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Norm group convergence for singular Schrödinger Operators

by

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ABSTRACT. – Let $\{V_n\}$, V be distributions in $\mathbf{R}^1$ with bounded Fourier transforms, and $\hat{V}_n(k) \underset{k \to \infty}{\to} \hat{V}(k)$ pointwise boundedly. Let $H_n = -\Delta + V_n$, $H = -\Delta + V$ be the corresponding Schrödinger operators. Then $H_n \to H$ in the sense of unitary group convergence, *i.e.*, for all $t \in \mathbf{R}^1$, $e^{it(-\Delta+V_n)} \to e^{it(-\Delta+V)}$ in norm. Thus the correspondence between (possibly distributional) potentials and Hamiltonians is continuous from the topology of pointwise bounded convergence (of Fourier transforms) to that of norm group convergence. As a consequence, we obtain norm group approximation of Hamiltonians with finitely many point interactions by local scaled short-range interactions. In addition, we consider infinitely many point interactions, approximation in the norm resolvent sense, and provide a counterexample to show that our results cannot be strengthened to include standard weaker notions of convergence for distributional potentials.

RÉSUMÉ. – Soient $\{V_n\}$, V des distributions sur $\mathbf{R}^1$, et supposons que leurs transformations de Fourier sont bornées, et $\hat{V}_n(k) \underset{n \to \infty}{\to} \hat{V}_n(k)$ de façon ponctuelle et bornée. On note $H_n = -\Delta + V_n$, $H = -\Delta + V$, les opérateurs de Schrödinger correspondants. Alors $H_n \to H$ dans le sens de la convergence des groupes unitaires, c'est-à-dire $e^{it(-\Delta+V_n)} \to e^{it(-\Delta+V)}$ en norme

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pour tout t réel. Donc la correspondance entre les potentiels (qui peuvent être des distributions) et les hamiltoniens est continue, de la topologie de convergence ponctuelle bornée (des transformations de Fourier) à la topologie de convergence en norme des groupes. On en déduit un résultat d'approximation (en norme des groupes) pour des hamiltoniens avec un nombre fini d'interactions ponctuelles, par des interactions de type local à courte portée. On considère aussi des hamiltoniens avec un nombre infini d'interactions ponctuelles, l'approximation dans le sens de la norme des résolvantes, et on montre par un contre-exemple que les conclusions de cet article ne sont pas vraies en général pour les notions usuelles de convergence des potentiels distributions

INTRODUCTION

The purpose of this paper is to establish norm group convergence of Hamiltonians with smooth local potentials to Hamiltonians involving point interactions in one dimension. We do so by considering the more general problem of approximation of Hamiltonians with distributional potentials, using the interaction representation approach to singular perturbations proposed by Segal [9]. Consequently, Hamiltonians with such potentials are natural as generators of time translations of Schrödinger systems, to the extent that they belong to an appropriate completion of the regular Schrödinger operators, made in the topology of norm (uniform) convergence of unitary groups. That is, for Schrödinger operators H_n and H, we define convergence $H_n \underset{n\to\infty}{\to} H$ if $e^{itH_n} \underset{n\to\infty}{\to} e^{itH}$ in the norm topology for all real t. Equivalently, the mapping from potentials V to Hamiltonians $-\Delta+V$ is continuous in this sense.

This has been done previously using other notions of convergence. It has been shown (*see* [1]) that Hamiltonians with distributional (delta) potentials can be defined as limits of operators with smooth potentials in the topology of resolvent convergence. Part of the interest in our results lies in the fact that they refer directly to convergence of unitary propagators e^{itH}, which connect Hamiltonians to time evolution.

There is already an extensive literature on this subject, culminating most recently in the monograph of Albeverio, Gesztesy, *et al.* (*cf.* [1], [5]). The previous result of particular interest to us is the approximation of the Hamiltonian $H=-\Delta+c\delta$ by scaled short-range Hamiltonians $H_\varepsilon=-\Delta+\varepsilon^{-2}\lambda(\varepsilon)V(\cdot/\varepsilon)$, as $\varepsilon\downarrow 0$ in the norm resolvent sense (analogous results for finitely, as well as infinitely many point interactions have also been obtained). Our results complement these, for in general norm resolvent convergence implies strong group convergence, but not norm group convergence. Our proof exploits the specific representation of the corresponding unitary groups as convergent time-ordered exponential series that was obtained in [6], yielding a form of continuity of the mapping $V\mapsto e^{it(-\Delta+V)}$ that implies our main result:

1. THEOREM. – *Let* $\{V_n\}$, V *be distributions such that* $\hat{V}\in L^\infty(\mathbf{R}^1)$, $\sup_n\|\hat{V}_n\|_\infty<\infty$, *and* $\hat{V}_n(k)\to\hat{V}(k)$ a.e. *on* $\mathbf{R}^1$. *Then for all* $t\in\mathbf{R}^1$, $e^{it(-\Delta+V_n)}\to e^{it(-\Delta+V)}$ *in the uniform operator topology.*

This result applies to the following special case [1, 2]: Assume $V\in L^1(\mathbf{R}^1)$, and let λ be real-analytic near the origin with $\lambda(0)=0$. For $0<\varepsilon\leqq\varepsilon_0$, define $V_\varepsilon(x)=\dfrac{\lambda(\varepsilon)}{\varepsilon^2}V((x-y)/\varepsilon)$, $y\in\mathbf{R}$. Then clearly $\hat{V}_\varepsilon(k)$ satisfy the hypotheses of our Theorem, and $\hat{V}_\varepsilon(k)\to\lambda'(0)\int V(x)\,dx e^{-iky}=\alpha e^{-iky}$. Consequently, $-\Delta+V_\varepsilon\to-\Delta+\alpha\delta_y$ in the sense of norm group convergence. This type of approximation specializes to standard Gaussian and square well approximations of δ functions.

Norm resolvent convergence is proved in [1, Theorem I.3.2.3], but we show in section 3 that it is an immediate consequence of pointwise bounded convergence of the potentials. It is clear that this may be extended to finitely many point interactions centered at $y_1, \ldots, y_n\in\mathbf{R}$.

In section 3, we discuss our results for infinitely many point interactions, which differ in several respects from those in [1]; for example, we must assume that the coupling constants form a convergent series in order to control the behavior of the Fourier transforms, thus precluding the possibility of gaps in the essential spectrum that occur, for example, in the case of periodic potentials. Moreover, we need not assume that the singularities of the potential are separated by a minimum positive distance.

In section 4, we give a counterexample to show that our results cannot be strengthened to the extent that convergence of potentials in two natural topologies on potentials which are weaker than that of pointwise bounded convergence fail to imply norm convergence for the corresponding unitary groups.

Remarks. – At this point we make some observations about the results in this paper and also some of their immediate implications:

1. We remark first that though we show here that pointwise bounded convergence of (Fourier transform of) potentials V_n to V implies unitary group convergence of the corresponding Schrödinger operators,

$e^{itH_n} \to e^{itH}$, this extends immediately to convergence of a larger class of functions of the operators. Specifically let f be the Fourier transform of an L^1 function or, more generally, of a bounded signed measure μ. Then if $V_n \to V$ boundedly, we have

$$f(H_n) - f(H) = \frac{1}{\sqrt{2\pi}} \int (e^{-ixH_n} - e^{-ixH})\, d\mu.$$

Thus, taking norms of both sides and using the finiteness of μ, it follows that $f(H_n) - f(H) \underset{n\to\infty}{\to} 0$ in norm (using the results in this paper on norm group convergence).

2. The type of convergence proved here, norm group convergence, is in a sense the strongest type of convergence possible for unbounded operators. Indeed, if $e^{itH_n} \to e^{itH}$ in the norm group sense and in addition the operators H_n, H commute and have a common domain, then this implies that $H_n \to H$ essentially uniformly, in the sense that

$$\sup_{\varphi \in \mathcal{D}(H);\ \|\varphi\| = 1} \|(H_n - H)\varphi\| \underset{n\to\infty}{\to} 0.$$

3. We remark that for $V_n \to V$ (in the sense that their Fourier transforms converge) pointwise boundedly it is sufficient that $\|V_n - V\|_{L^1} \to 0$, or, more generally, that V_n and V are finite signed measures on $\mathbf{R}^1$, and $V_n \to V$ weakly. Thus these last two types of convergence for potentials are subsumed by our results.

4. Finally, we compare the type of convergence for unbounded operators that we are considering to so-called generalized (graph) convergence [7]. It is shown in [7] that generalized convergence is equivalent to norm resolvent convergence (Theorem IV.2.23). Since norm group convergence is strictly stronger than norm resolvent convergence, our results show that pointwise bounded convergence for potentials implies generalized convergence for the corresponding Schrödinger operators. Hence the results here hold if norm group convergence is replaced by generalized convergence.

2. NORM GROUP CONVERGENCE

We begin by introducing the basic notion of convergence of potentials that we shall employ throughout. Let $\mathcal{D}(\mathbf{R}^1)$ denote the set of all distributions on $\mathbf{R}^1$, and $\tilde{\mathcal{D}}(\mathbf{R}^1) = \{ V \in \mathcal{D} \mid \hat{V} \in L^\infty(\mathbf{R}^1) \text{ and } \hat{V}(-k) = \hat{V}(k) \}$.

