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Retarded functional differential equations with white noise perturbations

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ABSTRACT. — Given a retarded functional differential equation, a stable equilibrium point and a bounded neighbourhood contained in its basin of attraction, one perturbs the differential equation with small white noise and states a large deviation result, namely, solutions starting in the neighbourhood very likely escape from it. The end of the escaping path can be described. The Euler-Lagrange equation corresponding to the extremals of the action-functional presents delay and advance in time. An example is completely studied.

RÉSUMÉ. — Soit un voisinage borné d'un point d'équilibre stable d'une équation différentielle avec retard contenu dans le bassin d'attraction de ce point d'équilibre. Nous montrons un résultat de grandes déviations concernant les perturbations de l'équation avec retard par un petit bruit blanc. Nous décrivons la fin du chemin de sortie probable du voisinage. C'est la solution d'une équation avec retard et avance.

0. INTRODUCTION

It is well known that small random perturbations can produce large deviations in the behavior of the trajectories of a deterministic dynamical system. In fact, let us consider as the unperturbed dynamical system an ordinary differential equation given by

$$\dot{x}(t)=b(x(t)) \tag{0.1}$$

and the perturbed system given by

$$\dot{X}^\varepsilon(t)=b(X^\varepsilon(t))+\varepsilon \dot{w}(t), \qquad t>0, \tag{0.2}$$

where $\{w(t), t \geqq 0\}$, $w(0)=0$, is the Brownian motion in $\mathbb{R}^n$ starting from 0, that is, (0.2) is a perturbation of (0.1) with a white noise. Given any $\varepsilon>0$ and $\varphi \in \mathbb{R}^n$, denote by $X^\varepsilon(t, \varphi)$ the solution of (0.2) satisfying $X^\varepsilon(0, \varphi)=\varphi$. Suppose that $D \subset \mathbb{R}^n$ is a bounded domain and $0 \in D$ is an asymptotically stable equilibrium (constant solution) of (0.1). Freidlin and Wentzel considered the action functional associated to (0.1):

$$S(\gamma)=\frac{1}{2}\int_{-T}^{0} |\dot{\gamma}-b(\gamma(t))|^2\,dt, \qquad T>0, \tag{0.3}$$

defined for paths $\gamma \in W^{1,2}[-T, 0]$ = set of all absolutely continuous functions $\gamma:[-T, 0] \to \mathbb{R}^n$ such that $|\dot{\gamma}| \in L^2[-T, 0]$. For each $\varphi \in \mathbb{R}^n$ they introduced the so-called quasipotential of (0.1) relatively to $0 \in D$:

$$V(\varphi)=\inf\{S(\gamma): T>0\,\gamma \in W^{1,2}[-T, 0], \gamma(0)=\varphi, \gamma(-T)=0\}$$

and proved [FW_1] that, if V (φ) has its minimum in a unique point $\varphi_0 \in \partial D$, then

$$\lim_{\varepsilon \to 0} \mathbb{P}\{|X^\varepsilon(\tau_\varepsilon, \bar{\varphi})-\varphi_0|<\delta\}=1 \tag{0.4}$$

for any $\bar{\varphi} \in D$ and $\delta>0$, where $\tau_\varepsilon=\inf\{t>0: X^\varepsilon(t, \bar{\varphi}) \in \partial D\}$. Here, $\mathbb{P}$ denotes the probability measure.

In [FW_2], the same authors have developed a theory of large deviations for Gaussian processes with values in Hilbert spaces; Azencott [Az] also considered processes with values in Banach spaces. As a general reference on large deviations we can mention the book by Deuschel and Stroock [D-S], 1989.

In the present paper, we consider a small stochastic perturbation of a retarded functional differential equation. This kind of equation defines a dynamical system in a Banach space but the solutions have a description in $\mathbb{R}^n$ with many self intersections. The perturbation is made with a white noise in $\mathbb{R}^n$ but, since the dynamical system defines a semigroup in a suitable Banach space, the deviations will be measured with the distance of the (not locally compact) considered space. That is, the solutions starting in a neighbourhood contained in the basin of attraction of an asymptotically stable equilibrium point, very likely escape from the neighbourhood, generalizing the Freidlin and Wentzel ideas. We will see that the Euler-Lagrange equations, which gives the extremal points of the action functional, turns out to be a functional differential equation with advance and delay in time. We were able to compute the minimum on the boundary of a suitable domain of the quasi-potential for the case of the linear equation $\dot{x}=-x(t-b)$.

I. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

Let C be the Banach space of all continuous paths $\varphi:[-1, 0]\to\mathbb{R}^n$ with respect to the norm $\|\varphi\|=\max\{|\varphi(\theta)|, -1\leqq\theta\leqq 0\}$, where $|\,.\,|$ is the Euclidean norm in $\mathbb{R}^n$. For a continuous function $x:[t_0-1, t_0+A)\to\mathbb{R}^n$, $A>0$, and a real number t in $[t_0, t_0+A)$, we write x_t to denote the element in C given by $x_t(\theta)=x(t+\theta), -1\leqq\theta\leqq 0$; then, the function $t\in[t_0, A)\mapsto x_t\in C$ is continuous.

We call retarded functional differential equation a relation of the form

$$\dot{x}(t)=f(x_t) \tag{1.0}$$

where $f:C\to\mathbb{R}^n$ is a continuous function (*see* [Ha]).

