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## Stationary solutions for the Swift-Hohenberg equation in non-uniform backgrounds

by

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**ABSTRACT.** — We prove the existence of stationary solutions for the Swift-Hohenberg equation

$$(\alpha(x)\varepsilon^2 - (1 + \partial_x^2)^2)u(x) - \beta(x)u^3(x) = 0,$$

in the case when  $\alpha$  and  $\beta$  are step functions taking different values for  $x < 0$  and  $x > 0$ . This setup corresponds to physical problems with different driving forces for  $x < 0$  and  $x > 0$ .

**RÉSUMÉ.** — On démontre l'existence de solutions stationnaires de l'équation de Swift-Hohenberg

$$(\alpha(x)\varepsilon^2 - (1 + \partial_x^2)^2)u(x) - \beta(x)u^3(x) = 0,$$

pour le cas où  $\alpha$  et  $\beta$  sont des fonctions d'escalier qui prennent deux valeurs différentes lorsque  $x < 0$  et  $x > 0$ . Ce cadre correspond à des problèmes physiques où les forces extérieures diffèrent pour  $x < 0$  et  $x > 0$ .

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<sup>(1)</sup> On leave from the University of Geneva.

## 1. INTRODUCTION

In this paper, we study the existence of solutions of the modified Swift-Hohenberg equation

$$(\alpha(x)\varepsilon^2 - (1 + \partial_x^2)^2)u(x) - \beta(x)u^3(x) = 0. \quad (1.1)$$

The bifurcation theory in the parameter  $\varepsilon$  is well established in the case of constant  $\alpha$  and  $\beta$ , [CE]. The purpose of this paper is to show that a two-parameter family of solutions exists in the case when  $\alpha > 0$  and  $\beta > 0$  are constant on the two half-spaces  $x < 0$  and  $x > 0$  but are not necessarily equal on the two sides of 0. We will show that such solutions exist when the values on the left and the right are not too different. Numerical experimentation shows that this restriction is unnecessary. We need it because we use the implicit function theorem, starting from the homogeneous case.

The Equation (1.1) with non-constant coefficients is interesting for the physics of extended systems with non-uniform driving forces such as the Couette-Taylor experiment for cylinders with non-constant radius, or Rayleigh-Bénard experiments with non-homogeneous driving forces or non-constant plate-distances. More recent experimental results, and theoretical work can be found in [R], [RP], [NAC], with references to the earlier literature.

When  $\alpha$  and  $\beta$  are independent of  $x$ , then it is known that at small amplitudes there is a two-parameter family of solutions of the form

$$u(x) \approx \frac{2c(\omega)\varepsilon}{\sqrt{\beta}} \cos(\omega x + \vartheta),$$

where  $\omega \approx 1$  and  $\vartheta$  are the two parameters, and  $c(\omega)$  is given by a relation of the form

$$3c(\omega)^2\varepsilon^2 + (1 - \omega^2)^2 \approx \alpha\varepsilon^2.$$

Our result is therefore that, in some intuitive sense, the number of parameters does not change as the problem becomes nonhomogeneous, but it will be seen that the relation between the parameters and the amplitude  $c$  is more interesting.

## 2. STATIONARY SOLUTIONS FOR HOMOGENOUS PROBLEMS

The basis for constructing solutions for the inhomogeneous problem is the existence of solutions of half-spaces which we will describe below. In this section, we restate the existence of stationary solutions [CE], Theorem 17.1. We formulate it here in a way which will be useful later.

PROPOSITION 2.1. — *Let  $a > 0$ ,  $b > 0$  be given. Then there is an  $\varepsilon_0 > 0$  such that the equation*

$$(a^2 \varepsilon^2 - (1 + \partial_x^2)^2) u(x) - b^2 u^3(x) = 0$$

*has a solution of the form*

$$u(x) = \frac{2c}{b} \varepsilon \cos(\omega x) + \mathcal{O}(\varepsilon^3), \tag{2.1}$$

