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On the chaotic behavior and the non-integrability of the four vortices problem

by

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ABSTRACT. – The paper is based in a survey talk given by the author in 1990 about the vortex model. Some brief comments are made about the works of Ziglin and Khanin on the chaotic behavior, non-integrability and quasi-periodicity of the vortex motions in the case of four points vortices; some other ideas about a new approach to non-integrability are also presented.

RÉSUMÉ. – Cet article est basé sur un exposé de revue donné par l'auteur en 1990 sur le modèle de vortex. On commente brièvement les travaux de Ziglin et Khanin sur le comportement chaotique, la non-intégrabilité et la quasi-périodicité du modèle dans le cas de quatre vortices ponctuels. On présente aussi d'autres idées sur une nouvelle approche de la non intégrabilité.

1. THE VORTEX MODEL

For detailed presentations about the vortex model in fluid mechanics we refer the reader to Chorin and Marsden (1979) and to Marchioro and Pulvirenti (1983).

Choose constants $K_1, \ldots, K_N$ and initial positions

$$\mathbf{x}_1=(x_1, y_1), \ldots, \mathbf{x}_N=(x_N, y_N)$$

in the plane. The velocity $\mathbf{u}_j(\mathbf{x}, t)$ of $\mathbf{x}=(x, y)\in\mathbf{R}^2$ due to the jth vortex is given by

$$\mathbf{u}_j(\mathbf{x}, t)=\left(-\frac{K_j}{2\pi}\frac{(y-y_j)}{|\mathbf{x}-\mathbf{x}_j|^2}, \frac{K_j}{2\pi}\frac{(x-x_j)}{|\mathbf{x}-\mathbf{x}_j|^2}\right), \tag{1}$$

if we ignore the other vortices. When all the vortices are moving, they produce the velocity field $\mathbf{u}(\mathbf{x}, t)=\sum_{j=1}^{N}\mathbf{u}_j(\mathbf{x}, t)$ and since each vortex ought to move as it was carried by the net velocity field of the other vortices, then each $\mathbf{x}_j$, $j=1, \ldots, N$, moves according to the equations:

$$\left.\begin{aligned}\dot{x}_j&=-\frac{1}{2\pi}\sum_{i\neq j}\frac{K_i(y_j-y_i)}{r_{ij}^2}\\ \dot{y}_j&=\frac{1}{2\pi}\sum_{i\neq j}\frac{K_i(x_j-x_i)}{r_{ij}^2}, \qquad r_{ij}=|\mathbf{x}_i-\mathbf{x}_j|,\end{aligned}\right\} \tag{2}$$

or, equivalently, for $i, j=1, \ldots, N$:

$$\left.\begin{aligned}K_j\dot{x}_j&=\frac{\partial H}{\partial y_j}\\ K_j\dot{y}_j&=-\frac{\partial H}{\partial x_j}, \qquad H=-\frac{1}{4\pi}\sum_{i\neq j}K_iK_j\log|\mathbf{x}_i-\mathbf{x}_j|.\end{aligned}\right\} \tag{3}$$

The construction of the velocity field $\mathbf{u}(\mathbf{x}, t)$ produces formal solutions of the Euler's equation in $\mathbf{R}^2$ and have the property that the classical circulation theorems are satisfied (*see op. cit.* above).

Poincaré and Kirchoff observed that in a new system of coordinates of $\mathbf{R}^{2N}$, (x'_α, y'_α), given by

$$\left.\begin{aligned}x'_\alpha&=\sqrt{|K_\alpha|}\,x_\alpha\\ y'_\alpha&=\sqrt{|K_\alpha|}\,\mathrm{sign}(K_\alpha)\,y_\alpha, \qquad \alpha=1, \ldots, N,\end{aligned}\right\} \tag{4}$$

the system of ordinary differential equations (3) becomes a Hamiltonian system. The phase space is an open dense set of $\mathbf{R}^{2N}$ since $r_{ij}\neq 0$ (collisions of vortices are not allowed).