For technical reasons, we will suppose that f is continuously differentiable and that

$$\|f\|_1=\max\{\sup_{\varphi\in C}|f(\varphi)|, \sup_{\varphi\in C}\|Df(\varphi)\|\}$$

is finite, where

$$\|Df(\varphi)\|=\sup\{|Df(\varphi)\psi|:\|\psi\|=1\}.$$

Equation (1.0) defines the semigroup solution $\phi(t))_{t\geqq 0}$ – the flow – on C by

$$\phi(t)\varphi=x_t, \qquad t\geqq 0, \qquad \varphi\in C$$

where x is the solution of (1.0) in $[0, +\infty)$ which starts in φ, *i.e.*, $x_0=\varphi$ or $x(\theta)=\varphi(\theta)$, for all θ in $[-1, 0]$. We know that $\phi(t):C\to C$ is continuously differentiable, $\phi(t+s)=\phi(t)\phi(s)$ for all t, $s\geqq 0$, $\phi(0)=$identity of C and also that for any fixed φ in C, and any ψ in C,

$$\lim_{t\to 0}\left[\frac{\partial}{\partial\varphi}(\phi(t)\varphi)\right]\psi=\psi \text{ in C}.$$

Let $W^{1,2}$ be the subset of all functions $\varphi : [-1, 0] \to \mathbb{R}^n$ absolutely continuous on $[-1, 0]$ such that $\dot{\varphi} \in L_2$ *i.e.*

$$\|\dot{\varphi}\|_{L_2} = \left[\int_{-1}^{0} |\dot{\varphi}(\theta)|^2 d\theta\right]^{1/2} < \infty.$$

With respect to the norm

$$\|\varphi\|_{1,2} = \left[|\varphi(0)|^2 + \int_{-1}^{0} |\dot{\varphi}(\theta)|^2 d\theta\right]^{1/2},$$

$W^{1,2}$ is a Hilbert space and the inclusion $W^{1,2} \to C$ is continuous. We know that $W^{1,2}$ is invariant under the flow $(\phi(t))_{t \geqq 0}$ and that $\phi(t) : W^{1,2} \to W^{1,2}$ remains continuously differentiable.

Later, we will suppose that the function $\varphi = 0$ in C is an equilibrium point of equation (1.0), that is, $f(0) = 0$, which attracts its small neighbourhoods. We assume that there exist positive constants K and a such that $\|x_t(\varphi)\| \leqq K e^{-at} \|\varphi\|$ for all $t \geqq 0$ and all φ in a sufficiently small neighbourhood of 0. The above estimate is obtained (*see* [Ha]) by considering the linearized variational equation about $\psi = 0 : \dot{y}(t) = f'(0) y_t$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $w(t) : \Omega \to \mathbb{R}^n$, $t \geqq 0$, be the Brownian motion in $\mathbb{R}^n$.

Let us consider the perturbed stochastic differential equation

$$\dot{X}^{\varepsilon}(t) = f(X_t^{\varepsilon} + \varepsilon \dot{w}(t), \qquad t > 0. \tag{1, ε}$$

A solution of this equation through $\varphi \in C$ at time $t = 0$ is a continuous random variable $X^{\varepsilon}(t, w)$, $t \geqq -1$, $w \in \Omega$ such that

$$X^{\varepsilon}(\theta, \omega) = \varphi(\theta)$$

(with probability one) for all $\theta \in [-1, 0]$ and such that for all $t \geqq 0$, we have

$$X^{\varepsilon}(t) = \varphi(0) + \int_{0}^{t} f(X_s^{\varepsilon})\, ds + \varepsilon w(t)$$

also with probability one.

We know that, for each $\varphi \in C$, there exists one and only one solution of equation $(1, \varepsilon)$ through φ defined for all $t \geqq 0$. We will prove that:

I.1. Proposition. — *Given an interval* $[0, T]$ *and a function* $\varphi \in C$, *then, during the time interval* $[0, T]$, *the solution* $X^{\varepsilon}(t)$ *of equation* $(1, \varepsilon)$ *through* φ *at time* $t = 0$ *very likely follows the solution of equation* (1.0) *through* φ *at time* $t = 0$; *more precisely, for any* $\delta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_{\varphi} \{ \sup_{t \in [-1, T]} X^{\varepsilon}(t) - x(t) | > \delta \} = 0$$

or, equivalently,

$$\lim_{\varepsilon \to 0} \mathbb{P}_\varphi \{ \sup_{t \in [0, T]} \| X_t^\varepsilon - x_t \| > \delta \} = 0,$$

where the subscript φ *indicates that the considered solutions start at* φ.

Proof. — It is easy to see that

$$\| X_t^\varepsilon - x_t \| \leqq \| f \|_1 \int_0^1 \| X_s^\varepsilon - x_s \| \, ds + \varepsilon \, | w(t) |,$$

so, Gronwall's inequality implies that

$$\| X_t^\varepsilon - x_t \| \leqq \varepsilon \, e^{T \| f \|_1} \sup_{t \in [0, T]} | w(t) |.$$

Using the classical inequality (*see* [F-W$_1$]):

$$\mathbb{P} \{ \sup_{0 \leqq t \leqq T} \varepsilon \, | w(t) | > \eta \} \leqq C_1 \, e^{-C_2/\varepsilon^2},$$

we find that

$$\mathbb{P}_\varphi \{ \sup_{0 \leqq t \leqq T} \| X_t^\varepsilon - x_t \| > \delta \} \leqq D_1 \, e^{-D_2/\varepsilon^2}$$

for appropriate constants C_1, C_2, D_1 and D_2; this last inequality clearly implies the conclusion of our proposition.