*for every  $\varepsilon$ ,  $0 < \varepsilon < \varepsilon_0$  and for every  $W$  satisfying  $W^2 < 1$ . The constants  $c$  and  $\omega$  are defined by*

$$\omega = \sqrt{1 + \sqrt{3} c \varepsilon W} \tag{2.2}$$

*and*

$$a^2 = 3 c^2 (1 + W^2). \tag{2.3}$$

*Remark.* — The above statement is a rewriting of Theorem 17.1 and of the bound (17.27-28) in [CE]. We have introduced an explicit dependence on the parameters  $a$  and  $b$ . This dependence follows at once from the case  $c = 1$ ,  $b = 1$  by replacing  $\varepsilon$  by  $c\varepsilon$  and  $u$  by  $u/b$ . The first transformation is really a change of scale in  $\varepsilon$  which we need to make explicit for the matching problem. In the assumptions of Theorem 17.1, we required that  $\omega^2$  be contained in some strict subinterval  $I$  of  $(2/5, 2)$ . This assumption is implied by the assumptions of Proposition 2.1. Namely, by (2.3), we see that  $c^2 < a^2/3$ . Thus, for sufficiently small  $\varepsilon$ , the resulting  $\omega$ , as defined by (2.2), lies in  $I$ . The allowed range of  $W$  is in fact much larger than what we assume in Proposition 2.1.

The periodic solutions described in Proposition 2.1 are not all stable and this phenomenon is well known as the “Eckhaus Instability”. In terms of the parameters in Proposition 2.1 the solutions are stable if

$$|W| < \frac{1}{\sqrt{2}},$$

and unstable if the above inequality is reversed. Looking at the relation (2.3), we see that the instability occurs for small amplitudes. It describes a possible sliding of wavelengths for small amplitudes. We will be interested in “large” amplitudes, *i. e.*, in  $|W| < 1/\sqrt{2}$ .

### 3. SOLUTIONS IN HALF-SPACES

In the region where the periodic solutions are stable, a *linear* analysis can be performed around these stable solutions and it shows that there exist solutions which decay exponentially on half-spaces towards the stable

solutions. This then implies by the contraction mapping principle that the same kind of solution exists for the *nonlinear* problem.

Let  $u_0$  denote one of the  $2\pi/\omega$ -periodic stationary solutions described by Proposition 2.1. We look now for solutions  $u = u_0 + v$  on the half-space  $x > 0$  and we require  $v(x) \rightarrow 0$  as  $x \rightarrow \infty$ . The equation for  $v$  is clearly

$$(a^2 \varepsilon^2 - (1 + \partial_x^2)^2 - 3b^2 u_0^2(x)) v(x) = 3b^2 u_0(x) v^2(x) + b^2 v^3(x). \quad (3.1)$$

We first study its linearization:

$$(a^2 \varepsilon^2 - (1 + \partial_x^2)^2 - 3b^2 u_0^2(x)) v(x) = 0. \quad (3.2)$$

This is a fourth order differential equation with periodic coefficients and therefore we can study its spectrum by the method of Bloch waves. The relevant ansatz is

$$v(x) = w(x) e^{-kx},$$

where  $w$  is a  $2\pi/\omega$ -periodic function (of small amplitude) and  $k \geq 0$ . Writing  $u_0$  and  $w$  as Fourier series,

$$u_0(x) = \sum_{n \in \mathbf{Z}} \eta_n e^{in\omega x}, \quad W(x) = \sum_{n \in \mathbf{Z}} w_n e^{in\omega x},$$

one can do perturbation theory in  $\varepsilon$  and the lowest order approximation is, by (2.1),

$$u_0(x) = \frac{c\varepsilon}{b} (e^{i\omega x} + e^{-i\omega x}) + \mathcal{O}(\varepsilon^3),$$

where  $c$  is defined in (2.3),  $a^2 = 3c^2(1 + W^2)$ . To the same order in  $\varepsilon$ , we find that the "modes"  $w_{\pm 1}$  decouple from the others and lead to the coupled system of equations:

$$\left. \begin{aligned} (a^2 \varepsilon^2 - (1 - (ik + \omega)^2)^2 - 6c^2 \varepsilon^2) w_{+1} - 3c^2 \varepsilon^2 w_{-1} &= 0, \\ (a^2 \varepsilon^2 - (1 - (ik - \omega)^2)^2 - 6c^2 \varepsilon^2) w_{-1} - 3c^2 \varepsilon^2 w_{+1} &= 0. \end{aligned} \right\} \quad (3.3)$$

It is useful to introduce the parametrization

$$ik = c\varepsilon \sqrt{3} K/2, \quad (3.4)$$

and to recall the relation

$$\omega = \sqrt{1 + \sqrt{3} c\varepsilon W}.$$

Then (3.3) takes the form of a matrix equation with matrix

$$A_K = 3c^2 \varepsilon^2 \begin{pmatrix} -2WK - K^2 - 1 & -1 \\ -1 & 2WK - K^2 - 1 \end{pmatrix}.$$

This matrix has a zero eigenvalue when  $K = 0$  or  $K = \pm i\sqrt{2 - 4W^2}$ . Thus,  $k$  is either real or purely imaginary and if we require the decay of  $v(x)$  as  $x \rightarrow \infty$ , *i. e.*, if we require that  $k$  is real, this means that we must have  $W^2 < 1/2$  as asserted above.