It is easy to see that, because of the symmetries of the Hamiltonian H, the system of equations (3) has the four first integrals:

$$I_1 = H, \qquad I_2 = \sum_{\alpha=1}^{N} K_\alpha x_\alpha,$$

$$I_3 = \sum_{\alpha=1}^{N} K_\alpha y_\alpha \quad \text{and} \quad I_4 = \sum_{\alpha=1}^{N} K_\alpha (x_\alpha^2 + y_\alpha^2).$$

For the Poisson bracket corresponding to the canonical symplectic 2-form of $\mathbf{R}^{2n}$: $\omega = \sum_{\alpha=1}^{N} dx'_\alpha \wedge dy'_\alpha$, one has $\{I_2^2 + I_3^2, I_4\} = 0$. In fact,

$$\{I_2^2 + I_3^2, I_4\} = \sum_{i=1}^{N} \left[\frac{\partial I_4}{\partial y'_i} \frac{\partial (I_2^2 + I_3^2)}{\partial x'_i} - \frac{\partial I_4}{\partial x'_i} \frac{\partial (I_2^2 + I_3^2)}{\partial y'_i} \right] = 0$$

[*see* Aref and Pomphrey (1982)]. The vortex systems, for $N=2$ and $N=3$, are then Liouville integrable. The existence of four independent integrals of motion suggest, at first sight, that the integrability of system (3) is possible in the case of $N=4$ vortices. But this is not, necessarily, the case because the integrals I_2, I_3 and I_4 do not commute, that is, their Poisson brackets do not vanish, since:

$$\{I_2, I_3\} = \sum_\alpha K_\alpha, \qquad \{I_2, I_4\} = 2I_3 \quad \text{and} \quad \{I_3, I_4\} = -2I_2.$$

The case of positive *intensities* ($K_\alpha > 0$, $\alpha = 1, \ldots, N$) implies that all solutions in $\mathbf{R}^{2N}$ are bounded (since $I_4 = \text{const.}$ is compact) and then defined for all time; in particular when $N=2$ or 3 the phase space has regions foliated by invariant tori, then all the solutions define quasi-periodic motions. Using carefully KAM theory, Khanin (1982) showed that in the phase space of any system with an arbitrary number of vortices there exists a set of initial conditions of positive measure for which the motions of vortices are quasi-periodic (*see* Khanin 1982).

Questions related to choatic behavior and non integrability of the four vortices problem will be considered in the next section.

In the paper the motions of vortices happen in an unbounded domain, that is, $\Gamma = \mathbf{R}^2$. The case of Γ bounded presents also many interesting questions; *see* for instance Dürr and Pulvirenti (1982), Marchioro and Pulvirenti (1983) and the survey paper Aref (1983).

2. THE PROBLEM OF FOUR VORTICES

Fix a system of coordinates in $\mathbf{R}^2$ and consider four vortices P_α of coordinates $\mathbf{x}_\alpha = (x_\alpha, y_\alpha)$, $\alpha = 1, \ldots, 4$. Assume $K_i = 1$ for $i = 1, 2, 3$ and

$K_4=\varepsilon>0$. The equations of motion are given by

$$\left.\begin{aligned} \dot{x}_i&=\frac{\partial H}{\partial y_i}=\frac{\partial H_0}{\partial y_i}+\varepsilon\frac{\partial H_1}{\partial y_i},\\ \dot{y}_i&=-\frac{\partial H}{\partial x_i}=-\frac{\partial H_0}{\partial x_i}-\varepsilon\frac{\partial H_1}{\partial x_i}, \qquad i=1, 2, 3,\\ \varepsilon\dot{x}_4&=\frac{\partial H}{\partial y_4}=\varepsilon\frac{\partial H_1}{\partial y_4}, \qquad \varepsilon\dot{y}_4=-\frac{\partial H}{\partial x_4}=-\varepsilon\frac{\partial H_1}{\partial x_4}, \end{aligned}\right\} \tag{5}$$

where

$$H=H_0+\varepsilon H_1, \qquad H_0=-\frac{1}{4\pi}(\log r_{12}^2+\log r_{23}^2+\log r_{13}^2),$$

