ANNALES DE L'I. H. P., SECTION A

MARZIO CASSANDRO
ANTONIO GALVES
PIERRE PICCO

Dynamical phase transitions in disordered systems : the study of a random walk model

Annales de l'I. H. P., section A, tome 55, nº 2 (1991), p. 689-705
<http://www.numdam.org/item?id=AIHPA_1991__55_2_689_0>

Ann. Inst. Henri Poincaré,

Vol. 55, n° 2, 1991, p. 689-705. *Physique théorique*

Dynamical phase transitions in disordered systems: the study of a random walk model

by

Marzio CASSANDRO

Dipartimento di Fisica, Università La Sapienza,
C.N.R.-G.N.S.M.-C.N.R.-G.N.F.M. Roma, Italy,

Antonio GALVES

Instituto de Matematica e Estatística,
Universidade de São Paulo, Brazil

and

Pierre PICCO

Centre de Physique Théorique,
C.N.R.S. Luminy, Marseille, France

ABSTRACT. — We study a random walk in a N dimensional hypercube. To each site of the hypercube we associate a random value according to a Bernouilli distribution. These values together with the "temperature" define the law of the waiting times of the walk at each site. We exhibit a transition between the high and low temperatures regime by studying the distance between two trajectoires starting from different initial conditions and subjected to the same noise.

Keywords : Random walk in random environment, dynamical phase transition, disordered system.

RÉSUMÉ. — Nous étudions une marche aléatoire sur un hypercube de dimension N. A chaque site de l'hypercube est associé une valeur aléatoire distribuée selon une loi de Bernoulli. Ces valeurs et la température définissent la loi des temps de sauts de la marche. Nous prouvons qu'il existe une transition entre des régimes à haute et basse température en étudiant la distance entre deux trajectoires initialement distinctes, quoique soumises au même bruit.

I. INTRODUCTION

Experiments and numerical simulations both suggest the existence of phase transitions in disordered systems. A characteristic feature of these systems seems to be the occurence of very long relaxation times and it is believed that this is due to the presence of a large number of metastable states.

A crucial point to study even numerically this kind of phenomena is to establish the relevant time scales. Furthermore Monte Carlo methods are based on the observation of the time evolution of finite Markov Chains. These chains, as long as they have only one closed set are ergodic (*i. e.* admit only one invariant probability measure). So that the problem is to find a criterium of dynamical phase transition which is not based on the multiplicity of the invariant states.

This can be done by studying the way the process approaches its invariant state and loses memory of its initial conditions.

A simple example will help understanding what we mean by that. Consider a simple random walk on the set $\{0, 1, 2, \ldots, N\}$ with probability p (and $1-p$ respectively) to jump one step to the right (to the left resp.) where 0 is an absorbing barrier and N is a reflecting barrier. For any $p<1$ regardless to the starting point the process always ends its motion by being trapped by 0. Therefore the Dirac measure concentrated on zero is the unique invariant state. Nevertheless the way the process approaches the final state depends in a crucial way on the value of the parameter p.

If $\frac{1}{2}<p<1$, the process has to fight against the drift to join the invariant state 0. This has the following consequence. If we call T_N the number of steps the process takes to reach 0 starting from N, then as N diverges, $T_N/\mathbb{E}(T_N)$ converges in law to a mean one exponential random variable. This amounts to say that this time is unpredictable. On the other hand if $p<\frac{1}{2}$ the drift pushes the motion towards the final invariant state. In this case $T_N/\mathbb{E}(T_N)$ converges in probability to the constant 1, that is the life time of the process is essentially deterministic if N is large enough.

It is also interesting to observe that in the first case $\mathbb{E}(T_N)$ grows exponentially with N, whereas in the second one it grows linearly.

These elementary results can be obtained by almost direct computations. In the first case, if we neglect the trivial case, where the starting point is already in the trap, the result follows from the fact that the process looses memory from its initial condition in a time which is much shorter than $\mathbb{E}(T_N)$. The second case follows from the Law of Large Numbers.

This transition from a deterministic to an unpredictable relaxation time is present in many models, for instance dynamical systems with a small random perturbation and Markovian systems with interacting particles], and defines a kind of dynamical phase transition.

In the context of disordered systems, B. Derrida in a recent series of papers ([1], [4], [5]) has suggested to study the dependence from the initial condition by comparing two time evolutions subjected to the same thermal noise.

Numerical simulations and heuristic arguments strongly suggests the existence of a low temperature phase where starting from two different configurations they remain at a finite distance from each other for a very long time, and a high temperature phase where their distance goes to zero in a much shorter time. As far as we know, all these studies are non rigorous and in particular the time scales involved are not explicitly discussed.

Here we perform this program in a rigorous way with one of the simplest non trivial models of disordered system. We consider a simple random walk taking values on the hypercube $\{-1, +1\}^N$, where N is a positive natural number (*i. e.* the set of all possible configurations of N spins taking the values -1 or $+1$). Traps are randomly distributed in the following way: each point of the hypercube (*i. e.*, each spin configuration) is choosen to be a trap independently of the others with probability $1/N^\gamma$, where γ is a real positive number. Once we have fixed this random medium (the choice of the traps), the time evolution is defined in the following way. If the process is in the configuration σ at time t, the probability to leave it at time $t+1$ is $1/(1+N^\beta)$ if σ is a trap, otherwise it is $1/2$.

Once the jump is decided a spin is choosen with probability $1/N$ and flipped. In this definition β is a real positive number ("the inverse temperature"). To implement this program we suitably couple two processes evolving in the same random medium, with the law described above, in such a way that once they meet they remain together forever.

From elementary arguments of the theory of Markov chains, it follows that they will eventually meet. Therefore the natural question is: how long does it take for the processes to meet?

Let us call T_N the random time in which this occurs. In the absence of traps $T_N/N \log N$ converges to 1 in probability, as N diverges and this is the content of proposition III.1.

