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An overview on large deviations of the empirical measure of interacting particle systems

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ABSTRACT. — We present the main ideas in the theory of large deviations for the empirical measure of interacting particle systems. To illustrate these ideas, we consider the superposition of a symmetric simple exclusion process and a Glauber dynamic on a torus of finite macroscopic volume. We obtain a lower and an upper bound which coincide in a dense subset of the space of trajectories.

Key words : Infinite particle systems, large deviations, empirical measure.

RÉSUMÉ. — Nous présentons les principales idées de la théorie des grandes déviations de la mesure empirique de systèmes à une infinité de particules interagissantes. Pour les illustrer, nous étudions une superposition de l'exclusion simple symétrique et d'une dynamique de Glauber sur un tore à volume macroscopique fini. Nous obtenons une borne supérieure et inférieure qui sont égales sur un ensemble dense de l'espace des trajectoires.

Classification A.M.S. : Primary 60 K 35.

1. INTRODUCTION

The purpose of this paper is to present the main ideas in proving large deviation principles for the empirical measure of interacting particle systems. To illustrate these ideas, we will consider a process closely related to reaction-diffusion equations. It is a Markov process where two dynamics are superposed. The first one, the so called symmetric simple exclusion process can be informally described as follows. Consider a countable space X and symmetric transition probabilities $p(k, j)$ on X. Initially, we distribute particles on X in such a way that there is at most 1 particle at each site. Each particle waits, independently from the others, a mean 1 exponential time at the end of which, if this particle is on k, it chooses a site j with probability $p(k, j)$. If the chosen site j is unoccupied, the particle jumps to j. Otherwise it stays at k.

Superposed with this process we consider a Glauber dynamics: for each site k of X, consider two non negative function b_k and d_k on the configurations η of particles on X. Assume that each one of these functions depends on the configurations only through a finite number of sites. At each site k of X and for each configuration η, if the site is unoccupied, a particle is created at a rate $b_k(\eta)$. If it is occupied, the particle is destroyed at a rate $d_k(\eta)$.

The exclusion process is accelerated in order to obtain a diffusion phenomen on some space rescaling.

We present a method, quite general, proposed by Kipnis, Olla and Varadhan in [KOV], to prove large deviations principles for the empirical measure of interacting particle systems.

In the middle of the seventies, Donsker and Varadhan presented a new approach to the theory of large deviations for Markov processes (*see*, for instance, [DV1]-[DV3]). They succeed to obtain an elegant proof of a large deviation principle for a wide class of Markov processes. In some sense their proof follows the same ideas of the classical proof of large deviations of the mean of a sequence of i. i. d. random variables. The same ideas apply to interacting particle systems as we will see in sections 3-5.

We first recall how to prove a large deviation principle for the mean of i. i. d. random variables. There are two main steps. Let $\{Y_i\}$ be real i. i. d. random variables and suppose, to avoid technicals difficulties, that these variables have exponential moments of all orders ($E[\exp\{\theta Y_1\}]<\infty$ for all $\theta\in\mathbf{R}$) and that for every $K\in\mathbf{R}$, $P[Y_1>K]>0$ and $P[Y_1<K]>0$.

The first main difficulty is to find how the random variables Y_i behave on the set $(1/N)\sum_{1\leq i\leq N} Y_i=\alpha$. In other words, for every $\alpha\in\mathbf{R}$, we have to study the distribution of Y_1 given that $(1/N)\sum_{1\leq i\leq N} Y_i=\alpha$.

For $\alpha\in\mathbf{R}$, consider i.i.d. random variables $\{Y_i(\alpha)\}$ such that for every bounded measurable function f,

$$E[f(Y_1(\alpha))]=E_\alpha[f(Y_1)]=(1/M(\alpha))\,E[\exp\{\alpha Y_1\}f(Y_1)]$$

where $M(\alpha)=E[\exp\{\alpha Y_1\}]$. A computation shows that under regularity conditions, the distribution of Y_1 given that $(1/N)\sum_{1\leqq i\leqq N} Y_i=\alpha$ converges in law to $Y_1(\alpha)$ (*see* [DF] for some results and a list of references).

Therefore, to estimate $P[(1/N)\sum_{1\leqq i\leqq N} Y_i=\alpha]$, we have to compare the distribution of $(Y_1, \ldots, Y_N)$ with that of $(Y_1(\alpha), \ldots, Y_N(\alpha))$. This is the main idea in the proof of large deviations.

We first consider the upper bound part of the large deviation principle. Let $K\subset\mathbf{R}$ be compact and for an integer N, let $\bar{Y}_N$ denotes the mean of the first N r.v. Y_i: $\bar{Y}_N=(1/N)\sum_{1\leqq i\leqq N} Y_i$. Then, by the definition of the random variables $Y_i(\alpha)$,

$$\begin{aligned}P[\bar{Y}_N\in K]&=E[\mathbf{1}_{\{\bar{Y}_N\in K\}}]\\&=E_\alpha[e^{-N\{\alpha\bar{Y}_N-\log M(\alpha)\}}\mathbf{1}_{\{\bar{Y}_N\in K\}}]\\&\leqq E_\alpha[e^{-N\inf_{x\in K}\{\alpha x-\log M(\alpha)\}}\mathbf{1}_{\{\bar{Y}_N\in K\}}].\end{aligned}$$

Since $E_\alpha[\mathbf{1}_{\{\bar{Y}_N\in K\}}]$ is bounded by 1, minimizing with respect to all the perturbations considered, we obtain that

$$\limsup_{N\to\infty}\frac{1}{N}\log P[\bar{Y}_N\in K]\leqq-\sup_{\alpha\in\mathbf{R}}\inf_{x\in K}\{\alpha x-\log M(\alpha)\}.$$

Here appears a technical difficulty, we have to exchange the order of the supremum and of the infimum in the formula above.

By Hölder's inequality, $\log M(\alpha)$ is a convex function. Moreover, since $E[\exp\{\theta Y_1\}]<\infty$ for all $\theta\in\mathbf{R}$, $\log M(\alpha)$ is continuous. Therefore $f(\alpha, x)=\alpha x-\log M(\alpha)$ is a concave function in the first variable and convex in the second. Since f is continuous, by Theorem 4.2′ of [S], we can exchange the supremum and the infimum in the formula above.

In this way, we prove the upper bound part of the large deviation principle and we obtain a variational formula for the rate function:

$$\limsup_{N\to\infty}\frac{1}{N}\log P[\bar{Y}_N\in K]\leqq-\inf_{x\in K} I(x),$$

where $I(x)=\sup_{\alpha\in\mathbf{R}}\{\alpha x-\log M(\alpha)\}$.

The second main difficulty consists in to obtain an explicit expression for this rate function. If $\log M(\alpha)$ is strictly convex, since for every $K\in\mathbf{R}$, $P[Y_1>K]>0$ and $P[Y_1<K]>0$, for every $x\in\mathbf{R}$ there exists a unique $\alpha(x)$

such that $I(x)=\alpha(x)x-\log M(\alpha(x))$. $\alpha(x)$ is given by the equation $x=M'(\alpha(x))/M(\alpha(x))$. In particular

$$E_{\alpha(x)}[Y_1]=\frac{E[Y_1 e^{\alpha(x)Y_1}]}{E[e^{\alpha(x)Y_1}]}=\frac{M'(\alpha(x))}{M(\alpha(x))}=x.$$

So that for every $x\in\mathbf{R}$, there exist a perturbation $Y_1(\alpha(x))$ which has a Radon-Nikodym derivative with respect to the original r. v. Y_1 and has mean equal to x.