In order to estimate the probability that a solution X_t^ε of the equation (1, ε) belongs to a neighourhood of a fixed path $\{ x(t), T_1 \leqq t \leqq T_2 \}$, we introduce the action functional associated to the random processes X_t^ε and a quasipotential extending Freidlin and Wentzell's construction for perturbed vector fields [*see* F-W$_1$].

In a similar way as Freidlin and Wentzell do for vector fields, let us define the action functional associated to the processes X_t^ε by:

$$S(\gamma) = \frac{1}{2} \int_{T_1}^{T_2} | \dot{\gamma}(t) - f(\gamma_t) |^2 \, dt,$$

for $\gamma \in W^{1, 2} [T_1, T_2]$, which, when $f = 0$, reduces to the usual action functional associated to the Brownian motion.

Without loss of generality, we can assume that either $T_1 = 0$ and $T_2 > 0$ or $T_2 = 1$ and $T_1 < 0$.

Let us denote by ρ_T the distance between two continuous functions x and y in $C[-1, T]$:

$$\rho_T(x, y) = \max_{t \in [-1, T]} | x(t) - y(t) |$$

and, for $\Phi \subset C[-1, T]$, let us define:

$$\rho_T(x, \Phi) = \inf \{ \rho_T(x, y) : y \in \Phi \}.$$

Let us state, without proof, a straightforward extension of a theorem of Freidlin and Wentzell [F-W$_1$], which will be used several times:

I.2. THEOREM. — *Let* $X^\varepsilon(\varphi)$ *be a solution of the perturbed equation* $(1, \varepsilon)$. *Then, given* $T>1$, $\delta>0$, $\beta>0$ *and* $s_0>0$ *we have:*

(*a*) *There exists* $\varepsilon_0>0$ *such that for all* $\varepsilon\in(0, \varepsilon_0)$ *the inequality*

$$\mathbb{P}\{\rho_T(X^\varepsilon(\varphi), \gamma)<\delta\}\geqq\exp\{-\varepsilon^{-2}(S(\gamma)+\beta)\}$$

hold for all $\varphi\in W^{1,2}$ *and* $\gamma\in W^{1,2}[-1, T]$ *such that* $\gamma_0=\varphi$ *and* $S(\gamma)\leqq s_0$.

(*b*) *There exists* $\varepsilon_0>0$ *such that for all* $\varepsilon\in(0, \varepsilon_0)$, *all* $s\in[0, s_0]$ *and all* $\varphi\in W^{1,2}$, *we have*

$$\mathbb{P}\{\rho_T(X^\varepsilon(\varphi), \Phi(s, \varphi))\geqq\delta\}\leqq\exp\{-\varepsilon^{-2}(s-\beta)\}$$

where $\Phi(s, \varphi)=\{\gamma\in W^{1,2}[-1, T]:\gamma_0=\varphi$ *and* $S(\gamma)\leqq s\}$.

The *quasipotential* of equation (1.0) with respect to the origin 0 in $W^{1,2}$ is, by definition, the functional

$$V(\varphi)=V(0, \varphi)=\inf\{S(\gamma):T_1<T_2, \gamma\in W^{1,2}[T_1-1, T_2], \gamma_{T_1}=0, \gamma_{T_2}=\varphi\}.$$

It is clear that $V(0, \varphi)\geqq 0$ for all $\varphi\in W^{1,2}$ and that $V(0, \varphi)$ is continuous in φ. Moreover, if 0 is an equilibrium of equation (1, 0), then $V(0, 0)=0$ and, since extending backwards γ by zero does not increase the action, one can assume $T_1=-\infty$.

The name quasipotential comes from the fact that for gradient systems in $\mathbb{R}^n$, $\dot{x}(t)=-\operatorname{grad} U(x(t))$, with 0 as an attractor, the quasipotential is twice the potential U, if we stay in the basin of the attractor.

The following theorem, as in the non retarded case, studies the exit from a bounded domain D contained in the basin of an attracting equilibrium. Since, under our hypotheses, the solution operator of the retarded functional differential equation is compact, one can extend each step used by [F-W$_1$] in the proof. The extension, although not completely standard, can be done in a natural way and we will ommit the details.

Let $\tau_\varepsilon=\inf\{t>0:X_t^\varepsilon\in\partial D\}$; as in the nonretarded case, it can be shown to be almost surely finite and moreover, we have:

I.3. THEOREM. — *Let* 0 *be an asymptotically stable equilibrium of equation* (1, 0) *and let* $D\subset C$ *be a bounded connected open neighbourhood of* 0, *the closure of which admits a* δ_0*-neighbourhood* D_{δ_0} *contained in the basin of* 0. *Let us suppose also that* $\bar{D}$ *and* $\bar{D}_{\delta_0}$ *are strictly contracted by the flow of the nonperturbed system, that is,* $\varphi_t(\bar{D})\subset D$ *and* $\phi_t(\bar{D}_{\delta_0})\subset D_{\delta_0}$. *Let us suppose, moreover, that there exists a unique point* $\varphi_0\in\partial D$ *minimizing the quasipotential* $V(0, \varphi)$ *on* ∂D.

Then, for any $\delta>0$ *and any* $\varphi\in D$ *we have:*

$$\lim_{\varepsilon\to 0}\mathbb{P}_\varphi\{\|X_{\tau_\varepsilon}^\varepsilon-\varphi_0\|<\delta\}=1.$$

II. THE ACTION FUNCTIONAL

From now on, we will restrict ourselves to random perturbations of equations of the following type:

$$\dot{x}(t)=f(x_t) \tag{2.1}$$

where $f:\mathrm{C}\to\mathbb{R}^n$ is given by

$$f(\varphi)=\mathrm{F}(\varphi(0),\,\varphi(-1))-\int_{-1}^{0} a(-\theta)\,g(\varphi(\theta))\,d\theta. \tag{2.2}$$

We suppose that $\mathrm{F}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}^n$ are bounded C^1 functions with bounded derivatives and that $a:[0,\,1]\to\mathbb{R}$ is of class C^2.

