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Elementary acceleration and multisummability I ⁽¹⁾

by

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Lorsqu'il suit le bon *rayon* vers la périphérie, le promeneur peut découvrir...
André Hardellet, « Périphérie »

This paper is extracted from the contents of a forthcoming book by the same authors [MR 3]. Paragraphs 1 to 3 joined to chapter 2 of [MR 2] form a more or less self-contained set. We recall basic definitions about *Borel-summability* (Borel [Bo 1], [Bo 2]), and its natural generalization *k-summability* (Leroy [Le], Nevanlinna [Ne], Ramis [Ra 1]). We describe the “*elementary acceleration*” introduced by Ecalle [E 4] and different summability operators related to it. If one compares to [E 4] our description is slightly modified in order to fit with our “*geometric*” interpretations [MR 2], [MR 3]. In paragraph 4 as an example of application we give a “*natural*”, *simple* and *general*, definition of *Stokes multipliers* ⁽²⁾, using a result ⁽³⁾ of Ramis [Ra 3] (cf. also [Ra 2]), and derive a new proof of a theorem of Ramis ([Ra 4], [Ra 5]) about the computation of the *differential Galois group* of a linear differential equation. As a byproduct we get ⁽⁴⁾

⁽¹⁾ Part I of this paper contains paragraphs 1 to 4 (a preliminary manuscript version has been distributed during a Luminy Conference, in september 1989); paragraphs 5 and 6 will appear in *Elementary acceleration and multisummability II*. The second author has exposed part II at a R.C.P. 25 meeting dedicated to R. Thom (Strasbourg, 1989). See also [LR 3].

⁽²⁾ Compare with the program of [Me]. Relations between our description of *Stokes phenomenon* and the *cohomological approach* [Ma 3], [Ma 4], [Si], [De 3], [J], [BJL], [BV], will be explained in 4.

⁽³⁾ The main steps of one proof of this result, using Gevrey asymptotic expansions technics, are detailed in paragraph 5. Cf. also [LR 1] for another approach.

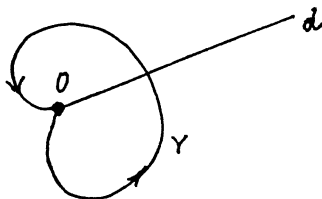
⁽⁴⁾ *Multisummability* (in its *analytical* formulation or its “*wild-Cauchy*” formulation) is *not necessary* in order to obtain this description (which can be derived from *Malgrange-Sibuya* results using algebraic tools from [LR 1]) but it allows an interesting presentation, fundamental for *non-linear* extensions.

also a description of the *meromorphic classification* of *meromorphic linear differential equations* on a Riemann surface by the finite dimensional *linear representations* of a “*wild fundamental group*”. (This is a *natural* generalization of the *Riemann-Hilbert correspondence*.) Paragraph 6 is *very sketchy*; we describe “*infinitesimal neighbourhoods*” of *analytic geometry* (following an idea of *Deligne* [De4]) and sheaves of “*analytic functions*” (*weakly* analytic and *wild* analytic functions) on these neighbourhoods. Afterwards we are able to give “*geometric interpretations*” of *elementary acceleration, summability and Stokes phenomena* ⁽⁵⁾ and to get various generalizations (the sum of a formal power series is now a *wild* analytic function) important for extensions to *non-linear* situations.

1. BOREL SUMMABILITY, BOREL AND LAPLACE TRANSFORMS

We denote by B_d the Borel transform in the direction d .

$$B_d f(\xi) = \mathbf{f}(\xi) = \frac{1}{2i\pi} \int_{\gamma} f(x) (e^{\xi/x} dx/x^2).$$



This formula makes sense with “good” hypothesis on f [MR 2]. We will omit d and write Bf if $B_d f$ is independent of d (up to analytic continuation).

If $\hat{\varphi}$ is a convergent power series ($\hat{\varphi} \in \mathbb{C}\{\xi\}$), we will denote by $\varphi(\xi) = S\hat{\varphi}(\xi)$ its *sum* on a “small disc” centered at zero.

If f is an analytic function in a “small disc” centered at zero, or, more generally, in a “small sector” bisected by the direction d , we will denote by $\cdot_d f$ its *analytic continuation* (if it exists) along d . In the following, when we write $\cdot_d f$, we will *always* suppose that $\cdot_d f$ is defined on a *sector* bisected by d with infinity radius.

⁽⁵⁾ Partially based upon a *cohomological version* of Phragmén-Lindelöf theorem. (Similar and more precise results have been obtained by Malgrange [Ma 5], using Fourier transform; cf. also Il’Yashenko’s Luminy Conference lectures.) The first cohomological version of Phragmén-Lindelöf theorem is due to Lin [Li].

Operators S and $\cdot_d$ are clearly *injective homomorphisms of differential algebras* (laws being addition and multiplication, and derivation being $d/d\xi$ or $\xi^2 d/d\xi$) or of “convolution ⁽⁶⁾ differential algebras” (laws being addition and convolution, and derivation being multiplication by ξ).

If $\lambda > 0$ and $f(x) = x^\lambda$, we get

$$B_d f(\xi) = B f(\xi) = \xi^{\lambda-1}/\Gamma(\lambda); \text{ in particular, for } \lambda = n \in \mathbf{N}^*, f(x) = x^n$$

($n \in \mathbf{N}$):

$$B f(\xi) = \xi^{n-1}/\Gamma(n) = \xi^{n-1}/(n-1)!$$

If we introduce

$$B_d f = B_d f(\xi) d\xi; \text{ then for } f(x) = 1, \text{ we get as a natural generalization:} \\ B_d f = \delta \text{ (Dirac distribution).}$$

We can now define a “formal Borel transform” $\hat{B}$:

$$\text{For } \hat{f} \in \mathbf{C}[[x]], \hat{f}(x) = \sum_{n \geq 1} a_n x^n$$

$$\hat{f}(\xi) = \hat{B} \hat{f}(\xi) = \sum_{n \geq 1} a_n \xi^{n-1}/(n-1)!$$

This definition can be extended, replacing $\mathbf{N}$ as a set of indices for the expansion $\hat{f}$ by a more general semi-group (contained in $\mathbf{R}$): $\Lambda^* = \Lambda - \{0\}$,

$$\hat{f}(x) = \sum_{\lambda \in \Lambda^*} a_\lambda x^\lambda, \quad \hat{B} \hat{f}(\xi) = \sum_{\lambda \in \Lambda^*} a_\lambda \xi^{\lambda-1}/\Gamma(\lambda).$$

We will also use later formal expansions indexed by $\lambda \in \alpha + \mathbf{N}$ ($\alpha \in \mathbf{C}$), and the corresponding *generalized asymptotic expansions* (simply named “asymptotic expansions” in the following). (Regular parts of such expansions will be called “polynomials”).

LEMMA 1. — *We have an isomorphism of differential algebras:*

$$\begin{array}{ccc} \text{Differential algebra } ^{(7)} \mathbf{C}\{x\} & \xrightarrow{B} & \text{Convolution differential algebra} \\ \text{of convergent power series} & & \text{of entire functions} \\ \text{without constant term.} & & \text{of order } \leq 1. \end{array}$$

Let f be holomorphic with *exponential growth* of order ≤ 1 in a “small” sector bisected by the direction d (or, more generally, infinitely differentiable on d ⁽⁸⁾ with an *exponential growth* of order ≤ 1). We can define its

⁽⁶⁾ The convolution law is defined by $\varphi * \psi = \int_0^\xi \varphi(t) \psi(\xi - t) dt$ in the analytic case and $\hat{\varphi} * \hat{\psi}$ is deduced, in the formal case, from the identities $(\xi^{m-1}/\Gamma(m)) * (\xi^{n-1}/\Gamma(n)) = \xi^{m+n-1}/\Gamma(m+n)$.

⁽⁷⁾ The differential is $x^2 d/dx$.

⁽⁸⁾ A function “infinitely differentiable on d ” is infinitely differentiable on the right at zero, by convention.

Laplace transform along d :

$$f(x) = L_d f(x) = \int_d f(\xi) (e^{-\xi/x} d\xi)$$

If $f \in \mathbb{C}\{x\}$ (resp. f entire of order ≤ 1):

$$LB f = f \quad \text{and} \quad BL f = f.$$

With “good hypothesis”:

$$L_d B_d = \text{id} \quad \text{and} \quad B_d L_d = \text{id} \text{ [MR 2].}$$

Example. — For $f(\xi) = \xi^\mu (\mu > -1)$, we have $L f(x) = \Gamma(\mu + 1) x^{\mu+1}$.

Let $\hat{f}$ be a formal power series, of Gevrey order $(^9) 1$ ($\hat{f} \in \mathbb{C}[[x]]_1$). Then

$$\hat{B} \hat{f} = \hat{f} \in \mathbb{C}\{\xi\}.$$

If $f = S \hat{f}$ can be analytically extended along some direction d in a fonction $f = \cdot_d S \hat{f}$ which is analytic with exponential growth of order ≤ 1 on a small sector bisected by d , we can define

$$f_d(x) = L_d \cdot_d S \hat{f} = L_d \cdot_d S \hat{B} \hat{f}.$$

By definition f_d is the “Borel sum” of $\hat{f}$ in the direction d ($\hat{f}$ is Borel-summable in the direction d).

Clearly if $\hat{f} \in \mathbb{C}\{x\}$, $S \hat{B} = B$ and $f_d(x) = S \hat{f}(x)$. So $S_d = L_d \cdot_d S \hat{B}$ extends the operator S .

LEMMA 2. — The operator S_d is an injective morphism of differential algebras:

$$\begin{array}{ccc} \text{Differential algebra } (^{10}) \text{ of Borel} & \xrightarrow{S_d} & \text{Differential algebra } (^{11}) \text{ of germs of} \\ \text{summable series} & & \text{holomorphic functions} \\ \text{in the direction } d. & & \text{on sectors bisected by } d. \end{array}$$

So Borel-summability is “natural” (i. e. “Galois”).

Let $R > 0$ and d a direction.

Let $D_{R,d} = \{t \in \mathbb{C} \mid |\text{Arg } t - \text{Arg } d| < \frac{\pi}{2} \text{ and } \text{Re}(e^{i \text{Arg } d}/t) > 1/R\}$.

We denote by γ_R the boundary of $D_{R,d}$ oriented in the positive sense. We write

$$B_d f(\xi) = f(\xi) = \frac{1}{2i\pi} \int_{\gamma_R} f(x) (e^{\xi/x} dx/x^2), \text{ if } f(x) = o(x^2),$$

and $B_d f(\xi) = 1$, if $f(x) = x$. Then we have defined $B_d f$ for $f(x) = o(x)$.

(⁹) For definitions and notations see [MR 1].

(¹⁰) The differential is $x^2 dx$.

(¹¹) Idem.

Later we will need the "well known"

LEMMA 3. — *The map*

<i>Convolution differential algebra of functions infinitely differentiable on d with an exponential growth of order ≤ 1 at infinity.</i>	$\xrightarrow{L}$	<i>Differential algebra⁽¹²⁾ of functions analytic on open discs $D_{R;d}$ ($R > 0$ arbitrary), with an asymptotic expansion⁽¹³⁾ (without constant term) at zero.</i>
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is an isomorphism of differential algebras.

Let f be an analytic function on the open Borel-disc $D_{R;d}$ with an asymptotic expansion (without constant term) at zero. Then, using Fubini's theorem and the formula

$$L(e^{-\xi})(x) = \frac{x}{x+1}, \text{ we get easily } LBf = f \quad (\text{see [Bo 2]}).$$

Let f be infinitely differentiable on d with an exponential growth of order ≤ 1 at infinity. If $Lf = 0$, then $f = 0$ (using inversion of *Fourier transform*).

Now, from $L(BLf) = LB(Lf) = Lf$, we deduce $BLf = f$. That ends the proof of lemma 3.

2. k -SUMMABILITY, k -BOREL AND k -LAPLACE TRANSFORMS

Using B_d , L_d , $\cdot_d$, S and *ramification operators* ρ_k ($k > 0$) it is easy to build new operators $B_{k;d}$ and $L_{k;d}$ (and the formal operator $\tilde{B}_k$ corresponding to $B_{k;d}$):

We will use the notation ($k > 0$): $\rho_k f(x) = f(x^{1/k})$ (x is varying onto the Riemann surface of Logarithm); $\rho_{1/k} = \rho_k^{-1}$.

If d^k corresponds to d by the ramification ρ_k , we will set:

$$B_{k;d} = \rho_k^{-1} B_d \rho_k$$

and

$$L_{k;d} = \rho_k^{-1} L_d \rho_k.$$

We have (in general we will simplify our notations: $f_k = f$, $\xi_k = \xi$):

$$B_{k;d} f(\xi_k) = f_k(\xi_k) = \frac{1}{2i\pi} \int_{\gamma_k} f(x) (k e^{\xi_k/x^k} dx / x^{k+1})$$

$$L_{k;d} f_k(x) = f(x) = \int_I f_k(\xi_k) (k e^{-\xi_k/x^k} \xi_k^{k-1} d\xi_k).$$

⁽¹²⁾ Idem.

⁽¹³⁾ Uniform on closed subdiscs $\bar{D}_{R';d}$ ($R' < R$).

The operator $L_{k;d}$ can be applied to functions holomorphic with an exponential growth of order $\leq k$ on a small sector bisected by d and an asymptotic expansion at the origin (indexed by the set $1-k+\mathbb{N}$). These functions form a k -convolution differential algebra:

The k -convolution $\star_k$ is defined by:

$$\mathbf{f}_k \star_k \mathbf{g}_k = \rho_k^{-1} ((\rho_k \mathbf{f}_k) \star (\rho_k \mathbf{g}_k)).$$

Operations are: $+$, $\star_k$, and derivation $\partial_k = B_k(x^2 d/dx) L_k$ (∂_k will be explicetely described later; ∂_1 is multiplication by $-\xi$).

LEMMA 4. — *We have an isomorphism of differential algebras:*

$$\begin{array}{ccc} \text{Differential algebra } (\mathbb{C}\{x\}, x^2 d/dx) & \xrightarrow{B_k} & k\text{-convolution differential algebra:} \\ \text{of convergent power series} & & \xi^{1-k} \{ \text{entire functions} \\ \text{vanishing at 0.} & & \text{of order } \leq k \}. \end{array}$$

We will use the following notations:

$\mathbb{C}[[x]]_{1/k}$ is the differential algebra of formal power series of Gevrey order $1/k$ (Gevrey level k)⁽¹⁴⁾;

$\mathbb{C}\{x\}_{1/k;d}$ is the differential algebra of formal power series k -summable in the direction d (definition is given just below);

$\mathbb{C}\{x\}_{1/k}$ is the differential algebra of k -summable series (that is of formal power series k -summable in every direction but perhaps a finite number).

Let $\hat{f} \in \mathbb{C}[[x]]_{1/k}$. Then $\hat{\mathbf{f}}_k = \hat{B}_k \hat{f} \in \mathbb{C}\{\xi_k\}$. If $\mathbf{f}_k = S \hat{\mathbf{f}}_k$ can be analytically extended along some direction d in a function $\cdot_d \mathbf{f}_k = \cdot_d S \hat{\mathbf{f}}_k$ analytic with exponential growth of order $\leq k$ on a small sector bisected by d , we can set:

$$f_{k;d}(x) = L_{k;d} \cdot_d S \hat{\mathbf{f}}_k = L_{k;d} S \hat{B}_k \hat{f}.$$

By definition $f_{k;d}$ is the “ k -sum” of $\hat{f}$ in the direction d ($\hat{f}$ is k -summable in the direction d). It is clear that $S_{k;d} = L_{k;d} \cdot_d S \hat{B}_k$ extends the operator S (defined for $\hat{f} \in \mathbb{C}\{x\}$).

LEMMA 5. — *The operator $S_{k;d}$ is an injective morphism of differential algebras:*

$$\begin{array}{ccc} \text{Differential algebra of } & S_{k;d} & \text{Differential algebra of germs of} \\ k\text{-summable series} & \longrightarrow & \text{holomorphic functions} \\ \text{in the direction } d. & & \text{on sectors bisected by } d. \end{array}$$

So k -summability is “natural” (i. e. “Galois”).

We have built a one parameter family ($k \in \mathbb{R}, k > 0$) of summation processes. We will now compare these processes for different values of the

⁽¹⁴⁾ Notations of [MR 2]. (Be careful, these notations differ from those of [Ra 1], [Ra 2], [Ra 7].)

parameter $k > 0$: if a formal power series is summable by two processes then the two sums are equal, but this is quite exceptional because k_1 -summability and k_2 -summability for $k_1 \neq k_2$ requires in some sense very different conditions. More precisely:

PROPOSITION 1. — Let $k, k' > 0$ with $k < k'$ and $\hat{f} \in \mathbb{C}[[x]]$ k -summable and k' -summable in the direction d . Then:

$$(i) \ S_{k;d} \hat{f} = S_{k';d} \hat{f};$$

(ii) The power series $\hat{f}$ is k' -summable in every direction d' with $\arg d' \in]\arg d - \pi/2k + \pi/2k', \arg d + \pi/2k - \pi/2k'[$ and the sums $S_{k';d'} \hat{f}$ glue together by analytic continuation;

(iii) The power series $\hat{f}$ is k'' -summable in every direction d'' with $\arg d'' \in]\arg d - \pi/2k + \pi/2k'', \arg d + \pi/2k - \pi/2k''[$, for $k < k'' < k'$.

$$\text{Moreover } S_{k'';d''} \hat{f} = S_{k';d''} \hat{f}.$$

PROPOSITION 2. — Let $k, k' > 0$ with $k < k'$ and $\hat{f} \in \mathbb{C}[[x]]_{1/k'}$. If $\hat{f}$ is k -summable, then $\hat{f}$ is a convergent power series

$$(i.e. \ \mathbb{C}[[x]]_{1/k'} \cap \mathbb{C}\{x\}_{1/k} = \mathbb{C}\{x\}).$$

This result, announced in [Ra 2], is proved in [Ra 5] (for a particular case and example, see [RS 1]).

From such a result it is easy to understand that summation operators $S_{k;d}$ (with d and $k > 0$), if very useful, are not sufficient if one wants to deal with quite simple situations as “non generic” linear algebraic differential equations:

A formal power series solution of a “generic” linear algebraic equation is k -summable for some $k > 0$ [Ra 2], [MR 2], [MR 3]. Let now $\hat{f}_1, \hat{f}_2 \in \mathbb{C}[[x]]$ be divergent power series, where $\hat{f}_1$ is k_1 -summable and $\hat{f}_2$ k_2 -summable ($k_1 \neq k_2$). Then $\hat{f} = \hat{f}_1 + \hat{f}_2$ is divergent (proposition 2) and there exists no $k > 0$ such that $\hat{f}$ is k -summable (proposition 1 and 2). If we suppose moreover that there exists $D_1, D_2 \in \mathbb{C}[x][d/dx]$ such that $D_1 \hat{f}_1 = 0$, $D_2 \hat{f}_2 = 0$, then there exists $D \in \mathbb{C}[x][d/dx]$ such that $D \hat{f} = 0$ (for an explicit example see [RS 1]).

Any formal power series solution of any analytic linear differential equation **can be summed** using a “blend” of a finite set of processes of k -summability (cf. 4, 6, infra). The corresponding values for k are computable using a Newton polygon [Ra 1], [Ra 7]. We get in this way a process of summability (consisting in replacing each formal power series in the blend by its k -sum choosing the “good” k). This method gives an injective morphism of differential algebras but is purely theoretical (i.e. not explicit). This motivates the introduction of a more general tool, that is multisummability. Multisummability [due to Ecalle] ⁽¹⁵⁾ is effective and a “blend” of

⁽¹⁵⁾ It is a particular case of his concept of “accelerosummability”.

k -summable power series is multisummable. Here we have slightly modified *Ecalte's* presentation in order to be as near as possible of our *geometric description* of multisummability ⁽¹⁶⁾ (cf. 6, infra).

3. ACCELERATION AND MULTISUMMABILITY

We will introduce here only a *very elementary acceleration* (for a more general theory cf. *Ecalte* [E 4]). It is sufficient for our applications (and easy to generalize along the same lines [MR 3]). Following *Ecalte*, *accelerating operators* are *first* defined using *Laplace, Borel and ramification operators*; afterwards we get an equivalent definition using an *integral formula*. The *important fact* is that this integral formula lead to a *natural extension of the domain* of the corresponding operator.

Let $\alpha \geq 1$. Formally the operator ρ_α of α -acceleration is the *conjugate* of the *ramification operator* ρ_α by the *Laplace transform*:

$$\rho_\alpha = L^{-1} \rho_\alpha L = B \rho_\alpha L.$$

The operator ρ_α is an *isomorphism of differential algebras*, therefore the operator ρ_α is an *isomorphism of convolution differential algebras*. More precisely:

$\rho_\alpha = L_d^{-1} \rho_\alpha L_d$, and:

Convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order ≤ 1 at infinity and an asymptotic expansion at zero.	$\rho_\alpha \rightarrow$	Convolution differential algebra of analytic functions on sectors with opening $> \pi(\alpha - 1)$, bisected by d^α with an exponential growth of order ≤ 1 at infinity and an "asymptotic expansion" at zero ⁽¹⁷⁾ .
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is an *isomorphism*.

As ρ_α the operator ρ_α moves the direction d . It is useful to introduce operators of "*normalized acceleration*" not moving d :

$$A_\alpha = \rho_{1/\alpha} \rho_\alpha = \rho_\alpha^{-1} L^{-1} \rho_\alpha L = (L \rho_\alpha)^{-1} \rho_\alpha L = B_\alpha L.$$

Then A_α is the *commutator* of $B = L^{-1}$ and $\rho_{1/\alpha} = \rho_\alpha^{-1}$.

⁽¹⁶⁾ As *analytic continuation* along rays starting from the origin across the "*analytic halo*".

⁽¹⁷⁾ This asymptotic expansion is in powers of $x^{1/\alpha}$.

The operator A_α gives an *isomorphism* of “convolution” differential algebras:

Convolution differential algebra of analytic functions, on sectors bisected by d with an exponential growth of order ≤ 1 at infinity and an asymptotic expansion at zero.	$A_\alpha \rightarrow$	α -convolution differential algebra of analytic functions, on sectors, with opening $> \pi/\beta = \pi \frac{\alpha-1}{\alpha}$, bisected by d with an exponential growth of order $\leq \alpha$ at infinity and an “asymptotic expansion” at zero.
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For the proof of this statement see below the more general case of $A_{k',k}$.

If necessary we will denote more precisely the operator A_α by $A_{\alpha;d}$.

The operator A_α is clearly related to level 1. We need now to introduce similar operators for arbitrary levels $k > 0$. Let $k' > k$, $\alpha = k'/k$, we will denote:

$$A_{k',k} = \rho_{1/k} A_\alpha \rho_k = (\rho_k)^{-1} (\rho_{k'/k})^{-1} L^{-1} \rho_{k'/k} L \rho_k$$

$$A_{k',k} = (\rho_{k'})^{-1} L^{-1} \rho_{k'/k} L \rho_k = (\rho_{k'})^{-1} L^{-1} \rho_{k'} (\rho_k)^{-1} L \rho_k.$$

The operator $A_{k',k}$ gives an *isomorphism* of “convolution” differential algebras

k -convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order $\leq k$ at infinity and an “asymptotic expansion” at zero	$A_{k',k} \rightarrow$	k' -convolution differential algebra of analytic functions, on sectors with opening $> \pi/\kappa = \pi \frac{k'-k}{kk'}$, bisected by d with an exponential growth of order $\leq k'$ at infinity and an “asymptotic expansion” at zero.
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If necessary we will denote more precisely the operator $A_{k',k}$ by $A_{k',k;d}$. We have:

$$A_{k',k}(f * g) = \rho_k^{-1} L^{-1} \rho_{k'/k} L \rho_k \rho_k^{-1} ((\rho_k f) * (\rho_k g))$$

$$A_{k',k}(f * g) = \rho_k^{-1} L^{-1} \rho_{k'/k} L ((\rho_k f) * (\rho_k g))$$

$$A_{k',k}(f * g) = \rho_k^{-1} L^{-1} \rho_{k'/k} ((L \rho_k f) (L \rho_k g))$$

$$A_{k',k}(f * g) = \rho_k^{-1} (L^{-1} (\rho_{k'/k} L \rho_k f)) * (L^{-1} (\rho_{k'/k} L \rho_k g))$$

$$A_{k',k}(f * g) = A_{k',k} f * A_{k',k} g.$$

In order to prove that $A_{k',k}$ is an *isomorphism* it suffices to remark that L_d is an *isomorphism* between the convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order ≤ 1 at infinity and an asymptotic expansion at zero, and the differential algebra of functions analytic on sectors with opening $> \pi$

bisected by d , and with an asymptotic expansion (without constant term) at zero.

It is natural to set:

$$\begin{aligned} \mathbf{A}_{\infty, k} &= \mathbf{L}_k \\ \mathbf{A}_{\infty, 1} &= \mathbf{L}. \end{aligned}$$

We have

$$\mathbf{A}_{k, 1} = \mathbf{A}_k \quad \text{and} \quad \mathbf{A}_{k, k} = \text{id}.$$

Let $k'' > k' > k > 0$. When the formula makes sense we get:

$$\mathbf{A}_{k'', k'} \mathbf{A}_{k', k} = \mathbf{A}_{k'', k}.$$

We will use later the above formula to *extend* the operator $\mathbf{A}_{k'', k}$:

The first step is to extend the domain of the operator $\mathbf{A}_{k', k}$ and the second to replace $\mathbf{A}_{k', k}$ in the formula by $\cdot_d \mathbf{A}_{k', k; d}$: $\mathbf{A}_{k'', k'; d} \cdot_d \mathbf{A}_{k', k; d} = \mathbf{A}_{k'', k', k; d}$ (*definition*).

More generally, let $k_1 > k_2 > \dots > k_r > 0$. When the formula makes sense, we get:

$$\mathbf{A}_{k_1, k_2} \mathbf{A}_{k_2, k_3} \dots \mathbf{A}_{k_{r-1}, k_r} = \mathbf{A}_{k_1, k_r}.$$

With this formula we will later *extend* the operator $\mathbf{A}_{k_1, k_r}$, using extensions of the operators

$$\mathbf{A}_{k_i, k_{i+1}; d} \quad (i = 1, \dots, r-1)$$

and

$$\mathbf{A}_{k_1, k_2; d} \cdot_d \mathbf{A}_{k_2, k_3; d} \dots \cdot_d \mathbf{A}_{k_{r-1}, k_r; d} = \mathbf{A}_{k_1, k_2, \dots, k_r; d} \quad (\text{definition}).$$

Let $k' > k$, when the formula make sense we get:

$$\mathbf{L}_{k'} \mathbf{A}_{k', k} = \mathbf{L}_k \quad (\text{or } \mathbf{A}_{\infty, k'} \mathbf{A}_{k', k} = \mathbf{A}_{\infty, k}).$$

So we can *extend* the operator $\mathbf{L}_k$ using $\mathbf{L}_{k'} \cdot_d \mathbf{A}_{k', k}$. Then

$$\begin{aligned} \text{id} &= \mathbf{L}_k \mathbf{B}_k = \mathbf{L}_{k'} \mathbf{A}_{k', k} \mathbf{B}_k \\ S &= \mathbf{L}_{k'} \mathbf{A}_{k', k} S \hat{\mathbf{B}}_k, \text{ and, more generally, for } k_1 > k_2 > \dots > k_r: \\ S &= \mathbf{L}_{k_1} \mathbf{A}_{k_1, k_2} \dots \mathbf{A}_{k_{r-1}, k_r} S \hat{\mathbf{B}}_{k_r}. \end{aligned}$$

Then it is natural to *extend* the domain $\mathbf{C}\{x\}$ of the *summation* operator S using the new summation operator (along the direction d):

$$S_{k_1, k_2, \dots, k_r; d} = \mathbf{L}_{k_1; d} \cdot_d \mathbf{A}_{k_1, k_2; d} \dots \cdot_d \mathbf{A}_{k_{r-1}, k_r; d} \cdot_d S \hat{\mathbf{B}}_{k_r}$$

(In this formula we have written $\mathbf{A}_{k_i, k_{i+1}; d}$ for an *extension* of $\mathbf{A}_{k_i, k_{i+1}}$; d that we will define precisely below.)