In the symmetric exclusion process this second main difficulty is overcomed with a Riesz representation theorem on some Sobolev space (*cf.* [KOV], Lemma 5.1). In the case considered in this paper, the proof is left to a forthcoming paper [JLV]. We just obtain an explicit expression for the rate function for smooth trajectories.

With this explicit expression for the upper bound rate function, we are ready to prove the lower bound. Just remark that since $E_{\alpha(x)}[Y_1]=x$, $\bar{Y}_N(\alpha(x))\to x$ in probability.

Let $V\subset\mathbf{R}$ be an open set and for a fixed $\delta>0$ and $x_0\in V$, $x_0\neq E[Y_1]$, let $V_{x_0}\subset V$ an open neighborhood of x_0 such that, $|y-x_0|<|\delta/\alpha(x_0)|$ for every $y\in V_{x_0}$. Then,

$$\begin{aligned}P[\bar{Y}_N\in V]&\geqq P[\bar{Y}_N\in V_{x_0}]\\&=E_{\alpha(x_0)}[e^{-N\{\alpha(x_0)\bar{Y}_N-\log M(\alpha(x_0))\}}\mathbf{1}_{\{\bar{Y}_N\in V_{x_0}\}}]\\&\geqq e^{-N\{\alpha(x_0)x_0-\log M(\alpha(x_0))+\delta\}}E_{\alpha(x_0)}[\mathbf{1}_{\{\bar{Y}_N\in V_{x_0}\}}].\end{aligned}$$

Therefore, since $\bar{Y}_N(\alpha(x))\to x$ in $P_{\alpha(x_0)}$-probability, letting $\delta\to 0$, we obtain that

$$\liminf_{N\to\infty}\frac{1}{N}\log P[\bar{Y}_N\in V]\geqq -\{\alpha(x_0)x_0-\log M(\alpha(x_0))\}=-I(x_0).$$

Since this is true for every $x_0\in V$, $x_0\neq E[Y_1]$, we have proved the lower bound part of the large deviation principle.

We have seen that to obtain the lower bound part of the large deviation principle, we need first to prove a law of large numbers for the perturbed r. v. In the case of particle systems, this result will require some work. In section 3, we will obtain hydrodynamical limits for the perturbed processes necessary to the lower bound. We shall remark that the law of large numbers obtained in section 3 and the method proposed by Guo, Papnicolau and Varadhan in [GPV] to derive it constitute in themselves intersting results.

In the proof of large deviations for interacting particle systems, appears another important difficulty. When writing the Radon-Nikodym derivative of the perturbed process with respect to the original one, appear expressions which can not be expressed in terms of the empirical measure.

Therefore, we have to obtain a result which enables us to substitute these expressions by functions of the empirical measure. A strong result which allows this substitution was obtained by Kipnis, Olla and Varadhan in [KOV] and is stated here as Theorem 3.1.

We present in the next sections the main ideas of the proofs. We slightly change the proofs of [KOV] in order to follow as closely as possible the proof of large deviations for i.i.d. random variables just presented. Moreover, instead of considering the empirical density, we will prove large deviations for the empirical measure, a more natural functional. In section 2 we present the results and establish the notation. In section 3 we obtain the law of large numbers required for the lower bound and in the last two sections we prove the upper and lower bound parts of the large deviations.

The main interest in proving such a result for the simple exclusion process superposed with a Glauber dynamics is to understand the nature of small deviations from the hydrodynamic trajectory for non conservative systems (*see* [J] for an interesting discussion on the subject).

Extensions of the results presented here, detailed proofs and a study of the rate function, will appear in [JLV].

2. NOTATION AND RESULTS

In this section we establish the notation and state the main results. To avoid technical difficulties, our state space will be of finite macroscopical volume. Nevertheless, adjusting the results of [L] to our context, we can extend the large deviations results obtained here to the infinite volume case. Throughout this paper, for an integer N, T_N will denote the torus with N points: $T_N = \{k/N;\ 0 \leqq k \leqq N-1\}$ and T the torus $T = \{0 \leqq x < 1\}$. Our state space $\{0, 1\}^{T_N}$ will be denoted by $\mathbf{X}_N$, while the configurations of $\mathbf{X}_N$ will be denoted by greek letters η and ξ. In this way, for $0 \leqq k \leqq N-1$, $\eta(k) = 1$ if there is a particle on k for the configuration η and $\eta(k) = 0$ otherwise.

For $k, j \in \mathbf{N}$, we denote by $C^{k, j}([0, 1] \times T)$ the functions on $[0, 1] \times T$ which have k continuous derivatives with respect to the time variable and j continuous derivatives with respect to the space variable. $C_e^{k, j}([0, 1] \times T)$ denotes the subset of $C^{k, j}([0, 1] \times T)$ of functions $\rho(t, x)$ for which there exists $\varepsilon > 0$ such that $\varepsilon < \rho < 1 - \varepsilon$.

Let b, d: $\mathbf{X}_N \to \mathbf{R}_+$ be fixed cylindrical functions, *i.e.*, two functions which depend only on a finite number of sites. For $k \in \mathbf{Z}$, let $\tau_k : \mathbf{X}_N \to \mathbf{X}_N$ be the translation by k on $\mathbf{X}_N$: $\tau_k \eta$ is the configuration obtained from η such that $\tau_k \eta(j) = \eta(k+j)$ for every $j \in \mathbf{Z}$, where the sum is taken in modulus N. We extend the translations to the functions on $\mathbf{X}_N$ and to the measures in the natural way: for every real continuous function f, $(\tau_k f)(\eta)=f(\tau_k \eta)$ and for every probability μ, $\int f d(\tau_k \mu)=\int (\tau_k f)\, d\mu$.

We consider a superposition of the symmetric simple exclusion process and Glauber dynamics on $\mathbf{X}_N$. This process (η_t) was informally described in the introduction. It is the unique strongly continuous Markov process whose generator acts on function as

$$\begin{aligned} L_N f(\eta)=\frac{N^2}{2} \sum_{|j-k|=1} [f(\eta^{k,j})-f(\eta)] \\ +\sum_k \tau_k b(\eta)[f(\eta_k)-f(\eta)]+\sum_k \tau_k d(\eta)[f(\eta^k)-f(\eta)], \end{aligned} \tag{2.1}$$

where for $0\leqq k, j\leqq N-1$, $\eta^{k,j}$, η^k and η_k are the configurations:

$$\eta^{k,j}(i)=\begin{cases}\eta(i) & \text{if } i\neq j, k,\\ \eta(j) & \text{if } i=k,\\ \eta(k) & \text{if } i=j,\end{cases} \qquad \eta_k(i)=\begin{cases}\eta(i) & \text{if } i\neq k\\ 1 & \text{if } i=k\end{cases} \qquad \eta^k(i)=\begin{cases}\eta(i) & \text{if } i\neq k\\ 0 & \text{if } i=k\end{cases} \tag{2.2}$$

and τ_k are the translations defined in the last paragraph.

Given a real function γ on T such that $0\leqq\gamma\leqq 1$, we denote by ν_γ^N the product measure on $\mathbf{X}_N$ with marginals given by $\nu_\gamma^N\{\eta(k)=1\}=\gamma(k/N)$, $0\leqq k\leqq N-1$. We identify the constant ρ with the real constant function ρ on T. Also for $0\leqq\rho\leqq 1$ and φ a cylindrical function, let

$$\tilde{\varphi}(\rho)=\int \varphi\, d\nu_\rho. \tag{2.3}$$

We denote by P_N^γ the probability on the path space $D([0, 1], \mathbf{X}_N)$ corresponding to the process with generator L_N given by (2.1) and with initial measure ν_γ^N.