This suggests that $N \log N$ is a natural scale to look for any kind of dynamical phase transition for our model.

In the case $\gamma > 1$ there are not enough traps to produce a dynamical phase transition. In fact the fraction of spins which are different in the two configurations becomes neglegiable before one of the processes reaches a trap for the first time. This is the content of theorem II.1.

The interesting case is $\gamma<1$ where dynamical phase transition occurs. In the high temperature phase ($\beta<\gamma$) the fraction of time each one of the processes spends in a trap is negligeable (actually this fraction is upperbounded by $1/N^{\gamma-\beta}$). This is the content of proposition IV.1. As a consequence the distance between the two processes becomes negligeable at time $N \log N$.

In the low temperature phase ($\beta>\gamma$) most of the time the processes are trapped and therefore the distance between them remains essentially constant for a time of order $N^{1+\delta}$, for a suitable $\delta>0$. This is the content of the proposition VI.1. As a consequence we finally obtain theorem II.2.

This paper is organized in the following way: in section 2 we define the model and state the results.

In section 3 we establish some estimates independent of β that single out the relevant time scale.

In section 4 we present a new construction of our stochastic process that allows to control the randommess introduced by the energies.

Sections 5 and 6 deal respectively with the proof of the high temperatures and law temperatures results.

We conclude and state some open questions in section 7.

II. DEFINITIONS AND MAIN RESULTS

The set of spin configurations with N spins ± 1 is denoted by $\mathscr{H}_N=\{-1,+1\}^N$. σ and ξ are generic elements of $\mathscr{H}_N$: $\sigma\equiv(\sigma_1,\ldots,\sigma_N)$. Given $j\in\{1,\ldots,N\}$ and $\sigma\in\mathscr{H}_N$ we denote by σ^j the spin configuration obtained from σ by flipping the spin at the site j, that is:

$$(\sigma^j)_i=\begin{cases} \sigma_i & \text{if } i\neq j \\ -\sigma_i & \text{if } i=j \end{cases}.$$

Given two configurations σ and ξ we define their distance by

$$d(\sigma,\xi)=\frac{1}{2N}\sum_{i=1}^{N}|\sigma_i-\xi_i|.$$

Let M be a random subset of $\mathscr{H}_N$ defined on a probability space $(\bar{\Omega}_N, \bar{\mathscr{A}}_N, \bar{\mathbb{P}}_N)$ and having Bernoulli distribution with density $1/N^\gamma$ where γ is a positive number *i. e.* for any subset $F\subset\mathscr{H}_N$

$$\bar{\mathbb{P}}_N(M\cap F=\varnothing)=\left(1-\frac{1}{N^\gamma}\right)^{|F|}$$

where $|F|$ is the cardinal of F.

Given $\bar{\omega}\in\bar{\Omega}_N$, $\sigma\in\mathscr{H}_N$, $i\in\{1,2,\ldots,N\}$ we define a quantity $p(\bar{\omega},\sigma)$ in the following way:

$$p(\bar{\omega},\sigma)=(1+e^{\beta\sigma_i(E(\sigma))})^{-1}$$

where

$$E(\sigma)=\begin{cases}-\log N & \text{if } \sigma\in M\\ 0 & \text{if } \sigma\notin M\end{cases}$$

and β is a positive number, the inverse temperature.

Even if $p(\bar{\omega},\sigma)$ is "i" dependent we shall omit to mention it in the notation.

We consider the following stochastic time evolution in $\mathscr{H}_N$: given a configuration $\sigma(t)$ at time t we choose an index $i\in\{1,\ldots,N\}$ with probability $1/N$ and we update the spin σ_i to the value $+1$ with probability $p(\bar{\omega},\sigma(t))$.

A realisation of this stochastic time evolution can be obtained in the following way: we introduce two independent sequences of independent random variables $I(t)$ and $U(t)$ where $t=1,2,\ldots$ They are defined on a new probability space $(\Omega_N,\mathscr{A}_N,\mathbb{P}_N)$. The random variables $I(t)$, taking values in the set $\{1,2,\ldots,N\}$, are identically distributed and for any $k\in\{1,2,\ldots,N\}$, $\mathbb{P}_N(I(t)=k)=1/N$.

The random variables $U(t)$ are identically distributed with uniform distribution on $[0,1]$ *i. e.* $\mathbb{P}_N(U(t)<u)=u$ for any $u\in[0,1]$.

We fix $\bar{\omega}\in\bar{\Omega}_N$, given an initial configuration $\sigma(0)$ and $\omega\in\Omega$, we define $\sigma(t,\omega)$ for all $t\geqq 1$ in the following way:

$$\sigma_i(t,\omega)=\begin{cases}\sigma_i(t-1,\omega) & \text{if } I(t,\omega)\neq i,\\ +1 & \text{if } I(t,\omega)=i \text{ and } U(t,\omega)<p(\bar{\omega},\sigma(t-1)),\\ -1 & \text{if } I(t,\omega)=i \text{ and } U(t,\omega)\geqq p(\bar{\omega},\sigma(t-1)).\end{cases}$$

We remark that $\sigma(t)$ performs a random walk on $\mathscr{H}_N$ in a random environment with traps (namely the configurations belonging to M) which slow down the process.