The domain of definition of the operator $\mathbf{A}_{k', k; d}$ is the set
 $\{\text{analytic functions on sectors bisected by } d \text{ with an exponential growth of order } \leq k \text{ at infinity and an asymptotic expansion at zero}\}.$

We will now see that there exists a *natural extension* of this operator to the *larger domain*

{ analytic functions on sectors bisected by d with an exponential growth of order $\leq \kappa = \frac{kk'}{k' - k}$ at infinity and an "asymptotic expansion" at zero };

$$1/k' + 1/\kappa = 1/k; \quad \kappa = k \frac{k'}{k' - k} > k.$$

It is clearly sufficient to understand how to *extend* the operator $A_{\alpha; d}$ ($\alpha > 1$) defined on the domain

{ analytic functions on sectors bisected by d with an exponential growth of order ≤ 1 at infinity and an "asymptotic expansion" at zero } to the *larger domain*

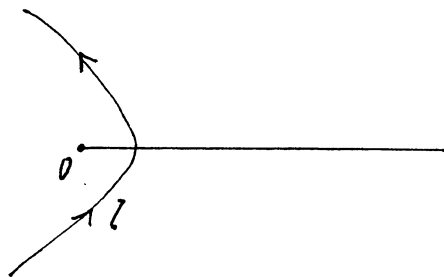
{ analytic functions on sectors bisected by d with an exponential growth of order $\leq \beta = \frac{\alpha}{\alpha - 1}$ at infinity and an "asymptotic expansion" at zero }.

$$1/\alpha + 1/\beta = 1.$$

This is done using an *integral formula* for $A_{\alpha; d}$ discovered by *Ecalé* [E 4]:

We introduce a family of "*special functions*" C_{α} ($\alpha > 1$), the "*accelerating functions*":

$$C_{\alpha}(t) = \frac{1}{2i\pi} \int_l e^{u - tu^{1/\alpha}} du; \text{ the path } l \text{ being a Hankel contour:}$$



It is easy to see that C_{α} is an *entire function* and to compute its analytic expansion at the origin:

$$C_1 = \frac{1}{\pi} \sum_{n \geq 0} \sin \frac{n\pi}{\beta} \frac{\Gamma(1 + n/\alpha)}{\Gamma(1 + n)} t^n;$$

$$\text{with } 1/\alpha + 1/\beta = 1.$$

Example:

$$\alpha = \beta = 2; \quad \text{then } C_2(t) = \frac{1}{2\sqrt{\pi}} t e^{-t^2/4}.$$

Functions C_α are *resurgent* at ∞ [E 4], [MA], [C]. If $\alpha \in \mathbf{Q}$ these functions are related to *Mejer G-functions* and *solutions of linear differential equations* (cf. below “*Formulae about accelerating functions*”).

LEMMA 6 ([E 4], [MR 3])⁽¹⁸⁾

Let $\beta > 1$, and $\alpha = \frac{\beta}{\beta-1}$. Let $0 < \theta < \frac{\pi}{\beta}$.

Let $V_\theta = \left\{ t \in \mathbf{C} \mid |\operatorname{Arg} t| < \frac{\theta}{2} \right\}$. Then (on V_θ):

$$|C_\alpha(t)| \leq \frac{K_\alpha}{\sqrt{\cos \beta\theta}} |t^{\beta/2} e^{-(t/c_\alpha)^\beta}|; \quad \text{with } K_\alpha > 0 \quad \text{and } c_\alpha = \beta(\alpha-1)^{1/\alpha}.$$

PROPOSITION 3. — Let $\alpha > 1$. Let $A_{\alpha;d} = (L_{d^\alpha} \rho_\alpha)^{-1} \rho_\alpha L_d$ and φ be an analytic function on a sector bisected by d and with an asymptotic expansion at zero (or, more generally an infinitely differentiable function on d with an exponential growth of order ≤ 1 at infinity). Then

$$A_{\alpha;d} \varphi(\zeta) = \zeta^{-\alpha} \int_d C_\alpha(\xi/\zeta) \varphi(\xi) d\xi.$$

DEFINITION 1. — Let $\alpha > 1$ and φ an infinitely differentiable function on ⁽¹⁹⁾ a direction d . If the integral $\int_d C_\alpha(\xi/\zeta) \varphi(\xi) d\xi$ exists, we will say that φ is α -accelerable in the direction d .

The operator $A_{\alpha;d} = (L_{d^\alpha} \rho_\alpha)^{-1} \rho_\alpha L_d$ is defined on the domain {analytic functions on sectors bisected by d with an exponential growth of order ≤ 1 at infinity and an asymptotic expansion at the origin}, but we have $\beta > 1$ and the operator $\varphi \rightarrow \int_d C_\alpha(\zeta/\xi) \varphi(\xi) d\xi$ is defined on the **larger** domain {analytic functions on sectors bisected by d with an exponential growth

⁽¹⁸⁾ More precisely, using saddlepoint method, it is possible to get an *asymptotic expansion* for the function C_α on the sector V_θ (and even in $|\operatorname{Arg} t| < \pi/2$), cf. [HL], p. 45, [Bak], p. 84, [MR 3].

⁽¹⁹⁾ The function φ is defined on $d - \{0\}$. We do not suppose it differentiable at the origin.

of order $\leq \beta$ at infinity and an asymptotic expansion at the origin }.
 (More generally a function infinitely differentiable on d with an exponential growth of order $\leq \beta$ at infinity is α -accelerable.)

Then we get from *proposition 3* the searched *extension* for the operator $A_{\alpha; d}$. (In the following we will still denote this extension by $A_{\alpha; d}$.)

Now using

$A_{k', k; d} \psi(\zeta) = \zeta^{-k'} \int_d \psi(\xi) C_\alpha(\xi^k/\zeta^k) k \xi^{k-1} d\xi$ where ψ is analytic on a sector bisected by d , with an exponential growth of order ≤ 1 at infinity, it is possible to extend the operator $A_{k', k; d}$ to the *larger domain*

{ analytic functions on sectors bisected by d with an exponential growth

$$\text{of order } \leq \kappa = \frac{kk'}{k' - k}$$

at infinity and an asymptotic expansion at the origin }.

We can define now the notion of $(k_1, k_2, \dots, k_r)$ -summability in a direction d and the corresponding summability operator $S_{k_1, k_2, \dots, k_r; d}$. (In the following definition operators $A_{k_i, k_{i+1}; d}$ must be interpreted in the *extended sense*, that is as *integral operators*.)

DEFINITION 2. — Let $k_1 > k_2 > \dots > k_r > 0$ and a direction d . A formal power series $\hat{f} \in \mathbb{C}[[x]]$ is called $(k_1, k_2, \dots, k_r)$ -summable in the direction d if the following conditions are satisfied:

- (0) $\hat{f} \in \mathbb{C}[[x]]_{1/k_r}$.
- (1) $S \hat{B}_{k_r} \hat{f}$ can be *analytically extended* along d to a function $\cdot_d S \hat{B}_{k_r} \hat{f}$ analytic on a sector bisected by d with an *exponential growth of order* $\leq \frac{k_{r-1} k_r}{k_{r-1} - k_r}$.
- (2) $A_{k_{r-1}, k_r; d} \cdot_d S \hat{B}_{k_r} \hat{f}$ can *analytically extended* along d to a function $\cdot_d A_{k_{r-1}, k_r; d} \cdot_d S \hat{B}_{k_r} \hat{f}$ with an *exponential growth of order* $\leq \frac{k_{r-2} k_{r-1}}{k_{r-2} - k_{r-1}}$.
-
- (i) $A_{k_{r-i+1}, k_{r-i+2}; d} \dots \cdot_d A_{k_{r-1}, k_r; d} \cdot_d S \hat{B}_{k_r} \hat{f}$ can be *analytically extended* along d to a function $\cdot_d A_{k_{r-i+1}, k_{r-i+2}; d} \dots \cdot_d A_{k_{r-1}, k_r; d} \cdot_d S \hat{B}_{k_r} \hat{f}$ with an *exponential growth of order* $\leq \frac{k_{r-i} k_{r-i+1}}{k_{r-i} - k_{r-i+1}}$.
-
- (r) $A_{k_1, k_2; d} \dots \cdot_d A_{k_{r-1}, k_r; d} \cdot_d S \hat{B}_{k_r} \hat{f}$ can be *analytically extended* along d to a function $\cdot_d A_{k_1, k_2; d} \dots \cdot_d A_{k_{r-1}, k_r; d} \cdot_d S \hat{B}_{k_r} \hat{f}$ with an *exponential growth of order* $\leq k_1$.

If a formal power series $\hat{f} \in \mathbb{C}[[x]]$ is $(k_1, k_2, \dots, k_r)$ -summable in the direction d , then:

$L_{k_1, d} \cdot d \cdot A_{k_1, k_2; d} \cdot \dots \cdot d \cdot A_{k_{r-1}, k_r; d} \cdot d \cdot S \hat{B}_{k_r} \hat{f}$ is defined and analytic in a sector bisected by d .

We will set

$$S_{k_1, k_2, \dots, k_r; d} = L_{k_1; d} \cdot d \cdot A_{k_1, k_2; d} \cdot \dots \cdot d \cdot A_{k_{r-1}, k_r; d} \cdot d \cdot S \hat{B}_{k_r};$$

$$S_{k_1, k_2, \dots, k_r; d} \hat{f} \text{ is the } (k_1, k_2, \dots, k_r)\text{-sum of } \hat{f} \text{ in the direction } d.$$

If $\hat{f} \in \mathbb{C}[[x]]$ is $(k_1, k_2, \dots, k_r)$ -summable in the direction d , we will write it

$$f \in \mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d}.$$

If $\hat{f} \in \mathbb{C}[[x]]$ is $(k_1, k_2, \dots, k_r)$ -summable in all directions, but perhaps a finite number, we will write it

$$\hat{f} \in \mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r}, \text{ and say that } \hat{f} \text{ is } (k_1, k_2, \dots, k_r)\text{-summable.}$$

LEMMA 7. — Let $k_1, k_2, \dots, k_r > 0$ and let d be a given direction. Then

(i) $\mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d}$ and $\mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r}$ are differential subalgebras of $\mathbb{C}[[x]]$;

(ii) The subalgebra of $\mathbb{C}[[x]]$ generated by the differential algebras $\mathbb{C}\{x\}_{1/k_1; d}, \mathbb{C}\{x\}_{1/k_2; d}, \dots, \mathbb{C}\{x\}_{1/k_r; d}$ is a differential subalgebra of $\mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d}$. Moreover if

$$f = \sum_{i \in I} f_{i, 1} \dots f_{i, r}, \text{ with } I \text{ finite and } f_{ij} \in \mathbb{C}[[x]]_{1/k_j; d} \text{ (} i \in I, \text{ and } j = 1, \dots, r),$$

then

$$S_{k_1, k_2, \dots, k_r; d} f = \sum_{i \in I} S_{k_1; d} f_{i, 1} \dots S_{k_r; d} f_{i, r} \text{ in particular the analytic function } \sum_{i \in I} S_{k_1; d} f_{i, 1} \dots S_{k_r; d} f_{i, r} \text{ is independent of the "decomposition" } \sum_{i \in I} f_{i, 1} \dots f_{i, r} \text{ of the formal power series }^{(20)}.$$

PROPOSITION 4. — Let $k' > k > 0$. The operator $A_{k', k}$ interpreted in the extended sense (that is as an integral operator) gives an **injective morphism** of "convolution" differential algebras:

k -convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order $\leq \kappa = \frac{kk'}{k' - k}$ at infinity, and with an "asymptotic expansion" at zero.

k' -convolution differential algebra of analytic functions on sectors with opening $> \pi/\kappa = \pi \frac{k' - k}{kk'}$ and arbitrary radius bisected by d , and with an "asymptotic expansion" at zero.

⁽²⁰⁾ This was proved in [Ra 5] using a different method, answering a question of [Ra 2].

Let f and g be infinitely differentiable (as functions of a real variable) on d with complex values. If f and g have a growth of order $\leq k$ (in particular if f and g have a compact support) we have

$$\begin{aligned} \mathbf{A}_{k',k}(f \star_k g) &= \rho_{k'}^{-1} L^{-1} \rho_{k'/k} L((\rho_k f) \star (\rho_k g)) \\ \mathbf{A}_{k',k}(f \star_k g) &= \rho_{k'}^{-1} L^{-1} \rho_{k'/k} ((L \rho_k f)(L \rho_k g)) \\ \mathbf{A}_{k',k}(f \star_k g) &= \mathbf{A}_{k',k} f \star_{k'} \mathbf{A}_{k',k} g. \end{aligned}$$

We get the *same* formula when f and g have *only* a growth $\leq k$ by a *density argument*. Then $\mathbf{A}_{k',k}$ is a morphism of “convolution differential algebras”.

The proof of *injectivity* is a little more subtle. We need a little bit of Ecalle’s “*deceleration theory*” [E 4]:

We have (definition)

$$\mathbf{A}_\alpha^{-1} = \mathbf{D}_\alpha = (\rho_\alpha L)^{-1} L \rho_\alpha = L^{-1} \rho_\alpha^{-1} L \rho_\alpha$$

and

$$\mathbf{A}_{k',k}^{-1} = \mathbf{D}_{k',k} = \rho_{k'}^{-1} L^{-1} \rho_{k/k'} L \rho_{k'} \text{ (formally } \mathbf{D}_{k',k} = \mathbf{A}_{k',k} \text{)}.$$

There exists *integral formulae* for the operators of “*normalized deceleration*” $\mathbf{D}_\alpha, \mathbf{D}_{k',k}$. To get them we need a new family of “special functions” $C^\alpha (\alpha > 1)$, the “*decelerating functions*”:

$$C^\alpha(t) = \int_{\mathbf{R}^+} e^{-u+tu^{1/\alpha}} u^{-1/\beta} du.$$

It is easy to see that C^α is an *entire function* and to compute its *analytic expansion at zero*:

$$\sum_{n \geq 0} \frac{\Gamma((n+1)/\alpha)}{\Gamma(n+1)} t^n.$$

Example:

$\alpha = \beta = 2$; then $C^2(t) = 2e^{t^2/4} \int_{-t/2}^{+\infty} e^{-u^2} du$. This function is related to “*error functions*” ⁽²¹⁾:

$$\operatorname{Erfc}(\sigma) = \frac{2}{\sqrt{\pi}} \int_{\sigma}^{+\infty} e^{-v^2} dv = 1 - \operatorname{Erf}(\sigma).$$

Functions C^α are *resurgent* at ∞ [E 4], [Ma 8], [C]. If $\alpha \in \mathbf{Q}$ these functions are related to *Mejer G-functions* and *solutions of linear differential equations* (cf. below “*Formulae about decelerating functions*”).

⁽²¹⁾ The function C^3 is simply related to Airy function Ai and to Bessel function $K_{1/3}$ (cf. [Bak], p. 98).

Ecalles's functions C^α are particular cases ⁽²²⁾ of *Faxén's* integrals:

$$Fi(\mu, \nu; t) = \int_{\mathbf{R}^+} e^{-u+\nu u^\lambda} u^{\mu-1} du \quad (\text{see [O 1], [Fa], [BHL]})$$

$$Fi(\alpha^{-1}, \alpha^{-1}; t) = C^\alpha(t).$$

There is in fact a very interesting family of functions:

$$F_{P; \pm}(\alpha; \beta; y) = \int_{\gamma_{\pm}} e^{P(v^\alpha) \pm \nu y} v^\beta dv; \quad \text{with } \alpha \in \mathbf{R}, \beta \in \mathbf{C}, P \in \mathbf{C}[w],$$

and $\gamma_{\pm}$ a convenient path.

There are many occurrences of particular cases of these functions in the literature: the main sources are *arithmetic* (in connection with exponential sums: cf. the *Hardy-Littlewood's* paper on Waring's problem [HL] ⁽²³⁾, and more recently works of *N. Katz* [Ka 4], *Deligne*, ...), *physic* (*Airy*, *Kelvin*, *Brillouin* ⁽²⁴⁾, ...), *analysis* (study of accelerating and decelerating functions, study of Laplace transform: cf. [Ma 5]), and *probabilities* (up to variable and function rescalings, *stable densities* are real parts of accelerating functions, cf. [Fe], p. 548). If $\alpha \in \mathbf{Q}$ the function $F_{P; \pm}(\alpha; \beta; y)$ is solution of a differential equation (obtained by a method similar to the derivation of Gauss-Manin connection). These functions ⁽²⁵⁾ would certainly deserve a thoroughful study.

LEMMA 8 [E 4], [MR 3] ⁽²⁶⁾.

Let $R' > 0$ and $\beta > 1$; we set $\alpha = \frac{\beta}{\beta-1}$.

Let $D'_{\beta, R'} = \{t \in \mathbf{C} \mid |\text{Arg } t| < \frac{\pi}{2\beta} \text{ and } \text{Re}(t^\beta) \geq 1/R'\}$. Then (on $D'_{\beta, R'}$):

$$|C^\alpha(t)| \leq K^\alpha R'^{\beta/2} |t^{\beta-1} e^{(t/c_\alpha)^\beta}|; \quad \text{with } K^\alpha > 0 \text{ and } c_\alpha = \beta(\alpha-1)^{1/\alpha}.$$

This Lemma is proved using *saddlepoint method*.

DEFINITION 3. — Let $\alpha > 1$, $\beta = \frac{\alpha}{\alpha-1}$, $R > 0$, and a direction d .

Let ψ be a function analytic on the open β -Borel disc

$$D_{\beta, R; d} = \{t \in \mathbf{C} \mid |\text{Arg } \zeta - \text{Arg } d| < \frac{\pi}{2\beta}$$

⁽²²⁾ This was mentionned to us by *A. Duval*.

⁽²³⁾ Cf. also *Bakhoom* [Bak].

⁽²⁴⁾ Cf. also [AS], p. 1002.

⁽²⁵⁾ And the similar functions obtained when we replace the *Laplace* transform by the *Mellin* transform in the definition (cf. functions Γ_P studied in [Du]).

⁽²⁶⁾ More precisely it is possible, using saddlepoint method, to get an *asymptotic expansion* for the function C^α on the domain $D'_{\beta, R'}$ (cf. [MR 3]).

and

$$\operatorname{Re}(\zeta e^{-i \operatorname{Arg} d})^{-\beta} > 1/R^\beta\},$$

and continuous on the closure of $D_{\beta, R, d}$.

If we denote by γ_R the boundary of $D_{\beta, R, d}$ oriented in the positive sense, we will say that ψ is α -decelerable in the direction d if the integral

$$\varphi(\xi) = \frac{1}{2i\pi} \int_{\gamma_R} \psi(\zeta) \zeta^\alpha C^\alpha(\xi/\zeta) d\zeta/\zeta^2 \text{ exists} \quad (\text{for } \xi \in d, \text{ arbitrary}).$$

PROPOSITION 5. — Let $\alpha > 1$, $\beta = \frac{\alpha}{\alpha-1}$. Let ψ be an analytic function

on a sector, with opening $> \frac{\pi}{\beta}$, bisected by d , with exponential growth of order $\leq \alpha$ at infinity and an “asymptotic expansion” at zero. Then ψ is α -decelerable in the direction d and:

$$\mathbf{D}_\alpha \psi(\xi) = L^{-1} \rho_\alpha^{-1} L \rho_\alpha \psi(\xi) = \frac{1}{2i\pi} \int_{\gamma_R} \psi(\zeta) \zeta^\alpha C^\alpha(\xi/\zeta) d\zeta/\zeta^2.$$

If the function ψ is analytic on a sector V with opening $> \frac{\pi}{\beta}$, bisected by d , and if ψ is sufficiently flat at zero, that is if there exists $\lambda > 0$ such that

$$\psi = o(\zeta^{1+\beta-\alpha+\lambda}) \quad \text{on } V,$$

then it is α -decelerable in the direction d and $\mathbf{D}_\alpha \psi$ is analytic on a sector bisected by d , with an exponential growth of order $\leq \beta$ at infinity.

If a function ψ is analytic on $D_{\beta, R, d}$ and admits an “asymptotic expansion” at zero and if there exists a “polynomial” P such that $\psi = \psi_0 + P$, where ψ_0 is α -decelerable in the direction d , we will still say that ψ is α -decelerable in the direction d and we will write

$$\mathbf{D}_\alpha \psi = \mathbf{D}_\alpha \psi_0 + \mathbf{D}_\alpha P$$

(where $\mathbf{D}_\alpha P$ is computed “formally”: see formulae at the end of this paragraph).

The operator $\mathbf{D}_{\alpha; d} = L^{-1} \rho_\alpha^{-1} L \rho_\alpha$ is defined on the domain

$$\left\{ \begin{array}{l} \text{analytic functions on sectors with opening } > \frac{\pi}{\beta} \text{ bisected by } d, \\ \text{with an exponential growth of order } \leq \alpha \text{ at infinity} \\ \text{and an “asymptotic expansion” at the origin} \end{array} \right\}.$$

The operator $\psi \rightarrow \frac{1}{2i\pi} \int_{\gamma_R} \psi(\zeta) \zeta^\alpha C^\alpha(\xi/\zeta) d\zeta/\zeta^2$ is defined on the *larger* domain

$\left\{ \begin{array}{l} \text{analytic functions on sectors with opening } > \frac{\pi}{\beta} \text{ with arbitrary} \\ \text{radius bisected by } d, \text{ with an asymptotic expansion at the origin} \end{array} \right\}.$

So, *proposition 5* gives an extension for the operator $D_{\alpha, d}$.

LEMMA 9. — *The function C^α is α -accelerable in the direction $\mathbf{R}^+$ and*

$$A_\alpha C^\alpha(\zeta) = \zeta/\zeta^\alpha (1 - \zeta).$$

PROPOSITION 6. — *Let $\alpha > 1$, $\beta = \frac{\alpha}{\alpha - 1}$.*

(i) *If a function ψ is α -decelerable in the direction d , then $D_\alpha \psi$ is α -accelerable in the direction d and:*

$$A_\alpha D_\alpha \psi = \psi.$$

(ii) *If a function φ is infinitely differentiable on d , with an exponential growth of order $\leq \beta$ at infinity, then $A_\alpha \varphi$ is α -decelerable in the direction d and:*

$$D_\alpha A_\alpha \varphi = \varphi.$$

The proof of (i) is easy, using Fubini's theorem and *lemma 9*.

To prove (ii), using *lemma 3*, we first prove it when ψ is infinitely differentiable on d with an exponential growth of order ≤ 1 at infinity (in particular when ψ has a compact support). Then, for ψ with *only* an exponential growth of order $\leq \beta$, we conclude by a *density argument*.

From *proposition 5* (ii) we deduce the *injectivity* of $A_{\alpha, d}$. The *injectivity* of $A_{k', k; d}$ follows. That ends the proof of *proposition 6*.

The following result is *essential*:

THEOREM 1. — *Let $k_1 > k_2 > \dots > k_r > 0$, and d a given direction. Then the **summation operator***

$$\mathbf{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d} \xrightarrow{S_{k_1, k_2, \dots, k_r; d}} \begin{array}{l} \text{Differential algebra of germs} \\ \text{of analytic functions} \\ \text{on sectors bisected by } d. \end{array}$$

is an injective morphism of differential algebras.

Operators S and $\cdot_d$ are *isomorphisms* of differential algebras and of k -convolution differential algebras. Operator $\hat{B}_{k_r}$ is an *isomorphism* of differential algebras between the differential algebra $(x \mathbf{C}[[x]], x^2 d/dx)$ and the k_r -convolution differential algebra $(\xi^{1-k_r} \mathbf{C}[[x]], \partial_k)$. Operator L_{k_1} is an *isomorphism* between the convolution differential algebra of analytic

functions on sectors bisected by d with an exponential growth of order $\leq k_1$ at infinity and an "asymptotic expansion at zero", and the differential algebra of analytic functions on sectors with opening $> \pi/k_1$, bisected by d , and with an "asymptotic expansion" (without constant term) at zero. We can now end the proof of theorem 1, using *proposition 4* with $k' = k_{i-1}$, $k = k_i$ ($i = r, \dots, 2$).

In fact it follows from this proof that the image of the operator $S_{k_1, k_2, \dots, k_r; d}$ is contained in the differential algebra of analytic functions on sectors with opening $> \pi/k_1$, bisected by d , and with an asymptotic expansion (without constant term) at zero.

It is possible to extend *proposition 2*:

PROPOSITION 7. — Let $k' > k_1 > k_2 > \dots > k_r > 0$. Then:

$$\mathbf{C}[[x]]_{1/k'} \cap \mathbf{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r} = \mathbf{C}\{x\}.$$

PROPOSITION 8. — Let $k'_1, k'_2, \dots, k'_r > 0$ and $k''_1, k''_2, \dots, k''_{r'} > 0$. If

$$\{k_1, k_2, \dots, k_r\} = \{k'_1, k'_2, \dots, k'_r\} \cap \{k''_1, k''_2, \dots, k''_{r'}\},$$

with $k_1 > k_2 > \dots > k_r > 0$ ($r \leq r', r''$), then:

$$\mathbf{C}\{x\}_{1/k'_1, 1/k'_2, \dots, 1/k'_r} \cap \mathbf{C}\{x\}_{1/k''_1, 1/k''_2, \dots, 1/k''_{r'}} = \mathbf{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r}.$$

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k'_1, k'_2, \dots, k'_r)$ -summable, the smallest set $\{k_1, k_2, \dots, k_r\}$ (with $k_1 > k_2 > \dots > k_r > 0$) such that $\hat{f}$ is $(k_1, k_2, \dots, k_r)$ -summable, is a subset of $\{k'_1, k'_2, \dots, k'_r\}$ and depends only on $\hat{f}$. The numbers $k_1, k_2, \dots, k_r$ are *the singular levels* of $\hat{f}$:

$$\{k_1, k_2, \dots, k_r\} = \mathbf{N} \Sigma(\hat{f}) \subset]0, +\infty[\text{ (definition).}$$

The situation is very different if $\hat{f} \in \mathbf{C}[[x]]$, is $(k'_1, k'_2, \dots, k'_r)$ -summable in a direction d . It is easy to prove then that there exists $\varepsilon < 0$, such that $\hat{f}$ is $(k'_1 - \varepsilon', k'_2 - \varepsilon', \dots, k'_r - \varepsilon')$ -summable in the direction d for every $\varepsilon' \in]0, \varepsilon]$.

We identify the real analytic blow-up of the origin in the complex plane ⁽²⁷⁾ with the circle S^1 . Then we introduce the "analytic halo" of the origin in the complex plane:

$$\mathbf{HA}_0 =]0, +\infty] \times S^1 = \{(k, d)/k \in]0, +\infty], d \in S^1\} \text{ (definition).}$$

The complex plane with an analytic halo at zero is:

$$\mathbf{CH}_0 = \{0\} \cup \mathbf{HA}_0 \cup \mathbf{C}^* = ((\{ "0" \} \cup ["0, +\infty]") \cup]0, +\infty[\times S^1) / \mathcal{R};$$

where the relation $\mathcal{R}$ corresponds to the identification of $\{ "0" \} \times S^1$ with a point $\{ "0" \}$.

⁽²⁷⁾ If we use *polar coordinates* for the points of $\mathbf{C}^*$:

$$\mathbf{C}^* = \{(\rho, \theta)/\rho > 0, \theta \in S^1\} =]0, +\infty[\times S^1, \text{ this set corresponds to } \{0\} \times S^1.$$

On the set $\{\text{"0"}\} \cup]0, +\infty[\cup]0, +\infty[$ we put the ordering relation:

Ordinary ordering relation on $]0, +\infty[$ and "0" , $\rho > 0 > k$, if $\rho \in]0, +\infty[$, and $k \in \text{"0"}$ ($+\infty$ is identified with 0). We endow $\{\text{"0"}\} \cup \mathbf{H}\mathbf{A}_0 \cup \mathbf{C}^*$ with the corresponding topology (quotient of the product topology). We will consider $\{\text{"0"}\} \times S^1$ as the "*real blow up*" of 0 in $\mathbf{C}\mathbf{H}_0$ (that is the set of directions starting from 0 in $\mathbf{C}\mathbf{H}_0$).

The universal covering of $(S^1, 1)$ is $(\mathbf{R}, 0)$. We will interpret $\mathbf{H}\mathbf{A}_0 =]0, +\infty[\times (\mathbf{R}, 0)$ as the "*universal covering* of $\mathbf{H}\mathbf{A}_0$ pointed on the direction $\text{"R"}^+ \in \{\text{"0"}\} \times S^1$ ".

Let $U \subset S^1$ be an open arc. Let $k_1 > k_2 > \dots > k_r > 0$. If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \dots, k_r)$ -summable in every direction $d \in U$, then the sums $f_{k_1, k_2, \dots, k_r; d}$ glue together in a function f analytic on a "sector" with opening equal to

$$(\text{opening of } U + \pi/k_1).$$

If now $U \subset S^1$ is an open arc bisected by d , let

$$U^+ = \{d^+ \in U / \text{Arg } d^+ > \text{Arg } d\},$$

and

$$U^- = \{d^- \in U / \text{Arg } d^- < \text{Arg } d\}.$$

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \dots, k_r)$ -summable in every direction $d' \in U - \{d\}$, then we write

$$f_{k_1, k_2, \dots, k_r; d}^+ = S_{k_1, k_2, \dots, k_r; d}^+ \hat{f}$$

and

$$f_{k_1, k_2, \dots, k_r; d}^- = S_{k_1, k_2, \dots, k_r; d}^- \hat{f}$$

for the sums of $\hat{f}$ for $d^+ \in U^+$ and $d^- \in U^-$ respectively.

They are in particular defined on a common "sector" bisected by d , with opening equal to π/k_1 .