$$H_1=-\frac{1}{4\pi}(\log r_{14}^2+\log r_{24}^2+\log r_{34}^2)$$

and

$$r_{t,s}=|\mathbf{x}_t-\mathbf{x}_s|, \qquad t, s=1, \ldots, 4.$$

Recall that $x_i'=x_i$, $y_i'=y_i$, $x_4'=\sqrt{\varepsilon}\,x_4$ and $y_4'=\sqrt{\varepsilon}\,y_4$ are canonical coordinates in $\mathbf{R}^8$ and the system (5) becomes a Hamiltonian system in the new coordinates; the symplectic canonical form is

$$\omega=\sum_{i=1}^{3} dx_i \wedge dy_i+dx_4' \wedge dy_4'.$$

The *motions for the* 3-*equal vortices* are given by

$$\dot{x}_i=\frac{\partial H_0}{\partial y_i}, \qquad \dot{y}_i=-\frac{\partial H_0}{\partial x_i}, \qquad i=1, 2, 3, \tag{6}$$

and the *restricted* 4-*vortices problem* corresponds to the motions of three vortices with equal unit intensities and the fourth vortex not influenciating the three first ones. It is, precisely, given by system (5) as $\varepsilon\to 0$, that is, the decoupled system:

$$\left.\begin{aligned} \dot{x}_i&=\frac{\partial H_0}{\partial y_i}, \qquad \dot{y}_i=-\frac{\partial H_0}{\partial x_i}, \qquad i=1, 2, 3,\\ \dot{x}_4&=\frac{\partial H_1}{\partial y_4}, \qquad \dot{y}_4=-\frac{\partial H_1}{\partial x_4}. \end{aligned}\right\} \tag{7}$$

Ziglin, in two papers [Ziglin (1980) and Khanin (1982) – appendix], considered the question of nonintegrability of a problem of four point vortices. He started with system (7), that is, with the restricted 4-vortices problem. Let a_i, $i=1, 2, 3$ be the sides of the triangle determined by the three unit vortices, and A_i, $i=1, 2, 3$ be the angles opposite to the corresponding sides. Then the relative problem of the three vortices has the following equations derived from (6) or $(7)_1$:

$$\left.\begin{aligned}\dot{a}_1&=\frac{1}{2\pi}\left(\frac{\sin A_3}{a_2}-\frac{\sin A_2}{a_3}\right)\\ \dot{a}_2&=\frac{1}{2\pi}\left(\frac{\sin A_1}{a_3}-\frac{\sin A_3}{a_1}\right)\\ \dot{a}_3&=\frac{1}{2\pi}\left(\frac{\sin A_2}{a_1}-\frac{\sin A_1}{a_2}\right),\end{aligned}\right\}\tag{8}$$

and admits the two independent first integrals: $a_1 a_2 a_3 = c_1^3$ and $a_1^2 + a_2^2 + a_3^2 = c_2^2$.

Substituting $a_3 = c_1^3/a_1 a_2$ into (8) he obtained the system $\dot{a} = F(a, c_2)$, $a = (a_1, a_2)$, which has a center $a_0 = (c_1, c_1)$, an equilateral triangle; after this, he took the periodic solutions close to this elliptical fixed point which are given by a one-parameter family of periodic functions; choosing properly a small parameter for this family, he substituted these periodic functions in equations $(7)_2$ of the fourth vortex. This way he obtained a time dependent Hamiltonian system:

$$\frac{d\xi}{d\tau}=\frac{\partial F}{\partial \eta},\qquad \frac{d\eta}{d\tau}=-\frac{\partial F}{\partial \xi},\tag{9}$$

where $F = F(\xi, \eta, \tau, \mu) = F_0(\xi, \eta) + \mu F_1(\xi, \eta, \tau, \mu) + \ldots,$. The unperturbed Hamiltonian system ($\mu = 0$) is defined by $F_0(\xi, \eta)$ and the corresponding phase portrait has a homoclinic fixed point. As usual [*see* Holmes (1980)], for $\mu \neq 0$ it is necessary to examine system (9) in the extended phase space $\{\xi, \eta, \tau(\text{mod}\, 2\pi)\}$ and consider the Poincaré-map of the plane $\{\tau(\text{mod}\ 2\pi)\} = 0$ to itself, given by the cilindrical phase-flow. In general, for $\mu \neq 0$, the homoclinic orbit of the hyperbolic fixed point of this map will split in two transversal separatrices. In this case the perturbed Poincaré-map presents a chaotic behavior [see Moser (1973) and Smale (1967)] due to the occurrence of a horseshoe; in particular, no domain containing the closure of the trajectory of that transverse homoclinic point admits an analytic first integral for the Poincaré map. Thus, the proof of the non integrability reduces to checking that the separatrices intersect transversally. In Ziglin (1980) the author reduced the proof to the non vanishing of an integral (*see* Melnikov, 1963) and he succed in showing this, after making the evaluation by computer. In Khanin (1982 – appendix), Ziglin argüed only using continuity and stated the non-integrability of a four-vortices problem of intensities (K_1, K_2, K_3, K_4) sufficiently close to the (1, 1, 1, 0) of the restricted case.