Now we consider, for sake of definitness, $\sigma(0)\equiv(\sigma_i(0))_{i=1}^N$ with $\sigma_i(0)=+1$, $\forall i\in 1,\ldots,N$ and $\xi(0)\equiv(\xi_i(0))_{i=1}^N$ with $\xi_i(0)=-1$, $\forall i\in 1,\ldots$ and we construct both $\sigma(t)$ and $\xi(t)$ using the same ω, that is the same random choice of indices $I(t,\omega)$ and the same choice of $U(t,\omega)$ in the following way:

given $\sigma(t)$, $\xi(t)$ and $I(t+1)=i$ if

$$U(t+1)<p(\xi(t))\wedge p(\sigma(t))$$

we update the two spins σ_i and ξ_i to the value $+1$ if

$$p(\xi(t))\wedge p(\sigma(t))<U(t+1)<p(\xi(t))\vee p(\sigma(t))$$

and

$$p(\xi(t)) \wedge p(\sigma(t)) = p(\sigma(t)) \qquad [\text{resp.}\, p(\xi(t))];$$

we update the spin σ_i to the value -1 (resp. $+1$) and the spin ξ_i to the value $+1$ (resp. -1); if

$$p(\xi(t)) \vee p(\sigma(t)) < \mathrm{U}(t+1)$$

we update the two spins σ_i and ξ_i to the value -1.

We remark that with this algorithm both $\sigma(t)$ and $\xi(t)$ have the same law, the one described in the introduction.

We denote

$$\mathrm{D_N}(\bar{\omega}, t, \omega) = d(\sigma(\bar{\omega}, t, \omega), \xi(\bar{\omega}, t, \omega)).$$

All the random objects we consider are defined in the product space

$$(\bar{\Omega}_{\mathrm{N}}, \otimes \Omega_{\mathrm{N}}, \bar{\mathscr{A}}_{\mathrm{N}} \otimes \mathscr{A}_{\mathrm{N}}, \bar{\mathbb{P}}_{\mathrm{N}} \otimes \mathbb{P}_{\mathrm{N}}).$$

Even if all those objects depend on N, $\bar{\omega}$, ω we shall omit them in the notation if no confusion is possible.

With these definitions in hand we can state formally our theorems. The first one treats the case in which the density of traps is not large enough to induce a dynamical phase transition, whereas the second one treats the case in which such a phenomenon occurs.

THEOREM II.1. — *Given* $\gamma > 1$, *for any* $\beta \geqq 0$, $\varepsilon > 0$, $\eta > 0$

$$\lim_{\mathrm{N} \to \infty} \bar{\mathbb{P}}(\mathbb{P}(d(\mathrm{N} \log \mathrm{N}) > \varepsilon) > \eta) = 0.$$

THEOREM II.2. — *Given* $0 < \gamma < 1$,
if $\beta > \gamma$, *for any* δ *such that* $\beta - \gamma > \delta > 0$ *and any* $0 < \varepsilon < 1$, $\eta > 0$

$$\lim_{\mathrm{N} \to \infty} \bar{\mathbb{P}}(\mathbb{P}(d(\mathrm{N}^{1+\delta}) < 1 - \varepsilon) > \eta) = 0$$

if $\beta < \gamma$, *for any* $\varepsilon > 0$, $\eta > 0$

$$\lim_{\mathrm{N} \to \infty} \bar{\mathbb{P}}(\mathbb{P}(d(\mathrm{N} \log \mathrm{N}) > \varepsilon) > \eta) = 0.$$

The actual proof of these theorems will include explicit bounds in terms of ε, η and N for the involved probabilities.

Theorem II.1 will be proved at the end of section 3.

The first part of theorem II.2 will be proved in section 5, the second part in section 6.

III. ESTIMATES INDEPENDENT OF β

In order to justify our choice of the time scale we present here a preliminary result related to a random walk on $\mathscr{H}_{\mathrm{N}}$. This result correspond to $\gamma=+\infty$ *i. e.* the set M is empty and therefore $\sigma(t)$ and $\xi(t)$ are homogeneous random walk on $\mathscr{H}_N$, that is we take the same definition as before with $p(\sigma)=1/2$ for all σ. For these processes we define the following stopping time

$$T_N=\operatorname{Inf}(t>0, d(\sigma(t), \xi(t))=0).$$

We emphasize that in this case $d(\sigma(t), \xi(t))$ is a Markov chain on the set $[0, 1/N, 2/N, \ldots, 1]$, we call it $d^*(t)$ and its probability transitions are as follows:

$$\mathbb{P}\left(d^*(t+1)=\frac{x-1}{N}\middle| d^*(t)=\frac{x}{N}\right)=\frac{x}{N}$$

and

$$\mathbb{P}\left(d^*(t+1)=\frac{x}{N}\middle| d^*(t)=\frac{x}{N}\right)=1-\frac{x}{N}$$

if $x\in\{1, 2, \ldots, N\}$ and

$$\mathbb{P}(d^*(t+1)=0 \mid d^*(t)=0)=1.$$

PROPOSITION III.1 :

$$\lim_{N\to\infty}\frac{T_N}{N\log N}=1 \quad \textit{in Probability.}$$

Proof. – We first remark that

$$T_N=\sum_{k=1}^{N}\tau_k$$

where $\tau_1=1$ and τ_K for $K=2, \ldots, N$ are independent random variables with distribution

$$\mathbb{P}(\tau_k=n)=(p_k)^{n-1}(1-p_k), \qquad \text{for} \quad n=1, 2, 3, \ldots,$$

where $p_k=\dfrac{k-1}{N}$.

The time τ_K is exactly the number of steps the process $D^*(t)$ takes to go from $1-\dfrac{k-1}{N}$ to $1-\dfrac{k}{N}$, therefore

$$\mathbb{E}(T_N)=\sum_{i=1}^{N}\frac{1}{1-((i-1)/N)}$$

and we get easily

$$N\log(N-1)\leqq\mathbb{E}(T_N)\leqq N(1+\log N).$$

Let us now estimate

$$\mathbb{P}(T_N\leqq(1-\varepsilon)N\log N).$$

For any positive real number λ we have

$$\begin{aligned}\mathbb{P}(\mathrm{T_N} \leqq (1-\varepsilon)\,\mathrm{N}\log\mathrm{N}) &= \mathbb{P}(\exp-\lambda\,\mathrm{T_N} \geqq \exp-\lambda(1-\varepsilon)\,\mathrm{N}\log\mathrm{N})\\ &\leqq \exp[\lambda(1-\varepsilon)\,\mathrm{N}\log\mathrm{N}]\,\mathbb{E}(\exp-\lambda\,\mathrm{T_N})\\ &\leqq \exp[\lambda(1-\varepsilon)\,\mathrm{N}\log\mathrm{N}]\prod_{i=1}^{\mathrm{N}}\mathbb{E}(e^{-\lambda\tau_i}).\end{aligned}$$