The perturbed equation is

$$\dot{\mathrm{X}}^\varepsilon(t)=f(\mathrm{X}_t^\varepsilon)+\varepsilon\,\dot{w}(t). \tag{2.3}$$

The corresponding action functional is

$$\mathrm{S}(\gamma)=\frac{1}{2}\int_0^{\mathrm{T}}\left|\dot{\gamma}(t)-\mathrm{F}(\gamma(t),\,\gamma(t-1))+\int_{-1}^{0} a(-\theta)\,g(\gamma(t+\theta))\,d\theta\right|^2 dt.$$

We want to minimize S on the following class:

$$\mathscr{M}=\{\gamma\in\mathrm{C}[-1,\,\mathrm{T}]\cap\mathrm{W}^{1,2}[0,\,\mathrm{T}]:\gamma_0=\varphi,\ \gamma_{\mathrm{T}}=\bar{\varphi}\}$$

where φ and $\bar{\varphi}$ are given functions.

It is clear that we have to assume $\mathrm{T}\geqq 1$.

II.1. Proposition. — *A necessary condition for $\gamma\in\mathscr{M}$ to minimize* S *on $\mathscr{M}$ is that γ satisfies the following Euler-Lagrange equation:*

$$\begin{aligned}\frac{d}{dt}\mathrm{H}(\gamma)(t)&+[\mathrm{D}_1\mathrm{F}(\gamma(t),\,\gamma(t-1))]^*\,\mathrm{H}(\gamma)(t)\\&+[\mathrm{D}_2\mathrm{F}(\gamma(t+1),\,\gamma(t))]^*\,\mathrm{H}(\gamma)(t+1)\\&-[g'(\gamma(t))]^*\int_{-1}^{0} a(-\theta)\,\mathrm{H}(\gamma)(t-\theta)\,d\theta=0\end{aligned} \tag{2.4}$$

where

$$\mathrm{H}(\gamma)(t)=\dot{\gamma}(t)-\mathrm{F}(\gamma(t),\,\gamma(t-1))+\int_{-1}^{0} a(-\theta)\,g(\gamma(t+\theta)\,d\theta$$

(the star indicates the matrix transposition).

Recall that $\gamma_0=\varphi$ and $\gamma_{\mathrm{T}}=\bar{\varphi}$ and observe that the equation above which is retarded and advanced in time, is, as a matter of fact, a second order integro-differential equation. In spite of the fact that we do not know *a priori* that γ has first and second derivatives in $[0,\,\mathrm{T}-1]$, we remark that the function $H(\gamma)(t)$ is absolutely continuous in $[0, T-1]$ and $\frac{d}{dt}H(\gamma(.)) \in L_2[0, T-1]$. Later on we will show that γ is C^2 in $[0, T-1]$.

Proof. Let $h \in C^1[0, T-1]$ such that $h(0)=h(T-1)=0$ and put $h(\theta)=0$, $\theta \in [-1, 0]$, and $h(t)=0$, $t \in [T-1, T]$. We know that if $\gamma \in \mathscr{M}$ is a local minimum for S then $\frac{d}{d\lambda}S(\gamma+\lambda h)|_{\lambda=0}=0$ for all h as above.

Let us make explicit this condition:

$$\frac{d}{d\lambda}S(\gamma+\lambda h)|_{\lambda=0}=\int_0^T H^*(\gamma)(t)\Big[\dot{h}(t)-D_1 F(\gamma(t), \gamma(t-1))h(t)$$
$$-D_2 F(\gamma(t), \gamma(t-1))h(t-1)+\int_{-1}^0 a(-\theta)g'(\gamma(t+\theta))h(t-\theta)\,d\theta\Big]dt.$$

Since $h_0=h_T=0$ and after using integration by parts and inverting the integrals we obtain the following expression:

$$\frac{dS}{d\lambda}(\gamma+\lambda h)|_{\lambda=0}=\int_0^{T-1}\Big\{H^*(\gamma)(t)+\int_0^t\Big[H^*(\gamma)(s)D_1 F(\gamma(s), \gamma(s-1))$$
$$+H^*(\gamma)(s+1)D_2 F(\gamma(s+1), \gamma(s))$$
$$-\int_{-1}^0 a(-\theta)H^*(\gamma)(s-\theta)\,d\theta \,.\, g'(\gamma(s))\Big]ds\Big\}\dot{h}(t)\,dt$$

Since h is arbitrary, the result follows from Du-Bois-Reymond's lemma (*see* [A]) which says that if $\Phi:[a, b] \to \mathbb{R}^n$ is continuous and if $\int_a^b \Phi^*(t)\dot{h}(t)\,dt=0$ for all C^1 functions $h:[a, b] \to \mathbb{R}^n$ such that

$$h(a)=h(b)=0,$$

then Φ is constant on $[a, b]$.