We assume now  $W^2 < 1/2$  and we set

$$K = i \sqrt{2 - 4W^2}. \tag{3.5}$$

Then  $A_K$  has an eigenvector with eigenvalue 0 which is of the form

$$w_{-1} = (2WK + K^2 + 1) w_1, \tag{3.6}$$

Since we are interested in *real* functions  $v$ , we also require  $\bar{w}_1 = w_{-1}$  so that the Eq. (3.6) leads to the solution

$$w_1 = \rho e^{i\theta}$$

with

$$\theta = \frac{1}{2} \arccos(1 - 4W^2). \tag{3.7}$$

From this, we see that, to first order in  $\varepsilon$ ,  $w$  is given by

$$w(x) = 2\rho \cos(\omega x + \theta),$$

and therefore

$$v(x) = 2\rho \cos(\omega x + \theta) e^{-kx}, \tag{3.8}$$

with  $\rho$  arbitrary. Note that by (3.4), (3.5), we have

$$k = c\varepsilon \sqrt{3/2 - 3W^2}.$$

We now use the stable manifold theorem [CE], Theorem 21.2, to conclude that to the perturbative solution just given there corresponds a solution of the non-linear problem (3.1). In other words, we have

**PROPOSITION 3.1.** — *Let  $a > 0$ ,  $b > 0$  be given and let  $W^2 < 1/2$ . Then there are an  $\varepsilon_0 > 0$  and a  $t_0 > 0$  such that the equation*

$$(a^2 \varepsilon^2 - (1 + \partial_x^2)^2) u(x) - b^2 u^3(x) = 0.$$

*has, on  $x > 0$ , solutions for every  $\varepsilon$ ,  $0 < \varepsilon < \varepsilon_0$ , for every  $\omega$  of the form*

$$\omega^2 = 1 + \sqrt{3} c \varepsilon W,$$

*and for every  $t$ ,  $|t| < t_0$ . These solutions are of the form*

$$u(x) = \frac{1}{b} (2c\varepsilon \cos(\omega x) + 2t\varepsilon \cos(\omega x + \theta) e^{-kx} + \mathcal{O}(\varepsilon^3)). \tag{3.9}$$

*The constant  $c$  satisfies the relation*

$$a^2 = 3c^2(1 + W^2), \tag{3.10}$$

*$k$  is given by*

$$k = c\varepsilon \sqrt{3/2 - 3W^2},$$

*and  $\theta$  is given by (3.7).*

COROLLARY 3.2. — *The above statements hold also for  $x < 0$  but with (3.9) replaced by*

$$u(x) = \frac{1}{b} (2c\varepsilon \cos(\omega x) + 2t\varepsilon \cos(\omega x - \theta)) e^{kx} + \mathcal{O}(\varepsilon^3). \quad (3.11)$$

*Remark.* — The presence of  $\theta$  in (3.9), (3.11) shows a phase shift of the exponential part of the solution with respect to the periodic part.

#### 4. MATCHING OF SOLUTIONS

We now construct solutions for the differential equation (1.1) for the case of

$$\alpha(x) = \begin{cases} a_L^2 \varepsilon^2, & \text{when } x < 0 \\ a_R^2 \varepsilon^2, & \text{when } x \geq 0 \end{cases}, \quad \beta(x) = \begin{cases} b_L^2, & \text{when } x < 0 \\ b_R^2, & \text{when } x \geq 0 \end{cases}.$$

These solutions will be 4 times differentiable except at  $x = 0$  where they are 3 times differentiable. Our strategy consists in checking the existence of solutions at the linear level and then using the implicit function theorem to establish the existence of corresponding solutions to the nonlinear problem. The existence of the solution for the linear problem will follow from a transversality argument. It is useful to introduce the following notation for the solution on right ( $x > 0$ ), neglecting terms of order  $\mathcal{O}(\varepsilon^3)$ :

$$f_R(\varepsilon; W_R, t_R, x_R, x) = \frac{1}{b_R} (2\varepsilon c_R \cos(\omega_R(x + x_R)) + 2\varepsilon t_R \cos((x + x_R)\omega_R - s_R \theta_R) e^{s_R k_R(x + x_R)}), \quad (4.1)$$

where

$$\begin{aligned} \omega_R^2 &= 1 + \sqrt{3} c_R \varepsilon W_R, \\ \theta_R &= \frac{1}{2} \arccos(1 - 4W_R^2), \\ k_R &= c_R \varepsilon \sqrt{3/2 - 3W_R^2}, \\ s_R &= -1, \\ c_R^2 &= \frac{a_R^2}{3(1 + W_R^2)}, \quad c_R > 0. \end{aligned}$$