DEFINITION 1. — Let $\{V_n\}$, $V \in \tilde{\mathcal{D}}$. We say V_n converges to V *pointwise boundedly* (p. b.) if $\sup_n \|\hat{V}_n\|_\infty < \infty$, and $\hat{V}_n(k) \to \hat{V}_n(k)$ a.e.

For notational convenience, let $W_t(x)=\dfrac{e^{itx}-1}{ix}$, $t\in\mathbf{R}$; we shall suppress t when no confusion can arise. Constants will generically be denoted by C, though their value may change in the course of a proof.

The following basic result will be used in our main theorem.

BASIC LEMMA. – *If* V_k, $V\in\tilde{\mathcal{D}}$, *and* $V_k\to V$ *p. b., then*

$$\lim_{k\to\infty}\sup_{x\in\mathbf{R}}\int_{-\infty}^{\infty}|\hat{V}_k(x-y)-\hat{V}(x-y)||W_t(x^2-y^2)|\,dy=0.$$

Proof. – For $k\in\mathbf{Z}^+$, set

$$M_k(x)=\int_{-\infty}^{\infty}|\hat{V}_k(x-y)-\hat{V}(x-y)||W_t(x^2-y^2)|\,dy.$$

Choose $x_k\in\mathbf{R}$ so that

$$M(x_k)>\frac{1}{2}\sup_x M_k(x).$$

We will show that $\lim_{k\to\infty}M(x_k)=0$. Assume this does not occur. Then there exists an $\varepsilon>0$, and a subsequence $\{x_{k_i}\}$, for which $M(x_{k_i})>\varepsilon$. We consider two cases:

Case 1. – $|x_{k_i}|\leqq a$ for some $a\in\mathbf{R}$, and all i. Then we observe that

$$|W_t(x_{k_i}^2-y^2)|\leqq\frac{C}{|x_{k_i}^2-y^2|+1}\leqq\frac{C'}{1+y^2},$$

so that

$$\begin{aligned}M(x_{k_i})&\leqq C\int_{-\infty}^{\infty}\frac{1}{1+y^2}|\hat{V}_{k_i}(x_{k_i}-y)-\hat{V}(x_{k_i}-y)|\,dy\\&=C\int_{-\infty}^{\infty}\frac{1}{1+(x_{k_i}-y)^2}|\hat{V}_{k_i}(y)-\hat{V}(y)|\,dy\\&\leqq C\int_{-\infty}^{\infty}\frac{1}{1+y^2}|\hat{V}_{k_i}(y)-\hat{V}(y)|\,dy\\&\to 0\quad\text{as } k_i\to\infty,\end{aligned}$$

by the dominated convergence theorem, and we have a contradiction.

Case 2. — $|x_{k_{i_j}}| \to \infty$ for some sub-subsequence, which for simplicity we denote $\{x_{k_j}\}$. Then $M(x_{k_j}) \leqq C \int |W_t(x_{k_j}^2 - y^2)| \, dy$. But

$$\lim_{|x| \to \infty} \int_{-\infty}^{\infty} |W_t(x^2 - y^2)| \, dy = 2 \lim_{|x| \to \infty} \int_0^{\infty} |W_t(x^2 - y^2)| \, dy.$$

Provided $|x| > 1$, write

$$\int_0^{\infty} |W_t(x^2 - y^2)| \, dy \leqq C \left[\int_0^{\sqrt{x^2-1}} \frac{1}{x^2 - y^2} dy + \int_{\sqrt{x^2-1}}^{\sqrt{x^2+1}} 1 \, dy + \int_{\sqrt{x^2+1}}^{\infty} \frac{1}{y^2 - x^2} dy \right]$$

for a suitable constant. By an elementary calculation, we see that $\lim_{|x| \to \infty} \int_0^{\infty} |W_t(x^2 - y^2)| \, dy = 0$, again contradicting the assumption that this does not occur.

A simple corollary of the proof of our Basic Lemma (with a proof essentially identical) is:

COROLLARY:

$$\lim_{k \to \infty} \sup_x \int_{-\infty}^{\infty} \frac{|\hat{V}_k(x-y) - \hat{V}(x-y)|}{|x^2 - y^2| + 1} dy = 0.$$