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \dots, k_r)$ -summable, then $S_{k_1, k_2, \dots, k_r; d}^+ \hat{f}$ and $S_{k_1, k_2, \dots, k_r; d}^- \hat{f}$ are defined for every direction $d \in S^1$.

We can define along the same lines operators $L_{k_1; d}^\varepsilon$ and $A_{k_{j-1}, k_j; d}^\varepsilon$, for $\varepsilon \in \{1, -1\}$.

Using *decelerating* operators we get easily the *very important*:

LEMMA 10. — Let $k_1 > k_2 > \dots > k_r > 0$ and d a given direction. Then if $\hat{f} \in \mathbf{C}[[x]]$, $(k_1, k_2, \dots, k_r)$ -summable in every direction of $U - \{d\}$, the following conditions are equivalent:

- (i) $\hat{f}$ is $(k_1, k_2, \dots, k_r)$ -summable in the direction d ;
- (ii) $S_{k_1, k_2, \dots, k_r; d}^+ \hat{f} = S_{k_1, k_2, \dots, k_r; d}^- \hat{f}$ on a "sector" bisected by d .

Moreover if these conditions are satisfied, then

$$S_{k_1, k_2, \dots, k_r; d}^+ \hat{f} = S_{k_1, k_2, \dots, k_r; d}^- \hat{f} = S_{k_1, k_2, \dots, k_r; d} \hat{f}.$$

If the conditions of *lemma 10* are *not* satisfied we will say that d is a *singular direction* for the formal power series $\hat{f}$, and we will write $d \in \Sigma(\hat{f})$; the “*singular support*” $\Sigma(\hat{f})$ of $\hat{f}$ is clearly *finite*, and $\Sigma(\hat{f}) = \emptyset$ is equivalent to $\hat{f} \in \mathbb{C}\{x\}$. We will see below that the “*jump*” from

$S_{k_1, k_2, \dots, k_r; d}^+ \hat{f}$ to $S_{k_1, k_2, \dots, k_r; d}^- \hat{f}$ is a natural generalisation of the classical “*Stokes phenomenon*” for solutions of linear differential equations.

We will give below (*cf.* 6) a very natural *interpretation of multisummability*:

A formal power series $\hat{f} \in \mathbb{C}[[x]]$ is *multisummable in the direction d* (that is there exist $k_1 > k_2 > \dots > k_r > 0$ such that $\hat{f}$ is $(k_1, k_2, \dots, k_r)$ -summable in the direction d) if and only if it is “*analytic*” (“*wild analytic*”) in an “*infinitesimal disc*”⁽²⁸⁾ and can be “*extended analytically*” along d across the “*infinitesimal neighbourhood*”⁽²⁹⁾ in a wild analytic function on a sector bisected by d with a “non infinitesimal” radius $R > 0$.

Then, just like one can give a *direct* (that is *without* using Borel and Laplace transforms) definition of *Borel-summability* and *k-summability* using *Gevrey estimates* [Ra 2], [MR 1], [MR 2], [MR 3], it is also possible to give a *direct* (that is *without* any use of *Ecalles*’ acceleration operators) definition of *multisummability* using the *wild Cauchy theory* recently introduced by the authors [MR 3]. This “*geometric*” definition is *easier to check* in the *usual applications*. Conversely the “*analytic*” definition gives an “*explicit*” way for the computation of the sum (for instance if one has in mind *numerical computations*).

Let $U \subset S^1$ be an open arc bisected by d . Let $k_1 > k_2 > \dots > k_r > 0$ and let $\hat{f} \in \mathbb{C}[[x]]$ be $(k_1, k_2, \dots, k_r)$ -summable in every direction $d' \in U - \{d\}$. There is a natural way to generalize the sums $S_{k_1, k_2, \dots, k_r; d}^+ \hat{f}$ and $S_{k_1, k_2, \dots, k_r; d}^- \hat{f}$:

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$, with $\varepsilon_i \in \{1, -1\}$ ($i = 1, \dots, r$). We will say that $(d; \varepsilon)$ defines a “*path*”⁽³⁰⁾ We can now introduce the notion of $(k_1, k_2, \dots, k_r)$ -summability along the path $(d; \varepsilon)$:

DEFINITION 3. — Let $U \subset S^1$ be an open arc bisected by d . Let $k_1 > k_2 > \dots > k_r > 0$ and let $\hat{f} \in \mathbb{C}[[x]]$, be $(k_1, k_2, \dots, k_r)$ -summable in every direction $d' \in U - \{d\}$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$, with $\varepsilon_i \in \{1, -1\}$ ($i = 1, \dots, r$). We will say that $\hat{f}$ is $(k_1, k_2, \dots, k_r)$ -summable along the path $(d; \varepsilon)$ if

$$S_{k_1, k_2, \dots, k_r; d}^\varepsilon \hat{f} = L_{k_1, k_2; d}^{\varepsilon_1} A_{k_1, k_2; d}^{\varepsilon_2} \dots A_{k_{r-1}, k_r; d}^{\varepsilon_r} S \hat{B}_k \hat{f}$$

⁽²⁸⁾ The corresponding punctured disc has a radius $\geq “k” > “0”$ in the analytic halo at zero.

⁽²⁹⁾ This infinitesimal neighborhood is the union of zero and the analytic halo at zero.

⁽³⁰⁾ Later we will see that such a $(d; \varepsilon)$ corresponds to a *wild homotopy class of paths* in the analytic halo of the origin, avoiding “*singularities*” of f in this halo.

exists. Then $S_{k_1, k_2, \dots, k_r; d}^\varepsilon \hat{f}$ is the *sum of $\hat{f}$ along the path $(d; \varepsilon)$* .

THEOREM 2. — Let $k_1 > k_2 > \dots > k_r > 0$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$, with $\varepsilon_i \in \{1, -1\}$ ($i = 1, \dots, r$). Let d be a given direction. Then the **summation operator**

$$(k_1, k_2, \dots, k_r)\text{-summable along the path } (d; e) \xrightarrow{s_{k_1, k_2, \dots, k_r; d}^\varepsilon} \begin{array}{l} \text{Differential algebra of germs of} \\ \text{analytic functions on sectors} \\ \text{bisected by } d. \end{array}$$

power series $f \in \mathbb{C}[[x]]$

is an **injective morphism of differential algebras**

Comparison between $S_{k_1, k_2, \dots, k_r; d}^\varepsilon \hat{f}$ and $S_{k_1, k_2, \dots, k_r; d}^{\varepsilon'} \hat{f}$ for different $\varepsilon, \varepsilon'$ will give birth to a “*generalized Stokes phenomenon*”.

We will finish this paragraph with a small list of **useful formulae**:

Let $k, k', \lambda, \mu > 0$. Then:

$$\begin{aligned} \rho_k(x^\lambda) &= u^{\lambda/k}, & \rho_\alpha(\xi^\mu) &= \frac{\Gamma(1+\mu)}{\Gamma((1+\mu)/\alpha)} \zeta^{(1+\mu-\alpha)/\alpha} \\ B_k(x^\lambda) &= \xi^{\lambda-k}/\Gamma(\lambda/k), & L_k(\xi^\mu) &= \Gamma(1+\mu/k) x^{\mu+k} \\ A_\alpha(\xi^\mu) &= \frac{\Gamma(1+\mu)}{\Gamma((1+\mu)/\alpha)} \zeta^{1+\mu-\alpha} \\ A_{k',k}(\xi^\mu) &= \frac{\Gamma((k+\mu)/k)}{\Gamma((k+\mu)/k')} \zeta^{\mu+k-k'} \\ D_\alpha(\zeta^\nu) &= \frac{\Gamma(1+\nu/\alpha)}{\Gamma(\nu+\alpha)} \xi^{\nu-1+\alpha} \\ D_{k',k}(\zeta^\nu) &= \frac{\Gamma((k'+\nu)/k')}{\Gamma((k'+\nu)/k)} \xi^{\nu+k'-k} \\ \partial_k(\xi^\lambda) &= (\lambda+k) \Gamma(1+\lambda/k) \xi^{\lambda+1}/\Gamma((\lambda+1)/k+1) \\ &= ((\lambda+k) \Gamma(1+\lambda/k) \xi/\Gamma((\lambda+1)/k+1)) \xi^\lambda \\ \xi^\lambda \star \xi^\mu &= \frac{\Gamma(1+\lambda) \Gamma(1+\mu)}{\Gamma(1+\lambda+\mu)} \xi^{\lambda+\mu}. \\ \xi^\lambda \star_k \xi^\mu &= \frac{\Gamma(1+\lambda/k) \Gamma(1+\mu/k)}{\Gamma(2+(\lambda+\mu)/k)} \xi^{\lambda+\mu+k}. \end{aligned}$$

Formulae about accelerating and decelerating functions.

The following results were obtained recently (january 1990) by *A. Duval*:

$$\begin{aligned} C_3(t) &= i \sqrt{3} G_{0,2}^{2,0}((t/3)^3 |_{1/3, 2/3}^1); \\ C^2(t) &= \frac{1}{\sqrt{\pi}} G_{1,2}^{2,1}((t/2)^2 |_{0, 1/2}^{1/2}) = \psi(1/2, 1/2; t^2/4); \end{aligned}$$

G is a *Mejer G-function* [Lu].

$$C_{\alpha}(t) = \int_{+\infty}^{(0^-)} \frac{\Gamma(-s)}{\Gamma(-s/\alpha)} t^s ds \quad (\text{Hankel type contour around } \mathbf{R}^+),$$

$$C^{\alpha}(t) = \frac{1}{2i\pi} \int_{+\infty}^{(0^-)} \Gamma(-s) \Gamma\left(\frac{s+1}{\alpha}\right) (-t)^s ds.$$

If $\alpha = p/q$, with p and q positive integers, $q > p > 0$, $(p, q) = 1$:

$$C_{q/p}(t) = \frac{1}{2i\pi} \sqrt{pq(2\pi)^{p-q}} \int_{+\infty}^{(0^-)} \frac{\prod_{j=1, \dots, q-1} \Gamma(-s+j/q)}{\prod_{j=1, \dots, p-1} \Gamma(s+j/p)} (p^p (t/q)^q)^s ds$$

$$C_{q/p}(t) = \sqrt{pq(2\pi)^{p-q}} G_{p-1, q-1}^{q-1, 0} (p^p (t/q)^q)^{\frac{1}{1/q}, \frac{2}{2/q}, \dots, \frac{(p-1)}{(q-1)/q}};$$

$$C^{q/p}(t) = -\frac{ip^{p/q} \sqrt{q}}{\sqrt{p(2\pi)^{q+p}}} \int_{+\infty}^{(0^-)} \frac{\prod_{j=0, \dots, q-1} \Gamma(-s+j/q)}{\prod_{j=0, \dots, p-1} \Gamma(s+1/q+j/p)} (p^p (-t/q)^q)^s ds$$

$$C^{q/p}(t) = \frac{2\pi p^{p/q} \sqrt{q}}{\sqrt{p(2\pi)^{q+p}}} G_{p, q}^{q, p} (p^p (-t/q)^q)^{\frac{1}{0, 1/q}, \dots, \frac{2}{p-1/q}, \dots, \frac{1}{1-1/q}}.$$

Accelerating functions $C_{q/p}$ are solutions of the differential operators (respectively of order $q-1$ and q):

$$q \prod_{j=1, \dots, q-1} (\delta - j) - (-1)^{q-p} p t^q \prod_{j=1, \dots, p-1} \left(\frac{p}{q} \delta + j \right) \quad (\delta = t d/dt),$$

and

$$D^q - (-1)^{q-p} \prod_{j=1, \dots, p} \left(\frac{p}{q} t D + j \right) \quad (D = d/dt).$$

We get in particular, for $q=n$, $p=1$:

$$D^n + (-1)^n \left(\frac{1}{n} t D + 1 \right).$$

Decelerating functions $C^{q/p}$ are solutions of differential operators

$$D^q - \prod_{j=0, \dots, p-1} \left(\frac{p}{q} t D + \frac{p}{q} + j \right).$$

We get in particular, for $q=n$, $p=1$:

$$D^n - \frac{1}{n} (t D + 1).$$

4. STOKES MULTIPLIERS

Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbb{C}\{x\}[x^{-1}])$, be a germ of *meromorphic differential operator* at the origin of the complex plane $\mathbb{C}$.

It is well known [Ma 2] that Δ admits a **formal fundamental solution** ⁽³¹⁾ :

$$\hat{F}(x) = \hat{H}(u) u^{\nu L} e^{Q(1/u)},$$

with $u^{\nu} = x$ (for some $\nu \in \mathbb{N}^*$), $L \in \text{End}(n; \mathbb{C})$, $\hat{H} \in \text{GL}(n; \mathbb{C}[[u]][u^{-1}])$, and Q a diagonal matrix with entries in $u^{-1}\mathbb{C}[u^{-1}]$, invariant, up to permutations of the diagonal entries, by the transformation corresponding to $u \rightarrow e^{2i\pi/\nu}u$ ($x \rightarrow e^{2i\pi}x$) and satisfying $[e^{2i\pi\nu L}, Q] = 0$. (If $\nu = 1$ [L, Q]=0, and L can be supposed in *Jordan form*.)

If $Q = \text{Diag}\{q_1, q_2, \dots, q_n\}$, then the set $\{q_1, q_2, \dots, q_n\}$ is a subset of $u^{-1}\mathbb{C}[u^{-1}]$ which is *independent* of the choice of the fundamental solution $\hat{F}$ (ν is chosen *minimal*).

We will set $\{q_1, q_2, \dots, q_n\} = \mathbf{q}(Q) = \mathbf{q}(\Delta)$; the *set* $\mathbf{q}(\Delta)$ is clearly a *formal invariant* of Δ (invariant by the transformation $\mathbf{q}(\Delta)(u) \rightarrow \mathbf{q}(\Delta)(e^{2i\pi/\nu}u)$).

PROPOSITION 9. — *Let $k_1 > k_2 > \dots > k_r > 0$, and $\nu \in \mathbb{N}^*$. Let d be a fixed direction. Let $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}$, and $q_1, q_2, \dots, q_n \in x^{1/\nu}\mathbb{C}[x^{1/\nu}]$. Then the summation operator*

$$\mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d} \xrightarrow{S_{k_1, k_2, \dots, k_r; d}} \begin{array}{c} \text{Differential algebra of germs} \\ \text{of analytic functions} \\ \text{on sectors bisected by } d. \end{array}$$

can be uniquely extended to a summation operator (still denoted by $S_{k_1, k_2, \dots, k_r; d}$)

$$\mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d} \langle x^{\alpha_i}, e^{q_j}, \text{Log } x \rangle_{(i=1, \dots, m; j=1, \dots, n)} \rightarrow \begin{array}{c} \text{Differential algebra of germs} \\ \text{of analytic functions} \\ \text{on sectors bisected by } d. \end{array}$$

such that [a "branch" of Log x being fixed] ⁽³²⁾ :

$$S_{k_1, k_2, \dots, k_r; d}(x^{\alpha_i}) = e^{\alpha_i \text{Log } x}, \quad S_{k_1, k_2, \dots, k_r; d}(e^{q_j}) = e^{q_j},$$

and

$$S_{k_1, k_2, \dots, k_r; d}(\text{Log } x) = \text{Log } x.$$

This operator is an injective morphism of differential algebras.

It is easy to extend the definition of the operator $S_d = S_{k_1, k_2, \dots, k_r; d}$ to the elements of $\mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d} \langle x^{\alpha_i}, \text{Log } x \rangle$ ($i = 1, \dots, m$).

⁽³¹⁾ Cf. *infra* for a more precise description of $\hat{F}$ when $\nu \geq 2$ ("ramified case").

⁽³²⁾ Log x is "formal" in the "left expression" and is an actual function in the "right expression".

Then, using *asymptotic expansions* (the inverse of S_d , restricted to $\text{Im } S_d$, is the *asymptotic expansion operator* in the classical sense), we get

$$\mathbf{C}\{x\} \langle e^{qj} \rangle \cap \mathbf{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r; d} \langle x^{\alpha_i}, \text{Log } x \rangle = \mathbf{C}\{x\} \\ (i=1, \dots, m; j=1, \dots, n).$$

The result follows.

THEOREM 3. — Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

We denote by $k_1 > k_2 > \dots > k_r$ the positive (non zero) slopes of the Newton polygon of the (rank n^2) differential operator

$$\text{End } \Delta = d/dx - [A, \cdot].$$

Let $\hat{F}$ be a **formal fundamental solution** of Δ . Then there exists a “natural decomposition”⁽³³⁾.

$\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r$, where $\hat{H}_i \in \text{GL}(n; \mathbf{C}[[u]][u^{-1}])$ is k_i -**summable** as a “function” of x (i.e. $\forall k_i$ -summable as a “function” of u), for $i=1, \dots, r$, and such that

(i) $\hat{F}^i(u^\vee) = \hat{H}_i(u) \hat{H}_{i+1}(u) \dots \hat{H}_r(u) u^{\vee L} e^{Q(1/u)}$ is a **formal fundamental solution** of a meromorphic differential operator $\Delta_v^i = d/dx - A_v^i$, with

$$A_v^i \in \text{End}(n; \mathbf{C}\{u\}[u^{-1}]), \quad \text{for } i=1, \dots, r \quad (34);$$

(ii) If $\Sigma(\hat{F}) = \Sigma(\hat{H}) = \bigcup_{i=1, \dots, r} \Sigma(\hat{H}_i)$, $H_{i;d} = S_{k_i; d} \hat{H}_i$ (for $i=1, \dots, r$)

and

$$H_d = H_{1;d} H_{2;d} \dots H_{r;d},$$

then, for $d \notin \Sigma(\hat{H})$, and every determination of $\text{Log } x (u = e^{(\text{Log } x)/\vee})$ and $u^L = e^{L \text{Log } u}$:

$F_d(x) = H_d(u) u^{\vee L} e^{Q(1/u)}$ is an **actual analytic fundamental solution** of the operator Δ on a sector bisected by d .

From this result (using proposition 9) it is easy to deduce the

THEOREM 4. — Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$. Let $\hat{F}$ be a **formal fundamental solution** of Δ . If we denote by $\mathbf{C}\{x\}[x^{-1}] \langle \hat{F} \rangle$ the differential field generated, on $\mathbf{C}\{x\}[x^{-1}]$, by the

⁽³³⁾ Unique up to “natural” analytic transformations (see [Ra 4]); in particular, the matrices H_i are well defined up to analytic (in u) conjugation.

⁽³⁴⁾ Moreover the matrices A_v^i and $\hat{H}_{i+1}$ have a common “blockstructure” and Δ_v^i can be reduced by a transform “ $Y = \text{Exp}(Q_i)Z$ ” to a differential operator whose *Katz’s invariant* [De 1] is k_{i+1} ; Q_i being a diagonal matrix whose entries are monomials in u (fixed for each block) of degree $\vee k_i [J]$, [Ra 6].

entries of $\hat{F}$, then, for $d \notin \Sigma(\hat{F})$, the map

$$\mathbf{C}\{x\}[x^{-1}] \langle \hat{F} \rangle \rightarrow \begin{array}{l} \text{Differential field generated,} \\ \text{on } \mathbf{C}\{x\}[x^{-1}], \\ \text{by the analytic solutions} \\ \text{of the operator } \Delta \text{ in a germ} \\ \text{of sector bisected by } d, \end{array}$$

defined by "identity" on $\mathbf{C}\{x\}[x^{-1}]$ and $\hat{F} \rightarrow F_d$, is an **isomorphism of differential fields**.

We will first admit *theorem 3*, and will go back in 5 to some indications about its proof, after some applications. It is very easy to deduce *theorem 4* from *theorem 3*, using **multisummability** (other ways to do that are explained in [Ra 5], [Ra 6], and [De 4] ⁽³⁵⁾ :

From *theorem 3* and *lemma 7* we get

THEOREM 5. — Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$. Let $\hat{F}$ be a **formal fundamental solution** of Δ . We denote by $k_1 > k_2 > \dots > k_r$ the positive (non zero) slopes of the Newton polygon of the operator

$$\text{End } \Delta = d/dx - [A, \cdot].$$

Then $\hat{F}$ is $(k_1, k_2, \dots, k_r)$ -summable in every direction, but perhaps a finite number belonging to $\Sigma(\hat{F}) \subset S^1$.

Clearly (using *lemma 7*) the sums (in a common non singular direction) given by *theorems 2* and *4* are the same.

If $d \notin \Sigma(\hat{F})$, the operator $S_{k_1, k_2, \dots, k_r, d}$ is injective and *Galois-differential*. So *theorem 4* follows from *theorem 5*. Moreover we have got an "explicit" method of **summation of formal solutions of linear differential equations** ⁽³⁶⁾. It is interesting to remark that $k_1, k_2, \dots, k_r$ are **rational numbers**, so $k_i/k_{i-1} = \alpha_i \in \mathbf{Q}$ and C_{α_i} ($i=1, \dots, r$) is a solution of a linear differential equation; moreover all the functions written under $\int$ when we apply the successive computations of the resummation algorithm are *solutions of linear differential equations*. A consequence is that, for numerical computations, we can apply efficient algorithms in order to compute the successive *analytic continuations* $\cdot_d$ (Runge-Kutta algorithm, Chudnovskys algorithm [Chu], ...).

⁽³⁵⁾ The methods differs by the respective proportions of analysis and algebra used.

⁽³⁶⁾ There exists an algorithm for the explicit computation of the levels $k_1, k_2, \dots, k_r$ [Ma 2]. An effective computation is possible on a computer using the systems "Reduce", "Desir" and "D5" [Tou]. For the ("generic") one-level case there are efficient numerical algorithms of summation [Th]; for the multilevelled case, algorithms are studied by Thomann.

Let now $d \in \Sigma(\hat{F})$ be a *singular* direction:

Then (a “branch” of Logarithm being choosen)

$$S_{k_1, k_2, \dots, k_r; d}^+ \hat{F} \quad \text{and} \quad S_{k_1, k_2, \dots, k_r; d}^- \hat{F}$$

are (different) **actual fundamental solutions** of Δ , *analytic on a common sector bisected by d , with opening π/k_1* , on the Riemann surface of Logarithm. So we get $F_d^+ = F_d^- St_d$, with $St_d \in GL(n; \mathbb{C})$. By definition St_d is **the Stokes matrix** associated to the *formal fundamental solution* $\hat{F}$ of Δ , to the *direction d* , and to the *choice of branch of Logarithm*.

The operator $(S_{k_1, k_2, \dots, k_r; d}^+)^{-1} (S_{k_1, k_2, \dots, k_r; d}^-) = St_d$ is clearly a K -automorphism of the differential extension $\mathbb{C}\{x\}[x^{-1}] \langle \hat{F} \rangle$ (which is a *Picard-Vessiot extension* of $\mathbb{C}\{x\}[x^{-1}]$ associated to Δ [Kap], [Kol]), that is an element of the *Galois differential group*, clearly *independent* of the choice of $\hat{F}$. Later we will *systematically* write the operation of elements of St_d , and, more generally, of differential automorphisms, **on the right** (and ask the reader to be careful with the *ordering* of compositions...). We will also denote by St_d the induced automorphism (this automorphism depends on $d \in S^1$ and on the choice of branch of Logarithm ⁽³⁷⁾, that is on $\mathbf{d} \in (\mathbb{R}, 0)$ *universal covering* of $(S^1, 0)$ “above” d) of the $\mathbb{C}$ -vector space of *formal solutions* of Δ (the matrix of this automorphism in the basis formed by the columns of $\hat{F}$ is St_d). So the **Stokes matrix St_d is an element of the representation of “the” differential Galois group** $\text{Gal}_K(\Delta) = \text{Aut}_K K \langle \hat{F} \rangle$ ($K = \mathbb{C}\{x\}[x^{-1}]$) in $GL(n; \mathbb{C})$ given by the *formal fundamental solution* $\hat{F}$.

Here one must be very careful: *Stokes matrices defined by our method* (very near of *Stokes original method* [Sto] (cf. references and comments in [MR 2], chapter 3)) are “in” the Galois differential group, but this is in general *completely false* for “classical” *Stokes matrices*. Classical definition, starting from asymptotic expansions in Poincaré’s sense ⁽³⁸⁾, is “unnatural” and corresponds to a misunderstanding of the original Stokes ideas (Stokes was working by numerical computations with in mind something like an idea of “*exact asymptotic expansions*”).

Remark. — Stokes operators St_d and Stokes matrices St_d are **unipotent** (see *infra*), so we can define their *logarithms* st_d and st_d respectively (the idea of a systematical use of these logarithms seems essentially due to

⁽³⁷⁾ Up to conjugation by the “formal monodromy” (cf. *infra*).

⁽³⁸⁾ Asymptotic expansions in Poincaré’s sense must be replaced by “*transasymptotic expansions*” (Ecale’s terminology): the transasymptotic expansion map is the *inverse* of the summation map). Transasymptotic expansions can only make “*exponentially small jump*” on *singular lines* (“anti Stokes lines”), but Poincaré asymptotic expansions can only make “*jumps*” on “*Stokes lines*” (consequence of transasymptotic expansion “jumps”, in “*quadrate of phasis*”).

*Ecal*le in a more general context):

$$St_{\mathbf{d}} = \text{Exp } st_{\mathbf{d}} \quad \text{and} \quad St_{\mathbf{d}} = \text{Exp } st_{\mathbf{d}}.$$

Then

$$F_{\mathbf{d}} = F_{\mathbf{d}}^+ \text{Exp} \left(-\frac{1}{2} st_{\mathbf{d}} \right) = F_{\mathbf{d}}^- \text{Exp} \left(\frac{1}{2} st_{\mathbf{d}} \right),$$

and we can choose $F_{\mathbf{d}}$ as *sum* of $\hat{F}$ in the *singular direction* $\mathbf{d}$ (this idea is already in *Dingle's* book [Din]; this has been recently extended to *extremely general* situations by *Ecal*le: “*sommat*ion *médiane*”). If the differential operator Δ is *real*, if $\hat{G}$ is a *real formal fundamental solution*, and if $\mathbf{d} = \mathbf{R}^+$, then we can choose the *fundamental determination* of the Logarithm, and “*median sum*” $G_{\mathbf{d}}$ is *real* (this can be applied to *Airy* equation at infinity, cf. [MR 2], chapter 3). Moreover $st_{\mathbf{d}}$ is a *Galois derivation* (i. e. *commuting* with the *derivation* of the differential field) of the differential field $K \langle \hat{G} \rangle$, and $\text{Exp} \left(\frac{1}{2} st_{\mathbf{d}} \right) \in \text{Aut}_K K \langle \hat{G} \rangle$, then, when the reality conditions given above are satisfied, the map $\mathbf{R} \{x\} [x^{-1}] \langle \hat{G} \rangle \rightarrow \text{germs of real meromorphic functions at } 0 \in]0, +\infty[$, defined by

$$\hat{G} \rightarrow G_{\mathbf{d}} \text{ on } \hat{G},$$

and equal to S on $\mathbf{R} \{x\} [x^{-1}]$ is *an injective morphism of differential fields*.

The following generalization of a *Schlesinger's* theorem ⁽³⁹⁾ [Sch] was first proved in [Ra 4], [Ra 5], using a different method ⁽⁴⁰⁾.

THEOREM 6. — *Let $K = \mathbf{C} \{x\} [x^{-1}]$. Let $\Delta = d/dx - A$, with $A \in \text{End}(n; K)$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$. Let $\hat{F}$ be a formal fundamental solution of Δ . Let H be the subgroup of $\text{GL}(n; \mathbf{C})$ generated by the formal monodromy matrix $\hat{M}$, the exponential torus T , and the Stokes matrices of Δ associated to the given formal fundamental solution $\hat{F}$. Then the representation of the Galois differential group $\text{Gal}_K(\Delta)$ of Δ in $\text{GL}(n; \mathbf{C})$, given by $\hat{F}$, is the Zariski closure of H in $\text{GL}(n; \mathbf{C})$.*

Using “*Galois correspondence*” [Kap], it suffices to prove that the *invariant field* of H (that is the subfield of $K \langle \hat{F} \rangle$ consisting of the *invariant elements* by H) is K .

First we must define the “*formal monodromy*” and the “*exponential torus*” of Δ .

⁽³⁹⁾ Schlesinger's theorem is for the case of Fuchsian equations.

⁽⁴⁰⁾ A second proof has been given by *Deligne* using “*Tannakian*” ideas [De 4], and, during Luminy conference (september 1989), I have learned from *Y. Il'Yashenko* that he has also recently got another proof...

Replacing u by $ue^{2i\pi}$ in $\hat{F}(u)$, we get a (in general new) *fundamental solution* of the differential operator Δ :

$\hat{F}(ue^{2i\pi}) = \hat{F}(u) \hat{M}$, with $\hat{M} \in GL(n; \mathbb{C})$. By definition $\hat{M}$ is the *formal monodromy matrix* associated to Δ and to the fundamental solution $\hat{F}$. The corresponding element $\hat{M}$ of $\text{Aut}_{\mathbb{K}} \mathbb{K} \langle \hat{F} \rangle$ is clearly *independent* of the choice of $\hat{F}$ and is a *formal invariant* of Δ ; it is the *formal monodromy* of Δ . (We will later systematically write the operation of $\hat{M}$ on the *right*.)