In Holmes and Marsden (1982) the authors say that "the problem treated by Ziglin possesses some difficulties that cannot be avoided without a careful study"; also, according to Aref [*see* Aref (1983)], Marsden in a private communication say that "the proof of Ziglin is incomplete for some reasons, and similar criticism applies to the extension of the proof in Khanin (1982)".

In a joint work, in progress, with V. Moauro, P. Negrini and M. S. A. C. Castilla, we decided to come back to the question of the non-integrability for the problem of 4-vortices. From now on I will present some rough ideas of our work.

First of all, let O be the origin of $\mathbf{R}^2$, M_1 be the center of mass of $P_i = O + \mathbf{x}_i$, $i=1, 2, 3$, with unitary masses, and M_0 be the center of mass of the four vortices, the last one $P_4 = O + \mathbf{x}_4$ with mass $\varepsilon > 0$, that is,

$$M_0 - O = \left(\frac{\sum_{i=1}^{3} x_i + \varepsilon x_4}{3+\varepsilon}, \frac{\sum_{i=1}^{3} y_i + \varepsilon y_4}{3+\varepsilon} \right) = \frac{1}{3+\varepsilon}(I_2, I_3)$$

and

$$M_1 - O = \frac{1}{3}\left(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i \right).$$

Let us introduce now a new system of coordinates $(\sqrt{\varepsilon}\, x_4, \sqrt{\varepsilon}\, y_4; \theta_1, \tilde{p}_1; \theta_2, \tilde{p}_2; \theta_3, \tilde{p}_3)$ by:

$$P_1 - P_2 = \alpha \sqrt{\tilde{p}_1}\, e^{i\theta_1}$$
$$P_3 - M_1 = \beta \sqrt{\tilde{p}_2}\, e^{i\theta_2}$$
$$P_4 - M_1 = \gamma \sqrt{\tilde{p}_3}\, e^{i\theta_3},$$

where α, β, γ will be determined in order that the new coordinates will be canonical.

From the definitions of M_1 and M_0 we have $(P_1 - M_1) + (P_2 - M_1) + (P_3 - M_1) = 0$ and $3(M_1 - M_0) + \varepsilon (P_4 - M_0) = 0$ that imply

$$P_1 - M_0 = \frac{1}{2}\left[(P_1 - P_2) - (P_3 - M_1) - \frac{2\varepsilon}{3+\varepsilon}(P_4 - M_1) \right]$$
$$P_2 - M_0 = -\frac{1}{2}\left[(P_1 - P_2) + \frac{2\varepsilon}{3+\varepsilon}(P_4 - M_1) + (P_3 - M_1) \right]$$
$$P_3 - M_0 = (P_3 - M_1) - \frac{\varepsilon}{3+\varepsilon}(P_4 - M_1).$$

In cartesian coordinates we have then:

$$x_1 = \frac{1}{3+\varepsilon} I_2 + \frac{1}{2}\alpha \sqrt{\tilde{p}_1} \cos\theta_1 - \frac{1}{2}\beta \sqrt{\tilde{p}_2} \cos\theta_2 - \frac{\gamma\varepsilon}{3+\varepsilon}\sqrt{\tilde{p}_3} \cos\theta_3$$
$$y_1 = \frac{1}{3+\varepsilon} I_3 + \frac{1}{2}\alpha \sqrt{\tilde{p}_1} \sin\theta_1 - \frac{1}{2}\beta \sqrt{\tilde{p}_2} \sin\theta_2 - \frac{\gamma\varepsilon}{3+\varepsilon}\sqrt{\tilde{p}_3} \sin\theta_3$$