Now

$$\mathbb{E}(e^{-\lambda\tau_k}) = \frac{e^{-\lambda}}{1+(p_k/(1-p_k))(1-e^{-\lambda})}.$$

Since for any $x>0$ we have

$$(1+x)^{-1} \leqq \exp\left[-x+\frac{x^2}{2}\right],$$

we get

$$\prod_{i=1}^{\mathrm{N}}\mathbb{E}(e^{-\lambda\tau_i}) \leqq e^{-\lambda\mathrm{N}}\, e^{-(1-e^{-\lambda})\sum_{k=1}^{\mathrm{N}} p_k/(1-p_k)}\, e^{(1/2)(1-e^{-\lambda})^2\sum_{k=1}^{\mathrm{N}}(p_k/1-p_k)^2}.$$

Now since

$$\sum_{k=1}^{\mathrm{N}}\frac{p_k}{1-p_k} \geqq \mathrm{N}\log(\mathrm{N}-1)-(\mathrm{N}-1)$$

and

$$\sum_{k=1}^{\mathrm{N}}\left(\frac{p_k}{1-p_k}\right)^2 \leqq 2\,\mathrm{N}^2-2\,\mathrm{N}\log(\mathrm{N}+1)+(\mathrm{N}-1).$$

Choosing λ such that $1-e^{-\lambda}=\dfrac{3\varepsilon}{4}\,\dfrac{\log\mathrm{N}}{\mathrm{N}}$ we get after some easy estimates

$$\mathbb{P}(\mathrm{T_N} \leqq (1-\varepsilon)\,2\,\mathrm{N}\log\mathrm{N}) \leqq e^{-(\varepsilon^2/4)(\log\mathrm{N})^2}$$

On the other hand, for any $\lambda>0$

$$\mathbb{P}(\mathrm{T_N} \geqq (1+2\,\varepsilon)\,\mathrm{N}\log\mathrm{N}) \leqq e^{-\lambda(1+2\,\varepsilon)\,\mathrm{N}\log\mathrm{N}}\,\mathbb{E}(e^{\lambda\mathrm{T_N}}).$$

If $\lambda=\alpha/\mathrm{N}$ with $\alpha<1$, using

$$(1-x)^{-1} \leqq \exp\left[x+\frac{x^2}{2}\left(\frac{1}{1-x}\right)\right] \qquad \text{if} \quad x<1$$

we get

$$\begin{aligned}\mathbb{E}(e^{\lambda\mathrm{T_N}}) \leqq \exp\Bigg[\lambda\,\mathrm{N} &+ \sum_{k=1}^{\mathrm{N}}\frac{p_k}{1-p_k}(e^{\lambda}-1)\\ &+ \sum_{k=1}^{\mathrm{N}}\left(\frac{p_k}{1-p_k}\right)^2\frac{1}{2}(e^{\lambda}-1)^2\frac{1}{1-(p_k/(1-p_k))(e^{\lambda}-1)}\Bigg].\end{aligned}$$

It can be checked that for any $\varepsilon>0$

$$\operatorname*{Sup}_{k=1-N} \frac{1}{1-(p_k/(1-p_k))(e^\lambda-1)} < \frac{1}{1-\alpha(1+\varepsilon)}$$

if N is large enough.

After some easy estimates we get

$$\mathbb{P}(T_N \geqq (1+2\varepsilon) N \log N) \leqq e^{-\alpha\varepsilon \log N}$$

and this conclude the proof of proposition III.1.

According to proposition III.1, the time the coupled random walks take to meet in the absence of traps is typically $N \log N$. Therefore this is a natural time scale to study the distance between the random walks with the random environment. Now the question is: how long does it take for a random walk to reach a trap for the first time? This is the content of proposition III.2. Let us define the stopping time

$$S^\sigma = \operatorname{Inf}(t>0,\ \sigma(t) \in M).$$

PROPOSITION III.2. – *For any* $\gamma>\delta>0,\ \varepsilon>0$

$$\lim_{N\to\infty} \bar{\mathbb{P}}(\mathbb{P}(S^\sigma < N^{\gamma-\delta}) > \varepsilon) = 0.$$

Proof. – Using the Markov inequality

$$\bar{\mathbb{P}}(\mathbb{P}(S^\sigma < N^{\gamma-\delta}) > \varepsilon) \leqq \frac{1}{\varepsilon} \bar{\mathbb{E}}(\mathbb{P}(S^\sigma < N^{\gamma-\delta})) = \frac{1}{\varepsilon} \bar{\mathbb{E}}(\mathbb{P}(M(\bar{\omega}) \cap V(\omega,\bar{\omega}) \neq \varnothing))$$

where

$$V = \{\sigma(s);\ s=0,1,2,\ldots,N^{\gamma-\delta}\}.$$

In order to obtain an upperbound to the last expectation we introduce an auxiliary process $\tilde{\sigma}(t)$ defined in the following way:

$$\tilde{\sigma}(0) = \sigma(0)$$

and for any $t \geqq 1$

$$\tilde{\sigma}(t)_i = \tilde{\sigma}(t-1)_i, \qquad \text{if} \quad I(t) \neq i$$

and if $I(t)=i$

$$\tilde{\sigma}(t)_i = \begin{cases} +1, & \text{if} \quad U(t) \leqq \dfrac{1}{2} \\ -1, & \text{if} \quad U(t) > \dfrac{1}{2} \end{cases}.$$