We will now define the “*exponential torus*”.

Let $\hat{\mathbb{K}} = \hat{\mathbb{K}}_v \langle u^L, e^Q \rangle$ the differential field generated by $\hat{\mathbb{K}}_v = \mathbb{C}[[u]][u^{-1}]$ and the entries of the matrices u^L and e^Q .

Let $\hat{\mathbb{L}}_v = \hat{\mathbb{K}}_v \langle e^Q \rangle = \hat{\mathbb{K}}_v \langle e^{q_1}, e^{q_2}, \dots, e^{q_n} \rangle \subset \hat{\mathbb{K}}$.

If μ is the dimension of the (free) abelian $\mathbb{Z}$ -module $\mathbf{E}(\Delta) \subset u^{-1} \mathbb{C}[u^{-1}]$ generated by $q_1, q_2, \dots, q_n$, the Galois differential group $\text{Aut}_{\hat{\mathbb{K}}_v} \hat{\mathbb{L}}_v = \text{Aut}_{\mathbb{K}_v} \mathbb{L}_v$ is a torus $\mathcal{T}(Q) = \mathcal{T}_v(Q) = \mathcal{T}(\mathbf{q}(\Delta))$ isomorphic to $(\mathbb{C}^*)^\mu$ (clearly $\mu \leq n$). (We have set $\mathbb{K}_v = \mathbb{C}\{u\}[u^{-1}]$ and $\mathbb{L}_v = \mathbb{K}_v \langle e^Q \rangle$.)

We have $\hat{\mathbb{L}}_v \cap \hat{\mathbb{K}}_v \langle u^L \rangle = \hat{\mathbb{K}}_v$. Then $\mathcal{T}(Q)$ can be identified with a subgroup of $\text{Aut}_{\hat{\mathbb{K}}_v} \hat{\mathbb{K}}$ leaving $\hat{\mathbb{K}}_v \langle u^L \rangle$ fixed (still denoted by $\mathcal{T}(Q)$).

We have $\mathbb{K} \langle \hat{F} \rangle \subset \hat{\mathbb{K}}$, and $\mathbb{K} \langle \hat{F} \rangle$ are invariant by $\mathcal{T}(Q)$; so $\mathcal{T}(Q)$ can be identified with a subgroup of $\text{Aut}_{\mathbb{K}} \mathbb{K} \langle \hat{F} \rangle = \text{Gal}_{\mathbb{K}}(\Delta)$. This group is clearly independent of the choice of $\hat{F}$. By *definition* we call this group “the *exponential torus*” of Δ . It will be denoted by $T(\Delta)$ (it depends only on $\mathbf{q}(\Delta)$ and is a *formal invariant* of Δ). Its representation in $GL(n; \mathbb{C})$ given by the fundamental solution $\hat{F}$ will be denoted by $T = T(\Delta) = T(Q(\Delta))$ (and still named “*exponential torus*”).

Let $\mathbb{K}'_v = \mathbb{C}\{u\}_{1/v k_1, 1/v k_2, \dots, 1/v k_r}$. We have

$$\mathbb{K} \langle \hat{F} \rangle \subset \mathbb{K}'_v \langle u^L, e^Q \rangle = \mathbb{K}'.$$

Let now $\xi \in \mathbb{K} \langle \hat{F} \rangle$ be an element *invariant* by H (more precisely by the subgroup of $\text{Aut}_{\mathbb{K}} \mathbb{K} \langle \hat{F} \rangle$ corresponding to H). If $x = u^v$, then ξ is invariant by $\hat{M}^v$, that is by the *formal monodromy* “in u ”, so $\xi \in \mathbb{K}'_v \langle e^Q \rangle$. But ξ is also invariant by the *exponential torus* and $\xi \in \mathbb{K}'_v$. From the invariance of ξ by the *Stokes matrices* we deduce that the $(k_1, k_2, \dots, k_r)$ -summable power series ξ admits *no singular direction* (Lemma 10), so ξ is *convergent* and $\xi \in \mathbb{K}_v$. The action of the monodromy matrix $\hat{M}$ on $\xi \in \mathbb{K}_v$ is the same as the action of the (ordinary) Galois group $\text{Aut}_{\mathbb{K}} \mathbb{K}_v$ (isomorphic to $\mathbb{Z}/v\mathbb{Z}$), so ξ is invariant by $\text{Aut}_{\mathbb{K}} \mathbb{K}_v$ and $\xi \in \mathbb{K}$ (by the *ordinary Galois correspondence*). That ends the proof of *Theorem 5*.

Examples. — From fundamental systems of solutions at infinity ($z = x^{-1}$; $x = 0$) for *Airy* and *Kummer differential equations* it is possible to compute *formal monodromies*, *exponential tori* and *Stokes multipliers*. From these

results it is possible to compute the Galois differential groups of our differential equations ⁽⁴¹⁾. See [MR 3]).

For a deeper study of germs of analytic linear differential equations we need now a little “*toolbox*” ⁽⁴²⁾ (built with elementary linear algebra).

Let $E_v = x^{-1/v} \mathbb{C} \{x^{-1/v}\}$ ($v \in \mathbb{N}^*$) and $E = \bigcup_{v \in \mathbb{N}^*} E_v$. If

$$\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset E,$$

we denote by

$$\mathbf{E}(\mathbf{q}) = \mathbf{Z}_{q_1} + \mathbf{Z}_{q_2} + \dots + \mathbf{Z}_{q_n} \subset E$$

the sublattice of E generated by $q_1, q_2, \dots, q_n$. The smallest integer v such that $\mathbf{E}(\mathbf{q}) \subset x^{-1/v} \mathbb{C} \{x^{-1/v}\}$ is, by definition, the *ramification* of $\mathbf{q}$, or $\mathbf{E}(\mathbf{q})$. We have:

$$E = \bigcup_{\mathbf{q}} \mathbf{E}(\mathbf{q}) = \varinjlim_{\mathbf{q}} \mathbf{E}(\mathbf{q}).$$

We define an action of the (classical) Galois group $\text{Aut}_K K_v \approx \mathbb{Z}/v\mathbb{Z}$ on a sublattice E' of E_v , by

$q(x^{-1/v}) \rightarrow q(e^{-2i\pi/v} x^{-1/v})$ (corresponding to $x \rightarrow e^{-2i\pi} x$). If E' is *invariant* by this action we will say that E' is *Galois invariant*. The lattice $\mathbf{E}(\mathbf{q})$ is Galois invariant if and only if the set $\mathbf{q}$ is invariant by the corresponding action (Galois invariant).

If $q \in \mathbf{E}(\mathbf{q})$, its “*degree*” $\delta(q)$ is the rational number $m/v \in \frac{1}{v}\mathbb{Q}$, where m is the degree of q as a polynomial in $x^{1/v}$. There is a natural filtration of E by the degree, that is by the sublattices

$$E^m = \{q \in E / \delta(q) \leq m\}.$$

We identify the *universal covering* of $(S^1, 1)$ to $(\mathbb{R}, 0)$. By definition the “*front*” $\text{Fr}(q)$ of $q \in E$ ($q \neq 0$) is the subset of $(\mathbb{R}, 0)$ whose elements are the “*lines of maximal decrease*” of e^q (we will also call “*front*” the natural projection of this set on the v -covering of $(S^1, 1)$, identified with another copy of $(S^1, 1)$). The front of q depends clearly *only* on the monomial of maximal degree $\delta(q)$ of q . If $\mathbf{d}$ is a direction belonging to the *front* of q

⁽⁴¹⁾ “Classical computation” of the Galois differential group of Airy equation is in [Kap]; the computation of the Galois differential group of Kummer equations is, as far as we know, new (it is possible to do the computations “classical”, using improvements of Kovacic’s algorithm [Kov], [DLR], [MR 3]).

⁽⁴²⁾ A first version of these tools was first introduced by Balser, Jurkat, Lutz [BJL 1], [J]. In our presentation we have also used ideas of Deligne, Malgrange [De 3], [Ma 3], [Ma 4], Babbitt, Varadarajan [BV], and the systematic treatment of M. Loday-Richaud [LR 1].

(or of its projection on S^1), or if $q=0$, we will say that q is "carried" by $\mathbf{d}$.

If $x=u^v$, we write $K_v = \mathbf{C}\{u\}[u^{-1}]$, and $K_v = \mathbf{C}[[u]][u^{-1}]$.

Let $\mathbb{L}_v = K_v \langle e^{q_1}, e^{q_2}, \dots, e^{q_n} \rangle$, and $\hat{\mathbb{L}}_v = \hat{K}_v \langle e^{q_1}, e^{q_2}, \dots, e^{q_n} \rangle$. As above we write $\text{Aut}_{\hat{K}_v} \hat{\mathbb{L}}_v = \text{Aut}_{K_v} \mathbb{L}_v = \mathcal{T}(\mathbf{q})$.

To each $q \in \mathbf{E}(\mathbf{q})$ we can associate a *character* of the exponential torus $\mathcal{T}(\mathbf{q})$, that is a (continuous) homomorphism of groups (still denoted by q):

$$\begin{aligned} q: \mathcal{T}(\mathbf{q}) &\rightarrow \mathbf{C}^* \\ q: \theta &\rightarrow q(\theta), \end{aligned}$$

with

$$(e^q)\theta = q(\theta)e^q \quad (e^q \in \mathbb{L}_v \text{ and } \theta \text{ acts on } \mathbb{L}_v).$$

Let $(p_1, p_2, \dots, p_v)$ be a $\mathbf{Z}$ -basis of the lattice $\mathbf{E}(\mathbf{q})$

We get an isomorphism

$$\begin{aligned} (p_1, p_2, \dots, p_v): \mathcal{T}(\mathbf{q}) &\rightarrow (\mathbf{C}^*)^v \\ (p_1, p_2, \dots, p_v): \theta &\rightarrow (p_1(\theta), p_2(\theta), \dots, p_v(\theta)). \end{aligned}$$

In the following the *exponential lattice* $\mathbf{E}(\mathbf{q})$ will be identified with the *lattice of characters on the exponential torus* $\mathcal{T}(\mathbf{q})$.

Let $\mathbf{d} \in (\mathbf{R}, 0)$ [the universal covering of $(S^1, 1)$], we set

$\mathbf{E}_d(\mathbf{q}) = \{q \in \mathbf{E}(\mathbf{q}) / q \text{ is carried by } \mathbf{d}\}$; $\mathbf{E}_d(\mathbf{q})$ is a *semi-lattice* of $\mathbf{E}(\mathbf{q})$, and depends clearly only on the projection d of $\mathbf{d}$ on the v -covering of S^1 :

$$\mathbf{E}_d(\mathbf{q}) = \mathbf{E}_d(\mathbf{q}).$$

To the set $\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E}$, after the choice of an *ordering*, we associate the diagonal matrix $e^{\mathbf{Q}}$, with $\mathbf{Q} = \text{Diag}\{q_1, q_2, \dots, q_n\}$.

We will use *ordering relations* associated to a direction $\mathbf{d} \in (\mathbf{R}, 0)$:

$q \gg_d q'$, if and only if $q' - q \in \mathbf{E}_d(\mathbf{q})$ (i.e. $q' - q$ is carried by $\mathbf{d}$);

$q >_d q'$, if and only if $e^{q' - q}$ is infinitely flat on d ;

$q \geq_d q'$, if and only if $e^{q' - q}$ is bounded on d .

Clearly, if $q \gg_d q'$, then $q >_d q'$; and, if $q >_d q'$, then $q \geq_d q'$.

We will also use an *equivalence relation* on the space $\mathbf{E}$ associated to a rational number $k > 0$, $k \in \mathbf{Q}$:

$q =_k q'$ if and only if $\delta(q - q') < k$ [if $\delta(q - q') \geq k$, we will write $q \neq_k q'$].

To a rational number $k > 0$ we associate the *partition* of the set $\mathbf{q} = \{q_1, q_2, \dots, q_n\}$, defined by the relation $=_k$. This partition is named the "*k-partition*". The only "significant" values for k are in the set $\{k_1, k_2, \dots, k_r\} = \mathbf{N}\Sigma(\mathbf{q})$ of values taken by $\delta(q_i - q_j)$ ($q_i \neq q_j$). We will *always* suppose in the following that we have chosen an ordering on $q_1, q_2, \dots, q_n$ such that, for every $k > 0$, $k \in \mathbf{Q}$, the elements of each subset of the k -partition are *consecutive*. Then, there exists a *unique block-decomposition* (by definition the *k-block-decomposition*) of the matrix $\mathbf{Q}$ which is invariant by transposition and induces the k -partition on the

diagonal. For $k = k_1, k_2, \dots, k_r$ we get, by definition, the “iterated block-decomposition” (cf. [BJL 1], [J]). If a matrix A admits the same k -block-decomposition than Q , we will say that A admits a (Q, k) -block-structure. Moreover, a direction $\mathbf{d}$ being fixed, it is possible to choose an *indexation* (called by definition a $\mathbf{d}$ -indexation) of the elements q_i of $\mathbf{q}$ such that:

$$q_1 \leq_{\mathbf{d}} q_2 \leq_{\mathbf{d}} \dots \leq_{\mathbf{d}} q_n.$$

The corresponding ordering on $\mathbf{q}$ satisfies the above conditions; the corresponding iterated block-decomposition is named a $\mathbf{d}$ -iterated block-decomposition.

The set $\mathbf{q}$ and the direction $\mathbf{d}$ being fixed, and an *ordering* (perhaps depending on $\mathbf{d}$) being chosen on $\mathbf{q}$, the diagonal matrix Q is defined. To this matrix and a *fixed direction* $\mathbf{d} \in (\mathbf{R}, 0)$, we will associate families of subgroups of $GL(n; \mathbf{C})$, indexed by $k_m \in \{k_1, k_2, \dots, k_r\} = N\Sigma(\mathbf{q})$ (isotropy groups, and Stokes groups).

All these groups are *unipotent*. More precisely, if P is a matrix belonging to one of these group, all the diagonal terms of P are equal to 1, and $I - P$ is *nilpotent* (if the order on $\mathbf{q}$ corresponds to a $\mathbf{d}$ -indexation, then P is *upper-triangular*).

Let $\Lambda(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j, \text{ and } c_{ij} \neq 0, \text{ then } q_i <_{\mathbf{d}} q_j\}$; $\Lambda(Q; \mathbf{d})$ is a subgroup of $GL(n; \mathbf{C})$, named the *isotropy subgroup* in the direction $\mathbf{d}$. Let $\text{Sto}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j, \text{ and } c_{ij} \neq 0, \text{ then } q_i \leq_{\mathbf{d}} q_j\}$; $\text{Sto}(Q; \mathbf{d})$ is a subgroup of $\Lambda(Q; \mathbf{d})$, named the *Stokes subgroup* in the direction $\mathbf{d}$. Let now $k_m \in \{k_1, k_2, \dots, k_r\} = N\Sigma(\mathbf{q})$. We set:

$$\Lambda^{\geq k_m}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j \\ \text{and } c_{ij} \neq 0, \text{ then } q_i <_{\mathbf{d}} q_j \text{ and } q_i \neq_{k_m} q_j\};$$

$$\Lambda^{k_m}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j \\ \text{and } c_{ij} \neq 0, \text{ then } q_i <_{\mathbf{d}} q_j, q_i \neq_{k_m} q_j \text{ and } q_i =_{k_m-1} q_j\};$$

$$\Lambda^{< k_m}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j \\ \text{and } c_{ij} \neq 0, \text{ then } q_i <_{\mathbf{d}} q_j \text{ and } q_i =_{k_m} q_j\};$$

and

$$\text{Sto}^{\geq k_m}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j, \\ \text{and } c_{ij} \neq 0, \text{ then } q_i \leq_{\mathbf{d}} q_j \text{ and } q_i \neq_{k_m} q_j\};$$

$$\text{Sto}^{k_m}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j, \\ \text{and } c_{ij} \neq 0, \text{ then } q_i \leq_{\mathbf{d}} q_j, q_i \neq_{k_m} q_j \text{ and } q_i =_{k_m-1} q_j\};$$

$$\text{Sto}^{< k_m}(Q; \mathbf{d}) = \{C = (c_{ij})/\text{if } i=j, c_{ij}=1, \text{ and, if } i \neq j, \\ \text{and } c_{ij} \neq 0, \text{ then } q_i \leq_{\mathbf{d}} q_j \text{ and } q_i =_{k_m} q_j\}.$$

PROPOSITION 10. — *Let Q be a diagonal matrix with entries in $\mathbf{E}$, and $\mathbf{d} \in (\mathbf{R}, 0)$ be a fixed direction. Then, for every $k > 0$, $k \in \mathbf{Q}$, the four sequences*

$$\begin{aligned} \{\text{id}\} &\rightarrow \Lambda^{\geq k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \Lambda(\mathbf{Q}; \mathbf{d}) \rightarrow \Lambda^{< k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \{\text{id}\}, \\ \{\text{id}\} &\rightarrow \Lambda^{k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \Lambda^{\leq k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \Lambda^{< k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \{\text{id}\}, \\ \{\text{id}\} &\rightarrow \text{Sto}^{\geq k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \text{Sto}(\mathbf{Q}; \mathbf{d}) \rightarrow \text{Sto}^{< k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \{\text{id}\}, \\ \{\text{id}\} &\rightarrow \text{Sto}^{k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \text{Sto}^{\leq k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \text{Sto}^{< k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \{\text{id}\}, \end{aligned}$$

are split exact sequences of (algebraic) groups.

Maps are evident inclusions and evident “projections” (by “suppression” of some entries). The sequences are split by the inclusion maps $\Lambda^{< k_m}(\mathbf{Q}; \mathbf{d}) \rightarrow \Lambda(\mathbf{Q}; \mathbf{d})$, ...

Proposition 9 consists of “block variations” on the

LEMMA 11. — *Let D_n be the subgroup of $\text{GL}(n; \mathbf{C})$ of diagonal invertible matrices. Let T_n be the subgroup of $\text{GL}(n; \mathbf{C})$ of upper triangular invertible matrices. Let B_n be the subgroup of $\text{GL}(n; \mathbf{C})$ of upper triangular unipotent matrices. Then we have a split exact sequence of groups:*

$$\{\text{id}\} \rightarrow B_n \rightarrow T_n \rightarrow D_n \rightarrow \{\text{id}\}.$$

The map $T_n \rightarrow D_n$ is the evident “projection” (we replace by zero the off diagonal entries), and the map $B_n \rightarrow T_n$ is the natural injection; the natural inclusion $D_n \rightarrow T_n$ gives the splitting.

Then T_n is the semi-direct product of B_n and D_n . We will write

$$T_n = D_n \ltimes B_n;$$

$\Lambda(\mathbf{Q}; \mathbf{d})$ is the semi-direct product of $\Lambda^{\geq k_m}(\mathbf{Q}; \mathbf{d})$ and $\Lambda^{< k_m}(\mathbf{Q}; \mathbf{d})$, we will write

$$\Lambda(\mathbf{Q}; \mathbf{d}) = \Lambda^{< k_m}(\mathbf{Q}; \mathbf{d}) \ltimes \Lambda^{\geq k_m}(\mathbf{Q}; \mathbf{d}), \dots$$

LEMMA 12. — *If*

$$\{k_1, k_2, \dots, k_r\} = \{\delta(q_i - q_j)/i, j=1, \dots, n \text{ and } q_i - q_j \neq 0\}$$

$(k_1 > k_2 > \dots > k_r > 0)$, we have:

$$\Lambda(\mathbf{Q}; \mathbf{d}) = \Lambda^{k_r}(\mathbf{Q}; \mathbf{d}) \ltimes \Lambda^{k_{r-1}}(\mathbf{Q}; \mathbf{d}) \ltimes \dots \ltimes \Lambda^{k_1}(\mathbf{Q}; \mathbf{d}).$$

If $C \in \Lambda(\mathbf{Q}; \mathbf{d})$, there exists a unique decomposition:

$$C = C_r C_{r-1} \dots C_1, \quad \text{with } C_i \in \Lambda^{k_i}(\mathbf{Q}; \mathbf{d}).$$

We can now go back to *linear differential equations*. We need a more precise version of *theorem 3*. (Beware of the slight change of notation for H .)

Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of *meromorphic differential operator* at the origin of the complex plane $\mathbf{C}$.

The operator Δ admits a **formal fundamental solution**:

$$\hat{F}(x) = \hat{H}(x) x^L U e^{Q(1/u)},$$

with: $u^v = x$ (for some $v \in \mathbb{N}^*$), $L \in \text{End}(n; \mathbb{C})$, in *Jordan form*, $\hat{H} \in \text{GL}(n; \mathbb{C}[[x]][x^{-1}])$, Q a diagonal matrix with entries in $u^{-1} \mathbb{C}[u^{-1}]$, Galois invariant, unique up to permutations of the diagonal entries, and $U \in \text{End}(n; \mathbb{C})$ a “*universal*” matrix (depending only on Q) [BJL 1], [J] (v is chosen *minimal*).

Let $\hat{M} = U^{-1} e^{2i\pi L} U$. We have:

$$\hat{F}(e^{2i\pi} x) = \hat{H}(x) x^L U \hat{M} e^{Q(\exp(-2i\pi/v)/u)} = \hat{F}(x) \hat{M},$$

and

$$e^{Q(\exp(-2i\pi/v)/u)} = \hat{M}^{-1} e^{Q(1/u)} \hat{M}, \quad [\hat{M}^v, Q] = 0.$$

THEOREM 7. — Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbb{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbb{C}$.

We denote by $k_1 > k_2 > \dots > k_r$ the positive (non zero) slopes of the Newton polygon of the (rank n^2) differential operator

$$\text{End } \Delta = d/dx - [A, \cdot].$$

Let $\hat{F}$ be a **formal fundamental solution** of Δ as above. Then there exists a “*natural decomposition*” (unique up to “*meromorphic transforms*” [Ra 4])

$\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r$, where $\hat{H}_i \in \text{GL}(n; \mathbb{C}[[x]][x^{-1}])$, is k_i -**summable** for $i = 1, \dots, r$, and such that

(i) $\hat{F}^i(x) = \hat{H}_i(x) \hat{H}_{i+1}(x) \dots \hat{H}_r(x) x^L U e^{Q(1/u)}$ is a **formal fundamental solution** of a **meromorphic differential operator** $\Delta^i = d/dx - A^i$, with

$$A^i \in \text{End}(n; \mathbb{C}\{x\}[x^{-1}]), \quad \text{for } i = 1, \dots, r;$$

(ii) If $\Sigma(\hat{F}) = \Sigma(\hat{H}) = \bigcup_{i=1, \dots, r} \Sigma(\hat{H}_i)$, $H_{i;d} = S_{k_i;d} \hat{H}_i$ (for $i = 1, \dots, r$)

$$H_d = H_{1;d} H_{2;d} \dots H_{r;d},$$

then, for $d \notin \Sigma(H)$, and every determination of $\text{Log } x (u = e^{(\text{Log } x)/v})$ and $x^L = e^{L \text{Log } x}$.

$F_d(x) = H_d(x) x^L U e^{Q(1/u)}$ is an **actual analytic fundamental solution** of the operator Δ in a sector bisected by d [$d \in (\mathbb{R}, 0)$ “above” d corresponds to the given branch of Logarithm].

Moreover $\hat{H}^i$ admits a (Q, k_{i-1}) -block-structure ($i = 2, \dots, r$) and A^i admits a (Q, k_i) -block-structure ($i = 1, \dots, r$).

We define $F_d^i(x) = H_{i;d} H_{i+1;d} \dots H_{r;d} x^L U e^{Q(1/u)}$; $F_d^i(x)$ is an **actual analytic fundamental solution** of the operator Δ^i in a sector bisected by d ($i = 1, \dots, r$), and admits a (Q, k_{i-1}) -block-structure ($i = 2, \dots, r$).

We have:

$$F_d^i = H_{i;d} F_d^{i+1} \quad (i = 1, \dots, r-1), \text{ and we set } (i = 1, \dots, r):$$

$$H_{i;d}^+ F_{\mathbf{d}}^{i+1} = H_{i;d}^- F_{\mathbf{d}}^{i+1} S_{i;d}.$$

We have $S_{i;d} \in \text{GL}(n; \mathbf{C})$ ($i = 1, \dots, r$) and $St_{\mathbf{d}} = S_{r;d} S_{r-1;d} \dots S_{1;d}$.

LEMMA 13. — Let $\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E}$, and, after an ordering, let Q be the diagonal matrix $Q = \text{Diag}\{q_1, q_2, \dots, q_n\}$. Let $C \in \text{End}(n; \mathbf{C})$, and $\mathbf{d}$ a fixed direction [$\mathbf{d} \in (\mathbf{R}, 0)$]:

(i) The following conditions are equivalent:

(a) $e^Q C e^{-Q} = I + \Phi$, with Φ **infinitely flat** on $\mathbf{d}$.

(b) $C \in \Lambda(Q; \mathbf{d})$.

(ii) The following conditions are equivalent:

(a) $e^Q C e^{-Q} = I + \Phi$, with Φ **exponentially flat of order $\geq k$** on $\mathbf{d}$.

(b) $C \in \Lambda^{\geq k}(Q; \mathbf{d})$.

(iii) The following conditions are equivalent:

(a) $e^Q C e^{-Q} = I + \Phi$, with Φ **exponentially flat of order exactly k** on $\mathbf{d}$.

(b) $C \in \Lambda^k(Q; \mathbf{d})$.

(iv) The following conditions are equivalent:

(a) $e^Q C e^{-Q} = I + \Phi$, with Φ **exponentially flat of order $\geq k$ on an open sector with opening π/k , bisected by $\mathbf{d}$** .

(b) $e^Q C e^{-Q} = I + \Phi$, with Φ **exponentially flat of order exactly k on an open sector with opening π/k , bisected by $\mathbf{d}$** .

(c) $C \in \text{Sto}^k(Q; \mathbf{d})$.

THEOREM 8. — Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

We denote by $k_1 > k_2 > \dots > k_r$ the positive (non zero) slopes of the Newton polygon of the differential operator

$$\text{End } \Delta = d/dx - [A, \cdot].$$

Let $\hat{F}(x) = \hat{H}(x) x^L U e^{Q(1/u)}$, be a **formal fundamental solution** of Δ as above, and

$$\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r, \text{ a decomposition like in theorem 7.}$$

Let $S_{i;d} \in \text{GL}(n; \mathbf{C})$ ($i = 1, \dots, r$) defined as above. Then:

(i) $S_{i;d} \in \text{Sto}^{k_i}(Q; \mathbf{d})$ ($i = 1, \dots, r$).

(ii) $St_{\mathbf{d}} \in \text{Sto}(Q; \mathbf{d})$ and $St_{\mathbf{d}} = S_{r;d} S_{r-1;d} \dots S_{1;d}$ is the unique decomposition of $St_{\mathbf{d}}$ corresponding to

$$\Lambda(Q; \mathbf{d}) = \Lambda^{k_r}(Q; \mathbf{d}) \ltimes \Lambda^{k_{r-1}}(Q; \mathbf{d}) \ltimes \dots \ltimes \Lambda^{k_1}(Q; \mathbf{d}).$$

Assertion (i) is a consequence of lemma 13 (iv):

We have $(H_{i;d}^-)^{-1} H_{i;d}^+ = I + \Psi$, with Ψ **exponentially flat of order $\geq k_i$ on an open "sector" with opening π/k_i bisected by d** (H_i is k_i -summable).

We set

$$G_i = H_{i+1;d} \dots H_{r;d} x^L U;$$

it is clear that G_i and G_i^{-1} are analytic on an open "sector" with opening π/k_{i+1} ($\pi/k_{i+1} > \pi/k_i$) bisected by d , and admit a moderate growth at the origin on this sector. Then $e^Q S_{i;d} e^{-Q} = G_i(I + \Psi) G_i^{-1} = I + \Phi$, where Φ is exponentially flat of order $\geq k_i$ on an open "sector" with opening π/k_i , bisected by d . Assertion (ii) follows from (i) and lemma 12.

Stokes matrices $S_{i;d}$ are *a priori* defined in a transcendental way. Theorem 8 says that we can get them by an algebraic algorithm from the knowledge of St_d and Q . We will give later an "infinitesimal version" of this computation.

LEMMA 14. — Let $k'_1 > k'_2 > \dots > k'_r > k' > 0$. Let $d = \mathbf{R}^+$. Then:

$$e^{-1/x^{k'}} = L_{k'_1; d} A_{k'_1, k'_2; d} \dots A_{k'_{r-1}, k'; d} B_{k'; d} (e^{-1/x^{k'}}).$$

From this lemma and theorem 8, we get

THEOREM 9. — Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

We denote by $k_1 > k_2 > \dots > k_r$ the positive (non zero) slopes of the Newton polygon of the differential operator

$$\text{End } \Delta = d/dx - [A, \cdot].$$

Let $\hat{F}(x) = \hat{H}(x) x^L J e^{Q(1/x)}$ be a formal fundamental solution of Δ as above, and

$$\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r, \text{ a decomposition like in theorem 7.}$$

Let $S_{i;d} \in \text{GL}(n; \mathbf{C})$ ($i = 1, \dots, r$) defined as above.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$, and $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_r)$, with $\varepsilon_i, \varepsilon'_i \in \{1, -1\}$ ($i = 1, \dots, r$).