We have then, for all $t \in [0, T-1]$:

$$H^*(\gamma)(t)+\int_0^t\Big[H^*(\gamma)(s)D_1 F(\gamma(s), \gamma(s-1))$$
$$+H^*(\gamma)(s+1)D_2 F(\gamma(s+1), \gamma(s))$$
$$-\int_{-1}^0 a(-\theta)H^*(\gamma)(s-\theta)\,d\theta \,.\, g'(\gamma(s))\Big]ds=\text{const.}$$

Since the integrand belongs to $L_2[0, T-1]$, it follows that $H^*(\gamma)(t)$ is absolutely continuous on $[0, T-1]$. If we transpose the expression obtained after computing the derivative of the last equality we get the following Euler-Lagrange equation which holds for γ, almost everywhere in

$[0, T-1]$:

$$\frac{d}{dt}H(\gamma)(t)+[D_1 F(\gamma(t), \gamma(t-1))]^* H(\gamma)(t) + [D_2 F(\gamma(t+1), \gamma(t))]^* H(\gamma)(t+1) - [g'(\gamma(t))]^* \int_{-1}^{0} a(-\theta) H(\gamma)(t-\theta)\, d\theta = 0,$$

as claimed.

Let us now consider the existence of an absolute minimum for S in the class $\mathcal{M}$, which will imply the existence of solution for the equation (2.4) with boundary condition $\gamma_0=\varphi$ and $\gamma_T=\bar{\varphi}$.

Let $\mu=\inf\{S(\gamma):\gamma\in\mathcal{M}\}\geqq 0$ and $(\gamma_m)_{m\in\mathbb{N}}$ a minimizing sequence of elements in $\mathcal{M}$, that is, $S(\gamma_m)\to\mu$ as $m\to\infty$. Without loss of generality we may assume $S(\gamma_m)\leqq S(\gamma_1)$.

II.2. Proposition. – *There exists a subsequence, also denoted by* $(\gamma_m)_{m\in\mathbb{N}}$, *which converges uniformly in* $[0, T-1]$ *to a function* $\bar{\gamma}\in\mathcal{M}$ *such that* $S(\bar{\gamma})=\mu$.

Proof. – The idea is to show that $\mathcal{M}$ is compact in $C[0, T]$ and that S is lower semi-continuous on $\mathcal{M}$.

We start observing that there exist constants α and β such that $\alpha>0$ and

$$\frac{1}{2}\left|\dot{\gamma}_m(t)-F(\gamma_m(t), \gamma_m(t-1))+\int_{-1}^{0} a(-\theta)\, g(\gamma_m(t+\theta))\, d\theta\right|^2 \geqq \alpha\,|\dot{\gamma}_m(t)|^2-\beta.$$

This implies that

$$-\beta T+\alpha\int_0^T |\dot{\gamma}_m(t)|^2\, dt \leqq S(\gamma_m)\leqq A$$

and then

$$\|\dot{\gamma}_m\|_{L_2}\leqq B=\left[\frac{A+\beta T}{\alpha}\right]^{1/2} \quad \text{for all } m.$$

By the Arzela-Ascoli's Theorem, there exists a subsequence of (γ_m) which converges uniformly to a function $\bar{\gamma}\in C[0, T-1]$; extend $\bar{\gamma}$ to $[-1, T]$ by $\bar{\gamma}_0=\varphi$ and $\bar{\gamma}_T=\bar{\varphi}$. It is easy to show that $\bar{\gamma}$ is absolutely continuous in $[0, T-1]$.

To prove $\dot{\bar{\gamma}}\in L_2[0, T-1]$ take $t\in(0, T-1)$ and choose $h>0$ small enough such that $[t, t+h]\subset[0, T-1]$.

Using again Cauchy-Schwarz inequality we have

$$\left|\frac{\gamma_m(t+h)-\gamma_m(t)}{h}\right|^2 \leqq \frac{1}{h}\int_0^h |\dot{\gamma}_m(t+u)|^2\, du.$$

If we integrate between 0 and $(T-1-h)$ and by inversion in the order of integration one obtains

$$\int_0^{T-1-h}\left|\frac{\gamma_m(t+h)-\gamma_m(t)}{h}\right|^2 dt \leqq \int_0^h\left[\int_0^{T-1-h}|\dot{\gamma}_m(t+u)|^2\,dt\right]du$$

and since the last term is bounded by B^2, we pass to the limit as $m\to\infty$ and see that, for all $\tau<T-1$ and $h>0$ sufficiently small, we have

$$\int_0^\tau\left|\frac{\bar{\gamma}(t+h)-\bar{\gamma}(t)}{h}\right|^2 dt \leqq B^2.$$

Since $\bar{\gamma}$ is absolutely continuous in $[0, T-1]$, we use Fatou's theorem and obtain

$$\int_0^\tau |\dot{\bar{\gamma}}(t)|^2\,dt \leqq B^2.$$

But $\tau<T-1$ arbitrary implies $\dot{\bar{\gamma}}\in L_2[0, T-1]$ and $\|\dot{\bar{\gamma}}\|_{L_2}\leqq B$.

A standard argument shows now that $\dot{\gamma}_m$ converges weakly to $\dot{\bar{\gamma}}$ on $[0, T-1]$, in the sense that, for each $\psi\in L_2[0, T-1]$, we have $\lim_{m\to\infty} I_m=0$, where

$$I_m=\left|\int_0^{T-1}(\dot{\gamma}_m(t)-\dot{\bar{\gamma}}(t))^*\,\psi(t)\,dt\right|.$$

It is easy now to prove that the absolute minimum μ of S is achieved in $\bar{\gamma}$.