Let M be the space of subprobability measures on the torus T. For $\sigma\in M$ and $f\in C(T)$, define $\langle \sigma, f\rangle=\int_T f\, d\sigma$. Consider M endowed with the topology induced by C(T) with the duality $\langle\ ,\ \rangle$. In this metrizable topology M is compact.

Let λ denotes the Lebesgue measure on T and Let M_1 be the closed subspace of M defined as

$$M_1=\left\{\mu\in M;\ \mu\ll\lambda \text{ and } \frac{d\mu}{d\lambda}\leqq 1\right\}. \tag{2.4}$$

For $\sigma\in M_1$, when no confusion arises, we also denote by σ the density of σ with respect to λ.

Consider the empirical measure μ^N:

$$\mu_t^N = \frac{1}{N} \sum_{0 \leqq k \leqq N-1} \eta_t(k)\, \delta_{k/N}, \tag{2.5}$$

where for $x \in T$, δ_x is the probability measure concentrated on x.

This is the subprobability obtained from the configuration η_t assigning mass $1/N$ on each particle. Let $D([0, 1], M)$ be the path space of the process μ_t^N and Q_N^γ the probability on $D([0, 1], M)$ corresponding to the process μ_t^N with initial measure ν_N^γ.

In the proof of the large deviation principle for the empirical measure, two assumptions on the process are needed for technical reasons. For $0 \leqq \rho \leqq 1$, let

$$\begin{aligned} B(\rho) &= \nu_\rho [b(\eta)(1 - \eta(0))] \\ D(\rho) &= \nu_\rho [d(\eta)\, \eta(0)]. \end{aligned} \tag{2.6}$$

Troughout this paper we will assume that:

(A1): $B(\rho) - D(\rho)$ is linear.

(A2): B and D are concave functions.

Condition (A1) implies that the hydrodynamical equation of the process is a second order linear PDE:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \Delta \rho + B(\rho) - D(\rho).$$

In the process where particles are created and disappear in each site independently to the others, $b(\eta) = c_1 (1 - \eta(0))$ and $d(\eta) = c_2\, \eta(0)$, where c_1 and c_2 are non negative constants. In this case, $B(\rho) = c_1 (1 - \rho)$ and $D(\rho) = c_2\, \rho$. Therefore, assumptions (A1) and (A2) are satisfied. In [JLV], we extend the results presented here to processes which do not satisfy assumption (A1).

As it was explained in the introduction, to prove large deviations principles for Markov processes, we have to consider small perturbations of our original process. In our case, the perturbations are of the following type.

Let $H \in C^{1,2}([0, 1] \times T)$. Consider the unique strongly continuous Markov process whose generator acts on functions as:

$$\begin{aligned} L_{N,t}^H f(\eta) = \frac{N^2}{2} \sum_{|j-k|=1} & \eta(j)[1 - \eta(k)]\, e^{H(t, k/N) - H(t, j/N)} [f(\eta^{k,j}) - f(\eta)] \\ & + \sum_k \tau_k b(\eta)\, e^{H(t, k/N)} [f(\eta_k) - f(\eta)] \\ & + \sum_k \tau_k d(\eta)\, e^{-H(t, k/N)} [f(\eta^k) - f(\eta)], \end{aligned} \tag{2.7}$$

where $\eta^{k,j}$, η_k and η^k were defined in (2.2).

In this process, the particles do not evolve symmetrically on $\mathbf{X}_N$ as before but with a small time and space dependent drift $\frac{1}{N}\frac{\partial H}{\partial x}$. In the same way, the birth and death rates $e^{H}[1-\eta(0)]\,b(\eta)$ and $e^{-H}\eta(0)\,d(\eta)$ also depend on time and space and are related to the drift.

Let $P_N^{H,\gamma}$ and $Q_N^{H,\gamma}$ denote respectively the probability measure on the path space $D([0,1], \mathbf{X}_N)$ and $D([0,1], M)$ corresponding to the processes η_t and μ_t^N with generator given by (2.7) and with initial measure ν_γ^N.

We now introduce the rate functions. For $H \in C^{1,2}([0,1]\times T)$, let $J_H : D([0,1], M) \to \mathbf{R}$ be given by:

$$
\begin{aligned}
J_H(\mu) = {} & \langle m_1, H_1 \rangle - \langle m_0, H_0 \rangle \\
& - \int_0^1 dt \left\langle m_t, \frac{\partial H_t}{\partial t} + \frac{1}{2}\Delta H_t \right\rangle - \frac{1}{2}\int_0^1 dt \left\langle m_t(1-m_t), \left[\frac{\partial H_t}{\partial x}\right]^2 \right\rangle \\
& - \int_0^1 dt \{ \langle B(m_t), (e^{H_t}-1) \rangle + \langle D(m_t), (e^{-H_t}-1) \rangle \}, \qquad (2.8)
\end{aligned}
$$

if $\mu \in D([0,1], M_1)$, where m_t denotes $d\mu_t/d\lambda$ and where B and D have been introduced in (2.6) and M_1 in (2.4). $J_H(\mu) = \infty$ if $\mu \notin D([0,1], M_1)$.

It follows from the assumptions (A1) and (A2) that for each $H \in C^{1,2}([0,1]\times T)$, J_H is a convex lower semicontinuous function. Let $\tilde{I}_0 : D([0,1], M) \to [0,\infty]$ be given by

$$
\tilde{I}_0(\mu) = \sup_{H \in C^{1,2}([0,1]\times T)} J_H(\mu). \qquad (2.9)
$$

$\tilde{I}_0$ inherit the properties of convexity and lower semicontinuity of J_H.

For $\gamma : T \to [0,1]$, define $h_\gamma : D([0,1], M) \to \mathbf{R}_+$ in the following way,

$$
h_\gamma(\mu) = \sup_{g \in C_e(T)} \left\{ \left\langle m_0, \log\frac{g}{\gamma} \right\rangle + \left\langle (1-m_0), \log\frac{1-g}{1-\gamma} \right\rangle \right\} \qquad (2.10)
$$

if $\mu \in D([0,1], M_1)$, and $h_\gamma(\mu) = +\infty$ if $\mu \notin D([0,1], M_1)$.

A simple computation shows that for $\mu \in C_e([0,1], M_1)$ and $\gamma \in C_e(T)$,

$$
h_\gamma(\mu) = \left\langle m_0, \log\frac{m_0}{\gamma} \right\rangle + \left\langle (1-m_0), \log\frac{1-m_0}{1-\gamma} \right\rangle. \qquad (2.11)
$$

We are now ready to define the upper bound rate function. In section 4, we will see that once the perturbations have been fixed, this rate function given by a variational formula appears quite naturally. For $\gamma : T \to [0,1]$, let $\tilde{I}_\gamma : D([0,1], M) \to [0,\infty]$:

$$
\tilde{I}_\gamma = \tilde{I}_0(\mu) + h_\gamma(\mu). \qquad (2.12)
$$

Since h_γ is clearly convex and lower semicontinuous, $\tilde{I}_\gamma$ has also these properties. These properties will be important to show that the upper bound rate function just introduced and the lower bound rate function are equal.