Then, for every direction $d \in (\mathbf{R}, 0)$:

(i) $\hat{H}$ is $(k_1, k_2, \dots, k_r)$ -summable along the paths $(d; \varepsilon)$ and $(d; \varepsilon')$.

(ii) If $S_{k_1, k_2, \dots, k_r; d}^{\varepsilon} \hat{F} = S_{k_1, k_2, \dots, k_r; d}^{\varepsilon'} \hat{F} \text{St}_d^{\varepsilon, \varepsilon'}$, $\text{St}_d^{\varepsilon, \varepsilon'} \in \text{GL}(n; \mathbf{C})$, and if $\varepsilon = (-, -, \dots, -) = -: \text{ then}$

$$\text{St}_d^{\varepsilon, \varepsilon'} = \varepsilon'(S_{r;d}) \varepsilon'(S_{r-1;d}) \dots \varepsilon'(S_{1;d}),$$

with

$$\varepsilon'(S_{i;d}) = S_{i;d} \text{ if } \varepsilon'_i = +, \quad \text{and} \quad \varepsilon'(S_{i;d}) = I \text{ if } \varepsilon'_i = -.$$

(iii) If $\varepsilon = (-, -, \dots, -)$ and $\varepsilon' = (-, -, \dots, +, \dots, -)$, with a only at the index i , then $\text{St}_d^{\varepsilon, \varepsilon'} = S_{i;d}$, and $S_{i;d}$ is in the representation in $\text{GL}(n; \mathbf{C})$ of the differential Galois group $\text{Gal}_{\mathbf{K}}(\Delta)$ given by the fundamental formal solution $\hat{F}$ ($i = 1, \dots, r$).

We will write $S_{i;d} = \text{St}_{d; k_i}$.

Our aim now is to use preceding results and concepts to give a "purely combinatorial" description of the category of germs of meromorphic connections at the origin of the complex plane, as simple as possible. In "down

to earth terms” a germ of meromorphic connection is a germ of differential system *up to meromorphic equivalence* [De 1], [Ma 4], [MR 2]; so the searched combinatorial description is equivalent to a *meromorphic classification of germs of differential systems*.

Such a result is well known for the *regular singular* case. It is given by the **Riemann-Hilbert correspondance** [De 1], [Ka 2], [MR 2]:

Germ of Fuchsian connections at the origin of $\mathbb{C}$. *Finite dimensional linear representations of the local fundamental group* ⁽⁴³⁾.

Germ of meromorphic fuchsian differential operator Δ , up to meromorphic equivalence. $\rightarrow$ *Monodromy $M(\Delta)$ “around 0”.*

This map is **bijjective**, moreover it is an *equivalence of Tannakian categories* [Saa], [De Mi], [De 2]. The result is *false* if we *suppress* the *fuchsian hypothesis*.

The now “classical” meromorphic classification of germs of meromorphic differential operators is given in terms of *cohomology of sheaves of groups* (isotropy groups of a “normal form”) on S^1 [Si], [Ma 3], [Ma 4], [De 3], [MR 1] ⁽⁴⁴⁾. We have in mind a “better” description (particularly adapted to the computation of differential Galois groups), *extending the Riemann-Hilbert correspondence to the irregular case*, that is a description of connections in terms of **representations of groups**:

Germ of connections at the origin of $\mathbb{C}$. *Finite dimensional linear representations of the local “wild fundamental group”.*

Germ of meromorphic differential operator Δ , up to meromorphic equivalence. $\rightarrow$ *???*

We will call “Gevrey front” of $q \in \mathbf{E}$ ($q \neq 0$) the set

$$\text{Gfr } q = \{(\mathbf{d}, k) / \mathbf{d} \in \text{Fr } q, k = \delta(q)\} \subset \tilde{\mathbf{H}}\tilde{\mathbf{A}}_0,$$

universal covering of the analytic halo $\mathbf{H}\mathbf{A}_0$.

We write

$$\begin{aligned} \text{Fr}(\mathbf{q}) &= \bigcup_{ij} \text{Fr } q_{ij} \quad (q_{ij} = q_i - q_j \neq 0), \\ \text{Gfr}(\mathbf{q}) &= \bigcup_{ij} \text{Gfr } q_{ij}, \end{aligned}$$

and denote by $\Sigma(\mathbf{q})$ the *projection* on S^1 of $\text{Fr}(\mathbf{q})$.

⁽⁴³⁾ Generated by a *loop* turning “one time” around the origin and isomorphic to $\mathbf{Z}$.

⁽⁴⁴⁾ We will recall this description in part 5.

We define an action of the group $(\hat{\gamma}_0)$ generated ⁽⁴⁵⁾ by $\hat{\gamma}_0$ on the (non abelian) free group generated by the γ_d 's ($d \in \text{Fr}(\mathbf{q})$) by

$$\hat{\gamma}_0: \gamma_d \rightarrow \gamma_{\exp(-2i\pi)d}(\exp(-2i\pi)). \text{ is a translation of } -2\pi \text{ in } (\mathbf{R}, 0)).$$

We denote by $\Pi(\mathbf{q})$ the corresponding *semi-direct* product

$$\Pi = (\hat{\gamma}_0) \ltimes \left(\bigstar_{d \in \text{Fr}(\mathbf{q})} (\gamma_d) \right).$$

In $\Pi(\mathbf{q})$ we have $\hat{\gamma}_0 \gamma_d \hat{\gamma}_0^{-1} = \gamma_{\exp(-2i\pi)d}$.

We define an action of the free group $(\hat{\gamma}_0)$ generated by $\hat{\gamma}_0$ on the (non abelian) free group generated by the γ_a 's ($a \in \text{Gfr}(\mathbf{q})$) by

$$\hat{\gamma}_0: \gamma_a \rightarrow \gamma_{\exp(-2i\pi)a}(a = (d, k), \exp(-2i\pi)a = (\exp(-2i\pi)d, k)).$$

We denote by $G\Pi(\mathbf{q})$ the corresponding *semi-direct* product

$$G\Pi(\mathbf{q}) = (\hat{\gamma}_0) \ltimes \left(\bigstar_{a \in \text{Gfr}(\mathbf{q})} (\gamma_a) \right).$$

In $G\Pi(\mathbf{q})$ we have $\hat{\gamma}_0 \gamma_a \hat{\gamma}_0^{-1} = \gamma_{\exp(-2i\pi)a}$.

The groups $\Pi(\mathbf{q})$, and $G\Pi(\mathbf{q})$ are “first approximations” of the “wild local fundamental group” ⁽⁴⁶⁾. We can identify $\Pi(\mathbf{q})$ to a subgroup of $G\Pi(\mathbf{q})$ by

$$\gamma_d = \gamma_{a_r} \gamma_{a_{r-1}} \cdots \gamma_{a_1} \quad (a_i = (d, k_i); i = 1, \dots, r).$$

We will obtain below a classification in terms of *linear representations* of these groups ⁽⁴⁷⁾. Unfortunately there are *conditions* (“*Stokes conditions*”) on these representations in order that *they come from a connection*. That is *unsatisfying*: we want a “*wild fundamental group*” whose *all* finite dimensional linear representations come from a connection, just like for the Riemann-Hilbert correspondance. We will be led to the “good” group $\pi_{1,s}(\mathbf{C}^*, 0)$ by a “*Fourier analysis*” of the (Galois differential) “*unfolding*” of the *Stokes phenomena* under the *adjoint action of the exponential torus*. Moreover we will see that this approach gives ⁽⁴⁸⁾ a *very natural* interpretation of *Ecalé's resurgence* [E 4].

Let $\Delta = d/dx - A$, with $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

⁽⁴⁵⁾ Here $\hat{\gamma}_0$ and the γ_d 's, γ_a 's are “labels”; later $\hat{\gamma}_0$ and γ_a will be interpreted as *loops* turning around respectively 0 and a .

⁽⁴⁶⁾ The terminology “wild π_1 ” (in french “ π_1 -sauvage”) was suggested to the second author by B. Malgrange for the group $G\Pi$ [Ma 7].

⁽⁴⁷⁾ If we consider “*isoformal*” families, that is if we fix the “*formal form*”. If we leave it free, we need to “add” a representation of the “*formal fundamental group*”.

⁽⁴⁸⁾ With paragraph 6 tools this approach will lead us to an essentially “*geometric*” description of the resurgence where *Laplace transform* and *convolution* no longer play the central characters... The second author was led to this description particularly by Malgrange's account of a part of *Ecalé's* work [Ma 8].

Let $\hat{F}(x) = \hat{H}(x) x^L U e^{Q(1/u)}$, be a **formal fundamental solution** of Δ as above. We set

$F_0(x) = x^L U e^{Q(1/u)}$ (Hermite formal fundamental solution [J]).

For $\hat{P} \in GL(n; \mathbb{C}[[x]][x^{-1}])$, we set

$$A^{\hat{P}} = \hat{P} A \hat{P}^{-1} + \frac{d\hat{P}}{dx} \hat{P}^{-1}$$

and $\Delta^{\hat{P}} = d/dx - A^{\hat{P}}$, and we say that the differential operators Δ and $\Delta^{\hat{P}}$ are **formally equivalent**. If $P \in GL(n; \mathbb{C}\{x\}[x^{-1}])$, we will say that the differential operators Δ and Δ^P are **analytically equivalent**. We have $(\Delta^{\hat{P}_2})^{\hat{P}_1} = \Delta^{\hat{P}_1 \hat{P}_2}$.

It is easy to check [BJL 1] that F_0 is a fundamental solution of a rational differential operator $\Delta_0 = d/dx - A_0$, with $A_0 \in \text{End}(n; \mathbb{C}(x)[x^{-1}])$, which is formally equivalent to Δ ($\Delta = \Delta_0^{\hat{H}}$).

We will write

$$\mathcal{J}_0(\hat{F}) = \{C \in GL(n; \mathbb{C}) / C \hat{F} = \hat{F} C\},$$

and $\mathcal{J}(F) = \{C \in GL(n; \mathbb{C}) / \text{there exists } \hat{G} \in GL(n; \mathbb{C}[[x]][x^{-1}]) \text{ such that } \hat{G}\hat{F} = \hat{F}C\}$; $\mathcal{J}_0(\hat{F})$ and $\mathcal{J}(\hat{F})$ are algebraic subgroups of $GL(n; \mathbb{C})[BV]$, and $\mathcal{J}_0(\hat{F}) \subset \mathcal{J}(\hat{F})$.

We will write:

$\mathcal{J}(\Delta) = \{\hat{G} \in GL(n; \mathbb{C}[[x]][x^{-1}]) / \text{there exists } C \in GL(n; \mathbb{C}) \text{ such that } \hat{G}\hat{F} = \hat{F}C\}$; $\mathcal{J}(\Delta)$ is a subgroup of $GL(n; \mathbb{C}[[x]][x^{-1}])$. It is easy to check that $\mathcal{J}(\Delta_0)$ is a subgroup of $GL(n; \mathbb{C}(x)[x^{-1}])$ containing $\mathcal{J}_0(F_0)$. It is clear that $\Delta^{\hat{G}} = \Delta$ is equivalent to $\hat{G} \in \mathcal{J}(\Delta)$ ($\mathcal{J}(\Delta)$ is independant of the choice of $\hat{F}$).

We leave now Δ_0 fixed and we want to **classify, up to meromorphic equivalence**, all the **meromorphic differential operators** Δ **formally equivalent** to Δ_0 . Moreover we are also interested in the classification of the “**marked pairs**” $(\Delta, \hat{H})$ such that $\Delta^{\hat{H}} = \Delta_0$.

To a differential operator Δ **formally equivalent** to Δ_0 (a **fundamental solution** F_0 of Δ_0 being **fixed**) we can associate representations $\rho_{\text{irr}}(\Delta)$ of the groups $\Pi(\mathfrak{q})$ and $G\Pi(\mathfrak{q})$ in $GL(n; \mathbb{C})$ defined by:

$$\rho_{\text{irr}}(\Delta)(\hat{\gamma}_0) = \hat{M}, \quad \rho_{\text{irr}}(\Delta)(\gamma_d) = \text{St}_d(\Delta), \quad \rho_{\text{irr}}(\Delta)(\gamma_a) = \text{St}_{d,k}(\Delta)$$

($a = (d, k)$). (We use the formulae:

$$\hat{M} \text{St}_d(\Delta) \hat{M}^{-1} = \text{St}_{\exp(-2i\pi)d}(\Delta), \quad \text{and} \quad \hat{M} \text{St}_a(\Delta) \hat{M}^{-1} = \text{St}_{\exp(-2i\pi)a}(\Delta).)$$

These representations are clearly submitted to the **constraints**:

$$\rho_{\text{irr}}(\Delta)(\gamma_d) \in \text{Sto}(\mathfrak{Q}; \mathfrak{d}), \quad \text{and} \quad \rho_{\text{irr}}(\Delta)(\gamma_a) \in \text{Sto}^k(\mathfrak{Q}; \mathfrak{d}) = (a = (\mathfrak{d}, k)).$$

We will name these conditions “**Stokes conditions**”. These representations are defined up the action (by conjugacy) of $\mathcal{J}(F_0)$: if $\hat{F} = \hat{H}F_0$ is a formal fundamental solution of Δ , C an element of $\mathcal{J}(F_0)$, and $\hat{G} \in GL(n; \mathbb{C}\{x\}[x^{-1}])$ the corresponding element of $\mathcal{J}(\Delta_0)$, then

$\hat{F}C = \hat{H}F_0C = \hat{H}\hat{G}F_0$ is also a formal fundamental solution of Δ . These representations do *not* change if we replace Δ by a *meromorphically* equivalent operator ($\hat{H}$ is then changed in $P\hat{H}$, with $P \in GL(n; \mathbb{C}\{x\}[x^{-1}])$, and $\rho_{\text{irr}}(\Delta)$ depends *only* on the connection ∇ associated to Δ and of the *choice* of F_0 ; we can set $\rho_{\text{irr}}(\nabla) = \rho_{\text{irr}}(\Delta)$.

THEOREM 10. — *Let Δ_0 be a fixed differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0 . We set $\mathbf{q} = \mathbf{q}(Q)$, and denote by n the rank of Δ_0 .*

(i) *The natural map*

$$\begin{array}{ccc} \text{Meromorphic connections } \nabla & & \text{Representations} \\ \text{formally} & \xrightarrow{\rho_{\text{irr}}} & \text{of the group } G\Pi(\mathbf{q}) \\ \text{equivalent to } \nabla_0. & & \text{in } GL(n; \mathbb{C}) \text{ satisfying} \\ & & \text{Stokes conditions, up to} \\ & & \text{the action of } \mathcal{I}(F_0). \end{array}$$

$$\nabla \rightarrow \rho_{\text{irr}}(\nabla)$$

is a bijection.

(ii) *The natural map*

$$\begin{array}{ccc} \text{Meromorphic connections } \nabla & & \text{Representations} \\ \text{formally} & \xrightarrow{\rho_{\text{irr}}} & \text{of the group } \Pi(\mathbf{q}) \\ \text{equivalent to } \nabla_0. & & \text{in } GL(n; \mathbb{C}) \text{ satisfying} \\ & & \text{Stokes conditions, up to} \\ & & \text{the action of } \mathcal{I}(F_0). \end{array}$$

is a bijection.

This result is *non trivial*. We deduce its proof from the (non trivial...) *classification of isoformal meromorphic connections* in the form given by *Malgrange and Sibuya*, [Ma 3], [Si] ⁽⁴⁹⁾. We need before to recall some definitions and results (we will return to this topic in more details in 5). In the following we will systematically consider a function f (with values in a $\mathbb{C}$ -vector space) holomorphic on an *open sector* V as an “object” on the *open arc* U corresponding to V in S^1 (the real analytic blow-up of the origin in $\mathbb{C}$) as in [Ma 3]. We define in this way on S^1 the sheaf $\mathcal{A}$ of holomorphic functions (with values in $\mathbb{C}$) on sectors, admitting an asymptotic expansion at the origin (with Taylor expansion in $\mathbb{C}[[x]][x^{-1}]$). We denote by Λ_1 the subsheaf of $\text{End}(n; \mathcal{A})$ of germs of analytic matrices which are *asymptotic to identity*; Λ_1 is a sheaf of (non abelian) *groups*. If $\mathcal{F}$ is a sheaf on S^1 we will denote by $\mathcal{F}_d$ its fiber at $d \in S^1$.

⁽⁴⁹⁾ The first general classification (after the work of *Birkhoff* for the “generic case”) is in [BJL 2].

THEOREM 11 (Malgrange, Sibuya [Ma 3], [Si]). — *There exists a natural isomorphism*

$$\mathrm{GL}(n; \mathbb{C}\{x\}[x^{-1}]) \setminus \mathrm{GL}(n; \mathbb{C}[[x]][x^{-1}]) \xrightarrow{\mu} H^1(S^1; \Lambda_1).$$

We recall the definition of the *Malgrange-Sibuya map* μ :

Let $\mathbf{U} = \{U_i\}_{i \in I}$ be a finite open covering of S^1 by open arcs. We suppose that $U_i \cap U_j \cap U_k = \emptyset$, if $i, j, k \in I$ are distinct ⁽⁵⁰⁾.

Let $\hat{A} \in \mathrm{GL}(n; \mathbb{C}[[x]][x^{-1}])$. By *Borel-Ritt theorem* [Wa] we can “represent” $\hat{A}$ by a collection $\{A_i\}_{i \in I}$ (A_i being a *holomorphic matrix* on an open sector V_i corresponding to U_i admitting $\hat{A}$ as *asymptotic expansion* at the origin).

We consider $\{A_i\}_{i \in I}$ as a 0-cochain [with values in $\mathrm{GL}(n; \mathcal{A})$] and we take its *coboundary*

$\delta = \{A_j^{-1} A_i\}_{ij \in I} \in Z^1(\mathbf{U}; \mathrm{GL}(n; \mathcal{A}))$. We have $\delta \in Z^1(\mathbf{U}; \Lambda_1)$ (A_i and A_j have the *same asymptotic expansion* $\hat{A}$). We write $A_j^{-1} A_i = A_{ij}$.

By *definition* $\mu(\hat{A})$ is the image of δ in $H^1(S^1; \Lambda_1)$. If $P \in \mathrm{GL}(n; \mathbb{C}\{x\})$, and $\hat{B} = P\hat{A}$, we can choose $A_i = PA_i$; then $\mu(\hat{B}) = \mu(\hat{A})$. In the following we will set

$I = [1, \dots, p]$ (where “ $p+1=1$ ”), the bijection between I and $[1, \dots, p]$ being chosen such that $U_{i, i+1} = U_i \cap U_{i+1} \neq \emptyset$ ($i=1, \dots, p$) and such that the bisecting lines of the arcs $U_{i, i+1}$ *turn clockwise* when i increases.

If $\Sigma = \{d_1, d_2, \dots, d_p\} \subset S^1$, we will say that the covering $\mathbf{U}$ is “*adapted*” to Σ if

$$U_{i, i+1} = U_i \cap U_{i+1} \cap \Sigma = \{d_i\} \quad (i=1, \dots, p).$$

Let $k_1 > k_2 > \dots > k_r > 0$. Let $A \in \mathrm{GL}(n; \mathbb{C}\{x\}_{1/k_1, 1/k_2, \dots, 1/k_r}[x^{-1}])$.

If ⁽⁵¹⁾ $\Sigma = \Sigma(\hat{A}) = \{d_1, d_2, \dots, d_p\}$, we can build a covering $\mathbf{U} = \{U_i\}_{i \in I}$ *adapted* to Σ , where $U_i \cap U_{i+1}$ is *bisected* by d_i with *opening* $\leq \pi/k_1$ ($i=1, \dots, p$); such a covering is said to be *k_1 -adapted* to Σ . We can choose

$A_i = S_{k_1, k_2, \dots, k_r; d_i} \hat{A}$ (where $d \in U_i$ is arbitrary ⁽⁵²⁾ between d_i and d_{i+1} ; $i=1, \dots, p$). Then the 1-cocycle

$\mathbf{St}(\mathbf{U}; \hat{A}) = \{A_{i+1}^{-1} A_i\}_{i \in I}$ is well defined; the image of $\mathbf{St}(\mathbf{U}; \hat{A})$ in $H^1(S^1; \Lambda_1)$ is clearly $\mu(\hat{A})$. We will denote by $\mathbf{St}(\hat{A})$ the 1-cocycle $\mathbf{St}(\mathbf{U}; \hat{A})$ up to the choice of $\mathbf{U}$ (satisfying our hypothesis), and *identify* it to $\{(A_{i, i+1})_{d_i}\}_{i \in I}$.

⁽⁵⁰⁾ We will make this hypothesis for *all* the coverings in the following.

⁽⁵¹⁾ More generally we can also take $\Sigma(A) \subset \Sigma$ *finite*.

⁽⁵²⁾ The values of A_i obtained for the different d glue together by analytic continuation in an analytic matrix still denoted by A_i .

If $\mathbf{U}$ is an open covering of S^1 , and $\mathcal{F}$ a sheaf of groups on S^1 , we denote by

$$i_{\mathbf{U}}: Z^1(\mathbf{U}; \mathcal{F}) \rightarrow H^1(S^1; \mathcal{F}) \text{ the natural map.}$$

Let $k > 0$. We denote by $\Lambda^{\geq k}$ the subsheaf of Λ_1 of germs $I + \Phi$ where Φ is *exponentially flat of order $\geq k$* .

DEFINITION 5. — Let $k > 0$. Let $\Sigma = \{d_1, d_2, \dots, d_p\} \subset S^1$, and an open covering $\mathbf{U}$ “adapted” to Σ . A 1-cochain $\delta \in C^1(\mathbf{U}; \Lambda_1)$ is said to be “ k -summable”, if $\delta = \{A_{i, i+1}\}_{i \in I}$ with $A_{i, i+1} \in \Gamma(U_{i, i+1}; \Lambda^{\geq k})$, and if each $A_{i, i+1}$ can be (uniquely of course) “analytically” extended to an element of $\Gamma(V_{i, i+1}; \Lambda^{\geq k})$ where $V_{i, i+1}$ is an open arc of $(\mathbf{R}, 0)$ with *opening* π/k “containing” $U_{i, i+1}$ ($i=1, \dots, p$).

We will denote by $H^{1; \geq k}(S^1; \Lambda^{\geq k}) \subset H^1(S^1; \Lambda_1)$ the subset consisting of the images of k -summable 1-cocycles.

THEOREM 12 (Martinet-Ramis [MR 1], I-6. — Let $k > 0$.

(i) *The Malgrange-Sibuya isomorphism*

$$\mathrm{GL}(n; \mathbf{C} \{x\} [x^{-1}]) \setminus \mathrm{GL}(n; \mathbf{C} [[x]] [x^{-1}]) \xrightarrow{\mu} H^1(S^1; \Lambda_1).$$

induces an isomorphism

$$\mathrm{GL}(n; \mathbf{C} \{x\} [x^{-1}]) \setminus \mathrm{GL}(n; \mathbf{C} \{x\}_{1/k} [x^{-1}]) \xrightarrow{\mu} H^{1; \geq k}(S^1; \Lambda^{\geq k}).$$

(ii) *If $\delta \in Z^1(\mathbf{U}; \Lambda^{\geq k})$ is a k -summable 1-cocycle, then*

$$\mathbf{St}(\mathbf{U}; \mu^{-1} i_{\mathbf{U}}(\delta)) = \delta.$$

Let now Δ be a differential operator. We denote by $\Lambda(\Delta)$ the sheaf (on S^1) of solutions of $\mathrm{End} \Delta$ and by $\Lambda_1(\Delta)$ the subsheaf of solutions of $\mathrm{End} \Delta$ which are *asymptotic to identity*; $\Lambda_1(\Delta_0)$ is a subsheaf of Λ_1 .

Let now Δ_0 be a differential operator with a fundamental solution $F_0 = x^{\mathbf{L}} U e^{Q(1/\mu)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0 , and write $\mathbf{q} = \mathbf{q}(Q)$; $N\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\}$ is the set of values taken by $\delta(q_i - q_j)$ ($q_i \neq q_j$), and n the rank of Δ_0 . We write as above $\mathrm{End} \Delta_0 = d/dx - [A_0, \cdot]$.

Let $\mathbf{d} \in (\mathbf{R}, 0)$ be a direction and $d \in S^1$ its projection. To the choice of $\mathbf{d} \in (\mathbf{R}, 0)$ corresponds a “branch” of Logarithm and a “sum” $F_{0, \mathbf{d}}$ of $F_0 = x^{\mathbf{L}} U e^{Q(1/\mu)}$, which is analytic on an open sector bisected by d .

The map

$$\begin{aligned} \lambda_{\mathbf{d}}: \quad & \mathrm{GL}(n; \mathbf{C}) \rightarrow \Lambda(\Delta_0)_{\mathbf{d}} \\ \lambda_{\mathbf{d}}: \quad & \mathbf{C} \rightarrow F_{0, \mathbf{d}} \mathbf{C}(F_{0, \mathbf{d}})^{-1} \end{aligned}$$

is an *isomorphism of groups*.

Let

$$\Lambda(\Delta_0; \mathbf{d}; F_0) = \lambda_{\mathbf{d}}(\Lambda(Q; d))$$

$$\begin{aligned}\Lambda^k(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\Lambda^k(Q; d)) \\ \Lambda^{\geq k}(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\Lambda^{\geq k}(Q; d)) \\ \Lambda^{< k}(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\Lambda^{< k}(Q; d)).\end{aligned}$$

It is easy to see that $\Lambda(\Delta_0; \mathbf{d}; F_0)$, $\Lambda^k(\Delta_0; \mathbf{d}; F_0)$, $\Lambda^{\geq k}(\Delta_0; \mathbf{d}; F_0)$, and $\Lambda^{< k}(\Delta_0; \mathbf{d}; F_0)$ do **not** depend on the choice of F_0 and $\mathbf{d}$; moreover $\Lambda(\Delta_0; \mathbf{d}; F_0) = \Lambda_1(\Delta_0)_d$. We can set:

$$\begin{aligned}\Lambda^k(\Delta_0; \mathbf{d}; F_0) &= \Lambda^k(\Delta_0)_d, \Lambda^{\geq k}(\Delta_0; \mathbf{d}; F_0) = \Lambda^{\geq k}(\Delta_0)_d, \\ \Lambda^{< k}(\Delta_0; \mathbf{d}; F_0) &= \Lambda^{< k}(\Delta_0)_d.\end{aligned}$$

All these groups ⁽⁵³⁾ are subgroups of $\Lambda(\Delta_0)_d$ and when the direction d varies we get subsheaves $\Lambda^k(\Delta_0)$, $\Lambda^{\geq k}(\Delta_0)$, and $\Lambda^{< k}(\Delta_0)$ of $\Lambda_1(\Delta_0)$. (When d moves the groups remain "in general" the "same". They can "jump" only for a finite set of values of d , the "Stokes lines".)

Let

$$\begin{aligned}\text{Sto}(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\text{Sto}(Q; d)) \\ \text{Sto}^k(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\text{Sto}^k(Q; d)) \\ \text{Sto}^{\geq k}(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\text{Sto}^{\geq k}(Q; d)) \\ \text{Sto}^{< k}(\Delta_0; \mathbf{d}; F_0) &= \lambda_{\mathbf{d}}(\text{Sto}^{< k}(Q; d)).\end{aligned}$$

It is easy to see that $\text{Sto}(\Delta_0; \mathbf{d}; F_0)$, $\text{Sto}^k(\Delta_0; \mathbf{d}; F_0)$, $\text{Sto}^{\geq k}(\Delta_0; \mathbf{d}; F_0)$, and $\text{Sto}^{< k}(\Delta_0; \mathbf{d}; F_0)$ do **not** depend on the choice of F_0 and $\mathbf{d}$. We can set:

$$\begin{aligned}\text{Sto}(\Delta_0; \mathbf{d}; F_0) &= \text{Sto}(\Delta_0)_d, \text{Sto}^k(\Delta_0; \mathbf{d}; F_0) \\ &= \text{Sto}^k(\Delta_0)_d, \text{Sto}^{\geq k}(\Delta_0; \mathbf{d}; F_0) = \text{Sto}^{\geq k}(\Delta_0)_d, \\ \text{Sto}^{< k}(\Delta_0; \mathbf{d}; F_0) &= \text{Sto}^{< k}(\Delta_0)_d.\end{aligned}$$

If $d \notin \Sigma(\Delta_0)$, then $\text{Sto}(\Delta_0)_d$ reduces to identity.