We remark that $\tilde{\sigma}(t) = \sigma(t)$, for all $t \leqq S^\sigma$. In particular we have

$$\bar{\mathbb{E}}\mathbb{P}\{M(\bar{\omega}) \cap V(\omega,\bar{\omega}) = \varnothing\} = \bar{\mathbb{E}}\mathbb{P}\{M(\bar{\omega}) \cap \tilde{V}(\omega) = \varnothing\}$$

where

$$\tilde{V} = \{\sigma(s) : s \leqq N^{\gamma-\delta}\}.$$

Since the auxiliary process $\tilde{\sigma}(t)$ is homogeneous, $\tilde{V}$ depends only on ω, not on $\bar{\omega}$. Therefore we are allowded to use Fubini-Tonelli theorem to get

$$\bar{\mathbb{E}}\mathbb{P}\{M(\bar{\omega})\cap\tilde{V}(\omega)=\varnothing\}=\mathbb{E}\bar{\mathbb{P}}\{M(\bar{\omega})\cap\tilde{V}(\omega)=\varnothing\}=\mathbb{E}\left[\left(\left(1-\frac{1}{N^{\gamma}}\right)^{|\tilde{V}|}\right)\right]$$

Now we use the elementary fact that $|\tilde{V}|\leqq N^{\gamma-\delta}$ to obtain the lower bound

$$\mathbb{E}\left[\left(\left(1-\frac{1}{N^{\gamma}}\right)^{|\tilde{V}|}\right)\right]\geqq\left(1-\frac{1}{N^{\gamma}}\right)^{N^{\gamma-\delta}}$$

Therefore we get:

$$\bar{\mathbb{E}}(\mathbb{P}(S^{\sigma}\langle N^{\gamma-\delta})\rangle\varepsilon)\leqq\frac{1}{\varepsilon}\left[\left(1-\left(1-\frac{1}{N^{\gamma}}\right)^{N^{\gamma-\delta}}\right)\right]\leqq\frac{C}{\varepsilon}\,\frac{N^{\gamma}}{N^{\gamma-\delta}}$$

for some positive constant and this ends the proof of Proposition III.2.

The idea of the proof of theorem II.1 is the following: since for $\gamma>1$ the time each random walk takes to reach M is longer than $N\log N$, they meet before realising they are in a trapped environment.

Proof of Theorem II.1. – We want to show that if $\gamma>1$

$$\lim_{N\to\infty}\bar{\mathbb{P}}(\mathbb{P}(d(N\log N)>\varepsilon)>\eta)=0.$$

Using Markov inequality we get

$$\bar{\mathbb{P}}(\mathbb{P}(d(N\log N)>\varepsilon)>\eta)\leqq\frac{1}{\eta}\bar{\mathbb{E}}(\mathbb{P}\,d(N\log N)>\varepsilon)).$$

Now

$$\begin{aligned}&\bar{\mathbb{E}}(\mathbb{P}(d(N\log N)>\varepsilon))\\&\qquad\leqq\bar{\mathbb{E}}(\mathbb{P}(d(N\log N)>\varepsilon,\ S^{\sigma}>N^{\gamma-\delta},\ S^{\xi}>N^{\gamma-\delta}))\\&\qquad\qquad+2\,\bar{\mathbb{E}}(\mathbb{P}(S^{\sigma}<N^{\gamma-\delta})).\end{aligned}$$

Taking δ small enough in order that $\gamma-\delta>1$ we get

$$\bar{\mathbb{E}}(\mathbb{P}(d(N\log N)>\varepsilon,\ S_N^{\sigma}>N^{\gamma-\delta},\ S_N^{\xi}>N^{\gamma-\delta}))\leqq\mathbb{P}(d^*(N\log N)>\varepsilon).$$

In the set $\{S_N^{\sigma}>N^{\gamma-\delta}\}\cap\{S_N^{\xi}>N^{\gamma-\delta}\}$ neither the process $\sigma(t)$ nor $\xi(t)$ had met the set M. Therefore we have reconstructed the process $d^*(N\log N)$ using the $I(t)$ and $U(t)$ and we remark that

$$p(\sigma(s-1))=p(\xi(s-1))=1/2$$

for any $s<\mathrm{Min}(S^{\sigma},S^{\xi})$.

Now using Proposition III.1, $\mathbb{P}(d^*N\log N)>\varepsilon)$ goes to zero and using Proposition III.2 $\bar{\mathbb{E}}(\mathbb{P}(S^{\sigma}\leqq N^{\gamma-\delta}))$ goes to zero.

IV. THE NUMBER OF EFFECTIVE JUMPS

In this section we give another construction of the process $\sigma(t)$ which allows us to control the effective number of jumps performed in a given time interval. The first step is to construct the embedded chain $(\zeta(k))$.

We take as the starting point $\zeta(0)=\sigma(0)$ and define $\zeta(k)$, $k=1, 2, \ldots$, as a homogeneous random walk taking values in $\mathscr{H}_N$, defined on a probability space $(\Omega_0, \mathscr{A}_0, \mathbb{P}_0)$ with transition probability given by

$$\mathbb{P}_0(\zeta(k+1)=\eta^i \mid \zeta(k)=\eta)=1/\mathrm{N}$$

for all $k\geqq 0$, $i\in\{1, 2, \ldots, \mathrm{N}\}$ and $\eta\in\mathscr{H}_N$.