$$x_2=\frac{1}{3+\varepsilon}I_2-\frac{1}{2}\alpha\sqrt{\tilde{p}_1}\cos\theta_1-\frac{1}{2}\beta\sqrt{\tilde{p}_2}\cos\theta_2-\frac{\gamma\varepsilon}{3+\varepsilon}\sqrt{\tilde{p}_3}\cos\theta_3$$

$$y_2=\frac{1}{3+\varepsilon}I_3-\frac{1}{2}\alpha\sqrt{\tilde{p}_1}\sin\theta_1-\frac{1}{2}\beta\sqrt{\tilde{p}_2}\sin\theta_2-\frac{\gamma\varepsilon}{3+\varepsilon}\sqrt{\tilde{p}_3}\sin\theta_3$$

$$x_3=\frac{1}{3+\varepsilon}I_2+\beta\sqrt{\tilde{p}_2}\cos\theta_2-\frac{\gamma\varepsilon}{3+\varepsilon}\sqrt{\tilde{p}_3}\cos\theta_3$$

$$y_3=\frac{1}{3+\varepsilon}I_3+\beta\sqrt{\tilde{p}_2}\sin\theta_2-\frac{\gamma\varepsilon}{3+\varepsilon}\sqrt{\tilde{p}_3}\sin\theta_3$$

$$x_4=\frac{1}{3+\varepsilon}I_2+\frac{3\gamma}{3+\varepsilon}\sqrt{\tilde{p}_3}\cos\theta_3$$

$$y_4=\frac{1}{3+\varepsilon}I_2+\frac{3\gamma}{3+\varepsilon}\sqrt{\tilde{p}_3}\sin\theta_3.$$

The preservation of the symplectic form is given by the equalities

$$\omega=\varepsilon\,dx_4\wedge dy_4+\sum_{i=1}^{3}dx_i\wedge dy_i=\varepsilon\,dx_4\wedge dy_4+\sum_{i=1}^{3}d\tilde{p}_i\wedge d\theta_i.$$

Making the origin O to be equal to the center of mass M_0 and fixing $\alpha=2$, $\beta=\frac{2\sqrt{3}}{3}$ and $\gamma=\sqrt{\frac{2(3+\varepsilon)}{3\varepsilon}}$, we see that the new coordinates are canonical.

Remark that $\tilde{p}_3=\frac{3\varepsilon}{2(3+\varepsilon)}|P_4-M_1|^2$, that is, $\tilde{p}_3=O(\varepsilon)$.

The distances between vortices are given by:

$$r_{12}^2=4\tilde{p}_1$$

$$r_{13}^2=\tilde{p}_1+3\tilde{p}_2-2\sqrt{3}\sqrt{\tilde{p}_1\tilde{p}_2}\cos(\theta_1-\theta_2)$$

$$r_{23}^2=\tilde{p}_1+3\tilde{p}_2+2\sqrt{3}\sqrt{\tilde{p}_1\tilde{p}_2}\cos(\theta_1-\theta_2)$$

$$r_{14}^2=\tilde{p}_1+\frac{1}{3}\tilde{p}_2+\frac{2}{3}\,\frac{(3+\varepsilon)}{\varepsilon}\tilde{p}_3-\frac{2}{\sqrt{3}}\sqrt{\tilde{p}_1\tilde{p}_2}\cos(\theta_1-\theta_2)$$
$$-2\sqrt{\frac{2}{3}}\sqrt{\frac{3+\varepsilon}{\varepsilon}}\sqrt{\tilde{p}_1\tilde{p}_3}\cos(\theta_1-\theta_3)$$
$$+\frac{2\sqrt{2}}{3}\sqrt{\frac{3+\varepsilon}{\varepsilon}}\sqrt{\tilde{p}_2\tilde{p}_3}\cos(\theta_2-\theta_3)$$