From proposition 10 and lemma 12, we get

PROPOSITION 11. — Let $d \in S^1$ and $k > 0$. Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L \cup e^{Q(1/u)}$. We set $\mathbf{q} = \mathbf{q}(Q)$, and $N\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\}$ ($k_1 > k_2 > \dots > k_r$). Then:

(i) The four sequences

$$\begin{aligned}\{\text{id}\} &\rightarrow \Lambda^{\geq k}(\Delta_0)_d \rightarrow \Lambda(\Delta_0)_d \rightarrow \Lambda^{< k}(\Delta_0)_d \rightarrow \{\text{id}\}, \\ \{\text{id}\} &\rightarrow \Lambda^k(\Delta_0)_d \rightarrow \Lambda^{\geq k}(\Delta_0)_d \rightarrow \Lambda^{< k}(\Delta_0)_d \rightarrow \{\text{id}\}, \\ \{\text{id}\} &\rightarrow \text{Sto}^{\geq k}(\Delta_0)_d \rightarrow \text{Sto}(\Delta_0)_d \rightarrow \text{Sto}^{< k}(\Delta_0)_d \rightarrow \{\text{id}\}, \\ \{\text{id}\} &\rightarrow \text{Sto}^k(\Delta_0)_d \rightarrow \text{Sto}^{\geq k}(\Delta_0)_d \rightarrow \text{Sto}^{< k}(\Delta_0)_d \rightarrow \{\text{id}\},\end{aligned}$$

are split exact sequences of groups.

$$(ii) \quad \Lambda(\Delta_0)_d = \Lambda^{k_r}(\Delta_0)_d \ltimes \Lambda^{k_r-1}(\Delta_0)_d \ltimes \dots \ltimes \Lambda^{k_1}(\Delta_0)_d.$$

⁽⁵³⁾ It is possible to give a "direct" definition of these groups using Deligne I -filtered structures (or Stokes structures) [Ma 4], [De 3], [De 4].

THEOREM 13 (Malgrange, Sibuya, Babbitt-Varadarajan [Ma 3], [Si], [BV])
 Let Δ_1 be a meromorphic differential operator. We denote by ∇_1 the meromorphic connection defined by Δ_1 . Let Δ_0 be a differential operator with a fixed fundamental solution $F_0 = x^L \cup e^{Q(1/u)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0 . Then:

(i) There is a natural isomorphism $v = v_{\nabla_1}$:

Marked pairs (∇, ξ) , where
 ∇ is a meromorphic connections
 which is **formally equivalent** $\xrightarrow{v} H^1(S^1; \Lambda(\Delta))$
 to ∇_1 and ξ is an **isomorphism**
 between ∇ and ∇_1 .

(ii) If $\nabla_1 = \nabla_0$ the natural isomorphism v induces an isomorphism:

Meromorphic connections ∇
 which are $\xrightarrow{v} \mathcal{F}(\Delta_0) \setminus H^1(S^1; \Lambda(\Delta_0))$
formally equivalent to ∇_0 .

[The group $\mathcal{F}(\Delta_0)$ is acting by **conjugacy** on the sheaf $\Lambda(\Delta_0)$.]

DEFINITION 6. — Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L \cup e^{Q(1/u)}$. We set

$$\mathbf{q} = \mathbf{q}(\mathbf{Q}), \quad N\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\},$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2, \dots, d_p\}$ the projection of $\text{Fr}(\mathbf{q})$ on S^1 . Let $\mathbf{U} = \{U_i\}_{i \in I}$ be an open covering k_1 -adapted to $\Sigma(\mathbf{q})$. Then, a 1-cochain

$$\delta \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$$

is called a “**Stokes cochain**” if

$$\delta = \{A_{i, i+1}\}_{i \in I} \quad (I = \{1, \dots, p\})$$

with

$$(A_{i, i+1})_{d_i} \in \text{Sto}(\Delta_0)_{d_i} \quad (i = 1, \dots, p).$$

Let $\mathbf{d} \in \text{Fr}(\mathbf{q})$, let d be its projection on S^1 , and let ρ be a representation of $\Pi(\mathbf{q})$ in $\text{GL}(n; \mathbb{C})$. It is easy to check that $\lambda_{\mathbf{d}}(\rho(\gamma_{\mathbf{d}})) \in \Lambda(\Delta_0)_d$ depends only on $d \in S^1$.

LEMMA 15. — Let Δ_0 be a fixed differential operator with a fundamental solution $F_0 = x^L \cup e^{Q(1/u)}$.

We set $\mathbf{q} = \mathbf{q}(\mathbf{Q})$, $N\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\}$ ($k_1 > k_2 > \dots > k_r$), and denote by $\Sigma(\mathbf{q})$ the projection of $\text{Fr}(\mathbf{q})$ on S^1 . Let $\mathbf{U} = \{U_i\}_{i \in I}$, be an open covering which is k_1 -adapted to $\Sigma(\mathbf{q})$.

The natural map

Representations of $\Pi(\mathbf{q})$ in $\text{GL}(n; \mathbb{C}) \xrightarrow{z_{\mathbf{U}}} \{\text{Stokes cocycles of } Z^1(\mathbf{U}; \Lambda(\Delta_0))\}$

$$\rho \xrightarrow{z_{\mathbf{U}}} “\{\lambda_{\mathbf{d}}(\rho(\gamma_{\mathbf{d}}))\}” \quad (d \in \Sigma(\mathbf{q}))$$

is a **bijection**.

THEOREM 14. — Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We set

$$\mathbf{q} = \mathbf{q}(\mathbf{Q}), \quad \mathbf{N}\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\} \quad (k_1 > k_2 > \dots > k_r),$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2, \dots, d_p\}$ the projection of $\text{Fr}(\mathbf{q})$ on S^1 . Let $\mathbf{U} = \{U_i\}_{i \in \mathbb{N}}$ be an open covering k_1 -adapted to $\Sigma(\mathbf{q})$. Then:

(i) Let ⁽⁵⁴⁾:

$$\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r,$$

where $\hat{H}_i \in \text{GL}(n; \mathbb{C})[[x]][x^{-1}]$ is k_i -summable for $i = 1, \dots, r$. We suppose that $F = H x^L U e^{Q(1/u)}$ is a **formal** fundamental solution of a **meromorphic** differential operator Δ . Then the 1-cocycle $\mathbf{St}(\mathbf{U}; \hat{H})$ is a **Stokes cocycle**.

(ii) Let $\delta \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ be a **Stokes cocycle**. Then, $\Lambda(\Delta_0) \subset \Delta_1$, $\delta \in Z^1(\mathbf{U}; \Lambda_1)$, and if $\hat{H} = \mu^{-1} i_U(\delta)$:

(a) $\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r$, where $\hat{H}_i \in \text{GL}(n; \mathbb{C})[[x]][x^{-1}]$ is k_i -summable for $i = 1, \dots, r$;

(b) $\hat{F} = \hat{H} x^L U e^{Q(1/u)}$ is a **formal** fundamental solution of a **meromorphic** differential operator Δ which is **formally** equivalent to Δ_0 .

Moreover: $\delta = \mathbf{St}(\mathbf{U}; \hat{H}) = \mathbf{St}(\mathbf{U}; \mu^{-1} i_U(\delta))$, and if ∇ is the meromorphic connection associated to Δ , then $v(\nabla) = i_U(\delta)$.

(iii) Let $\alpha \in H^1(S^1; \Lambda(\Delta_0))$. Then there exist **one and only one** **Stokes cocycle** $\delta \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$ such that $\alpha = i_U(\delta)$ (that is, representing α) ⁽⁵⁵⁾.

We will first prove assertion (i).

Using the construction of theorem 10, we can associate to $\hat{F} = \hat{H} F_0$ a representation $\rho(H)$ of $\Pi(\mathbf{q})$ in $\text{GL}(n; \mathbb{C})$, satisfying Stokes conditions. We have $\mathbf{St}(\mathbf{U}; \hat{H}) = z_U(\rho(\hat{H}))$, and $\mathbf{St}(\mathbf{U}; \hat{H})$ is a **Stokes cocycle**.

We will **admit** assertion (ii), for a moment.

Assertion (iii) follows easily from assertions (ii) and (iii):

Let $\alpha \in H^1(S^1; \Lambda(\Delta_0))$. From theorem 13, we get a meromorphic connection $\nabla = v^{-1}(\alpha)$, which is formally equivalent to ∇_0 . We choose a differential operator Δ representing ∇ ; then there exists a fundamental solution $\hat{F} = \hat{H} F_0$ of Δ , with $\hat{H} \in \text{GL}(n; \mathbb{C})[[x]][x^{-1}]$. From theorem 7 we get a decomposition

$$\hat{H} = \hat{H}_1 \hat{H}_2 \dots \hat{H}_r,$$

⁽⁵⁴⁾ It is important to notice that this definition is stated in such a way that it is **not** necessary to know theorem 5 or theorem 7 to apply it (see footnote below). Of course one can also apply it in the situation of theorem 5 or theorem 7...

⁽⁵⁵⁾ Assertion (iii) is due to M. Loday-Richaud [LR 1]. Her proof is **completely different**: she gives an **explicit algebraic algorithm** in order to **compute explicitly** δ from α . She uses *Malgrange-Sibuya* theory but **not** Gevrey asymptotics and multisummability; so it is possible, using her result and noting that assertions (i) and (ii) are proved here **without any use of** theorem 5 or theorem 7, to get a **new proof of theorem 7** [LR 1]. Cf. also [BV].

where $\hat{H}_i \in \text{GL}(n; \mathbf{C})[[x]][x^{-1}]$ is k_i -summable for $i=1, \dots, r$.

We have $\rho(\hat{H}) = \rho_{\text{irr}}(\nabla)$. Let $z_U(\rho(\hat{H})) = \delta \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$. We have $i_U(\delta) = \alpha$, and δ is a Stokes cocycle representing α .

It remains to prove *unicity*. Let $\delta \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$, with $i_U(\delta) = \alpha$. From assertion (ii) we get $\delta = \mathbf{St}(\mathbf{U}; \mu^{-1} i_U(\delta)) = \mathbf{St}(\mathbf{U}; \mu^{-1}(\alpha))$, but $\mathbf{St}(\mathbf{U}; \mu^{-1}(\alpha))$ depends only on α ; *unicity* of δ follows.

Before we prove assertion (ii) we will give some consequences of theorem 14.

PROPOSITION 12. — Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We set

$$\mathbf{q} = \mathbf{q}(\mathbf{Q}), \quad N\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\} \quad (k_1 > k_2 > \dots > k_r),$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2, \dots, d_p\}$ the projection of $\text{Fr}(\mathbf{q})$ on S^1 . Let $\mathbf{U} = \{U_i\}_{i \in I}$ be an open covering which is k_1 -adapted to $\Sigma(\mathbf{q})$. Then the natural map

$$\begin{array}{ccc} \text{Representations of the group } \Pi(\mathbf{q}) \\ \text{in } \text{GL}(n; \mathbf{C}), \text{ satisfying the} & & \rightarrow H^1(S^1; \Lambda(\Delta_1)) \\ \text{Stokes conditions} & & \\ \rho \rightarrow z_U(\delta) & & \end{array}$$

is a **bijection** commuting with the action of $(\mathcal{J}(F_0); \mathcal{J}(\Delta_0))$.

Theorem 10 follows from theorem 13 and proposition 12.

It remains now to prove assertion (ii) of theorem 14.

Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We set

$$\mathbf{q} = \mathbf{q}(\mathbf{Q}), \quad N\Sigma(\mathbf{q}) = \{k_1, k_2, \dots, k_r\} \quad (k_1 > k_2 > \dots > k_r),$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2, \dots, d_p\}$ the projection of $\text{Fr}(\mathbf{q})$ on S^1 . Let $\mathbf{U} = \{U_i\}_{i \in I}$ ($I = \{1, \dots, p\}$), be an open covering which is k_1 -adapted to $\Sigma(\mathbf{q})$.

Let $\delta \in (\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ be a Stokes cocycle. Then, $\Lambda(\Delta_0) \subset \Lambda_I$, $\delta \in Z^1(\mathbf{U}; \Lambda_I)$. Let $\hat{H} = \mu^{-1} i_U(\delta)$. We will prove that δ is a Stokes cocycle by a **descending induction** on $i=r, r-1, \dots, 1$.

Our induction hypothesis is:

(Hyp i) Let $\delta^i = \{A_{i, i+1}\}_{i \in I} \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ be a Stokes cocycle **satisfying**:

$$(A_{i, i+1})_{d_i} \in \text{Sto}^{k_i}(\Delta_0)_{d_i} \quad (i=1, \dots, p; \text{Sto}^{\leq k_i} = \text{Sto}^{< k_{i-1}}, \text{ if } i > 1, \text{ and } \text{Sto}^{\leq k_1} = \text{Sto}).$$

Then, if $\hat{H}^i = \mu^{-1} i_U(\delta^i)$:

(a_i) $\hat{H}^i = \hat{H}_i \hat{H}_{i+1} \dots \hat{H}_r$, where $\hat{H}_j \in \text{GL}(n; \mathbf{C}[[x]][x^{-1}])$ is k_j -summable for $j=i, \dots, r$.

(b_i) $\hat{F}^i = \hat{H}^i x^L U e^{Q(1/u)}$ is a formal fundamental solution of a meromorphic differential operator Δ^i which is **formally** equivalent to Δ_0 .

Moreover:

$\delta^i = \mathbf{St}(\mathbf{U}; \hat{H}^i) = \mathbf{St}(\mathbf{U}; \mu^{-1} i_U(\delta^i))$, and, if ∇^i is the meromorphic connection associated to Δ^i , then $v(\nabla^i) = i_U(\delta^i)$

Assertion (ii) is (Hyp 1).

We will first prove (Hyp r).

Let $\delta^r = \{A_{i, i+1}\}_{i \in I} \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ be a Stokes cocycle with:

$$(A_{i, i+1})_{d_i} \in \text{Sto}^{k_r}(\Delta_0)_{d_i}.$$

We have (for $d_i \in (\mathbf{R}, 0)$ "above" d_1)

$\lambda_{d_i}^{-1} (A_{i, i+1})_{d_i} = C_{d_i; r}$, or $(A_{i, i+1})_{d_i} = F_{0, d_i} C_{d_i; r} (F_{0, d_i})^{-1}$; if $V_{i, i+1}$ is the open arc of $(\mathbf{R}, 0)$ bisected by d_i , with opening π/k_r , then $C_{d_i; r} \in \text{Sto}^{k_r}(Q; d)$, and $F_{0, d_i} C_{d_i; r} (F_{0, d_i})^{-1}$ is the germ of a function belonging to $\Gamma(V_{i, i+1}; \Lambda^{\geq k_r})$. So the 1-cocycle δ^i is k_r -summable. It follows from theorem 12 that $\hat{H}^r = \hat{H}_r$ is k_r -summable and (a_r) is proved; (b_r) follows from theorem 13.

We suppose now that (Hyp j) is true for $r \geq j \geq i > 1$, and will prove (Hyp i-1).

Let $\delta^{i-1} = \{A_{i, i+1}^{i-1}\}_{i \in I} \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ be a Stokes cocycle with:

$$(A_{i, i+1}^{i-1})_{d_i} \in \text{Sto}^{\leq k_{i-1}}(\Delta_0)_{d_i}.$$

Let $\{C_{d_i}^{i-1}\} = zU^{-1}(\delta^{i-1})$. We have $C_{d_i}^{i-1} \in \text{Sto}^{\leq k_{i-1}}(Q; d)$, and from the decomposition (Lemma 12):

$$\Lambda^{\leq k_{i-1}}(Q; d) = \Lambda^{k_r}(Q; d) \times \Lambda^{k_{r-1}}(Q; d) \times \dots \times \Lambda^{k_{i-1}}(Q; d),$$

we get, for $C_{d_i}^{i-1} \in \Lambda^{\leq k_{i+1}}(Q; d)$, a decomposition:

$$C_{d_i}^{i-1} = C_{d_i; r} C_{d_i; r-1} \dots C_{d_i; i-1}, \quad \text{with } C_{d_i; j} \in \Lambda^{k_j}(Q; d)$$

(j = r, ..., i-1).

We have

$$C_{d_i}^{i-1} = C_{d_i}^i C_{d_i; i-1},$$

with $C_{d_i}^i \in \Lambda^{\leq k_i}(Q; d)$, and $C_{d_i; i-1} \in \Lambda^{k_{i-1}}(Q; d)$.

We have $(A_{i, i+1}^i)_{d_i} = \lambda_{d_i}(C_{d_i}^i)$ (which is independant of the choice of $d_i \in (\mathbf{R}, 0)$ "above" d_i), and $\delta^i = \{A_{i, i+1}^i\}_{i \in I} \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$. If $\hat{H}^i = \mu^{-1} i_U(\delta^i)$; then $\delta^i = \mathbf{St}(\mathbf{U}; \hat{H}^i)$.

It we set:

$$S_{k_i, k_{i+1}, \dots, k_r, d_i}^+ \hat{H}^i = H_{d_i}^{i+},$$

and

$$S_{k_i, k_{i+1}, \dots, k_r, d_i}^- \hat{H}^i = H_{d_i}^{i-},$$

we get:

$$(H_{d_i}^{i-})^{-1} H_{d_i}^{i+} = (A_{i, i+1}^i)_{d_i} = \lambda_{d_i} (C_{d_i}^i),$$

or

$$H_{d_i}^{i+} F_{0, d_i} = H_{d_i}^{i-} F_{0, d_i} C_{d_i}^i.$$

We set

$$(B_{i, i+1})_{d_i} = H_{d_i}^{i+} \lambda_{d_i} (C_{d_i, i-1}) (H_{d_i}^{i+})^{-1}.$$

Let $V_{i, i+1}^i$ and $V_{i, i+1}^{i-1}$ be open arcs of $(\mathbf{R}, 0)$ bisected by d_i with respective openings π/k_i and π/k_{i-1} ($V_{i, i+1}^{i-1}$ is contained in $V_{i, i+1}^i$). Then the germ $\lambda_{d_i} (C_{d_i, i-1})$ is the germ at d_i of a function $B_{i, i+1}^i$ belonging to $\Gamma(V_{i, i+1}^{i-1}; \Lambda^{\geq k_{i-1}})$ (this follows from $C_{d_i, i-1} \in \Lambda^{k_{i-1}}(Q; d)$). The germ $H_{d_i}^{i+}$ is the germ at d_i of a function H^{i+} belonging to $\Gamma(V_{i, i+1}^i; \Lambda)$ asymptotic to $\hat{H}^i$ on $V_{i, i+1}^i$ (and, *a fortiori*, on $V_{i, i+1}^{i-1}$). We conclude that the germ $(B_{i, i+1})_{d_i}$ is the germ at d_i of a function $B_{i, i+1}$ belonging to $\Gamma(V_{i, i+1}^{i-1}; \Lambda^{\geq k_{i-1}})$.

We have built a k_{i-1} -summable cochain $\beta = \{B_{i, i+1}\}_{i \in \mathbf{I}}$. We check easily that

$$\beta \in Z^1(\mathbf{U}; \Lambda(\Delta^i)).$$

Then it follows from *theorem 12* that $\hat{H}_{i-1} = \mu^{-1} i_U(\beta)$ is k_{i-1} -summable, and from *theorem 13* (i) that $(\Delta^i)^{\hat{H}_{i-1}} = \Delta^{i-1}$ (*definition* of Δ^{i-1}) is a *meromorphic* differential operator. We define

$$\hat{H}^{i-1} = \hat{H}_{i-1} \hat{H}^i = \hat{H}_{i-1} \hat{H}_i \dots \hat{H}_r.$$

Then

$$\Delta^{i-1} = (\Delta^i)^{\hat{H}_{i-1}} = (\Delta_0^{\hat{H}^i})^{\hat{H}_{i-1}} = \Delta_0^{\hat{H}_{i-1} \hat{H}^i} = \Delta_0^{\hat{H}^{i-1}},$$

and

$$\hat{F}^{i-1} = \hat{H}^{i-1} x^L \cup e^{Q(1/u)}$$

is a *formal fundamental solution* of the *meromorphic* differential operator Δ^{i-1} , *formally equivalent* to Δ_0 .

Let

$$H_{d_i, i-1}^+ = S_{k_{i-1}, \dots, k_r; d_i}^+ \hat{H}_{i-1} \quad \text{and} \quad H_{d_i, i-1}^- = S_{k_{i-1}, \dots, k_r; d_i}^- \hat{H}_{i-1}.$$

We get:

$$\begin{aligned} H_{d_i, i-1}^+ H_{d_i}^{i+} F_{0, d_i} &= H_{d_i, i-1}^- H_{d_i}^{i-} F_{0, d_i} C_{d_i}^i C_{d_i, i-1} \\ H_{d_i, i-1}^+ H_{d_i}^{i+} F_{0, d_i} &= H_{d_i, i-1}^- H_{d_i}^{i-} F_{0, d_i} C_{d_i}^{i-1}; \\ H_{d_i}^{i-1+} F_{0, d_i} &= H_{d_i}^{i-1-} F_{0, d_i} C_{d_i}^{i-1}. \end{aligned}$$

Then $\delta^{i-1} = \mathbf{St}(\mathbf{U}; \hat{H}^{i-1}) = \mathbf{St}(\mathbf{U}; \mu^{-1} i_U(\delta^{i-1}))$. We have got (Hyp $i-1$) and *assertion* (ii) of *theorem 14* is proved by induction. That *concludes* the proof of *theorem 14*.

Examples. — To illustrate the preceding constructions, it is possible to compute the “wild groups” and their representations for *Airy equation* and *Kummer equations*. This is a simple reformulation of computations of [MR 2], chapter 3.

Remark. — For $\mathbf{d} \in (\mathbf{R}, 0)$ the “label” $\gamma_{\mathbf{d}} \in \Pi(\mathbf{q})$ will later (see 6, infra) correspond to a *loop* pointed at “ $\mathbf{R}^+$ ” = (“0”, $\mathbf{R}^+$) $\in \{“0”\} \times (\mathbf{R}, 0)$ (“ $\mathbf{R}^+$ ” is a point belonging to the universal covering of the real blow-up of the origin $\{“0”\} \times S^1$ in the analytic halo).

We start from “ $\mathbf{R}^+$ ” and go (along $\{“0”\} \times (\mathbf{R}, 0)$) to

$$(“0”, \mathbf{d}) \in \{“0”\} \times (\mathbf{R}, 0).$$

Then we turn clockwise around “ $]0, +\infty]$ ” $\times \{\mathbf{d}\}$ onto the universal covering of $\mathbf{C}^*$ with an analytic halo at zero and go back to (“0”, $\mathbf{d}$). Afterwards we return to “ $\mathbf{R}^+$ ” (along $\{“0”\} \times (\mathbf{R}, 0)$). Then groups $\Pi(\mathbf{q})$ and $G\Pi(\mathbf{q})$ are interpreted as “wild fundamental groups *pointed at*

$$“\mathbf{R}^+” \in \{“0”\} \times S^1.”$$

The *Stokes operator* $St_{\mathbf{d}}(\Delta_0)$ corresponds to a “wild monodromy” along the loop $\gamma_{\mathbf{d}}$ for the vector space of “germs of solutions of the differential operator Δ at “ $\mathbf{R}^+$ ””, *modulo an isomorphism* between this linear space and the linear space of *formal* solutions of Δ_0 (in order to get this isomorphism we use the “analyticity” of $\hat{H}$ near 0 in the analytic halo and choose the principal determination for the Logarithm: $\Delta = \Delta_0^H$). The “wild connections” induced by ∇_0 and ∇ in a “small” sector of the universal covering of the analytic halo bisected by $\mathbf{R}^+$ are *isomorphic* (H is a wild analytic function in such a sector), then the representation $\rho_{\text{irr}}(\nabla)$ in $GL(n, \mathbf{C})$, up to the action of $\mathcal{T}(F_0)$, can be interpreted as a *representation* of $\Pi(\mathbf{q}(\nabla))$ in the linear group $GL(\mathbf{V})$ of the vector space $\mathbf{V} = \text{Sol}_{\mathbf{R}^+}(\nabla)$ of *germs of horizontal sections of ∇* “at “ $\mathbf{R}^+$ ”” $\in \{“0”\} \times S^1$ ” ($\mathbf{V}$ can be identified with a subspace of $(K\langle x^L \rangle \otimes_K \hat{K}\langle e^Q \rangle)^n$, where $K\langle x^L \rangle$ is identified with a space of germs of meromorphic functions on sectors bisected by $\mathbf{R}^+$, and a *class modulo $\mathcal{T}(F_0)$* corresponds to a *uniquely determined* representation in $GL(\mathbf{V})$). Finally we get a “wild monodromy” (which does not depend on the choices of Δ_0 and F_0). This “wild monodromy” expresses the “difference” between ∇ and ∇_0 . In fact we want to understand ∇ *independantly* of ∇_0 . In order to do that we will first translate ∇_0 in terms of *linear representation*.

Let

$$\mathbf{E} = \bigcup_{\mathbf{q}} \mathbf{E}(\mathbf{q}) = \varinjlim_{\mathbf{q}} \mathbf{E}(\mathbf{q}).$$

Let $\mathcal{T}(\mathbf{q})$ be the exponential torus associated to

$$\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E} \quad (\mathcal{T}(\mathbf{q}) = \text{Aut}_{K_v} \mathbb{L}_v).$$

To natural injections

$$\mathbf{E}(\mathbf{q}) \rightarrow \mathbf{E}$$

correspond natural projections

$$\mathcal{T}(\mathbf{q}) \rightarrow \mathcal{T}.$$

We write $\mathcal{T} = \varprojlim_{\mathbf{q}} \mathcal{T}(\mathbf{q})$. By definition $\mathcal{T}$ is the *exponential torus*; it is

a *commutative group*. The algebraic tori $\mathcal{T}(\mathbf{q})$ are endowed with the Zariski topology, and $\mathcal{T}$ is endowed with the corresponding inverse limit topology.

LEMMA 16. — (i) Let $\kappa: \mathcal{T} \rightarrow \mathbf{C}^*$ be a continuous homomorphism of groups. Then there exists a uniquely determined $\mathbf{q} \in \mathbf{E}$, such that κ is equal to the composition of the natural projection $\mathcal{T} \rightarrow \mathcal{T}(\mathbf{q})$ ($\mathbf{q} = \{q\}$) and of the character $q: \mathcal{T}(\mathbf{q}) \rightarrow \mathbf{C}^*$. (We will identify κ and q .)

(ii) Let V be a finitely dimensional $\mathbf{C}$ -vector space ($n = \dim_{\mathbf{C}} V$), and $\theta: \mathcal{T} \rightarrow \mathrm{GL}(V)$ be a continuous homomorphism of groups. Let $G = \theta(\mathcal{T})$.

Then there exists a basis of V such that the subgroup G of $\mathrm{GL}(V)$, identified by the choice of this basis to $\mathrm{GL}(n; \mathbf{C})$, is *diagonal*. If $\varphi_1, \varphi_2, \dots, \varphi_n: G \rightarrow \mathbf{C}^*$ are the corresponding homomorphisms of groups [if $g \in G$, $\varphi_1(g)$ is the first entry of g on the diagonal...], and if q_i is associated to $\kappa_i = \varphi_i \theta$, like in (i) it is possible to associate to κ the set $\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E}$, which is *independent* of the choice of the basis of V , and θ is the composition of the natural projection $\mathcal{T} \rightarrow \mathcal{T}(\mathbf{q})$ and of

$$(q_1, q_2, \dots, q_n): \mathcal{T}(\mathbf{q}) \rightarrow \mathrm{GL}(n; \mathbf{C}) = \mathrm{GL}(V).$$

For $\tau \in \mathcal{T}$, $\theta(\tau) = \mathrm{Diag}(q_1(\tau), q_2(\tau), \dots, q_n(\tau))$.

In the situation of lemma 16 (ii), we will write $\mathbf{q} = \mathbf{q}_\theta$. From a given $\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E}$ we get a diagonal representation $\theta: \mathcal{T} \rightarrow \mathrm{GL}(n; \mathbf{C})$, uniquely determined up to conjugacy, such that $\mathbf{q} = \mathbf{q}_\theta$.

Let ∇ be a *formal connection*. There exists a uniquely determined (up to conjugacy) representation ⁽⁵⁶⁾ $\theta: \mathcal{T} \rightarrow \mathrm{GL}(n; \mathbf{C})$, such that $\mathbf{q}(V) = \mathbf{q}_\theta$.

Let $F_0(x) = x^L \cup e^{Q(1/u)}$, with $u^\nabla = x$, be a formal fundamental solution of a differential operator whose associate formal connection is ∇ [$\mathbf{q}(V)$ is then the set of the diagonal entries of $Q = \mathrm{Diag}(q_1, q_2, \dots, q_n)$]. Using F_0 we will identify the space V of horizontal sections of ∇ with $\mathbf{C}^n$.

Let $(\hat{\gamma}_0)$ be the free group generated by $\hat{\gamma}_0$. We define an action of the group $(\hat{\gamma}_0)$ on the lattice $\mathbf{E}$ by

$q \hat{\gamma}_0(u) = q(e^{-2i\pi/\nabla} u)$, and an action of the group $(\hat{\gamma}_0)$ on the exponential torus $\mathcal{T}$ by $\hat{\gamma}_0 \tau(q) = \tau(q \hat{\gamma}_0)$, for arbitrary $\tau \in \mathcal{T}$ and $q \in \mathbf{E}$.