We remark that

$$\mathbb{P}_0(\zeta(k+1)=\eta \mid \zeta(k)=\eta)=0.$$

The second step is the introduction of the sequence of times the process takes to perform the jumps described by the embedded chain: let τ_k^q; $k=1, 2, \ldots$ where q belongs to the set $\left\{\frac{1}{2}, \frac{1}{1+\mathrm{N}^\beta}\right\}$ be a sequence of independent geometric random variables defined on a probability space $(\Omega_1, \mathscr{A}_1, \mathbb{P}_1)$ with distribution probability given by

$$\mathbb{P}_1(\tau_k^q=n)=(1-q)^{n-1}q, \qquad n=1, 2, 3, \ldots$$

For $k=1, 2, 3\ldots$ let us take $q_k=1/2$, if $\zeta(k-1)\notin \mathrm{M}$ and $q_k=\dfrac{1}{1+\mathrm{N}^\beta}$, if $\zeta(k-1)\in\mathrm{M}$. Now we define the random process $\hat{\sigma}(t)$ in the following way:

$$\hat{\sigma}(t)=\begin{cases}\zeta(0)=\sigma(0) & \text{if } \ 0\leqq t<\tau_1^{q_1},\\ \zeta(k) & \text{if } \ \sum_{j=1}^{k}\tau_j^{q_j}\leqq t<\sum_{j=1}^{k+1}\tau_j^{q_j}.\end{cases}$$

We remark that this process, defined on the probability space $(\bar{\Omega}\otimes\Omega_0\otimes\Omega_1, \bar{\mathscr{A}}\otimes\mathscr{A}_0\otimes\mathscr{A}_1, \bar{\mathbb{P}}\otimes\mathbb{P}_0\otimes\mathbb{P}_1)$, has the same law that $\sigma(t)$.

The following proposition will be used in the proof of the $\beta<\gamma$ case.

Proposition IV.1. — *For any t, β, γ*

$$\bar{\mathbb{E}}\left(\mathbb{E}_0\,\mathbb{E}_1\left(\sum_{s=0}^{t}\mathbf{1}_{\{\hat{\sigma}(s)\in\mathrm{M}\}}\right)\right)\leqq t\,\frac{(\mathrm{N}^\beta+1)}{\mathrm{N}^\gamma}$$

Proof. — Let us define $\theta(0)=0$ and

$$\theta(k)=\sum_{j=1}^{k}\tau_j^{q_j}, \qquad \text{for} \quad k\geqq 1.$$

Then we get:

$$\sum_{s=0}^{t} \mathbf{1}_{\{\hat{\sigma}(s)\in M\}} = \sum_{k=0}^{t} \mathbf{1}_{\{\zeta(k)\in M\}} \sum_{s=k}^{t} \mathbf{1}_{\{\theta(k)\leqq s<\theta(k+1)\}}$$
$$\leqq \sum_{k=0}^{t} \mathbf{1}_{\{\zeta(k)\in M\}} (\theta(k+1)-\theta(k)) = \sum_{k=0}^{t} \tau_{k+1}^{q_{k+1}} \mathbf{1}_{\{\zeta(k)\in M\}}.$$

Therefore

$$\bar{\mathbb{E}}\left(\mathbb{E}_0\,\mathbb{E}_1\left(\sum_{s=0}^{t} \mathbf{1}_{\{\hat{\sigma}(s)\in M\}}\right)\right) \leqq \sum_{k=0}^{t} \bar{\mathbb{E}}\{\mathbb{E}_0[\mathbf{1}_{\{\zeta(k)\in M\}}\,\mathbb{E}_1(\tau_{k+1}^{q_{k+1}})]\}.$$

Now we remark that

$$\mathbf{1}_{\{\zeta(k)\in M\}}\,\mathbb{E}_1(\tau_{k+1}^{q_{k+1}}) = \frac{\mathbf{1}_{\{\zeta(k)\in M\}}}{q_{k+1}}.$$

This last expression is bounded from above by $(1+N^{\beta})\,\mathbf{1}_{\{\zeta(k)\in M}$.

Now using Fubini-Tonelli, we get

$$\bar{\mathbb{E}}(\mathbb{E}_0(\mathbf{1}_{(\zeta(k)\in M)})) = \mathbb{E}_0(\bar{\mathbb{E}}(\mathbf{1}_{(\zeta(k)\in M)})) = \sum_{\eta\in\mathscr{H}_N} \mathbb{P}_0(\zeta(k)=\eta)\,\bar{\mathbb{P}}(\eta\in M) = \frac{1}{N^{\gamma}}$$

from which the result follows.

Now we will prove two lemmata useful for the $\beta > y$ case.

LEMMA IV.2. – *Let* $\mathscr{K} = \operatorname{Inf}(k>0 : \zeta(k)\in M)$.
If $\gamma<1$ *then for any* $\rho>0$, *such that* $\gamma+\rho<1$

$$\bar{\mathbb{E}}\mathbb{P}_0(\mathscr{K} > N^{\gamma+\rho}) \leqq e^{-(1/2)N^{\rho}}\, e^{-((1-\gamma)/4)N^{\gamma}\log N}.$$

Proof:

$$\bar{\mathbb{E}}(\mathbb{P}_0(\mathscr{K} > N^{\gamma+\rho}))$$
$$= \bar{\mathbb{E}}(\mathbb{P}_0(\{\zeta(0),\ \ldots,\ \zeta(N^{\gamma+\rho})\}\cap M=\varnothing)) = \mathbb{E}_0\left(1-\frac{1}{N^{\gamma}}\right)^{C_{N^{\gamma+\rho}}}$$

where

$$C_{N^{\gamma+\rho}} = |\{\zeta(0),\ \ldots,\ \zeta(N^{\gamma+\rho})\}|.$$

The previous espression is bounded from above by

$$\left(1-\frac{1}{N^{\gamma}}\right)^{(1/2)N^{\gamma+\rho}} + \mathbb{P}_0\left(C_{N^{\gamma+\rho}} < \frac{1}{2}N^{\gamma+\rho}\right).$$

We remark that to have $C_t \geqq \alpha t$ we must flip at least αt spins. Therefore for any $0<\alpha<1$ and any $t>0$ the following inequalities hold:

$$\mathbb{P}_0(C_t<\alpha t) \leqq \sum_{k=0}^{\alpha t} \binom{t}{k} (k/N)^{t-k}$$
$$\leqq \alpha t\left(\frac{\alpha t}{N}\right)^{(1-\alpha)t} \binom{t}{t/2} \leqq \exp\left[(1-\alpha)t\log\frac{\alpha t}{N} + t\log 2\right].$$

Choosing $t=\mathrm{N}^{\gamma+\rho}$ it follows that

$$\mathbb{P}_0\left(\mathrm{C}_{\mathrm{N}^{\gamma+\rho}}<\frac{1}{2}\mathrm{N}^{\gamma+\rho}\right)\leqq e^{-(1/4)(1-\gamma)\mathrm{N}^\gamma \log \mathrm{N}}$$

and this ends the proof of the lemma.