$$r_{24}^2=\tilde{p}_1+\frac{1}{3}\tilde{p}_2+\frac{2}{3}\frac{(3+\varepsilon)}{\varepsilon}\tilde{p}_3+\frac{2}{\sqrt{3}}\sqrt{\tilde{p}_1\tilde{p}_2}\cos(\theta_1-\theta_2)$$
$$+2\sqrt{\frac{2}{3}}\sqrt{\frac{3+\varepsilon}{\varepsilon}}\sqrt{\tilde{p}_1\tilde{p}_3}\cos(\theta_1-\theta_2)$$
$$+\frac{2\sqrt{2}}{3}\sqrt{\frac{3+\varepsilon}{\varepsilon}}\sqrt{\tilde{p}_2\tilde{p}_3}\cos(\theta_2-\theta_3)$$
$$r_{34}^2=\frac{4}{3}\tilde{p}_2+\frac{2}{3}\frac{(3+\varepsilon)}{\varepsilon}\tilde{p}_3-\frac{4\sqrt{2}}{3}\sqrt{\frac{3+\varepsilon}{\varepsilon}}\sqrt{\tilde{p}_2\tilde{p}_3}\cos(\theta_2-\theta_3)$$

The Hamiltonian function H becomes

$$-4\pi H=-4\pi H_0-4\pi\varepsilon H_1=[\log r_{12}^2 r_{13}^2 r_{23}^2](\tilde{p}_1,\tilde{p}_2,\theta_1-\theta_2)$$
$$+\varepsilon[\log r_{14}^2 r_{24}^2 r_{34}^2](\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\theta_1-\theta_2,\theta_1-\theta_3,\theta_2-\theta_3,\varepsilon),$$

then, the Hamiltonian system of energy H has a first integral $\tilde{p}_1+\tilde{p}_2+\tilde{p}_3$ and, of course, depends on ε as a real parameter.

Choosing a new system of coordinates:
$(\sqrt{\varepsilon}x_4, \sqrt{\varepsilon}y_4; \varphi_1, p_1; \varphi_2, p_2; \varphi_3, p_3)$ defined by

$$p_1=\tilde{p}_1,\qquad p_2=\tilde{p}_1+\tilde{p}_2+\tilde{p}_3,\qquad p_3=\tilde{p}_3$$
$$\varphi_1=\theta_1-\theta_2,\qquad \varphi_2=\theta_2,\qquad \varphi_3=\theta_3-\theta_2,$$

and since

$$dp_1\wedge d\varphi_1+dp_2\wedge d\varphi_2+dp_3\wedge d\varphi_3$$
$$=d\tilde{p}_1\wedge(d\theta_1-d\theta_2)+(d\tilde{p}_1+d\tilde{p}_2+d\tilde{p}_3)\wedge d\theta_2+d\tilde{p}_3\wedge(d\theta_3-d\theta_2)$$
$$=d\tilde{p}_1\wedge d\theta_1+d\tilde{p}_2\wedge d\theta_2+d\tilde{p}_3\wedge d\theta_3,$$

one sees that the new coordinates are also canonical and the new Hamiltonian $\tilde{H}$ does not depend on φ_2, that is:

$$\tilde{H}(p_1,p_2,p_3,\varphi_1,\varphi_3,\varepsilon)\overset{\text{def}}{=}\tilde{H}_0+\varepsilon\tilde{H}_1$$

where

$$\tilde{H}_0=H_0(p_1,p_2-p_1-p_3,\varphi_1)$$

and $\tilde{H}_1 = H_1(p_1, p_2 - p_1 - p_3, p_3, \varphi_1, \varphi_1 - \varphi_3, -\varphi_3, \varepsilon)$. Then

$$\left.\begin{aligned} \dot{p}_1 &= \frac{\partial \tilde{H}}{\partial \varphi_1} = \frac{\partial \tilde{H}_0}{\partial \varphi_1} + \varepsilon \frac{\partial \tilde{H}_1}{\partial \varphi_1} \\ \dot{\varphi}_1 &= -\frac{\partial \tilde{H}}{\partial p_1} = -\frac{\partial \tilde{H}_0}{\partial p_1} - \varepsilon \frac{\partial \tilde{H}_1}{\partial p_1} \\ \dot{p}_2 &= 0 \\ \dot{\varphi}_2 &= -\frac{\partial \tilde{H}}{\partial p_2} = -\frac{\partial \tilde{H}_0}{\partial p_2} - \varepsilon \frac{\partial \tilde{H}_1}{\partial p_2} \\ \dot{p}_3 &= \frac{\partial \tilde{H}}{\partial \varphi_3} = \varepsilon \frac{\partial \tilde{H}_1}{\partial \varphi_3} \\ \dot{\varphi}_3 &= -\frac{\partial \tilde{H}}{\partial p_3} = -\frac{\partial \tilde{H}_0}{\partial p_3} - \varepsilon \frac{\partial \tilde{H}_1}{\partial p_3}. \end{aligned}\right\} \tag{10}$$

The system above has the coordinate p_2 as a first integral.