Now, we introduce the lower bound rate function I_γ. It may seems that the way we define this lower rate function is artificial. Nevertheless, we will see at the end of section 4 that once the upper bound rate function is defined, the lower bound rate function is automatically obtained, at least on some subset of regular paths of $D([0,1], M_1)$.

For $\rho \in C_e^{2,3}([0,1]\times T)$, there exist a unique $H \in C^{1,2}([0,1]\times T)$ such that

$$\frac{\partial \rho}{\partial t}=\frac{1}{2}\Delta\rho-\frac{\partial}{\partial x}\left[\frac{\partial H}{\partial x}\rho(1-\rho)\right]+B(\rho)e^{H}-D(\rho)e^{-H}. \tag{2.13}$$

Let $I_0 : C_e^{2,3}([0,1]\times T) \to \mathbf{R}_+$ be given by

$$\begin{aligned}I_0(\rho)=\frac{1}{2}\int_0^1 dt\left\langle \rho_t(1-\rho_t), \left[\frac{\partial H_t}{\partial x}\right]^2\right\rangle \\ +\int_0^1 dt\langle B(\rho_t), 1-e^{H_t}+H_t e^{H_t}\rangle \\ +\int_0^1 dt\langle D(\rho_t), 1-e^{-H_t}-H_t e^{-H_t}\rangle,\end{aligned}$$

where $H \in C^{1,2}([0,1]\times T)$ is the solution of (2.13). In section 4, we will see that for $\rho \in C_e^{2,3}([0,1]\times T)$, $\tilde{I}_0(\rho)=I_0(\rho)$ and it is easy to see that $C_e^{2,3}([0,1]\times T)$ is dense in $C([0,1], M_1)$ endowed with the Skorohod topology. We extend I_0 to the whole $D([0,1], M_1)$ in the natural way to obtain a lower semicontinuous function:

$$I_0(\rho)=\lim_{\varepsilon\to 0}\ \inf_{\mu \in B(\rho,\varepsilon)\cap C_e^{2,3}([0,1]\times T)} I_0(\mu),$$

where $B(\rho, \varepsilon)$ is the ball centered at ρ with radius ε.

Since $C([0,1], M_1)$ is a closed subset of $D([0,1], M)$ for the Skorohod topology, we see from this definition that $I_0(\rho)=\infty$ if $\rho \notin C([0,1], M_1)$. On the other hand it is not hard to prove that $\tilde{I}_0(\rho)=\infty$ if $\rho\notin C([0,1], M_1)$ (*see* [JLV]).

Since $\tilde{I}_0 \equiv I_0$ on $C_e^{2,3}([0,1]\times T)$, from the definition of I_0 and from the lower semi-continuity of $\tilde{I}_0$, we obtain that $\tilde{I}_0(\rho)\leqq I_0(\rho)$ for every $\rho\in D([0,1], M)$. Moreover since I_0 and $\tilde{I}_0$ coincide in a convex dense subset of $D([0,1], M)$, and since $\tilde{I}_0$ is a convex function, I_0 is also convex. We can now introduce the lower bound rate function. Let $I_\gamma : D([0,1], M) \to [0,\infty]$; $I_\gamma = I_0 + h_\gamma$.

In a forthcoming paper [JLV], we will see that $\tilde{I}_0 \equiv I_0$. Since the proof of this result is too long, we will skip this point and just remark that $\tilde{I}_0$ and I_0 coincide on a dense subset of $D([0,1], M)$.

We are now ready to state the main theorems of this paper. We saw in the introduction that to prove the lower bound part of a large deviation principle, we need first to obtain a law of large numbers for the modified processes considered. In our context, this means that we have to prove that the empirical measures μ^N for the process with generator $L^H_{N,t}$ and starting from the initial measure ν^γ_N converge in some sense to a trajectory μ. This is the content of Theorem 1. In the next sections, we prove the following theorems.

THEOREM 1 (HYDRODYNAMICAL LIMITS). – *Let* $H \in C^{1,2}([0, 1] \times T)$ *and* $\gamma \in C_e(T)$. *Then,* $Q^{H,\gamma}_N \Rightarrow Q$, *where* Q *is the probability measure concentrated on the deterministic trajectory* ρ, *where* ρ *is the unique bounded weak solution of*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \Delta \rho - \frac{\partial}{\partial x}\left[\frac{\partial H}{\partial x} \rho(1-\rho)\right] + B(\rho) e^{H} - D(\rho) e^{-H}$$
$$\rho(0, .) = \gamma$$

and $\Rightarrow$ *denotes the weak convergence of probabilities.*

THEOREM 2 (LARGE DEVIATIONS UPPER BOUND). – *Let* $\gamma \in C_e(T)$. *Then, for every compact set* $K \subset D([0, 1], M)$,

$$\limsup_{N \to \infty} \frac{1}{N} \log Q^\gamma_N[K] \leqq - \inf_{\mu \in K} \tilde{I}_\gamma(\mu).$$

THEOREM 3 (LARGE DEVIATIONS LOWER BOUND). – *Let* $\gamma \in C_e(T)$. *Then, for every open set* $V \subset D([0, 1], M)$,

$$\liminf_{N \to \infty} \frac{1}{N} \log Q^\gamma_N[V] \geqq - \inf_{\mu \in V} I_\gamma(\mu).$$

3. HYDRODYNAMICAL LIMITS

In this section, we obtain the law of large numbers needed in the proof of the lower bound part of the large deviation principle. We begin the section stating a theorem of Kipnis, Olla and Varadhan. It allows to substitute local quantities by macroscopic ones. When deriving hydrodynamical limits and proving the large deviation principles we obtain local quantities which can not be expressed in terms of the empirical measure. This next theorem enables to exchange them by functions of the empirical measure.

THEOREM 3.1 (Kipnis, Olla, Varadhan). — *Let* $H \in C([0,1] \times T)$, $\gamma : T \to [0,1]$ *and* φ *a cylindrical function. Define*

$$V_{N,\varepsilon}^{H,\varphi}(t,\eta) = \frac{1}{N}\sum_{i=1}^{N} H\left(t, \frac{i}{N}\right) + \left[\tau_i \varphi(\eta) - \tilde{\varphi}\left(\frac{1}{2\varepsilon N+1}\sum_{|i-j| \leq N\varepsilon} \eta(j)\right)\right].$$

Then, for every $\delta > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P_N^{H,\gamma}\left[\left|\int_0^1 V_{N,\varepsilon}^{H,\varphi}(t,\eta_t)\,dt\right| > \delta\right] = -\infty,$$

where $\tilde{\varphi}$ *is defined in* (2.3).

We refer the reader to [KOV] for a proof of this result in the finite volume case and to [L] for an extension to infinite volume.

With this theorem, with classical results on PDE's and well known criteriums of compactness on probabilities' spaces, we prove Theorem 1. The proof if divided in two steps. Fix $H \in C^{1,2}([0,1] \times T)$ and $\gamma \in C_e(T)$. We first prove that the sequence of probabilities $Q_N^{H,\gamma}$ on $D([0,1], M)$ is weakly relatively compact. Then we show that every limit point Q of the sequence $Q_N^{H,\gamma}$ is concentrated on trajectories $(\mu_t)_{0 \leq t \leq 1}$ such that $\mu_t(dx) = m(t,x)\,dx$ for $0 \leq t \leq 1$, where m is a weak solution of

$$\left.\begin{aligned} \frac{\partial m}{\partial t} = \frac{1}{2}\Delta m - \frac{\partial}{\partial x}\left[\frac{\partial H}{\partial x} m(1-m)\right] + B(m)e^{H} - D(m)e^{-H} \\ m(0,x) = \gamma(x). \end{aligned}\right\} \tag{3.1}$$

Since $H \in C^{1,2}([0,1] \times T)$, equation (3.1) has a unique bounded weak solution. This can be seen adapting the proofs of propositions 3.4 and 3.5 of [O] to our case. Therefore every converging subsequence of $Q_N^{H,\gamma}$ converges to the same limit Q. From the first part, it follows that the sequence $Q_N^{H,\gamma}$ converges to Q, the probability on $D([0,1], M)$ concentrated on the deterministic trajectory whose density is the bounded weak solution of (3.1).