The definitions for the solution on the left are analogous, except that the index R is replaced by L and that the sign  $s_L$  is  $s_L = +1$ . We impose the continuity conditions

$$F_p \equiv \partial_x^p f_L|_{x=0} - \partial_x^p f_R|_{x=0} = 0 \quad \text{for } p = 0, \dots, 3. \quad (4.2)$$

This is a system of 4 equations in the 6 unknowns  $W_L, t_L, x_L, W_R, t_R, x_R$ . We now show that the rank of this system is 4, when  $a_L = a_R$  and  $b_L = b_R$ . (By continuity, this will continue to be the case when  $a_L$  and  $a_R$  resp.  $b_L$  and  $b_R$  are close to each other.) To do this, we perform yet another (linear) change of variables, namely

$$\begin{aligned} \xi_1 &= \frac{1}{2}(W_R + W_L), & \xi_2 &= \frac{1}{2}(x_R + x_L), \\ \xi_3 &= \frac{1}{2}(W_R - W_L), & \xi_4 &= \frac{1}{2}(x_R - x_L), \\ \xi_5 &= t_L, & \xi_6 &= t_R. \end{aligned}$$

Denote by  $T$  the map from  $(\epsilon; \xi_1, \dots)$  to  $(\epsilon; W_L, \dots)$ , defined by this transformation.

We show the existence of transversality by perturbing the problem  $a_L = a_R = a, b_L = b_R = 1$ . In the new coordinates, we know from Proposition 2.1 that there is a two-parameter family of solutions for

$$\xi = (\xi_1, \dots, \xi_6) = (W, x, 0, 0, 0, 0),$$

where  $W^2 < 1/2$  and  $x \in \mathbf{R}$  are arbitrary (this is the translation 9). Consider now the natural decomposition in  $\mathbf{R}^6: \xi = (y, z)$ , with  $y \in \mathbf{R}^2$  and  $z \in \mathbf{R}^4$ . We want to apply the inverse function theorem to the function  $F = (F_1 \circ T, \dots, F_4 \circ T)$  to solve the equation

$$F(\epsilon; y, z(\epsilon; y)) = 0.$$

Since we know that  $(\epsilon; W, x, 0, 0, 0, 0)$  is a zero of  $F$ , for all  $x$  it suffices to show that the derivative

$$\det(D_z F)(\epsilon; W, x, 0, 0, 0, 0)$$

does not vanish. A lengthy calculation, which can be done with computer algebra, shows that the leading order of the determinant is

$$\det(D_z F)(\epsilon; W, x, 0, 0, 0, 0) = C \epsilon^6 a^4 (1 - 2W^2)^{5/2} + \mathcal{O}(\epsilon^8), \quad (4.3)$$

for some  $C \neq 0$ . This means that we have the

LEMMA 4.1. — *Let  $W_0 < 1/\sqrt{2}$ . There is an  $\epsilon_0 > 0$  such that for all  $\epsilon$  satisfying  $0 < \epsilon < \epsilon_0$  for all  $|W| < W_0$  and for all  $x \in \mathbf{R}$ , the determinant  $\det(D_z F)(\epsilon; W, x, 0, 0, 0, 0)$  is non-zero.*

We now apply the Implicit Function Theorem to the function  $F$  and we get the

LEMMA 4.2. — *Let  $W_0 < 1/\sqrt{2}$ . There is an  $\epsilon_1 > 0$  such that for all  $\epsilon$  satisfying  $0 < \epsilon < \epsilon_1$  for all  $0 < W < W_0$  and for all  $x \in \mathbf{R}$ , the continuity conditions (4.2) can be satisfied.*