III. EXAMPLE

Let us apply the foregoing results to the problem of exit from a domain attracted to 0, in the case of the dynamical system defined by the scalar linear retarded differential difference equation:

$$\dot{x}(t)=-x(t-b). \tag{3.0}$$

We know [Ha] that the condition $0\leqq b<\frac{\pi}{2}$ is a necessary and sufficient condition to ensure that 0 is an asymptotically stable equilibrium of equation (3.0); in fact, this condition is equivalent to assume that all roots of the characteristic equation

$$\lambda+e^{-\lambda b}=0$$

verify $\operatorname{Re}\lambda<0$.

The action functional corresponding to equation (3.0) is given by

$$S(\gamma)=\frac{1}{2}\int_{-T}^{b}[\dot{\gamma}(t)+\gamma(t-b)]^2\,dt,\qquad \gamma\in W^{1,2}[-T,b],\tag{3.1}$$

and the Euler-Lagrange equations for the extremals of S are given by

$$\dot{\gamma}(t)+\gamma(t-b)=H(t),\qquad t\in(-T,b)\tag{3.2.1}$$
$$\dot{H}(t)-H(t+b)=0,\qquad t\in(-T,0].\tag{3.2.2}$$

We will compute the quasipotential of equation (3.1) with respect to the origin.

As noted before, we may suppose that $T=\infty$.

Equation (3.2.2) is an advanced difference-differential equation, which becomes a retarded equation by performing the change of independent variable $t\mapsto -t$. Therefore, by [B-T], given $\psi\in L_2[-b,0]$, and $\xi\in\mathbb{R}$, we can solve (3.2.2) to find a unique function $H:(-\infty,b]\to\mathbb{R}$ which satisfies: 1) $H(b+\theta)=\psi(\theta)$ for almost all $\theta\in[-b,0]$; 2) $H(0)=\xi$; 3) H is absolutely continuous on $(-\infty,0]$, and, 4) for almost all t in $(-\infty,0]$, $\dot{H}(t)=H(t+b)$.

With $H=H(t,\xi,\psi)$ so determined, we solve equation (3.2.1) in $(-\infty,0]$ with initial condition $\gamma_{-\infty}=0$. We get a function $\gamma:(-\infty,+b]\to\mathbb{R}$ which is absolutely continuous on $(-\infty,0]$.

Of course, γ depends upon ψ and ξ; the relations

$$\gamma_b=\varphi,\qquad \varphi\in W^{1,2}\tag{3.3}$$

allow us determine ψ and ξ uniquely as functions of φ. In fact, from the variation of constants formula [Ha], we have

$$\gamma(t)=\int_{-\infty}^{t}X(t-s)\,H(s,\psi,\xi)\,ds,\qquad t\in(-\infty,b),\tag{3.4}$$

where X is the fundamental solution:

$$\begin{aligned}&\dot{X}(t)=-X(t-b),\qquad t>0\\&X(0)=1\\&X(t)=0,\qquad t<0.\end{aligned}$$

Given $\psi\in L_2$, let $J(t)$ be the solution of

$$\dot{J}(t)=-J(t-b),\quad t>0,\qquad J(0)=0,\qquad J_0=\tilde{\psi},$$

where $\tilde{\psi}$ is defined by $\tilde{\psi}(\theta)=\psi(-b-\theta)$ for all $\theta\in[-b,0]$.

It is easy to see that ([Ha], p. 22, formula 6.2):

$$J(t)=-\int_{-b}^{0}X(t+u)\,\psi(u)\,du,\qquad t>0$$

and that

$$H(t,\psi,\xi)=X(-t)\,\xi+J(-t),\qquad t\in(-\infty,0).$$

By equation (3.2.1) we have for $t \in [-b, 0]$:

$$\psi(t-b) = \dot{\varphi}(t-b) + \int_{-\infty}^{t-b} X(t-b-s)\left[X(-s)\xi - \int_{-b}^{0} X(-s+u)\psi(u)\,du\right]ds$$

or

$$\psi(\theta) = \dot{\varphi}(\theta) + \int_{-\theta}^{\infty} X(t+\theta)X(t)\,dt\,\xi - \int_{-b}^{0}\left[\int_{-\theta}^{\infty} X(t+\theta)X(t+u)\,dt\right]\psi(u)\,du. \quad (3.5)$$

Define the function $a(\theta)$, $\theta \in [-b, 0]$, by

$$a(\theta) = \int_{0}^{\infty} X(t-\theta)X(t)\,dt.$$

Then, we can write equation (3.4) as

$$\psi(\theta) = \dot{\varphi}(\theta) + a(\theta)\xi - \int_{-b}^{0} a(-|\theta-u|)\psi(u)\,du. \quad (3.6)$$

Letting $t=0$ in equation (3.4), we get

$$\varphi(-b) = a(0)\xi - \int_{-b}^{0} a(u)\psi(u)\,du. \quad (3.7)$$

Since $a(0)>0$, we can solve equation (3.7) for ξ so that equation (3.6) can be written as

$$\psi(\theta) = \dot{\varphi}(\theta) + \frac{a(\theta)}{a(0)}\varphi(-b) - \int_{-b}^{0} K(\theta, u)\psi(u)\,du \quad (3.8)$$

where $K(\theta, u) = a(-|\theta-u|) - \dfrac{a(\theta)a(u)}{a(0)}$.

We now compute the function $a(\theta)$.