The proof of the weak relative compactness of the sequence $Q_N^{H,\gamma}$ is standard since we consider M endowed with the weak toplogy. There exist simple criterias to prove it (*see* [K]).

To prove that Q is concentrated on weak solutions of (3.1), we proceed as follows. We first show that Q a.s., μ_t is absolutely continuous with respect to the Lebesgue measure λ for every $0 \leq t \leq 1$ and $m_t = \dfrac{d\mu_t}{d\lambda}$ is bounded by 1. This is simple to prove since for every $G : T \to \mathbf{R}$ Riemann integrable,

$$|\langle \mu_t^N, G \rangle = |(1/N)\sum G(k/N)\,\eta_t(k)| \leq (1/N)\sum |G(k/N)| \to \int |G(x)|\,dx.$$

To show that the density $m_s=\frac{d\mu_s}{d\lambda}$ is a weak solution of (3.1), we refer the reader to [KOV] since the proof of this result in our context is similar. The idea is to consider a test function $G\in C^{1,2}([0,1]\times T)$ and an expression which on the one hand converges to 0 and on the other hand converges to

$$-\langle m_1, G_1\rangle+\langle m_0, G_0\rangle$$
$$+\int_0^1 dt\left\{\left\langle m_t, \frac{\partial G_t}{\partial t}+\frac{1}{2}\Delta G_t\right\rangle+\left\langle m_t(1-m_t), \frac{\partial G_t}{\partial x}\frac{\partial H_t}{\partial x}\right\rangle\right.$$
$$\left.+\langle B(m_t), e^{H_t}G_t\rangle-\langle D(m_t), e^{-H_t}G_t\rangle\right\}. \quad (3.2)$$

This is realized considering the martingale

$$M_r^G=\frac{1}{N}\sum_{|k-j|=1}\int_0^r\left[G\left(t,\frac{k}{N}\right)-G\left(t,\frac{j}{N}\right)\right]$$
$$\times d\left(S_t^{j,k}-\frac{N^2}{2}\int_0^t \eta_s(j)[1-\eta_s(k)]\,ds\right)$$
$$+\frac{1}{N}\sum_k\int_0^r G\left(t,\frac{k}{N}\right)d\left(B_t^k-\int_0^t[1-\eta_s(k)]\,\tau_k b(\eta_s)\,ds\right)$$
$$+\frac{1}{N}\sum_k\int_0^r G\left(t,\frac{k}{N}\right)d\left(D_t^k-\int_0^t \eta_s(k)\,\tau_k d(\eta_s)\,ds\right), \quad (3.3)$$

where, for each $0\leqq j,\ k\leqq N-1$, $S_t^{j,k}$ is the number of jumps of particles from site j to site k from time 0 to time t, B_t^k is the number of particles created on k between time 0 and time t and D_t^k is the number of particles which disappeared on k between time 0 and time t.

First, computing the quadratic-variation process associated to this martingale, we show that it converges to 0. Then, rewriting the expression (3.3) in a convenient way and applying Theorem 3.1 to subsitute local terms by macroscopic ones, we prove that this martingale converges to (3.2). Details can be found in [KOV].

4. LARGE DEVIATIONS, UPPER BOUND

Following the ideas of [DV1], [DV2], [DV3], Kipnis, Olla and Varadhan proved in [KOV] a large deviation principle for the empirical density in the symmetric simple exclusion process. The main difficulties were to find the perturbations of the process needed to prove the lower bound part, the way to substitute local quantities by macroscopic ones and identify the upper bound rate functionnal given by a variational formula with the lower bound rate functional. The second problem was solved with the superexponential inequality presented in this paper in Theorem 3.1 and the third one via a Riesz' representation theorem in some Sobolev space.

In the case considered here, where the exclusion process is superposed with a Glauber dynamics, the perturbations needed are easily guessed from the previous work on the exclusion process. Theorem 3.1 allows also in our situation the substitution of local terms by functions of the empirical measure. Finally, since some exponential terms prevent from the use of the Riesz representation theorem, we will have to adopt a somehow undirect method. This last part is left to [JLV]. We do not prove here that the upper and lower rate functions are equal. We will only show that they coincide in a dense subset of D ([0, 1], M).

In the sake of clearness we present here a detailed proof. Before going trough the technical difficulties, we present the main ideas. One should remark that these ideas are exactly the same as those of the proof of large deviations for the mean of i.i.d. random variables.

We first fix a compact set K on the path space D ([0, 1], M) and consider small Markov perturbations of our process. In our context these perturbations are processes with generators given by (2.7). Then, we compute the Radon-Nikodym derivative of the probability on the trajectory space D([0, 1], M) of the perturbed process with respect to the probability on the same space of the original process. Applying Theorem 3.1, we substitute local terms which appear in the derivative by functions of the empirical measure. The last step consists on to bound above the derivative in the compact set K considered and minimize it with respect to all perturbations. In this way, we obtain a variational upper bound for $\limsup (1/N) \log Q_N^\gamma [K]$. There are at this point technical problems to exchange inf's and sup's but this can be donne with Sion's results on concave-convexe functions [S] or with an argument which appears in [V] (Lemma 11.3).

More precisely, fix $K \subset D([0, 1], M)$ compact, $H \in C^{1,2}([0, 1] \times T)$ and $\sigma \in C_e(T)$. We compute the Radon-Nikodym derivative of $P_N^{H,\sigma}$ with respect to P_N^γ. We have that:

$$\begin{aligned}\frac{dP_N^{H,\gamma}}{dP_N^\gamma} = \exp\Bigg\{ \sum_{|k-j|=1} \int_0^1 \left[H\left(t, \frac{k}{N}\right) - H\left(t, \frac{j}{N}\right) \right] dS_t^{j,k} \\ + \sum_k \int_0^1 H\left(t, \frac{k}{N}\right) d(B_t^k - D_t^k) \\ - \frac{N^2}{2} \sum_{|k-j|=1} \int_0^1 (e^{[H(t, k/N) - H(t, j/N)]} - 1)\, \eta_t(j)\, [1 - \eta_t(k)]\, dt \\ - \sum_k \int_0^1 (e^{H(t, k/N)} - 1)\, [1 - \eta_t(k)]\, \tau_k b(\eta_t)\, dt \\ - \sum_k \int_0^1 (e^{-H(t, k/N)} - 1)\, \eta_t(k)\, \tau_k d(\eta_t)\, dt \Bigg\}\end{aligned}$$

and

$$\frac{d\nu_\sigma^N}{d\nu_\gamma^N}=\prod_k\left(\frac{\sigma(k/N)}{\gamma(k/N)}\right)^{\eta(k)}\left(\frac{1-\sigma(k/N)}{1-\gamma(k/N)}\right)^{1-\eta(k)}.$$