With an analogous procedure, we obtain the equations corresponding to the problem of three equal unitary vortices. But it is an easy matter to get them just making $\varepsilon = 0$ and $\tilde{p}_3 = O(\varepsilon) = 0$ in the first four equations of system (10):

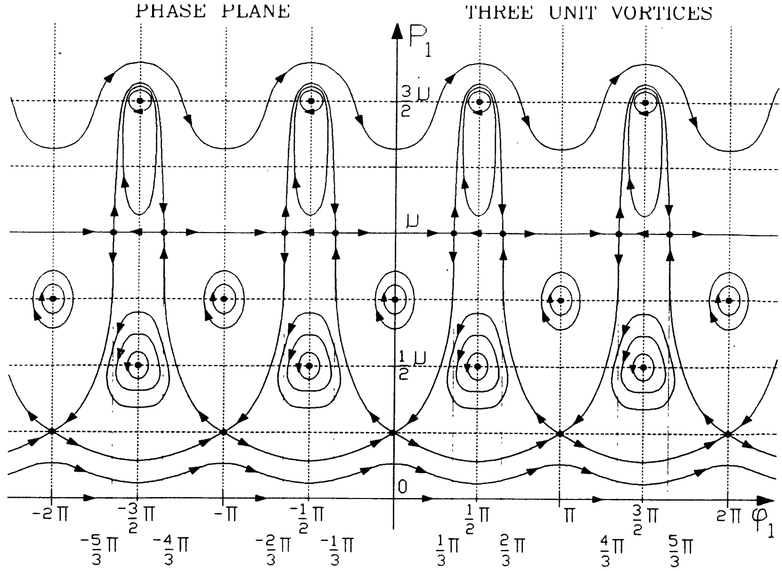
$$\left.\begin{aligned} \dot{p}_1 &= \frac{1}{\pi} e^{4\pi H_0} [-24 p_1^2 (p_2 - p_1) \cos \varphi_1 \sin \varphi_1] \\ \dot{\varphi}_1 &= -\frac{1}{\pi} e^{4\pi H_0} [(24 p_1 p_2 - 36 p_1^2) \cos^2 \varphi_1 \\ &\qquad - (3 p_2 - 2 p_1)^2 + 4 p_1 (3 p_2 - 2 p_1)] \\ p_2 &= \mu = \text{const.} \\ \dot{\varphi}_2 &= -\frac{1}{\pi} e^{4\pi H_0} [12 p_1^2 \cos^2 \varphi_1 - 6 p_1 (3 p_2 - 2 p_1)] \end{aligned}\right\} \tag{11}$$

where

$$\begin{aligned} H_0 &= -\frac{1}{4\pi} \log r_{12}^2 r_{13}^2 r_{23}^2 \\ &= -\frac{1}{4\pi} \log \{ 4 p_1 [(3 p_2 - 2 p_1)^2 - 12 p_1 (p_2 - p_1) \cos^2 \varphi_1] \}. \end{aligned}$$

The system (11) is decoupled; if φ_2 is a new time, (11) is equivalent to the system given by $\frac{dp_1}{d\varphi_2}$ and $\frac{d\varphi_1}{d\varphi_2}$ which has the same orbits as

$$\left.\begin{aligned} \dot{p}_1 &= -24 p_1^2 (\mu - p_1) \cos \varphi_1 \sin \varphi_1 \\ \dot{\varphi}_1 &= -[(24 p_1 \mu - 36 p_1^2) \cos^2 \varphi_1 \\ &\quad - (3\mu - 2 p_1)^2 + 4 p_1 (3\mu - 2 p_1)]. \end{aligned}\right\} \tag{12}$$



The phase space of (12) is given by figure for $p_1 < \mu$; it can be thought as an unperturbed system and presents heteroclinic connections as well as regular tori in regions where the KAM theory can be applied.