It follows from the definition that

$$\dot{a}(\theta) = -\int_{0}^{\infty} \dot{X}(t-\theta)X(t)\,dt = \int_{0}^{\infty} X(t-\theta-b)X(t)\,dt = a(-b-\theta).$$

Therefore, $\ddot{a}(\theta) = -a(\theta)$. Since $a(-b) = 1/2$, we have

$$a(\theta) = \frac{1+\sin b}{2\cos b\cos\beta}\cos(\theta-\beta) \text{ where } \beta = \frac{\pi}{4} - \frac{b}{2}$$

Also, the kernel $K(\theta, u)$ has the expression

$$K(\theta, u) = -\frac{\cos(\theta+b)\sin u}{\cos b} \quad \text{for} \quad -b \leqq \theta \leqq u \leqq 0$$

$$K(\theta, u) = -\frac{\cos(u+b)\sin\theta}{\cos b} \quad \text{for} \quad -b \leqq u \leqq \theta \leqq 0.$$

Equation (3.8) now becomes the Volterra equation

$$\begin{aligned}\psi(\theta) = \dot{\varphi}(\theta) + \frac{\cos(\theta-\beta)}{\cos\beta}\varphi(-b) \\ + \int_{-b}^{\theta} \frac{\sin\theta}{\cos b}\cos(u+b)\psi(u)\,du + \int_{\theta}^{0}\frac{\cos(\theta+b)}{\cos b}\sin u\,\psi(u)\,du. \qquad (3.9)\end{aligned}$$

Which has a unique solution $\psi \in L_2$ for each $\varphi \in W^{1,2}$.

We now compute the quasipotential relative to the origin.

From what has been proved above, the quasipotential $V(\varphi)$ is given by

$$V(\varphi) = \frac{1}{2}\int_{-\infty}^{b} |\dot{\gamma}(t) + \gamma(t-b)|^2\,dt$$

where $\gamma(t) = x(-t)$, $x(t)$ being the solution of

$$\dot{x}(t) = -x(t-b), \quad t > 0, \qquad x_0 = \tilde{\varphi}.$$

In fact, $\gamma_b = \varphi$ and $\gamma_{-\infty} = 0$ and γ satisfies the variational equations (3.2.1) and (3.2.2).

We have

$$V(\varphi) = \frac{1}{2}\int_{-b}^{\infty} |\dot{x}(t) - x(t+b)|^2\,dt,$$

$$\begin{aligned}V(\varphi) &= \frac{1}{2}\int_{-b}^{\infty} |\dot{x}(t)|^2\,dt - \int_{-b}^{\infty}\dot{x}(t)x(t+b)\,dt + \frac{1}{2}\int_{-b}^{\infty}|x(t+b)|^2\,dt \\ &= \frac{1}{2}\int_{-b}^{0}|\dot{\varphi}(\theta)|^2\,dt + \frac{1}{2}\int_{0}^{\infty}|x(t-b)|^2\,dt \\ &\quad - \left[x(t)x(t+b)\,|_{-b}^{\infty} - \int_{-b}^{\infty} x(t)\dot{x}(t+b)\,dt\right] + \frac{1}{2}\int_{-b}^{\infty}|x(t+b)|^2\,dt \\ &= \frac{1}{2}\int_{-b}^{0}|\dot{\varphi}(\theta)|^2\,d\theta - \frac{1}{2}\int_{-b}^{0}|\varphi(\theta)|^2\,d\theta + \varphi(0)\varphi(-b).\end{aligned}$$

Let $\mathscr{A}: W^{1,2} \to W^{1,2}$ be the linear operator defined by

$$\mathscr{A}\varphi(\theta) = \frac{1}{2}\varphi(\theta) + \frac{1}{2}\left[-\theta\varphi(0) + \varphi(-b) - \int_{-b}^{0}(1-u)\varphi(u)\,du\right] + \frac{1}{2}\int_{-b}^{\theta}(\theta-u)\varphi(u)\,du$$

for all $\theta \in [-b, 0]$.

It is clear that

$$\frac{d}{d\theta}\mathscr{A}\varphi(\theta) = \frac{1}{2}\dot{\varphi}(\theta) + \frac{1}{2}\int_{-b}^{\theta}\varphi(u)\,\mathrm{du} - \frac{1}{2}\varphi(0)$$

and

$$\mathscr{A}\varphi(0) = \frac{1}{2}\left[\varphi(0) + \varphi(-b) - \int_{-b}^{0}\varphi(u)\,du\right].$$

Recall that the inner product in $W^{1,2}$ is given by

$$\langle \varphi, \psi \rangle = \varphi(0)\psi(0) + \int_{-b}^{0}\dot{\varphi}(\theta)\dot{\psi}(\theta)\,d\theta,$$

for all φ, ψ in $W^{1,2}$.

A straitforward computation shows that:

PROPOSITION III.1. — *The operator $\mathscr{A}$ defined above is symmetric and*

$$V(\varphi) = \langle \varphi, \mathscr{A}\varphi \rangle$$

for all φ in $W^{1,2}$.

Let m be the infimum of $V(\varphi)$ over the unit sphere $\|\varphi\|_{1,2} = 1$. Then

$$V(\varphi) \geqq m\|\varphi\|_{1,2}^{2}$$

for all $\varphi \in W^{1,2}$.

THEOREM III.2. — *If $b \in (0, \pi/2)$, the constant m is strictly positive.*

Proof. — We know that m belongs to the spectrum $\sigma(A)$ of A; therefore, it suffices to prove that $\sigma(A)$ is contained in $(0, \infty)$ or, equivalently, that the resolvent set $\rho(A)$ of A contains the interval $(-\infty, 0]$.

Let λ be a real number, $\lambda \leqq 0$, and let us solve the equation

$$(\mathscr{A} - \lambda I)\varphi = \psi$$

for φ in $W^{1,2}$, where ψ is any given element in $W^{1,2}$.