After some simplifications, the Radon-Nikodym derivative can be rewritten as

$$\begin{aligned}\frac{d\mathrm{P}_N^{H,\gamma}}{d\mathrm{P}_N^\gamma}=\exp N\Big\{\langle\mu_1^N, H_1\rangle-\langle\mu_0^N, H_0\rangle-\int_0^1\Big\langle\mu_t^N, \frac{\partial H_t}{\partial t}+\frac{1}{2}\Delta H_t\Big\rangle dt\\ -\frac{1}{2N}\sum_k\int_0^1\eta_t(k)[1-\eta_t(k+1)]\Big[\frac{\partial H}{\partial x}\Big(t,\frac{k}{N}\Big)\Big]^2 dt\\ -\frac{1}{N}\sum_k\int_0^1[1-\eta_t(k)]\tau_k b(\eta_t)[e^{H(t,k/N)}-1]\,dt\\ -\frac{1}{N}\sum_k\int_0^1\eta_t(k)\tau_k d(\eta_t)[e^{-H(t,k/N)}-1]\,dt+O\Big(\frac{1}{N}\Big)\Big\},\end{aligned}$$

and

$$\frac{d\nu_\sigma^N}{d\nu_\gamma^N}=\exp N\Big\{\Big\langle\mu_0^N, \log\frac{\sigma}{\gamma}\Big\rangle+\Big\langle 1-\mu_0^N, \log\frac{1-\sigma}{1-\gamma}\Big\rangle\Big\}.\tag{4.1}$$

We see that there are some terms in the derivative $\frac{d\mathrm{P}_N^{H,\gamma}}{d\mathrm{P}_N^\gamma}$ which at first sight can not be expressed in terms of the empirical measure. Nevertheless, Theorem 3.1 enables us to substitute these terms by functions of the empirical measure. Indeed, let $C:[0, 1]\to[0, 1]$ be the function $C(x)=x(1-x)$ and for ε, $\delta>0$, let $B_{N,\varepsilon}^{H,\delta}$ be the set

$$\begin{aligned}\Big\{(\eta_t);\ \Big|\sum_k\int_0^1\Big[\frac{\partial H}{\partial x}\Big(t,\frac{k}{N}\Big)\Big]^2\Big(\eta_t(k)[1-\eta_t(k+1)]\\ -C\Big(\frac{1}{2N\varepsilon+1}\sum_{|i-k|\le N\varepsilon}\eta_t(i)\Big)\Big)dt\Big|<\delta N,\\ \Big|\sum_k\int_0^1(e^{H(t,k/N)}-1)\Big([1-\eta_t(k)]\tau_k b(\eta_t)\\ -B\Big(\frac{1}{2N\varepsilon+1}\sum_{|i-k|\le N\varepsilon}\eta_t(i)\Big)\Big)dt\Big|<\delta N,\\ \Big|\sum_k\int_0^1(e^{-H(t,k/N)}-1)\Big(\eta_t(k)\tau_k d(\eta_t)\\ -D\Big(\frac{1}{2N\varepsilon+1}\sum_{|i-k|\le N\varepsilon}\eta_t(i)\Big)\Big)dt\Big|<\delta N\Big\}.\end{aligned}$$

From Theorem 3.1, we know that for every $\delta>0$,

$$\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\frac{1}{N}\log P_N^\gamma[(B_{N,\varepsilon}^{H,\delta})^c]=-\infty \tag{4.2}$$

Therefore,

$$\limsup_{N\to\infty}\frac{1}{N}\log Q_N^\gamma[K]=\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\frac{1}{N}\log P_N^\gamma[(\mu^N\in K)\cap B_{N,\varepsilon}^{H,\delta}]. \tag{4.3}$$

On $B_{N,\varepsilon}^{H,\delta}$, the Radon-Nikodym derivative $\dfrac{dP_N^{H,\gamma}}{dP_N^\gamma}$ can be bounded above by expressions involving only the empirical measure. In fact, for N sufficiently large,

$$\begin{aligned}\frac{dP_N^\gamma}{dP_N^{H,\gamma}}\leqq\exp -N\Big\{&-4\delta-K(H)\varepsilon+\langle(\mu_1^N*\alpha_\varepsilon),H_1\rangle-\langle(\mu_0^N*\alpha_\varepsilon),H_0\rangle\\ &-\int_0^1 dt\Big\{\Big\langle(\mu_t^N*\alpha_\varepsilon),\frac{\partial H_t}{\partial t}+\frac{1}{2}\Delta H_t\Big\rangle+\frac{1}{2}\Big\langle C(\mu_t^N*\alpha_\varepsilon),\Big[\frac{\partial H_t}{\partial x}\Big]^2\Big\rangle\Big\}\\ &-\int_0^1 dt\{\langle B(\mu_t^N*\alpha_\varepsilon),e^{H_t}-1\rangle+\langle D(\mu_t^N*\alpha_\varepsilon),e^{-H_t}-1\rangle\}\Big\},\end{aligned} \tag{4.4}$$

where $K(H)$ is a constant which depends only on H, $*$ denotes the convolution and for $\varepsilon<1/2$, $\alpha_\varepsilon:\mathbf{R}\to\mathbf{R}_+$ is the function $\alpha_\varepsilon(x)=(1/2\varepsilon)\,1_{\{-\varepsilon\leqq x\leqq\varepsilon\}}$. Since for every integer N sufficiently large, and every $0\leqq t\leqq 1$, $\mu_t^N*\alpha_\varepsilon$ is absolutely continuous with respect to λ and the density is bounded by 1, we have from (4.4) that on $B_{N,\varepsilon}^{H,\delta}$,

$$\frac{dP_N^\gamma}{dP_N^{H,\gamma}}\leqq\exp -N\{J_H(\mu^N*\alpha_\varepsilon)-4\delta-K(H)\varepsilon\}, \tag{4.5}$$

for N sufficiently large, where J_H was defined in (2.8).

Therefore, from (4.1) and (4.5),

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\frac{1}{N}\log P_N^\gamma[(\mu^N\in K)\cap B_{N,\varepsilon}^{H,\delta}]\\ &\quad=\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\frac{1}{N}\log P_N^{H,\sigma}\Big[\frac{d\nu_\gamma^N}{d\nu_\sigma^N}\frac{dP_N^\gamma}{dP_N^{H,\gamma}}\mathbf{1}_{(\mu^N\in K)\cap B_{N,\varepsilon}^{H,\delta}}\Big]\\ &\quad\leqq\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\frac{1}{N}\log P_N^{H,\sigma}[\mathbf{1}_{(\mu^N\in K)\cap B_{N,\varepsilon}^{H,\delta}}\\ &\quad\times\sup_{\mu\in K}e^{-N\{J_H(\mu*\alpha_\varepsilon)+\langle\mu_0,\log\sigma/\gamma\rangle+\langle 1-\mu_0,\log(1-\sigma)/(1-\gamma)\rangle-4\delta-K(H)\varepsilon\}}].\end{aligned} \tag{4.6}$$

This last expression is bounded above by

$$\limsup_{\varepsilon\to 0}\sup_{\mu\in K} -\Big\{\Big\langle \mu_0, \log\frac{\sigma}{\gamma}\Big\rangle + \Big\langle 1-\mu_0, \log\frac{1-\sigma}{1-\gamma}\Big\rangle + J_H(\mu * \alpha_\varepsilon) - 4\delta - K(H)\varepsilon\Big\}$$
$$+\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P_N^{H,\sigma}[(\mu^N\in K)\cap B_{N,\varepsilon}^{H,\delta}]. \quad (4.7)$$