The perturbed system (10) can be reduced to a planar time-dependent Hamiltonian system. The idea will be presented in the sequel. Consider the motions on a level of energy $\tilde{H}(p_1, p_2 = \text{const.}, p_3, \varphi_1, \varphi_3, \varepsilon) = h$ and try to solve (locally) this equation in p_3. For this we compute the derivative $\dfrac{\partial \tilde{H}}{\partial p_3} = \dfrac{\partial \tilde{H}_0}{\partial p_3} + \varepsilon \dfrac{\partial \tilde{H}_1}{\partial p_3}$ and assume it is non zero; then one obtains, by the implicit function theorem, the function

$$p_3 = F(p_1, p_2 = \text{const.}, \varphi_1, \varphi_3, \varepsilon, h). \tag{13}$$

One can show that system (10), with $p_2 = \text{const.}$ and $\tilde{H} = h$, can be reduced to (13) and (14):

$$\left.\begin{aligned} \frac{dp_1}{d\varphi_3} &= \frac{\partial F}{\partial \varphi_1}(p_1, p_2 = \text{const.}, \varphi_1, \varphi_3, \varepsilon, h) \\ \frac{d\varphi_1}{d\varphi_3} &= -\frac{\partial F}{\partial p_1}(p_1, p_2 = \text{const.}, \varphi_1, \varphi_3, \varepsilon, h), \end{aligned}\right\} \tag{14}$$

where φ_3 is considered as a new time. In fact, if we assume (13), from the identity

$$\tilde{H}(p_1, p_2 = \text{const.}, F(p_1, p_2 = \text{const.}, \varphi_1, \varphi_3, \varepsilon, h), \varphi_1, \varphi_3, \varepsilon) = h$$

we obtain, by differentiation with respect to p_1 and φ_1:

$$\left.\begin{aligned}\frac{\partial F}{\partial \varphi_1}&=-\frac{\partial \tilde{H}}{\partial \varphi_1}\Big/\frac{\partial \tilde{H}}{\partial p_3}=-\left(\frac{\partial \tilde{H}_0}{\partial \varphi_1}+\varepsilon\frac{\partial \tilde{H}_1}{\partial \varphi_1}\right)\Big/\left(\frac{\partial \tilde{H}_0}{\partial p_3}+\varepsilon\frac{\partial \tilde{H}_1}{\partial p_3}\right)\\ \frac{\partial F}{\partial p_1}&=-\frac{\partial \tilde{H}}{\partial p_1}\Big/\frac{\partial \tilde{H}}{\partial p_3}=-\left(\frac{\partial \tilde{H}_0}{\partial p_1}+\varepsilon\frac{\partial \tilde{H}_1}{\partial p_1}\right)\Big/\left(\frac{\partial \tilde{H}_0}{\partial p_3}+\varepsilon\frac{\partial \tilde{H}_1}{\partial p_3}\right)\end{aligned}\right\} \tag{15}$$

where p_3 is given by (13). This, (13) and the fact that φ_3 can be considered in (10) as a new time, gives us the equations (14).

Remark that the system (14) is a time dependent Hamiltonian planar system with variables p_1, φ_1 and is periodic in time φ_3; moreover, equations (15) show that for $\varepsilon=0$ the system (14) is autonomous.

The final comment is that if we examine (14) and (13) in the extended phase space $\{p_1, \varphi_1, \varphi_3 \pmod{2\pi}\}$, the Poincaré-map of the plane $\{\varphi \pmod{2\pi}\}=0$ to itself, given by the cylindrical phase-flow, presents, for $\varepsilon=0$, the same orbits as the phase space of Figure. The main point now is to complete the proof using the Melnikov method, showing that the saddle connection I-V (see *Fig.*) of the unperturbed Poincaré-map will split, after perturbation, in two transversal separatrices.

By a classical Poincaré result (*see* Kozlov, 1983), one can conclude that system (10) of four vortices is not analytically integrable for ε small.

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