⁽⁵⁶⁾ In the following all the representations are supposed to be continuous.

By definition the wild formal fundamental group $\pi_{1, sf}((\mathbf{C}^*, 0); "R^+")$ of $(\mathbf{C}^*, 0)$ pointed at " R^+ " is the semi-direct product

$$(\hat{\gamma}_0) \ltimes \mathcal{T} \text{ built from the action of } (\hat{\gamma}_0) \text{ on } \mathcal{T}.$$

Let $\hat{M} = U^{-1} e^{2i\pi L} U$ be the formal monodromy matrix associated to F_0 . We set

$$\begin{aligned} \hat{\rho}(\hat{V})(\hat{\gamma}_0) &= \hat{M}, \quad \text{and, for } \tau \in \mathcal{T}, \\ \hat{\rho}(\hat{V})(\tau) &= \text{Diag}(q_1(\tau), q_2(\tau), \dots, q_n(\tau)). \end{aligned}$$

We have

$$\begin{aligned} \hat{M}^{-1} Q(1/u) \hat{M} &= Q(e^{-2i\pi/v}/u) \\ \hat{M}^{-1} \text{Diag}(q_1, q_2, \dots, q_n) \hat{M} &= (q_1 \hat{\gamma}_0, q_2 \hat{\gamma}_0, \dots, q_n \hat{\gamma}_0) \\ \hat{M}^{-1} \text{Diag}(q_1(\tau), q_2(\tau), \dots, q_n(\tau)) \hat{M} &= (q_1 \hat{\gamma}_0(\tau), q_2 \hat{\gamma}_0(\tau), \dots, q_n \hat{\gamma}_0(\tau)) \\ \hat{\rho}(\hat{V})(\hat{\gamma}_0)^{-1} \hat{\rho}(\hat{V})(\tau) \hat{\rho}(\hat{V})(\hat{\gamma}_0) &= \hat{M}^{-1} \hat{\rho}(\hat{V})(\tau) \hat{M} = \hat{\rho}(\hat{V})(\hat{\gamma}_0 \tau). \end{aligned}$$

So we have defined a **linear representation**

$$\hat{\rho}(\hat{V}): \pi_{1, sf}((\mathbf{C}^*, 0); "R^+") = (\hat{\gamma}_0) \ltimes \mathcal{T} \rightarrow \text{GL}(n; \mathbf{C}),$$

associated to the **formal connection** $\hat{V}$. (Interpreted as a representation in $\text{GL}(\hat{V})$, where $\hat{V}$ is the vector space $\text{Sol}_{R^+}(\hat{V})$ of horizontal sections of $\hat{V}$, this representation is independant of the choice of F_0 .)

We will see now that, given a finite dimensional vector space $\hat{V}$ and a linear representation

$$\rho_1: \pi_{1, sf}((\mathbf{C}^*, 0); "R^+") \rightarrow \text{GL}(\hat{V}),$$

there exists a **unique formal connection** $\hat{V}$, such that $\rho_1 = \hat{\rho}(\hat{V})$ ($\hat{V}$ being identified with the vector space $\text{Sol}_{R^+}(\hat{V})$ of horizontal sections of $\hat{V}$).

We set $\rho_1(\hat{\gamma}_0) = \hat{M}$ and $\rho_1(\mathcal{T}) = T_1$. We set $\mathbf{q} = \mathbf{q}_0$; θ being the restriction of ρ_1 to $\mathcal{T}$, $\mathbf{q}$ is Galois invariant (it is invariant by the action of $\hat{M}$). We can choose a basis of V in such a way that T_1 is a diagonal group: $T_1 = \{ Q(\tau) = \text{Diag}(q_1(\tau), q_2(\tau), \dots, q_n(\tau)) / \tau \in \mathcal{T} \}$ ($\mathbf{q} = \{ q_1, q_2, \dots, q_n \}$, and $Q = \text{Diag}(q_1, q_2, \dots, q_n)$). Using a method of [BJL], [J], we can suppose moreover that we have chosen our basis such that $U \hat{M} U^{-1}$ is in Jordan form. Then let L be such that $e^{2i\pi L} = U \hat{M} U^{-1}$ (L is defined up to multiplication on the right by a diagonal matrix $\text{Diag}(m_1, m_2, \dots, m_n)$, $m_i \in \mathbf{Z}$). Then $F_0 = x^L U e^Q$ is a fundamental solution of a rational differential operator Δ_0 and the corresponding connection V_0 is *independant of the choice of the basis and of the integers m_i* . We have clearly $\rho(V_0) = \rho_1$ ($\mathbf{C}^n$ being identified by F_0 with the space of horizontal sections of V_0).

So we get

THEOREM 15. — *The natural map*

$$\text{Formal meromorphic connections} \xrightarrow{\hat{\rho}} \begin{array}{l} \text{finite dimensional} \\ \text{linear representations} \\ \text{of the group } \pi_{1, sf}((\mathbf{C}^*, 0); "R^+"). \end{array}$$

$$\hat{\nabla} \rightarrow \hat{\rho}(\hat{\nabla})$$

is an isomorphism.

This isomorphism is compatible with *sums, duality, tensor products, ...* It is an *isomorphism of Tannakian categories*.

If now ∇ is a germ of **meromorphic** connection, we get from ∇ *two* linear representations ($V = \text{Sol}_{\mathbf{R}^{+,\infty}}(\nabla)$) :

$$\hat{\rho}(\nabla): \pi_{1, sf}((\mathbf{C}^*, 0); "R^{+}") \rightarrow \text{GL}(V),$$

and

$$\rho_{\text{irr}}(\nabla): G\Pi(\mathbf{q}) \rightarrow \text{GL}(V).$$

The respective restrictions of these representations $\hat{\rho}(\nabla)$ and $\rho_{\text{irr}}(\nabla)$ to the respective subgroups $(\hat{\gamma}_0)$ of $\pi_{1, sf}((\mathbf{C}^*, 0); "R^{+}")$ and $G\Pi(\mathbf{q})$ are clearly equal.

Conversely, two linear representations

$$\rho_1: \pi_{1, sf}((\mathbf{C}^*, 0); "R^{+}") \rightarrow \text{GL}(V),$$

and

$$\rho_2: G\Pi(\mathbf{q}) \rightarrow \text{GL}(V),$$

admitting *equal restrictions* to the subgroups

$$(\hat{\gamma}_0) \subset \pi_{1, sf}((\mathbf{C}^*, 0); "R^{+}") \quad \text{and} \quad (\hat{\gamma}_0) \subset G\Pi(\mathbf{q}),$$

being given, it is general impossible to find a germ of meromorphic connection ∇ such that $\hat{\rho}(\nabla) = \rho_1$ and $\rho_{\text{irr}}(\nabla) = \rho_2$: ρ_1 and ρ_2 must satisfy a "Stokes condition" [checked on $\text{GL}(V)$ in place of $\text{GL}(n; \mathbf{C})$; cf. theorem 10]; $V = \text{Sol}_{\mathbf{R}^{+,\infty}}(\nabla)$.

PROPOSITION 13. — Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. Let ∇_0 be the connection defined by Δ_0 .

Then the natural map

$$\begin{array}{l} \text{Germes of meromorphic} \\ \text{connections } \nabla \\ \text{formally equivalent to } \nabla_0. \end{array} \rightarrow \begin{array}{l} \text{Pairs of representations of the groups} \\ \pi_{1, sf}((\mathbf{C}^*, 0); "R^{+}") \text{ and } G\Pi(\mathbf{q}) \text{ in} \\ \text{GL}(V) \text{ coincident on the two} \\ \text{subgroups corresponding to } (\hat{\gamma}_0), \text{ such} \\ \text{that the first representation} \\ \text{corresponds to } \rho(\nabla_0), \\ \text{and satisfying "Stokes conditions".} \end{array}$$

$$\nabla \rightarrow (\rho(\nabla), \rho_{\text{irr}}(\nabla)),$$

is a **bijection** ($V = \text{Sol}_{\mathbf{R}^{+,\infty}}(\nabla)$).

The next step is now to build a new group $\pi_{1, s}((\mathbf{C}^*, 0); "R^{+}")$, the wild fundamental group of $(\mathbf{C}^*, 0)$, pointed at " R^{+} ", satisfying the following

properties:

(i) The wild fundamental group is a semi-direct product

$$\pi_{1,s}((C^*, 0); "R^+") = \pi_{1,sf}((C^*, 0); "R^+") \ltimes \mathcal{R}$$

$$\pi_{1,s}((C^*, 0); "R^+") = ((\hat{\gamma}_0) \ltimes \mathcal{T}) \ltimes \mathcal{R},$$

where $\mathcal{R}$ (the *resurgent group*) is the "exponential" of a free Lie algebra Lie $\mathcal{R}$ (the *resurgent Lie algebra*), with infinitely many generators.

(ii) To each germ ∇ of rank n meromorphic connection we can associate a linear representation ($V = \text{Sol}_{\cdot R^+}(\nabla)$):

$$\rho_s(\nabla): \pi_{1,s}((C^*, 0); "R^+") \rightarrow \text{GL}(V),$$

such that the restriction of $\rho_s(\nabla)$ to $\pi_{1,sf}((C^*, 0); "R^+")$ is $\hat{\rho}(\nabla)$, and such that, $\hat{\rho}(\nabla)$ being known, the knowledge of the restriction of $\rho_s(\nabla)$ to the resurgent group $\mathcal{R}$ is equivalent to the knowledge of the representation

$$\rho_{\text{irr}}(\nabla): G \Pi(\mathbf{q}) \rightarrow \text{GL}(V) (\mathbf{q} = \mathbf{q}(\nabla)).$$

(iii) If a finite dimensional representation $(^{57})$ of the wild fundamental group

$$\rho_0: \pi_{1,s}((C^*, 0); "R^+") \rightarrow \text{GL}(V), \text{ is given}$$

we denote by ρ_1 the restriction of ρ_0 to $\pi_{1,sf}((C^*, 0); "R^+")$, and

$$\rho_2: G \Pi(\mathbf{q}) \rightarrow \text{GL}(V)$$

the representation corresponding to the restriction of ρ_0 to the *resurgent group* $\mathcal{R}$ (and to the knowledge of $\rho_1, \dots$), with $\mathbf{q} = \mathbf{q}_{\rho_0}$. Then the pair (ρ_1, ρ_2) satisfies "Stokes conditions", so there exists (Proposition 13) a uniquely determined germ of meromorphic connection ∇ such that

$$(\rho(\nabla), \rho_{\text{irr}}(\nabla)) = (\rho_1, \rho_2),$$

and $(\rho(\nabla), \rho_{\text{irr}}(\nabla))$ can be recovered from the representation

$$\rho_s(\nabla): \pi_{1,s}((C^*, 0); "R^+") \rightarrow \text{GL}(V)$$

got from ∇ using the construction of (ii) ($V = \text{Sol}_{\cdot R^+}(\nabla)$).

Let $\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E}$, and, after ordering, let Q denote the diagonal matrix $\text{Diag}\{q_1, q_2, \dots, q_n\}$. Let $\mathcal{T}(\mathbf{q})$ be the exponential torus associated to $\mathbf{q}$, and let $T(Q)$ be its representation in $\text{GL}(n; \mathbf{C})$ given by Q .

An element $\tau \in \mathcal{T}(Q)$ is represented by the matrix

$$Q(\tau) = \text{Diag}(q_1(\tau), q_2(\tau), \dots, q_n(\tau)) \in T(Q) \subset \text{GL}(n; \mathbf{C}).$$

LEMMA 17. — Let $\mathbf{q} = \{q_1, q_2, \dots, q_n\} \subset \mathbf{E}$, and, after ordering, let Q be the diagonal matrix $Q = \text{Diag}(q_1, q_2, \dots, q_n)$. Let $C \in \text{End}(n; \mathbf{C})$,

⁽⁵⁷⁾ The restriction to $\mathcal{T}$ of such a representation will be *allway* supposed continuous in the following.

$C = (c_{ij}) (q_{ij} = q_i - q_j)$. Then:

(i) $\tau C \tau^{-1} = Q(\tau) C Q(\tau)^{-1} = (c_{ij} q_{ij}(\tau))$.

(ii) Let $q \in \mathbf{E}$, $q \neq 0$ and

$$C_q = (a_{ij}), \quad \text{with } a_{ij} = 0 \text{ if } q_i - q_j \neq q, \text{ and } a_{ij} = c_{ij} \text{ if } q_{ij} = q.$$

Then:

$$\tau C_q \tau^{-1} = Q(\tau) C_q Q(\tau)^{-1} = q(\tau) C_q.$$

(iii) Let $\text{Dia}(C)$ be the diagonal matrix with the same diagonal entries as C . Then

$$\tau C \tau^{-1} = \text{Dia}(C) + \sum_{i,j} q_{i,j}(\tau) C_{q_{i,j}} \quad (\text{with } C_q = 0 \text{ if } q = 0),$$

and such a decomposition is **uniquely** determined, i. e. if

$$\tau C \tau^{-1} = \text{Dia}(C) + \sum_{i,j} q_{i,j}(\tau) A_{q_{i,j}}, \quad \text{then } A_{q_{i,j}} = C_{q_{i,j}}.$$

(iv) Let $\mathbf{d} \in (\mathbf{R}, 0)$. If $C \in \text{Sto}(Q; \mathbf{d})$, then:

$$\tau C \tau^{-1} = I + \sum_q q(\tau) C_q, \text{ the sum being extended to } q = q_{i,j},$$

with $q_i \ll_{\mathbf{d}} q_j$,

$$\tau C \tau^{-1} = I + \sum_{q \in \mathbf{E}_{\mathbf{d}}(q)} q(\tau) C_q.$$

(v) Let $\mathbf{d} \in (\mathbf{R}, 0)$. If $C \in \text{Lie Sto}(Q; \mathbf{d})$, the Lie algebra of $\text{Sto}(Q; \mathbf{d})$, then:

$$\tau C \tau^{-1} = \sum q(\tau) C_q,$$

the sum being extended to $q = q_{i,j}$, with $q_i \ll_{\mathbf{d}} q_j$,

$$\tau C \tau^{-1} = \sum_{q \in \mathbf{E}_{\mathbf{d}}(q)} q(\tau) C_q.$$

The only non trivial point is *unicity* in (iii).

Let $(p_1, p_2, \dots, p_v)$ be a $\mathbf{Z}$ -basis of the lattice $\mathbf{E}(\mathbf{q})$.

We have an isomorphism

$$(p_1, p_2, \dots, p_v): \mathcal{T}(\mathbf{q}) \rightarrow (\mathbf{C}^*)^v$$

$$(p_1, p_2, \dots, p_v): \tau \rightarrow (p_1(\tau), p_2(\tau), \dots, p_v(\tau)).$$

We set $p_k(\tau) = \tau_k (k=1, \dots, v)$. Then each $q_{i,j}(\tau)$ is a *monomial* in the variables $\tau_k \in \mathbf{C}^*$ and the *distinct* $q_{i,j}(\tau)$ are *independent* on $\mathbf{C}$.

The decomposition (iii) appears as a “**Fourier decomposition**” of the “*unfolding*” $\tau C \tau^{-1}$ of the matrix C by the *adjoint action* of the *exponential torus* $\mathcal{T}(\mathbf{q})$.

Let $\Delta = d/dx - A$, where $A \in \text{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

Let $\hat{F}(x) = \hat{H}(x) x^L U e^{Q(1/u)}$ be a formal fundamental solution of Δ as above. We set

$F_0(x) = x^L U e^Q$, $q = q(Q)$, and we denote by n the rank of Δ .

Let $d \in \text{Fr}(q)$ and let $\text{St}_d(\Delta)$ be the corresponding Stokes matrix. For every $\tau \in \mathcal{T}$ the matrix $\tau \text{St}_d(\Delta) \tau^{-1}$ belongs to the image of the representation of $\text{Gal}_K(\Delta)$ in $\text{GL}(n; \mathbb{C})$ associated to $\hat{F}$, the matrix $\text{St}_d(\Delta)$ is *unipotent* and $\tau (\text{Log St}_d(\Delta)) \tau^{-1}$ belongs to the representation of $\text{Lie Gal}_K(\Delta)$ in $\text{End}(n; \mathbb{C})$ associated to $\hat{F}$, that is, it yields a *Galois derivation* of the field $K \langle \hat{F} \rangle$. Then it follows from Lemma 17 ($\text{St}_d(\Delta) \in \text{Sto}(Q; d)$) that we have a uniquely determined decomposition

$$\tau (\text{Log St}_d(\Delta)) \tau^{-1} = \sum q(\tau) \text{Log St}_d(\Delta)_q,$$

the sum being extended to $q = q_{i,j}$, with $q_i \leq_d q_j$, or

$$\tau (\text{Log St}_d(\Delta)) \tau^{-1} = \sum_{q \in E_d(q)} q(\tau) \text{Log St}_d(\Delta)_q,$$

where each $\text{Log St}_d(\Delta)_q$ belongs to the representation of $\text{Lie Gal}_K(\Delta)$ in $\text{End}(n; \mathbb{C})$ associated to $\hat{F}$, that is yields a *Galois derivation* of the field $K \langle \hat{F} \rangle$. We have performed a “Fourier analysis of the infinitesimal Stokes phenomena”.

THEOREM 16. — Let $\Delta = d/dx - A$, where $A \in \text{End}(n; \mathbb{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbb{C}$. We set $q = q(\Delta)$, and denote by n the rank of Δ . Then $\tau (\text{Log St}_d(\Delta)) \tau^{-1}$ belongs to $\text{Lie Gal}_K(\Delta)$ for each $d \in \text{Fr}(q)$, and we have a uniquely determined decomposition

$$\tau (\text{Log St}_d(\Delta)) \tau^{-1} = \sum q(\tau) \text{Log St}_d(\Delta)_q,$$

the sum being extended to $q = q_{i,j}$, with $q_i \leq_d q_j$, or

$$\tau (\text{Log St}_d(\Delta)) \tau^{-1} = \sum_{q \in E_d(q)} q(\tau) \text{Log St}_d(\Delta)_q,$$

with each $\text{Log St}_d(\Delta)_q$ belonging to $\text{Lie Gal}_K(\Delta)$.

Moreover

$$\tau (\text{Log St}_d(\Delta)_q) \tau^{-1} = q(\tau) \text{Log St}_d(\Delta)_q,$$

and

$$\hat{M} \text{St}_d(\Delta)_q \hat{M}^{-1} = \text{St}_{\exp(-2i\pi)_d}(\Delta)_q, \text{ for every } q \in E.$$

It is now natural to introduce the *free complex Lie algebra* $\text{Lie } \mathcal{R}$ generated by all “letters” $\Delta_{q,d}$ where (q, d) is such that $q \in E$ and $d \in \text{Fr } q$ (i.e. such that e^q is “maximally decaying” on d). We will name it the *resurgent Lie algebra* ⁽⁵⁸⁾.

⁽⁵⁸⁾ Because it contains all Ecalle’s resurgent algebras.

In the situation of *theorem 16* we get a *linear representation*

$$\begin{aligned} \text{Lie } \rho_{\text{res}}(\Delta): \text{ Lie } \mathcal{R} &\rightarrow \text{End}(n; \mathbb{C}) \\ \text{Lie } \rho_{\text{res}}(\Delta): \dot{\Delta}_{q, \mathbf{d}} &\rightarrow \text{Log St}_{\mathbf{d}}(\Delta)_q \quad \text{if } \mathbf{d} \in \text{Fr}(\mathbf{q}), \end{aligned}$$

and

$$\text{Lie } \rho_{\text{res}}(\Delta): \dot{\Delta}_{q, \mathbf{d}} \rightarrow 0, \quad \text{if } \mathbf{d} \notin \text{Fr}(\mathbf{q}).$$

We define an action of the *wild formal fundamental group*

$$\pi_{1, sf}(\mathbf{C}^*, 0); \text{ "R}^+ \text{") } = (\hat{\gamma}_0) \ltimes \mathcal{T}$$

on the *resurgent Lie algebra* *Lie* $\mathcal{R}$ by

$$\hat{\gamma}_0 \dot{\Delta}_{q, \mathbf{d}} \hat{\gamma}_0^{-1} = \dot{\Delta}_{q, \exp(-2i\pi)\mathbf{d}},$$

and

$$\tau \dot{\Delta}_{q, \mathbf{d}} \tau^{-1} = \tilde{q}(\tau) \dot{\Delta}_{q, \mathbf{d}}.$$

If we denote by $\hat{\rho}(\Delta)$ the representation

$$\hat{\rho}(\Delta): \pi_{1, sf}((\mathbf{C}^*, 0); \text{ "R}^+ \text{") } \rightarrow \text{GL}(n; \mathbb{C})$$

associated to the *formal connection* defined by the differential operator Δ , the above action is "*compatible*" with the pair of representations $(\hat{\rho}(\Delta), \text{Lie } \rho_{\text{res}}(\Delta))$ (*theorem 16*).

PROPOSITION 14. — *The natural map*

<i>Pairs of representations</i> $(\rho_1, L\rho)$ <i>of the group</i> $\pi_{1, sf}((\mathbf{C}^*, 0); \text{ "R}^+ \text{") }$ <i>in</i> $\text{GL}(n; \mathbb{C})$ <i>and of the Lie algebra</i> <i>Lie</i> $\mathcal{R}$ <i>in</i> $\text{End}(n; \mathbb{C})$ <i>which are</i> <i>"compatible" with the action</i> <i>of</i> $\pi_{1, sf}((\mathbf{C}, 0); \text{ "R}^+ \text{") }$ <i>on</i> <i>Lie</i> $\mathcal{R}$.	$\rightarrow$	<i>Pairs of representations of the groups</i> $\pi_{1, sf}((\mathbf{C}^*, 0); \text{ "R}^+ \text{") }$ <i>and</i> $\text{G}\Pi(\mathbf{q}_{p_1})$ <i>in</i> $\text{GL}(n; \mathbb{C})$ <i>which coincide on the two</i> <i>subgroups corresponding to</i> $(\hat{\gamma}_0)$ <i>and satisfy</i> Stokes conditions.
$(\rho_1, L\rho) \rightarrow (\rho_1, \rho_2)$		

where

$$\text{Log } \rho_2(\gamma_{\mathbf{d}}) = \sum q(\tau) L\rho(\dot{\Delta}_{q, \mathbf{d}}) \quad \text{for every } \mathbf{d} \in \text{Fr}(\mathbf{q}_{p_1}),$$

is a *bijection*.

From *Projections 13 and 14*, we get a *first version* of the “*wild Riemann Hilbert correspondence*”:

THEOREM 17. — *The natural map (where V is a finite dimensional space: $V = \text{Sol}_{\mathbf{R}^+}(\nabla)$)*

$$\begin{array}{ccc} \text{Pairs of representations of} & & \\ \text{the group } \pi_{1, sf}((\mathbf{C}^*, 0); \mathbf{R}^+) & & \\ \text{in } \text{GL}(V) & & \\ \text{Germes of meromorphic} & \rightarrow & \text{and of the Lie algebra Lie } \mathcal{R} \\ \text{connections} & & \text{in } \text{End}(V) \text{ which are “compatible”} \\ \nabla \text{ at the origin.} & & \text{with the action of } \pi_{1, sf}((\mathbf{C}^*, 0); \mathbf{R}^+) \\ & & \text{on Lie } \mathcal{R}. \\ \nabla \rightarrow (\hat{\rho}(\nabla), \text{Lie } \rho_{\text{res}}(\Delta)) \end{array}$$

is a *bijection*.

In order to get the wanted result, that is the *classification of germs of meromorphic connections in terms of representations of a group*, it only remains to replace the *resurgent Lie algebra Lie $\mathcal{R}$* by a *group*, the *resurgent group $\mathcal{R}$* (the “*exponential*” of *Lie $\mathcal{R}$*), and the *action of the wild formal fundamental group $\pi_{1, sf}((\mathbf{C}^*, 0); \mathbf{R}^+)$ on the Lie algebra Lie $\mathcal{R}$* by an *action of the same group on the group $\mathcal{R}$* . Then we will get a pair of representations $(\hat{\rho}(\nabla), \rho_{\text{res}}(\Delta))$ in $\text{GL}(n; \mathbf{C}) = \text{GL}(V)$ of the groups $\pi_{1, sf}((\mathbf{C}^*, 0); \mathbf{R}^+)$ and $\mathcal{R}$ respectively, *compatible with the action of the first group on the second, that is a representation of the semidirect product (defined by the same action)*

$$\pi_{1, sf}((\mathbf{C}^*, 0); \mathbf{R}^+) \ltimes \mathcal{R} \text{ in } \text{GL}(n; \mathbf{C}).$$

Let X be a *set*. We denote $[S]$ (LA 4.10) by L_X the *free complex Lie algebra* on X , by $\hat{L}_X$ its *completion*, by Ass_X the *complex associative algebra* on X , by $\hat{\text{Ass}}_X$ its *completion*, by $\hat{\mathcal{M}}_X$ the *ideal generated in $\hat{\text{Ass}}_X$ by X* , by $\Delta: \hat{\text{Ass}}_X \rightarrow \hat{\text{Ass}}_X \otimes \hat{\text{Ass}}_X$ the *diagonal map*, and by $\hat{G}_X$ the *set of $\beta \in I + \hat{\mathcal{M}}_X$ with $\Delta\beta = \beta \otimes \beta$* .

There is a *natural isomorphism*

$$\exp: \hat{\mathcal{M}}_X \rightarrow I + \hat{\mathcal{M}}_X.$$

We can *identify $\hat{L}_X$ with the set of primitive elements of $\hat{\text{Ass}}_X$* . Then by *restriction of the exponential* we get an *isomorphism*

$$\exp: \hat{L}_X \rightarrow \hat{G}_X.$$

By the *Campbell-Hausdorff formula* we get a *group structure on $\hat{G}_X$* .

If X is the *set of “labels” $\hat{\Delta}_{q, \mathbf{d}}$* , where $(q, \mathbf{d})$ is such that $q \in \mathbf{E}$ and $\mathbf{d} \in \text{Fr } q$, we write

$$\begin{array}{ll} \text{Lie } \mathcal{R} = L_X, & \mathcal{U}\mathcal{R} = \text{Ass}_X, \\ \mathcal{U}\hat{\mathcal{R}} = \hat{\text{Ass}}_X, & \mathcal{M}\hat{\mathcal{R}} = \hat{\mathcal{M}}_X, \hat{\mathcal{R}} = \hat{G}_X. \end{array}$$

We get isomorphisms

$$\begin{aligned}\exp: \mathcal{M}\hat{\mathcal{R}} &\rightarrow I + \mathcal{M}\hat{\mathcal{R}} \\ \exp: \operatorname{Lie}\hat{\mathcal{R}} &\rightarrow \hat{\mathcal{R}}.\end{aligned}$$

We denote by $\mathcal{R}$ the subgroup of $\hat{\mathcal{R}}$ generated by the image of $\operatorname{Lie}\hat{\mathcal{R}}$ by $\exp$; by definition $\mathcal{R}$ is the **resurgent group**.

LEMMA 16. — *We consider the action of the wild formal fundamental group $\pi_{1, sf}((\mathbf{C}^*, 0); "R^+")$ on the free Lie algebra $\operatorname{Lie}\hat{\mathcal{R}}$ defined by*

$$\begin{aligned}\hat{\gamma}_0 \Delta_{q, \mathbf{d}} \hat{\gamma}_0^{-1} &= \Delta_{q, \exp(-2i\pi)\mathbf{d}} \\ \tau \Delta_{q, \mathbf{d}} \tau^{-1} &= q(\tau) \Delta_{q, \mathbf{d}}.\end{aligned}$$

This action can be naturally extended to $\mathcal{U}\hat{\mathcal{R}}$ and we get (by restriction) an action on $\hat{\mathcal{R}}$, leaving $\mathcal{R}$ invariant, such that

$$\begin{aligned}\hat{\gamma}_0 \exp(\Delta_{q, \mathbf{d}}) \hat{\gamma}_0^{-1} &= \exp(\Delta_{q, \exp(-2i\pi)\mathbf{d}}) \\ \tau \exp(\Delta_{q, \mathbf{d}}) \tau^{-1} &= \exp(q(\tau) \Delta_{q, \mathbf{d}}).\end{aligned}$$

The **wild fundamental group** of the germ of $\mathbf{C}^*$ at the origin, pointed at " $\mathbf{R}^+$ ", is by definition the **semi-direct product**

$$\begin{aligned}\pi_{1, s}((\mathbf{C}^*, 0); "R^+") &= \pi_{1, sf}((\mathbf{C}^*, 0); "R^+") \ltimes \mathcal{R} \\ \pi_{1, s}((\mathbf{C}^*, 0); "R^+") &= ((\hat{\gamma}_0) \ltimes \mathcal{T}) \ltimes \mathcal{R}\end{aligned}$$

defined by the **action** of $\pi_{1, sf}((\mathbf{C}^*, 0); "R^+")$ on $\mathcal{R}$ introduced in lemma 16.