Let us write $1 - 2\lambda = \frac{1}{\mu^2}$ where $\mu \in (0, 1]$ and make the change of variable $\varphi = \mu^2(2\psi + \zeta)$.

Then, the above equation is equivalent to the system

$$\left.\begin{aligned}&\dot{\zeta}+\mu^2\int_{-b}^{\theta}\zeta=\mu^2\left[2\psi(0)+\zeta(0)+\zeta(0)-2\int_{-b}^{\theta}\psi\right]\\&\zeta(0)+\mu^2\zeta(-b)-\mu^2\int_{-b}^{0}\zeta=-\mu^2\left(2\psi(-b)-2\int_{-b}^{0}\psi\right).\end{aligned}\right\}\tag{3.10}$$

Since

$$\ddot{\zeta}+\mu^2\zeta=-2\mu^2\psi$$

we have, by the variation of parameters method:

$$\zeta(\theta)=(c_1+c(\theta))\cos\mu(\theta+b)+(c_2+d(\theta))\sin\mu(\theta+b)\tag{3.11}$$

where c_1 and c_2 are constants to be determined and $c(\theta)$ and $d(\theta)$ are functions given by

$$c(-b)=d(-b)=0$$

and

$$\dot{c}=2\mu\psi(\theta)\sin\mu(\theta+b),\qquad \dot{d}=-2\mu\psi(\theta)\cos\mu(\theta+b).$$

Taking (3.11) into (3.10) we arrive to the system in c_1 and c_2 given by

$$\left.\begin{aligned}&c_1\mu\cos\mu b-(1-\mu\sin\mu b)c_2\\&\qquad\qquad=-2\mu\psi(0)+\mu(c(0)\cos\mu b+d(0)\sin\mu b)\\&c_1(\cos\mu b+\mu^2-\mu\sin\mu b)+c_2(\sin\mu b+\mu\cos\mu b-\mu)=-2\mu^2\psi(-b)\\&\qquad\qquad-c(0)(\cos\mu b-\mu\sin\mu b)-d(0)(\sin\mu b+\mu\cos\mu b).\end{aligned}\right\}\tag{3.12}$$

Let D be the determinant of the coefficients of c_1 and c_2.
Then, $D=D(\mu)=2\mu^2+(1-\mu^2)\cos\mu b-(\mu+\mu^3)\sin\mu b$.

Let us prove that for any $\mu\in(0,1]$, $D>0$. Since $b\in\left(0,\frac{\pi}{2}\right)$, then $\cos\mu b>\cos b>0$ and $0<\sin\mu b<\sin b$; therefore

$$D\geqq 2\mu^2+(1-\mu^2)\cos b-(\mu+\mu^3)\sin b.$$

Letting τ be the tangent of $\frac{b}{2}$, we can write $D\geqq\frac{g(\mu)}{1+\tau^2}$, where

$$g(\mu)=1-\tau^2-2\tau\mu+(1+3\tau^2)\mu^2-2\tau\mu^3$$

and it is easy to see that $g(\tau)=(1-\tau^2)^2$ and

$$g(\mu)=2(\mu-\tau)^2(1-\mu\tau)+g(\tau).$$

Since $0<\tau<1$, we see immediately that $g(\mu)>0$ for any $\mu\in(0,1]$.

Therefore, we can solve (3.12) uniquely for c_1 and c_2 and, with ζ given by (3.11), we get a unique solution φ of $(\mathscr{A}-\lambda I)\varphi=\varphi$, for any $\psi\in W^{1,2}$, and any $\lambda\leqq 0$, depending linearly and continuously on ψ, which proves that $\rho(\mathscr{A})\supset(-\infty,0]$. This finishes the proof.

From Theorem III.2, we conclude that $\sqrt{V}$ is a hilbertian norm in $W^{1,2}$, which is equivalent to the previous one.

If we take D as the open ball with respect to the norm $\sqrt{V}$, with center c and radius R, where c and R are positive constants, $c<R$, then Theorem I.3 can be applied to the perturbed system

$$\dot{X}(t)^{\varepsilon} = -X^{\varepsilon}(t-b) + \varepsilon \dot{w}(t)$$

if $0<b<\pi/2$.

To show that fact, we must prove that the absolute minimum of $\sqrt{V(\varphi)}$ on $\partial D = \{\varphi \in W^{1,2} : V(\varphi - c) = R^2\}$ is achieved only at one point, namely, $\varphi = \left(1 - \dfrac{R}{\sqrt{V(c)}}\right) c$.

In fact, we have:

$$V(\varphi_0) = (R - \sqrt{V(c)})^2 \qquad \text{and} \qquad \text{if } \varphi \in \partial D$$

then

$$V(\varphi) = V(\varphi - c + c) = V(\varphi - c) + V(c) + 2\langle \varphi - c, c \rangle_V$$

where $\langle\ ,\ \rangle_V$ denotes the inner product associated to V; therefore

$$V(\varphi) \geqq R^2 + V(c) - 2\sqrt{V(\varphi - c)}\,.\,\sqrt{V(c)} = V(\varphi_0)$$

and the inequality is strict if $\varphi \neq \varphi_0$, which proves that the minimum of $\sqrt{V}$ on ∂D occurs only at φ_0.

As a final remark, we note that the method we have used can be applied to many other systems, for example, to the scalar linear integro-differential equation

$$\dot{x}(t) = -\int_{-r}^{0} a(-\theta)\, x(t+\theta)\, d\theta$$

with $a(t) = e^{-(t-r)} - 1$, $t \in [0, r]$.

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