LEMMA IV.3. – *Given* $\eta>0$ *then for any* $\alpha>\gamma+\eta$

$$\lim_{\mathrm{N}\to\infty} \bar{\mathbb{E}}\left(\mathbb{P}_0\left(\sum_{s=0}^{\mathrm{N}^\alpha}\mathbf{1}_{(\zeta(s)\in\mathrm{M})}\geqq\frac{\mathrm{N}^\alpha}{\mathrm{N}^{\gamma+\eta}}\right)\right)=1$$

Proof. – Let $\Delta=\mathrm{N}^{\gamma+\eta}$ and call B_k the event

$$\mathrm{B}_k=\{(\exists s\in[k\Delta,(k+1)\Delta[:\zeta(s)\in\mathrm{M}\}$$

using lemma IV.2

$$\bar{\mathbb{E}}\left(\mathbb{P}_0\left(\bigcup_{k=0}^{\mathrm{N}^{\alpha-(\gamma+\eta)}}\mathrm{B}_k^c\right)\right)\leqq\mathrm{N}^{\alpha-(\gamma+\eta)}e^{-\mathrm{N}^\eta}$$

and this proves the lemma.

V. THE CASE $\beta<\gamma$

We prove now the second part of Theorem II.2.

To simplify the notation let us call

$$\mathrm{X}(t)=(\sigma(t),\xi(t)),$$
$$\mathrm{B}=\mathrm{M}^c\times\mathrm{M}^c \text{ and } \mathrm{C}(t)=\{i\in\{1,\ldots,\mathrm{N}\}:\sigma_i(t)=\xi_i(t)\}.$$

Obviously $\mathrm{D}(t)=\mathrm{N}\,d(t)=\mathrm{N}-|\mathrm{C}(t)|$. We remark that in the time evolution of the set $\mathrm{C}(t)$ there are only three possibilities:

(i) $\mathrm{C}(t+1)=\mathrm{C}(t)$;

(ii) $\mathrm{C}(t+1)=\mathrm{C}(t)\setminus\{\mathrm{I}(t+1)\}$;

(iii) $\mathrm{C}(t+1)=\mathrm{C}(t)\cup\{\mathrm{I}(t+1)\}$.

In particular if $\mathrm{X}(t)\in\mathrm{B}$ we are in the third case.

Therefore

$$\mathrm{C}(t)\supset\bigcup_{\substack{s:\mathrm{X}(s-1)\in\mathrm{B}\\1\leqq s\leqq t}}\{\mathrm{I}(s)\}\setminus\bigcup_{\substack{s:\mathrm{X}(s-1)\notin\mathrm{B}\\1\leqq s\leqq t}}\{\mathrm{I}(s)\}$$

If we call

$$\mathscr{U}_1=\mathrm{Inf}(s\geqq 0:\mathrm{X}(s)\in\mathrm{B})$$

and for all $k\geqq 1$

$$\mathscr{U}_{k+1}=\mathrm{Inf}(s>\mathscr{U}_k:\mathrm{X}(s)\in\mathrm{B});$$

we can define the process Z_k by

$$Z_0 = N$$
$$Z_k = N - \left| \bigcup_{i=1}^{k} \{ I(\mathscr{U}_i) \} \right|, \quad \text{for all } k \geqq 1.$$

If we define

$$K(t) = \operatorname{Max}(k;\ \mathscr{U}_k \leqq t) \equiv \sum_{s=0}^{t-1} \mathbf{1}_{\{X(s) \in B\}}$$

we get

$$D(t) \leqq Z_{k(t)} + t - K(t).$$

Now it follows from proposition IV.1 that

$$\lim_{N \to \infty} \bar{\mathbb{P}}\left(\mathbb{P}\left(K(t) \leqq t\left(1 - \frac{1}{N^{\gamma-\beta-\varepsilon}}\right)\right) > \delta \right) = 0,$$

for ε small enough.

So that we get when N goes to ∞:

$$d(t) \leqq \frac{Z_{k(t)}}{N} + \frac{t}{N^{1+\gamma-\beta-\varepsilon}}.$$

Now we remark that (Z_k) is a Markov chain and that $\frac{1}{N} Z_k$ has the same law as $d^*(k)$, introduced in section III.
Therefore if

$$t = \frac{2 N \log N}{1 - (1/N^{\gamma-\beta-\varepsilon})}$$

using proposition III.1 $\frac{Z_{k(t)}}{N} \to 0$ in probability, and $\frac{t}{N^{1+\gamma-\beta-\varepsilon}}$ goes to zero, concluding the proof of the second part of Theorem II.2.

VI. THE CASE $\beta > \gamma$

The proof of the first part of Theorem II.2 will be a consequence of the following proposition.

PROPOSITION VI.1. — *Take* $\beta > \gamma$ *and* $\delta < \beta - \gamma$, *there exists* $\varepsilon_0 > 0$, *such that for any* $\varepsilon \in]0, \varepsilon_0[$

$$\lim_{N \to \infty} \bar{\mathbb{E}}(\mathbb{E}_0(\mathbb{P}_1(\mathscr{N}(N^{1+\delta}) > N^{1-\varepsilon}))) = 0,$$

where

$$\mathcal{N}(t)=\sum_{s=0}^{t-1} \mathbf{1}_{\{\hat{\sigma}(s)\neq\hat{\sigma}(s+1)\}}.$$

Proof. – We first remark that,

$$\{\mathcal{N}(t)\geqq n\}=\left\{\sum_{j=1}^{n} \tau_j^{q_j}\leqq t\right\}$$

where the $\tau_j^{q_j}$ are defined in section IV.