Therefore, from (4.3), (4.6) and (4.7), we obtain that

$$\limsup_{N\to\infty}\frac{1}{N}\log Q_N^\gamma[K] \leq \limsup_{\varepsilon\to 0}\sup_{\mu\in K}\Big\{-J_H(\mu*\alpha_\varepsilon) - \Big\langle \mu_0, \log\frac{\sigma}{\gamma}\Big\rangle - \Big\langle 1-\mu_0, \log\frac{1-\sigma}{1-\gamma}\Big\rangle\Big\}. \quad (4.8)$$

Let d_S be a metric on $D([0, 1], M)$ consistent with the Skorohod topology such that $d_S(\mu, \mu*\alpha_\varepsilon)\leq\varepsilon$ for every $\varepsilon>0$. For $U\subset D([0, 1], M)$ let

$$U^\varepsilon = \{\mu\in D([0, 1], M); d_S(\mu, U)<\varepsilon\}.$$

Then, for every $\varepsilon>0$, the r. h. s. of (4.8) is bounded above by

$$\sup_{\mu\in K^\varepsilon}\Big\{-J_H(\mu) - \Big\langle \mu_0, \log\frac{\sigma}{\gamma}\Big\rangle - \Big\langle 1-\mu_0, \log\frac{1-\sigma}{1-\gamma}\Big\rangle\Big\}.$$

Observe that up to this point we have not used the compactness of K. For an integer n, let V_i, $1\leq i\leq n$, be a finite open covering of K. Thus,

$$\limsup_{N\to\infty}\frac{1}{N}\log Q_N^\gamma[K] \leq \max_{1\leq i\leq n}\limsup_{N\to\infty}\frac{1}{N}\log Q_N^\gamma[V_i]$$
$$\leq \max_{1\leq i\leq n}\inf_{\substack{H\in C^{1,2}([0,1]\times T)\\ \sigma\in C_e(T)}}\sup_{\mu\in V_i^\varepsilon}\Big\{-J_H(\mu) - \Big\langle \mu_0, \log\frac{\sigma}{\gamma}\Big\rangle - \Big\langle 1-\mu_0 \frac{1-\sigma}{1-\gamma}\Big\rangle\Big\}.$$

Therefore, if we take the infimum over all finite open coverings of the compact K and over all $\varepsilon>0$, we obtain that:

$$\limsup_{N\to\infty}\frac{1}{N}\log Q_N^\gamma \log Q_N^\gamma[K]$$
$$\leq \inf_{\varepsilon}\inf_{\{V_i\}}\max_{1\leq i\leq n}\inf_{\substack{H\in C^{1,2}([0,1]\times T)\\ \sigma\in C_e(T)}}$$
$$\times\sup_{\mu\in V_i^\varepsilon}\left\{-J_H(\mu)-\left\langle\mu_0,\log\frac{\sigma}{\gamma}\right\rangle-\left\langle 1-\mu_0,\log\frac{1-\sigma}{1-\gamma}\right\rangle\right\}.$$

It is not hard to see that this last expression is equal to

$$\inf_{\{V_i\}}\max_{1\leq i\leq n}\inf_{\substack{H\in C^{1,2}([0,1]\times T)\\ \sigma\in C_e(T)}},$$
$$\times\sup_{\mu\in V_i}\left\{-J_H(\mu)-\left\langle\mu_0,\log\frac{\sigma}{\gamma}\right\rangle-\left\langle 1-\mu_0,\log\frac{1-\sigma}{1-\gamma}\right\rangle\right\},\qquad(4.9)$$

where the first infimum is taken over all finite open coverings of the compact K. We remarked just after the definition of J_H that for every $H\in C^{1,2}([0,1]\times T)$, J_H is lower semi-continuous. On the other hand, for every σ, $\gamma\in C_e(T)$, it is clear that

$$\mu\to\left\langle\mu_0,\log\frac{\sigma}{\gamma}\right\rangle-\left\langle 1-\mu_0,\log\frac{1-\sigma}{1-\gamma}\right\rangle$$

is continuous. Thus, we can apply Varadhan's method ([V], Lemma 11.3) to show that (4.9) is equal to

$$-\inf_{\mu\in K}\sup_{\substack{H\in C^{1,2}([0,1]\times T)\\ \sigma\in C_e(T)}}\left\{J_H(\mu)+\left\langle\mu_0,\log\frac{\sigma}{\gamma}\right\rangle\right.$$
$$\left.+\left\langle 1-\mu_0,\log\frac{1-\sigma}{1-\gamma}\right\rangle\right\}=-\inf_{\mu\in K}\tilde{I}_\gamma(\mu),$$

where $\tilde{I}_\gamma$ is give by (2.12).

The next two lemmas provide us with an explicit expression for the rate function $\tilde{I}_0$ on a dense subset of $D([0,1],M)$. We will see in [JLV] that if $\mu\notin C([0,1],M_1)$, then $\tilde{I}_0(\mu)=\infty$. Therefore on $C([0,1],M_1)^c$, $\tilde{I}_\gamma(\mu)=I_\gamma(\mu)=\infty$. On the other hand, in the next lemma we prove that on some subset of $C([0,1],M_1)$, $\tilde{I}_0$ and I_0 coincide. In the second lemma, we show that this subset is dense in $C([0,1],M_1)$. We omit the proof of the second, which will appear in [JVL].

LEMMA 4.1. — *Suppose that for every* $0 \leqq t \leqq 1$, $m_t = \dfrac{d\mu_t}{d\lambda}$ *exists and is bounded by* 1. *Suppose further that there exists* $H \in C^{1,2}([0, 1] \times T)$ *such that m is a weak solution of*

$$\frac{\partial m}{\partial t} = \frac{1}{2} \Delta m - \frac{\partial}{\partial x} \left[\frac{\partial H}{\partial x} m (1-m) \right] + B(m) e^{H} - D(m) e^{-H} \tag{4.10}$$

Then, $\tilde{I}_0(\mu) = I_0(\mu)$.

Proof. — Since μ is a weak solution of (4.10), we can simplify the expression of $\tilde{I}_0(\mu)$ and obtain that

$$\begin{aligned} \tilde{I}_0(\mu) = \sup_{G \in C^{1,2}([0,1] \times T)} \int_0^1 dt \Big\{ & \Big\langle m_t (1-m_t), \frac{\partial H_t}{\partial x} \frac{\partial G_t}{\partial x} \Big\rangle \\ & - \frac{1}{2} \Big\langle m_t (1-m_t), \Big[\frac{\partial G_t}{\partial x} \Big]^2 \Big\rangle + \langle B(m_t), 1 - e^{G_t} + e^{H_t} G_t \rangle \\ & + \langle D(m_t), 1 - e^{-G_t} - e^{-H_t} G_t \rangle \Big\}. \end{aligned}$$

This last expression is equal to:

$$\begin{aligned} & \int_0^1 dt \Big\{ \frac{1}{2} \Big\langle m_t (1-m_t), \Big[\frac{\partial H_t}{\partial x} \Big]^2 \Big\rangle \\ & \quad + \langle B(m_t), 1 - e^{H_t} + H_t e^{H_t} \rangle + \langle D(m_t), 1 - e^{-H_t} - H_t e^{-H_t} \rangle \Big\} \\ & \quad + \sup_{G \in C^{1,2}([0,1] \times T)} - \int_0^1 dt \Big\{ \Big\langle m_t (1-m_t), \frac{1}{2} \Big[\frac{\partial H_t}{\partial x} - \frac{\partial G_t}{\partial x} \Big]^2 \Big\rangle \\ & \qquad + \langle B(m_t), e^{G_t} - e^{H_t} + H_t e^{H_t} - G_t e^{H_t} \rangle \\ & \qquad + \langle D(m_t), e^{-G_t} - e^{-H_t} + G_t e^{-H_t} - H_t e^{-H_t} \rangle \Big\}, \end{aligned}$$

which is equal to $I_0(\mu)$, since $e^x - e^y + y e^y - x e^y \geqq 0$ for every $x, y \in \mathbf{R}$.