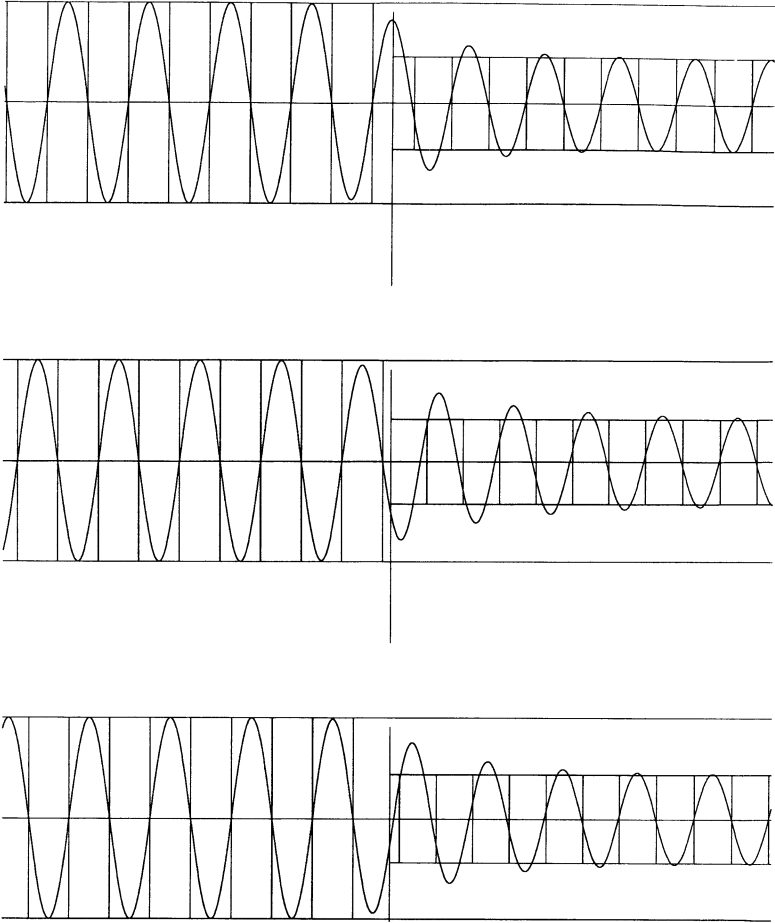


FIG. 1. — Three stationary solutions for the non-homogeneous equation. The parameters are  $a_R=0.5$ ,  $a_L=1.0$ ,  $\varepsilon=0.7$ ,  $c_L=0.577,35$ . The other parameters are adapted to make a smooth solution.

This means that the linearized equations have been matched at 0. We now want to match the solutions in the half-spaces and we apply again the Implicit Function Theorem.

**THEOREM 4.3.** — Fix  $a > 0$  and  $b > 0$ . Let  $W_0 < 1/\sqrt{2}$ . There is an  $a_0 > 0$ , a  $b_0 > 0$  and an  $\varepsilon_2 > 0$  such that the following holds: Let  $|a_L - a| < a_0$ ,  $|a_R - a| < a_0$ ,  $|b_L - b| < b_0$ ,  $|b_R - b| < b_0$ . For all  $\varepsilon$  satisfying  $0 < \varepsilon < \varepsilon_2$  for all  $|W| < W_0$  and for all  $x \in \mathbf{R}$ , the continuity conditions (4.2) can be satisfied for the equation

$$(\alpha(x)\varepsilon^2 - (1 + \partial_x^2)^2)u(x) - \beta(x)u^3(x) = 0$$

with

$$\alpha(x) = \begin{cases} a_L^2, & \text{when } x < 0 \\ a_R^2, & \text{when } x \geq 0 \end{cases}, \quad \beta(x) = \begin{cases} b_L, & \text{when } x < 0 \\ b_R, & \text{when } x \geq 0 \end{cases}.$$

*Remark.* — The steps to go from Lemma 4.2 to Theorem 4.3 are as follows. Consider first the case  $a = 1, b = 1$ . The determinant is a continuous function of the parameters  $a_L, a_R, b_L,$  and  $b_R,$  with an expression as in (4.3), so that  $\det(D_z F)(\varepsilon; W, x, 0, 0, 0, 0)/\varepsilon^6$  is uniformly bounded away from zero, when the  $a_L, \dots$  vary in small balls around 1. Therefore, the implicit function theorem applies uniformly in these balls. The case of arbitrary non-zero  $a$  and  $b$  follows now by scaling the variable  $\varepsilon$  and the amplitude of the solution.

## 5. EXAMPLE

In this section, we present a numerical example (cf. *Fig. 1*), in which we matched the *leading order solutions* for values of the parameters which are certainly outside of the validity of Theorem 4.3 as we prove it. The parameters have been chosen to make the following phenomena visible. First, one can see how the exponential part aids the interpolation of the two periodic solutions. Second, we see that as  $x$  is varied and the amplitude of the left solution is kept fixed, the amplitude and the wavelength of the right solution oscillates. Second, one should note that the speed of translation on the left is imposed (the three frames cover the three thirds of a period on the l.h.s.) and the r.h.s. moves a *different* distance in this period.

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