explicit formulae

Among the different isometries and other maps, introduced in Section 3, the following relations hold (the symbol W always denotes a Hilbert space isometry, and Id implies the inclusion (*i.e.*, identity map) :

$$\begin{array}{ccccc} \mathscr{H} & \xrightarrow{W_K} & \mathscr{H}_K & \xrightarrow{\mathrm{Id}_{\mathscr{H}_K}} & \tilde{\mathscr{H}} = \int_X^{\oplus} \mathscr{K}_x \, d\nu(x) \\ \downarrow \mathrm{Id}_{\mathscr{H}} & & \downarrow W|_{\mathscr{H}_K} & & \downarrow W \\ \mathscr{H} & \xrightarrow[\hat{W}_K]{} & \hat{\mathscr{H}}_K & \xrightarrow[\mathrm{Id}_{\hat{\mathscr{H}}_K}]{} & L^2(X, \nu; \mathbb{C}^n) \end{array}$$



From these diagrams and the various expressions given in Sections 2 and 3, the following are easily computed:

$$W_K^{-1}\Phi = \int_X A^{-1} F(x) \Phi_K(x) \, d\nu(x), \qquad \forall \Phi \in \mathscr{H}_K, \tag{A.16}$$

the integral being defined weakly, and

$$\hat{W}_K^{-1}\hat{\Phi} = \sum_{j=1}^{n} \int_X \Phi_{K,j}(y) A^{-1} \eta_y^j \, d\nu(y), \qquad \forall \hat{\Phi} \in \hat{\mathscr{H}}_K, \tag{A.17}$$

again the integral being defined weakly

Denoting the image of the OFS $\mathfrak{S}$ in (2.9) under $\hat{W}_K$ by $\hat{\mathfrak{S}}_K$, we have [*see* also (3.30)]:

$$\mathfrak{S} = W_K^{-1}\,\mathfrak{S}_K = \hat{W}_K^{-1}\,\hat{\mathfrak{S}}_K. \tag{A.18}$$

For the elements in these sets we adopt the notation

$$\left.\begin{aligned} \xi_x^i &= W_K\,\eta_x^i \in \mathfrak{S}_K, \\ \hat{\xi}_x^i &= \hat{W}_K\,\eta_x^i \in \hat{\mathfrak{S}}_K, \qquad \forall\,\eta_x^i \in \mathfrak{S}. \end{aligned}\right\} \tag{A.19}$$

Furthermore, $\hat{\xi}_x^i \in \hat{\mathfrak{S}}_K$ has the components

$$(\hat{\xi}_x^i)_j(y) = \langle \eta_y^j \mid \eta_x^i \rangle. \tag{A.20}$$

In terms of these, the maps $E_K(x)^*$, $\hat{E}_K(x)^*$ and $\hat{E}(x)^*$ become [*see* (2.14)]:

$$E_K(x)^* = \sum_{i=1}^{n} |\xi_x^i\rangle (v_i(x)|, \tag{A.21}$$

$$\hat{E}_K(x)^* = \sum_{i=1}^{n} |\hat{\xi}_x^i\rangle \langle e_i|, \tag{A.22}$$

$$\hat{E}(x)^* = \sum_{i=1}^{n} |\eta_x^i\rangle \langle e_i|, \tag{A.23}$$

where $e_1, e_2, \ldots, e_n$ is the canonical basis in $\mathbb{C}^n$:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0), \qquad \text{etc.} \end{aligned} \tag{A.24}$$

If $\{\mathcal{H}, F, A\}$ and $\{\mathcal{H}, F', A'\}$ are two reproducing triples, with both $F(x)$ and $F'(x)$ having the same finite rank n, $\forall\, x \in X$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\hat{W}_K} & \hat{\mathcal{H}}_K \\ \downarrow \mathrm{Id}_{\mathcal{H}} & & \downarrow \hat{W}_{K'K} = \hat{W}_{K'} \circ \hat{W}_K^{-1} \\ \mathcal{H} & \xrightarrow[\hat{W}_{K'}]{} & \hat{\mathcal{H}}_{K'} \end{array}$$

where K, K′ denote the reproducing kernels for F and F′, respectively. Explicitly

$$\left.\begin{aligned} (\hat{W}_{K'K}\,\hat{\Phi})_i(x) &= \sum_{j=1}^{n} \int_X \langle \eta_x'^i \mid A^{-1}\,\eta_y^j \rangle\, \Phi_{K,j}(y)\, d\nu(y), \\ &\forall\, \hat{\Phi} \in \hat{\mathcal{H}}_K, \end{aligned}\right\} \tag{A.25}$$

$\eta_x'^i$ ($i = 1, 2, \ldots, n$; $x \in X$) being the OFS for $\{\mathcal{H}, F', A'\}$.

ACKNOWLEDGEMENTS

One of us (STA) is indebted to the Institut de Physique Théorique, Université Catholique de Louvain, where most of this work was completed.

He would like to thank the Institute for its hospitality. The authors also benefitted at various times from discussions with A. Grossmann. H. D. Doebner, A. Unterberger, P. Michor, H. Feichtinger, S. Andersson and M. Holschneider.

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(*Manuscript received June* 8, 1990;
revised version received March 3, 1991.)