Now

$$\bar{\mathbb{E}}\mathbb{E}_0\left(\mathbb{P}_1\left(\sum_{j=1}^{N^{1-\varepsilon}} \tau_j^{q_j}\leqq N^{1+\delta}\right)\right)$$

$$\leqq \bar{\mathbb{E}}\mathbb{E}_0\left(\sum_{s=1}^{N^{1-\varepsilon}-1} \mathbf{1}_{\{\zeta(s)\in M\}}<\frac{N^{1-\varepsilon}}{N^{\gamma+\eta}}\right)$$

$$+\bar{\mathbb{E}}\mathbb{P}_1\left(\sum_{j=1}^{(N^{1-\varepsilon}-1)N^{-\gamma-\eta}} \tau_j^{q}<N^{1+\delta}\right)$$

where $q=\dfrac{1}{1+N^{\beta}}$ and $\eta\in]0, 1-\varepsilon-\gamma[$.

The first term in the right hand side of the previous inequality goes to zero by lemma IV.3, the second one also by direct computation if η and ε are such that

$$\beta-\gamma-\delta>\varepsilon+\eta.$$

This concludes the proof of proposition VI.1.

The proof of the first part of theorem II.2 follows from the following inequality

$$d(t)\geqq N-(\mathcal{N}^{\sigma}(t)+\mathcal{N}^{\xi}(t))$$

where for $x=\sigma, \xi$

$$\mathcal{N}^{x}(t)=\sum_{s=0}^{t-1} \mathbf{1}_{\{X(s)\neq X(s+1)\}}.$$

In fact

$$\bar{\mathbb{P}}(\mathbb{P}(d(t)\leqq N-N^{1-\varepsilon})>\delta)\leqq\bar{\mathbb{P}}(\mathbb{P}(\mathcal{N}^{\sigma}(t)+\mathcal{N}^{\xi}(t)\geqq N^{1-\varepsilon})>\delta)$$

$$\leqq\frac{1}{\delta}\left\{\bar{\mathbb{E}}\left(\mathbb{P}\left(\mathcal{N}^{\sigma}(t)\geqq\frac{N^{1-\varepsilon}}{2}\right)\right)+\bar{\mathbb{E}}\left(\mathbb{P}\left(\mathcal{N}^{\xi}(t)\geqq\frac{N^{1-\varepsilon}}{2}\right)\right)\right\}.$$

Since $\mathcal{N}^{\sigma}(t)$ and $\mathcal{N}^{\xi}(t)$ have the same law as $\mathcal{N}(t)$ introduced in proposition VI.1, this last upper bound goes to zero if $t=N^{1+\delta}$ and this ends the proof.

VII. CONCLUSION AND SOME OPEN QUESTIONS

As it is well known after Döeblin's pioneer work [3], studying the distance between two coupled time evolutions is a crucial point to characterize the relaxation time of the system.

In particular if we want to perform numerical simulations of the invariant states of the system by using some sort of Monte Carlo algorithm we must know how long the computer must run until we get a realistic picture. This is exactly the kind of question we study in this text. We only get a very partial answer.

First of all we did not say anything about the collapsing time in the low temperature case. By analogy with the results obtained in the context of metastability we conjecture that this time converges in law to an exponential random time, when suitably rescaled (*cf.* [6] for a survey of recent results of the so called pathwise approach to metastability).

In this paper we have considered a particular coupling between the two time evolutions. Therefore a second question is: how coupling dependent our results are? In particular in order to characterize the rate of convergence to the invariant state, we must look for an optimal coupling.

In the high temperature phase ($\beta<\gamma$) we have proved that the number of differences between the two processes is negligeable with respect to N. It would be natural to study the fluctuations of the distance.

Another open question is the study of the model for $\beta=\gamma$, where a different behaviour may show up.

Finally it is clear that the model we have presented is an extreme caricature of a disordered system. More realistic models should be treated. A first step in this direction is done in [2] which considers a Glauber dynamics associated to a kind of Random Energy Model.

ACKNOWLEDGEMENTS

We would like to thank V. Gayrard and G. Nappo for a careful reading of the preliminary version of this paper.

B.I.D./U.S.P. grant, CNPq 301301/79 grant, C.N.R. and FAPESP 90/1894-1 and 90/2250-0 grants are also aknowledged.

A. G. would like to thank C.P.T. Marseille-Luminy and Dipartimento di Fisica (Universita di Roma La Sapienza) for warm hospitality.

M. C. and P. P. would like to thank I.M.E.-U.S.P. (Sao Paulo) for warm hospitality.

REFERENCES

[1] M. N. Barber and B. Derrida, *J. Stat. Phys.*, Vol. **51**, 1988, p. 877.
[2] M. Cassandro, A. Galves and P. Picco, *Dynamical Phase Transition in a Disordered System: a Rigorous Analysis* (to appear).
[3] W. Doeblin, *Bull. Math. Soc. Roum. Sci.*, Vol. **39**, No. 1, p. 57 and Nos. 2, 3, 1937.
[4] B. Derrida, *Phys. Rep.*, Vol. **184**, 1989, p. 207.
[5] B. Derrida and G. Weisbush, *Europhys. Lett.*, Vol. **4**, 1987, p. 657.
[6] R. H. Schonmann, An Approach to Characterize Metastability and Critical Droplets in Stochastic Ising Models, *Ann. Inst. H. Poincaré, Phys. Théor.*, this issue.

(*Manuscript received January* 14, 1991; *revised version received March* 2*nd*, 1991.)