LEMMA 4.2. — *Let* $\mu \in C_e^{2,3}([0, 1] \times T)$. *Then there exist* $H \in C^{1,2}([0, 1] \times T)$ *such that* $m_t = \dfrac{d\mu_t}{d\lambda}$ *is a weak solution of* (4.10).

5. LARGE DEVIATIONS, LOWER BOUND

Let $V \subset D([0, 1], M)$ open. Fix $\delta > 0$ and $\mu \in V \cap C_e^{2,3}([0, 1] \times T)$. Since $\mu \in C_e^{2,3}([0, 1] \times T)$, from Lemma 4.2, there exists $H \in C^{1,2}([0, 1] \times T)$ such that $m = \frac{d\mu}{d\lambda}$ is a weak solution of (4.10).

Define $h_{\gamma, m_0} : D([0, 1], M) \to \mathbf{R}$ as

$$h_{\gamma, m_0}(\sigma) = \left\langle \sigma_0, \log \frac{m_0}{\gamma} \right\rangle + \left\langle 1 - \sigma_0, \log \frac{1 - m_0}{1 - \gamma} \right\rangle. \tag{5.1}$$

Since $\mu \in C_e^{2,3}([0, 1] \times T)$ and $\gamma \in C_e(T)$, h_{γ, m_0} is continuous for the Skorohod topology.

Let $\varepsilon_0(\delta) > 0$ such that if $\varepsilon < \varepsilon_0$, then

$$\limsup_{N \to \infty} \frac{1}{N} \log P_N^{\mu_0, H}[(B_{N,\varepsilon}^{H,\delta})^c] \leqq -1. \tag{5.2}$$

This is possible in view of Theorem 3.1.

Fix $0 < \varepsilon < \varepsilon_0$ and let $J_{H,\varepsilon} : D([0, 1], M) \to \mathbf{R}$,

$$\begin{aligned} J_{H,\varepsilon}(\sigma) = & \langle (\sigma_1 * \alpha_\varepsilon), H_1 \rangle - \langle (\sigma_0 * \alpha_\varepsilon), H_0 \rangle \\ & - \int_0^1 dt \left\{ \left\langle (\sigma_t * \alpha_\varepsilon), \frac{\partial H}{\partial t} + \frac{1}{2} \Delta H \right\rangle + \frac{1}{2} \left\langle C(\sigma_t * \alpha_\varepsilon), \left[\frac{\partial H}{\partial x} \right]^2 \right\rangle \right\} \\ & - \int_0^1 dt \{ \langle B(\sigma_t * \alpha_\varepsilon), e^{H_t} - 1 \rangle + \langle D(\sigma_t * \alpha_\varepsilon), e^{-H_t} - 1 \rangle \}. \end{aligned}$$

Since $H \in C^{1,2}([0, 1] \times T)$ and $\varepsilon > 0$, $J_{H,\varepsilon}$ is continuous. Therefore, there exists an open neighborhood $V_{\mu,\varepsilon} \subset V$ of μ such that if $\sigma \in V_{\mu,\varepsilon}$ then,

$$\begin{gathered} |J_{H,\varepsilon}(\sigma) - J_{H,\varepsilon}(\mu)| < \delta \\ |h_{\gamma, m_0}(\sigma) - h_{\gamma, m_0}(\mu)| < \delta. \end{gathered} \tag{5.3}$$

We can now prove Theorem 3. We have that:

$$\begin{aligned} Q_N^\gamma[V] &= P_N^\gamma[\mu^N \in V] \\ &\geqq P_N^\gamma[(\mu^N \in V_{\mu,\varepsilon}) \cap B_{N,\varepsilon}^{H,\delta}] \\ &= P_N^{\mu_0, H}\left[\frac{d\nu_\gamma^N}{d\nu_{\mu_0}^N} \frac{dP_N^\gamma}{dP_N^{H,\gamma}} \mathbf{1}_{(\mu^N \in V_{\mu,\varepsilon}) \cap B_{N,\varepsilon}^{H,\delta}} \right]. \end{aligned} \tag{5.4}$$

On $B_{N,\varepsilon}^{H,\delta}$, in the same way in which we have obtained (4.5), we get that for N sufficiently large,

$$\frac{dP_N^\gamma}{dP_N^{H,\gamma}} \geqq \exp - N\{ J_{H,\varepsilon}(\mu^N) + 4\delta + K(H)\varepsilon \}.$$

Since $\mu^N \in V_{\mu,\varepsilon}$, from (5.3) we have that

$$\frac{dP_N^{\gamma}}{dP_N^{H,\gamma}} \geqq \exp - N\{J_{H,\varepsilon}(\mu) + 5\delta + K(H)\varepsilon\}. \tag{5.5}$$

On the other hand, from (4.1) and (5.3), we have:

$$\frac{d\nu_{\gamma}^N}{d\nu_{\mu_0}^N} = \exp^{-N\{\langle \mu_0^N, \log m_0/\gamma\rangle + \langle 1-\mu_0^N, \log(1-m_0)/(1-\gamma)\rangle\}}$$

$$\geqq \exp^{-N\{\delta + h_{\gamma, m_0}(\mu)\}} \tag{5.6}$$

Since from (2.11) $h_{\gamma, m_0}(\mu) = h_\gamma(\mu)$, from (5.4), (5.5) and (5.6), we have that

$$\frac{1}{N}\log Q_N^{\gamma}[V] \geqq -\{6\delta + K(H)\varepsilon + h_\gamma(\mu) + J_{H,\varepsilon}(\mu)\}$$

$$+ \frac{1}{N}\log P_N^{\mu_0, H}[(\mu^N \in V_{\mu,\varepsilon}) \cap B_{N,\varepsilon}^{H,\delta}]. \tag{5.7}$$

From Theorem 1, we know that

$$\lim_{N\to\infty} P_N^{\mu_0, H}[\mu^N \in V_{\mu,\varepsilon}] = 1.$$

On the other hand, from (5.2), we have that:

$$\lim_{N\to\infty} P_N^{\mu_0, H}[\mu^N \in B_{N,\varepsilon}^{H,\delta}] = 1.$$

Hence, the second line of (5.7) converges to 0, when $N \uparrow \infty$. Letting $\delta \to 0$ (and consequently $\varepsilon \to 0$), we obtain that

$$\liminf_{N\to\infty} \frac{1}{N}\log Q_N^{\gamma}[V] \geqq -\{h_\gamma(\mu) + \limsup_{\varepsilon\to 0} J_{H,\varepsilon}(\mu)\}.$$

Since $\mu \in C_e^{2,3}([0, 1] \times T)$ and $H \in C^{1,2}([0, 1] \times T)$, it is easy to show that

$$\lim_{\varepsilon\to 0} J_{H,\varepsilon}(\mu) = J_H(\mu) = I_0(\mu).$$

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