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be $\mathbf{Z}$ -independent elements of $\operatorname{Lie}\hat{\mathcal{R}}$. Then the subgroup of $\hat{\mathcal{R}}$ generated by $\exp \alpha_1, \exp \alpha_2, \dots$, and $\exp \alpha_m$ is *isomorphic to the free group* generated by the m "letters" $\exp \alpha_1, \exp \alpha_2, \dots, \exp \alpha_m$. We get:

LEMMA 17. — *If $(\rho_1, L\rho_2)$ is a pair of representations of the group $\pi_{1, sf}((\mathbf{C}^*, 0); "R^+")$ in $\operatorname{GL}(n; \mathbf{C})$ and of the Lie algebra $\operatorname{Lie}\hat{\mathcal{R}}$ in $\operatorname{End}(n; \mathbf{C})$ "compatible" with the action of $\pi_{1, sf}((\mathbf{C}, 0); "R^+")$ on $\operatorname{Lie}\hat{\mathcal{R}}$, then there exists a unique representation*

$$\rho_2: \mathcal{R} \rightarrow \operatorname{GL}(n; \mathbf{C})$$

such that

$$\rho_2(\exp \alpha) = \exp L\rho_2(\alpha) \text{ for every } \alpha \in \operatorname{Lie}\hat{\mathcal{R}}.$$

This representation is compatible with the action of $\pi_{1, sf}((\mathbf{C}, 0); "R^+")$ on $\mathcal{R}$ defined in lemma 16.

We get the “*wild Riemann-Hilbert correspondence*”:

THEOREM 18. — *The natural map*

$$\begin{array}{ccc} \text{Germs of meromorphic} & & \text{Finite dimensional} \\ \text{connections } \nabla & \xrightarrow{\rho_s} & \text{linear representations} \\ \text{at the origin.} & & \text{of the} \\ & & \text{wild fundamental group} \\ & & \pi_{1,s}((C^*, 0); \mathbb{R}^+). \\ \nabla \rightarrow \rho_s(\nabla) \end{array}$$

is a *bijection*.

The *wild Riemann-Hilbert correspondence* in an equivalence of Tannakian categories.

Remarks. — 1. There are extensions of the *wild Riemann-Hilbert correspondence* to *non-linear situations* in relation with problems of *analytic classification* (germs of non linear analytic differential equations, germs of analytic diffeomorphisms, germs of analytic vector fields...) [MR 1], [E 3]. In these generalizations one gets statements which are similar to *theorem 17*. In the case of *differential equations* C^n is replaced by an analytic manifold, $\text{End}(n; C)$ by an analytic vector field and $\text{GL}(n; C)$ by the analytic pseudogroup of automorphisms of the manifold. *Theorem 18* takes a quite technical form...

2. In such situations *Ecalte* introduces “*hidden variables*” (“variables cachées”). We can easily describe [and extend ⁽⁶⁰⁾] his point of view using our technics:

Let ∇ be a germ of meromorphic connection and let $\rho_s(\nabla)$ be the corresponding representation by the *wild Riemann-Hilbert correspondence*. Let $X(\nabla)$ be the set of “labels” defined by

$$X(\nabla) = \{ \rho_s(\nabla)(\Delta_{q,d}) / q \in E \text{ and } d \in \text{Fr } q \}.$$

Then there are at most a finite number of values of (q, d) such that the matrix $\rho_s(\nabla)(\Delta_{q,d})$ is non zero. If this matrix is zero, we suppress the corresponding letter. It remains a finite subset $X'(\nabla)$. We write $\text{Ass}_{X'(\nabla)} = \mathcal{UR}(\nabla)$.

If f is a horizontal section of ∇ , then we set

$$X(\nabla; f) = \{ \rho_s(\nabla)(\Delta_{q,d})(f) / q \in E \text{ and } d \in \text{Fr } q \},$$

and denote by $X'(\nabla; f)$ the set of “labels” corresponding to $X'(\nabla)$. We write $\text{Ass}_{X'(\nabla; f)} = \mathcal{UR}(\nabla; f)$.

The idea is to interpret $\mathcal{UR}(\nabla; f)$ as a “*formal function*” on $\mathcal{UR}$ “*extending*” f . This “function” depends on *new (non commutative) variables*, the

⁽⁵⁹⁾ We recall that we suppose all the representations continuous on $\mathcal{T}$.

⁽⁶⁰⁾ Ecalte uses only particular “one-levelled” lattices.

“coordinates” of the elements of $\mathcal{U}\hat{\mathcal{R}}$. These “hidden variables” belongs to the dual of $\mathcal{U}\hat{\mathcal{R}}$. We will be more precise in part 6 below, and interpret $\mathcal{U}\hat{\mathcal{R}}(\mathbb{V}; f)$ as giving birth to a “formal function” on a **principal bundle with structure group $\hat{\mathcal{R}}$** , corresponding to an **actual function extending f** defined on a **principal bundle with structure group $\hat{\mathcal{R}}$** . Moreover there are *natural actions* of $\pi_{1,s}((\mathbb{C}^*, 0); “\mathbf{R}^+”)$ on all these objects.

3. The “Lie-algebra” $\text{Lie } \pi_{1,s}((\mathbb{C}^*, 0); “\mathbf{R}^+”)$ of the wild fundamental group is the **semi-direct product of Lie-algebras** ($\text{Lie } \pi_{1,s}((\mathbb{C}^*, 0); “\mathbf{R}^+”) = \text{Lie } \mathcal{T}$)

$$\text{Lie } \mathcal{T} \ltimes \text{Lie } \mathcal{R},$$

associated to the action of the commutative algebra (“Cartan algebra”) $\text{Lie } \mathcal{T}$ on the resurgent algebra $\text{Lie } \mathcal{R}$ defined by

$$[H, \hat{\Delta}_{q,d}] = q(H) \hat{\Delta}_{q,d},$$

for $H \in \text{Lie } \mathcal{T}$, where

$$q: \text{Lie } \mathcal{T} \rightarrow \mathbb{C}$$

is the infinitesimal map associated to

$$q: \mathcal{T} \rightarrow \mathbb{C}^*.$$

From the wild monodromy representation ρ_s we get a representation

$$\text{Lie } \rho_s: \text{Lie } \pi_{1,s}((\mathbb{C}, 0); “\mathbf{R}^+”) \rightarrow \text{End}(n; \mathbb{C}).$$

The restriction of this representation to $\text{Lie } \mathcal{R}$ is the map $\text{Lie } \rho_{\text{res}}$ of theorem 17. It corresponds to Ecalle’s “bridge equation” (“equation du pont”).

We will explain now how to change the “base point” “ $\mathbf{R}$ ” of the wild fundamental group $\pi_{1,s}((\mathbb{C}^*, 0); “\mathbf{R}^+”)$.

We will replace “ $\mathbf{R}^+$ ” by

$$“d” \in \{“0”\} \times S^1 (“0”, d) = “d”) \quad \text{or} \quad d \in \{“+\infty”\} \times S^1$$

(that we can identify with S^1 , the real analytic blow up of the origin in $\mathbb{C}$).

We fix “ d ” $\in \{“0”\} \times S^1$. Let “ c ” be an homotopy class of continuous paths on $\{“0”\} \times S^1$ with origin “ d ” and extremity “ $\mathbf{R}$ ” (corresponding to an homotopy class of paths c on S^1). We set

$$\pi_{1,s}((\mathbb{C}^*, 0); “d”) = \{“c” b “c”^{-1} / b \in \pi_{1,s}((\mathbb{C}^*, 0); “\mathbf{R}^+”) \},$$

and put on this set the evident structure of group; $\pi_{1,s}((\mathbb{C}^*, 0); “d”) is independent of the choice of c in a sense that we leave to the reader to explicit.$

Let now $d \in \{“+\infty”\} \times S^1$. We set

$$\pi_{1,s}((\mathbb{C}^*, 0); d) = \{(\gamma_d^-)^{-1} b \gamma_d^- / b \in \pi_{1,s}((\mathbb{C}^*, 0); “d”) \},$$

where the symbol γ_d^- corresponds to the *multisummation operator* S_d^- in “the” direction d^- (S_d^- is interpreted as an analytic continuation along γ_d^-). We put on $\pi_{1,s}((C^*, 0); d)$ the evident structure of group.

We can also set

$$\pi_{1,s}((C^*, 0); d) = \{ (\gamma_d^+)^{-1} b \gamma_d^+ / b \in \pi_{1,s}((C^*, 0); “d”) \};$$

there is a natural isomorphism between the two groups on the right side of our equalities.

We can now replace $\pi_{1,s}((C^*, 0); “R^+”)$ by $\pi_{1,s}((C^*, 0); “d”)$ or $\pi_{1,s}((C^*, 0); d)$ in *theorem 18* (by definition $\rho_s(V)(“c”)$ is the analytic isomorphism of solution spaces given by the analytic continuation of a fundamental solution F_0 of “the” *formal normal form* corresponding to V along c , $\rho_s(V)(\gamma_d^-)$ is the isomorphism of solution spaces given by S_d^-). Elements of $\pi_{1,s}((C^*, 0); d)$ are represented by linear permutations of *actual solutions* in a germ of sector bisected by d .

It is possible now to give a *global version* of our *wild fundamental group*.

Let X be a connected Riemann surface. Let $S = \{a_1, a_2, \dots, a_m\}$ be a finite subset of X , let x_0 be a base point in $X - S$, and, for each $i = 1, \dots, m$, let d_i be a fixed direction “starting from a_i ”. We choose homotopy classes of paths c_i (“in” $X - S$), with *origin* x_0 and *extremity* a_i , “arriving at a_i along the direction d_i ” ($i = 1, \dots, m$). We built, like above, groups

$$G_i = \{ c_i b c_i^{-1} / b \in \pi_{1,s}((C^*, 0); d_i) \}, \quad i = 1, \dots, m$$

(these groups are *independent* of the choice of c_i in a sense that we leave to the reader to explicit).

By *definition* ⁽⁶¹⁾ the *wild fundamental group* “of” $X - S$, *pointed at* x_0 , is

$$\pi_{1,s}(X - S, S; x_0) = G_1 * \dots * G_m \quad (\text{free product of groups}),$$

and the *wild fundamental group of* X is

$$\pi_{1,s}(X - \dots; \cdot) = \varprojlim_S \pi_{1,s}(X - S; S; \cdot).$$

(There are some trouble with marked point in the limit: we get rid of it as in the classical case...)

It is easy to prove the following results [we define $\rho_s(V)(c_i)$ as the analytic isomorphism of solutions spaces given by the analytic continuation

⁽⁶¹⁾ Be careful: the groups *depends on* X *and* S , not only on $X - S$.

along c_i]:

We have a **wild global Riemann-Hilbert correspondence**:

THEOREM 19. — *Let X be a connected Riemann surface. The natural map*

$$\begin{array}{ccc} \text{Meromorphic connections} & \xrightarrow{\rho_s} & \text{finite dimensional} \\ \text{on } X. & & \text{linear representations } (^{62}) \text{ of the} \\ & & \text{wild fundamental group} \\ & & \pi_{1,s}(X; \cdot). \\ \nabla \rightarrow \rho_s(\nabla) \end{array}$$

is a **bijection**.

The **wild global Riemann-Hilbert correspondence** is an equivalence of Tannakian categories.

We will call the map $\rho_s(\nabla)$ the **wild monodromy representation** of the connection ∇ .

Let $\rho_m(\nabla)$ be the (classical) **monodromy representation** of the connection ∇ (local or global case). It is possible to get $(^{63})$ the **actual monodromy representation** $\rho_m(\nabla)$ from the **wild monodromy representation** $\rho_s(\nabla)$. If X is a connected Riemann surface, we will write

$$\pi_1(X - \dots; \cdot) = \varprojlim_s \pi_1(X - S; \cdot) \quad (S \text{ finite subset of } X).$$

PROPOSITION 15. — (i) *Let $d \in S^1$ be a fixed direction. There exists a “natural” functor $\mathcal{D}$ from the tensor category of finite dimensional linear representations of $\pi_{1,s}((\mathbb{C}^*, 0); d)$ to the tensor category of finite dimensional linear representations of $\pi_1((\mathbb{C}^*, 0); d)$ such that*

$$\mathcal{D}(\rho_s(\nabla)) = \rho_m(\nabla)$$

for every germ of meromorphic connection ∇ at the origin.

This functor is defined by

$$\mathcal{D}(\rho) = \rho_1(\gamma_{d_1}) \cdots \rho_1(\gamma_{d_p}) \rho_1,$$

where (ρ_1, ρ_2) is the pair of representations in $\text{GL}(n; \mathbb{C})$ respectively of $\pi_{1,sf}((\mathbb{C}^*, 0); d)$ and $G\Pi(\mathbf{q}_{p_1})$ (pointed at d) associated to ρ ($\mathbf{q} = \mathbf{q}_{p_1}$, and $d_1, \dots, d_p$ are the directions of $\text{Fr}(\mathbf{q})$ contained in the interval $[0, 2\pi[\subset (\mathbb{R}, 0)$, **ordered** by the ordering relation induced by $\mathbb{R}$).

(ii) *Let X be a connected Riemann surface. There exists a “natural” functor $\mathcal{D}$ from the tensor category of finite dimensional linear representations of $\pi_{1,s}(X - \dots; \cdot)$ to the tensor category of finite dimensional linear*

$(^{62})$ We recall that we suppose all the representations *continuous* on $\mathcal{T}$.

$(^{63})$ In some sense π_1 is contained in a “completion” of $\pi_{1,s}$ and ρ_s can be extended to this completion “by continuity”. Then ρ_m is the restriction to π_1 of this extension.

representations of $\pi_1(\mathbf{X} - \dots; \cdot)$, such that

$$\mathcal{D}(\rho_s(\nabla)) = \rho_m(\nabla),$$

for every meromorphic connection ∇ .

We can reformulate *theorem 6* in a more “geometric form” (and extend it to the global case), replacing the actual monodromy representation by the wild monodromy representation in Schlesinger’s theorem.

THEOREM 20. — Let $K = \mathbb{C}\{x\}[x^{-1}]$. Let ∇ be a germ of meromorphic connection at the origin. We fix a $\mathbb{C}$ -basis of the space of horizontal sections on a germ of sector bisected by a given direction d and identify the Galois differential group $\text{Gal}_K(\nabla)$ with its corresponding representation in $\text{GL}(n; \mathbb{C})$.

Then $\text{Gal}_K(\nabla)$ is the Zariski closure of the image in $\text{GL}(n; \mathbb{C})$ of the wild monodromy representation

$$\rho_s(\nabla): \pi_{1,s}((\mathbb{C}^*, 0); d) \rightarrow \text{GL}(n; \mathbb{C}).$$

THEOREM 21. — Let $\mathbf{X}$ be a connected Riemann surface. Let $K_{\mathbf{X}}$ be the differential field of meromorphic functions on $\mathbf{X}$. Let ∇ be a meromorphic connection on $\mathbf{X}$, and $x_0 \in \mathbf{X}$ a regular point for ∇ . We fix a $\mathbb{C}$ -basis for the space of horizontal sections of ∇ on a germ of small “disc” centered at x_0 and we identify the Galois differential group $\text{Gal}_{K_{\mathbf{X}}}(\nabla)$ with its corresponding representation in $\text{GL}(n; \mathbb{C})$.

Then $\text{Gal}_{K_{\mathbf{X}}}(\nabla)$ is the Zariski closure of the image in $\text{GL}(n; \mathbb{C})$ of the wild monodromy representation

$$\rho_s(\nabla): \pi_{1,s}(\mathbf{X}; \cdot) \rightarrow \text{GL}(n; \mathbb{C}).$$

Examples and applications. — It is possible to *compute explicitly wild monodromy representations* for *generalized confluent hypergeometric differential equations* ⁽⁶⁴⁾ (using results of [DM]). These computations use elementary functions and Γ -function. It is possible to compute Galois differential groups of generalized confluent hypergeometric differential equations from these representations. This program is partially achieved [DM], [M1], [M2], [M3]. C. Mitschi has studied in particular *order seven* case and got, after N. Katz [K3], *generalized confluent hypergeometric differential equations of order seven admitting the exceptional group G_2 as Galois differential group* [M2], [M3].

⁽⁶⁴⁾ And for differential equations got from confluent hypergeometric equations by “elementary operations”, as, for instance, “Kummer pull-backs” [Kat 3], [M3]. (Differential equations satisfied by accelerating and decelerating functions, and more generally by Faxen’s integrals, correspond, when the parameters are rational numbers, to such pull-backs.)

From *theorem 18* (or *theorem 17*) it is also possible to get an interesting result for the “inverse problem” in differential Galois theory [Ra 8]:

THEOREM 22. — *Let L be a complex semi-simple Lie algebra. Let $L \rho$ be a finite dimensional representation of L . Then:*

(i) *There exists a rational differential equation D on $P^1(C)$, with singularities contained in $\{0, +\infty\}$, 0 being regular and $+\infty$ irregular, such that $\text{Gal}_{C(z)}(D)$ is Zariski connected and such that*

$\text{Lie Gal}_{C(z)}(D) \approx L \rho(L)$ (isomorphism of complex Lie-algebras).

(ii) *There exists a germ of meromorphic differential equation D at the origin such that $\text{Gal}_K(D)$ is Zariski connected and such that*

$$\text{Lie Gal}_K(D) \approx L \rho(L).$$

We will end this paragraph by a comparison between *N. Katz's* viewpoint and ours.

Let X^{an} be a compact connected Riemann surface. Let S be a fixed finite subset of X^{an} . We denote by $\text{D.E.}(X^{an}; S)$ the tensor category of meromorphic connections on X^{an} with singularities contained in S .

To each point z_0 of $X^{an} - S$ we can associate a fibre functor $\omega(z_0)$ of the tensor category $\text{D.E.}(X^{an}; S)$:

$$\omega(z_0)(V) = \{ \text{horizontal sections of } V \text{ on a germ of neighbourhood of } z_0 \}.$$

We will denote by $\pi_1^{\text{diff}}(X^{an} - S; S; z_0)$ the group $\text{Aut}^{an}(\omega(z_0))$ (automorphisms of the fibre functor $\omega(z_0)$).

There is a natural map

$$\pi_{1,s}(X^{an} - S; S; z_0) \rightarrow \pi_1^{\text{diff}}(X^{an} - S; S; z_0):$$

each element of $\pi_{1,s}(X^{an} - S; S; z_0)$ defines clearly an automorphism of the fibre functor $\omega(z_0)$.

Let Y be a smooth connected *C-scheme* such that the corresponding analytic variety is a connected Riemann surface $X^{an} - S = Y^{an}$. We denote by $\text{D.E.}(Y/C)$ the tensor category of algebraic connections on Y . There is a natural functor

$$\begin{aligned} \text{D.E.}(Y/C) &\rightarrow \text{D.E.}(X^{an}; S) \\ \nabla &\rightarrow \nabla^{an}. \end{aligned}$$

If X^{an} is compact it yields an equivalence of tensor categories between $\text{D.E.}(Y/C)$ and $\text{D.E.}(X^{an}; S)$.

We denote by $\pi_1^{\text{diff}}(Y/C; z_0)$ the group $\text{Aut}(\omega(z_0))$ [automorphisms of the fibre functor $\omega(z_0)$].

There is a natural morphism $\pi_1^{\text{diff}}(X^{an} - S; S; z_0) \rightarrow \pi_1^{\text{diff}}(Y/C; z_0)$. If X^{an} is compact it is an isomorphism. We get:

PROPOSITION 15. — *Let Y be a smooth connected C-scheme such that the corresponding analytic variety is a connected Riemann surface $X^{an} - S = Y^{an}$, where X^{an} is a Riemann surface and S a finite subset of X^{an} . Then*

$\pi_1^{\text{diff}}(\mathbf{Y}/\mathbf{C}; z_0)$ is an affine pro-algebraic $\mathbf{C}$ -group-scheme and there exists a natural homomorphism of groups

$$\pi_{1,s}(\mathbf{X}^{\text{an}} - \mathbf{S}; \mathbf{S}; z_0) \rightarrow \pi_1^{\text{diff}}(\mathbf{Y}/\mathbf{C}; z_0).$$

Even if $\mathbf{X}^{\text{an}}$ is compact this map is not onto. We ignore if it is injective. Anyway if $\mathbf{X}^{\text{an}}$ is compact π_1^{diff} appears as an “algebraic hull” of $\pi_{1,s}$, just like π_1^{diff} appears as an algebraic hull of π_1 in the fuchsian case.

If G is a linear algebraic group we will denote by G^0 the (Zariski) connected component of the identity.

If ∇ is a germ meromorphic connection at the origin we will denote by $\rho_{mf}(\nabla)$ the restriction to the subgroup $(\hat{\gamma}_0)$ of the representation $\hat{\rho}(\nabla)$, and by $G_m(\nabla)$ [resp. $G_{mf}(\nabla)$] the Zariski closure of the image of $\rho_m(\nabla)$ [resp. $\rho_{mf}(\nabla)$]. If ∇ is a meromorphic connection on a Riemann surface we will denote by $G_m(\nabla)$ the Zariski closure of the image of $\rho_m(\nabla)$.

Theorem 19 sounds quite abstract, however (using only algebraic methods) we can deduce from it quite interesting results. For instance we get easily a variant of a result of O. Gabber:

PROPOSITION 16. — (i) Let ∇ be a germ of meromorphic connection at the origin. Then the map

$$\pi_1((\mathbf{C}^*, 0); \mathbf{R}^+) \rightarrow \text{Gal}_{\mathbf{K}}(\nabla)/\text{Gal}_{\mathbf{K}}(\nabla)^0,$$

induced by the **monodromy** representation $\rho_m(\nabla)$, is a **surjection**.

(ii) Let ∇ be a germ of meromorphic connection at the origin. Then the map

$$(\hat{\gamma}_0) \rightarrow \text{Gal}_{\mathbf{K}}(\nabla)/\text{Gal}_{\mathbf{K}}(\nabla)^0,$$

induced by the **formal monodromy** representation $\rho_{mf}(\nabla)$, is a **surjection**.

(iii) Let ∇ be a meromorphic connection on a Riemann surface $\mathbf{X}$. Let $\mathbf{S}$ be a discrete subset of $\mathbf{X}$ containing all the singularities of ∇ . Then the map

$$\pi_1(\mathbf{X}^{\text{an}} - \mathbf{S}; \cdot) \rightarrow \text{Gal}_{\mathbf{K}_{\mathbf{X}}}(\nabla)/\text{Gal}_{\mathbf{K}_{\mathbf{X}}}(\nabla)^0,$$

induced by the **monodromy** representation $\rho_m(\nabla)$, is a **surjection**.

Proofs mimic Gabber’s proof [Kat 1] (1.2.5., p. 18). Proof of assertion (iii) is similar to proof of assertion (i), so we will prove only (i) and (ii). We denote by G the **finite group** $\text{Gal}_{\mathbf{K}}(\nabla)/\text{Gal}_{\mathbf{K}}(\nabla)^0$. Let ρ' be a faithful finite dimensional linear representation of G . Then $\rho' \rho_s(\nabla)$ is a finite dimensional linear representation of the **wild fundamental group** $\pi_{1,s}((\mathbf{C}^*, 0); \mathbf{R}^+)$, and, using theorem 19, we can **interpret** it as a meromorphic connection ∇' on $\mathbf{X}$, with singularities in $\mathbf{S}(\rho_s(\nabla)) = \rho' \rho_s(\nabla)$. Moreover ∇' belongs to the tensor category “generated” by ∇ .

We have a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} \pi_{1,s}((C^*, 0); "R^+") & \xrightarrow{\rho_s(V)} & \text{Gal}_K(V) \\ & \searrow \rho_s(V') & \downarrow \\ & & \text{Gal}_K(V') = G \\ & & = \text{Gal}_K(V)/\text{Gal}_K(V)^0. \end{array}$$

Using *proposition 14* it yields a new commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} \pi_1((C^*, 0); R^+) & \xrightarrow{\rho_m(V)} & \text{Gal}_K(V) \\ & \searrow \rho_m(V') & \downarrow \\ & & \text{Gal}_K(V') = G \\ & & = \text{Gal}_K(V)/\text{Gal}_K(V)^0. \end{array}$$

The Galois differential group $\text{Gal}_K(V')$ being *finite* the connection V' is *fuchsian* ⁽⁶⁵⁾, then the map $\rho_m(V')$ is *surjective*. Assertion (i) follows.

We have also a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} (\hat{\gamma}_0) & \xrightarrow{\rho_{mf}(V)} & \text{Gal}_K(V) \\ & \searrow \rho_{mf}(V') & \downarrow \\ & & \text{Gal}_K(V') = G \\ & & = \text{Gal}_K(V)/\text{Gal}_K(V)^0. \end{array}$$

The Galois differential group $\text{Gal}_K(V')$ being *finite* the connection V' is *fuchsian*. Then the map $\rho_m(V')$ is *surjective*, *actual monodromy* and *formal monodromy* can be identified, and the map $\rho_{lf}(V')$ is *also surjective*. Assertion (ii) follows.

PROPOSITION 17. — (i) *Let V be a germ of meromorphic connection at the origin. Then*

- (a) *If $G_m(V)$ is Zariski connected, then $\text{Gal}_K(V)$ is also Zariski connected.*
- (b) *If $G_{mf}(V)$ is Zariski connected, then $\text{Gal}_K(V)$ is also Zariski connected.*

(ii) *Let V be a meromorphic connection on a Riemann surface X . Then, if $G_m(V)$ is Zariski connected, then $\text{Gal}_{KX}(V)$ is also Zariski connected.*

Be careful, conversely $\text{Gal}_{KX}(V)$ can be connected and $G_m(V)$ or $G_{mf}(V)$ *not* connected. It is interesting to notice that we can decide if $G_{mf}(V)$ is *Zariski connected* using *purely algebraic* methods. This is *not* true in general

⁽⁶⁵⁾ If a connection is *not* fuchsian its Galois differential group contains a *non trivial* exponential torus and *cannot* be *finite*.

for the connectedness of $G_m(V)$, however there are exceptional (and interesting...) cases (see *examples* below).

Proposition 17 follows immediately from *proposition 16*. There is also a "more elementary" proof:

The *exponential tori* are *connected*, and, if S is a *Stokes multiplier* "in" $\text{Gal}_K(V)$, then S is *unipotent* and the one-parameter group $\{\exp(tS)/t \in \mathbb{C}\}$ is *connected and entirely contained in* $\text{Gal}_K(V)$. Then *exponential tori* are subgroups of $\text{Gal}_K(V)^0$ and *Stokes multipliers* belongs to $\text{Gal}_K(V)^0$. *Proposition 17* follows.

Example. — Following ideas of *N. Katz* [Kat 1] *proposition 17* yields elegant methods of computation of some Galois differential groups. Let V be a meromorphic connection on the Riemann sphere with singularities contained in $S = \{0, +\infty\}$, 0 being *regular or regular singular and* $+\infty$ *irregular*. We fix a base point $z_0 \in \mathbb{C}^*$. Monodromies around zero and infinity are inverse each other and *algebraically computable* (using *Frobenius algorithm*). We get in particular interesting situations when $G_m(V)$ is *Zariski connected* (especially when the monodromy around zero is *trivial*) and when the Newton polygon of V at infinity admits only a slope $k > 0$, where the rational number k is *not an integer*. Then the monodromy acts *non trivially* by conjugacy on the exponential torus and we get (even if it is not so evidence at first glance...) a lot on information on the connected group $\text{Gal}_{\mathbb{C}(z)}(V)$ (particularly in the *irreducible* case).

As an example of application of these ideas we will give a very easy computation of the Galois differential group for Airy equation $Dy = y'' - zy = 0$.

The (actual) monodromy of D is *trivial*, then $G_m(D) = \{id\}$ is *Zariski connected* and $\text{Gal}_{\mathbb{C}(z)}(D)$ is also *Zariski connected*. Using a formal fundamental system of solutions for D at infinity we identify $\text{Gal}_{\mathbb{C}(z)}(D)$ with a subgroup of $\text{GL}(2; \mathbb{C})$. The Wronskian of a fundamental system of solutions being constant we get more precisely a subgroup of $\text{SL}(2; \mathbb{C})$.

If our fundamental system of solutions is "well chosen" [MR 2], then the *exponential torus* is

$$T = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \middle| \alpha \in \mathbb{C}^* \right\},$$

and the *formal monodromy* matrix is

$$M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The formal monodromy matrix M acts *non trivially* on the exponential torus T (it permutes the characters). We have

$$TM = \left\{ \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} \middle| \beta \in \mathbb{C}^* \right\} \subset \text{Gal}_{\mathbb{C}(z)}(D),$$

and

$$T \cup TM \subset \text{Gal}_{\mathbb{C}(z)}(D).$$

But the *only connected subgroup* of $\text{SL}(2; \mathbb{C})$ containing $T \cup TM$ is $\text{SL}(2; \mathbb{C})$ itself. We get

$$\text{Gal}_{\mathbb{C}(z)}(D) = \text{SL}(2; \mathbb{C}).$$

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