Before proceeding to our main theorem, we recall some of the methods and terminology from [6], where the unitary group with self-adjoint generator $-\Delta + V$, for $V \in \tilde{\mathscr{D}}$, was obtained as a time-ordered exponential series in the interaction representation. Specifically, setting $H_0 = -\Delta$, $e^{it(H_0+V)} = e^{itH_0} \mathscr{F}^{-1} U(t, 0) \mathscr{F}$, where $\mathscr{F}$ is the L^2-Fourier transform, and $U(t, 0) = \sum_{n=0}^{\infty} (i)^n K_n(t, 0)$, with $K_n(t, 0)$ (bounded) integral operators defined below. In a formal sense that can be made precise by taking Fourier transforms, $\mathscr{F}^{-1} U(t, 0) \mathscr{F}$ is the unitary propagator for the time-dependent Schrödinger equation with Hamiltonian $H_I(t) = e^{-itH_0} V e^{itH_0}$. The unitary groups are then shown to depend continuously on V in the following sense:

(i) if $\hat{V}_n \to \hat{V}$ uniformly on $\mathbf{R}^1$, then $e^{it(H_0+V_n)} \to e^{it(H_0+V)}$ in the uniform operator topology ([6], Theorem 4.1), and

(ii) if $V_n \to V$ pointwise boundedly, then the corresponding groups converge in the strong operator topology ([6], Theorem 4.2).

Our main theorem strengthens the results of [6] for scaled short-range Hamiltonians whose Fourier transforms cannot converge in the sense (i).

From [6], we have several definitions and results (we suppress the dependence on t and V whenever they are not explicitly used):

$$K_0(x, y) = \delta(x-y)$$

(*i.e.*, K_0 corresponds to the identity operator);

$$K_1(t, x, y) = W_t(x^2-y^2)\hat{V}(x-y),$$

and for $n \geqq 2$,

$$K_n(t, x, y) = \int_0^t \int_{-\infty}^{\infty} e^{is(x^2-w^2)}\hat{V}(x-w)K_{n-1}(s, w, y)\,dw\,ds;$$

we suppress the t when no confusion can arise. In addition we define

$$\bar{K}_1(x, y, z) = W_t(x^2-y^2)\hat{V}(z-y),$$

and for $n \geqq 2$,

$$\bar{K}_n(x, y, z) = \int_0^t \int_{-\infty}^{\infty} e^{is(x^2-w^2)}\hat{V}(z-w)K_{n-1}(s, w, y)\,dw\,ds.$$

Note that if $x=z$, then $\bar{K}_n = K_n$. For all $n \geqq 1$, we have (with K^* the adjoint of K)

$$\sup_x \int_{-\infty}^{\infty} |K_n(x, y)|\,dy < \infty, \qquad \sup_x \int_{-\infty}^{\infty} |K_n^*(x, y)|\,dy < \infty. \tag{1}$$

where K_n^* is the adjoint of K_n, with kernel $K_n^*(x, y) = \overline{K_n(y, x)}$. Note that (*cf.* [6]) (1) guarantees that K is bounded. Finally, we also have

$$\sup_{x, z} \int_{-\infty}^{\infty} |\bar{K}_n(x, y, z)|\,dy < \infty$$

(*cf.* [6], Theorem 3.4).

The proof of our main theorem will proceed as follows. If K_{n, V_k} denotes the operator with kernel $K_n(x, y)$ (arising from potential V_k), we will show that when $V_k \to V$ boundedly, then $K_{n, V_k} \to K_{n, V}$ in norm. Because (*see* [6], Theorem 4.1)

$$\| K_{n, V_k} \| \leqq M c^{n-1} |t|^{n/2},$$

for t sufficiently small with M and c constants independent of n and k, it will follow that the corresponding time-ordered series converge in norm: $\lim_{k \to \infty} \| U_{V_k}(t, 0) - U_V(t, 0) \| = 0$. The extension to large values of $|t|$ is a consequence of the propagator relation $U(t, r)U(r, s) = U(t, s)$, and the unitarity of all operators involved, as discussed in [6]. Since our Basic Lemma already establishes the case $n=1$, we shall begin by considering in detail the case $n=2$, to illuminate the inductive proof that follows and form its basis step. First, we have a technical lemma.

Note. — Omitted limits of integration should be understood as $-\infty$ to ∞. Also, the kernel of K_1 should be $\dfrac{e^{-it(x^2-y^2)}-1}{-i(x^2-y^2)}\hat{V}(x-y)$, but we use $W_t(x^2-y^2)\hat{V}(x-y)$ for simplicity.

LEMMA 1. — *For all $x, z \in \mathbf{R}$, and $n \geqq 2$,*

$$\frac{1}{|x^2-z^2|}\int |\bar{K}_{n-1}(x, y, z)-K_{n-1}(z, y)|\,dy \leqq \frac{C}{|x^2-z^2|+1}, \qquad (2)$$

where C *may depend on n, but not on x and z.*

Proof. — If $|x^2-z^2|>1$, it suffices to note that the integral in (2) is bounded as a function of x and z, by (1). In the event $|x^2-z^2|\leqq 1$, recall from [6] that (2) is bounded by

$$\int\left|\int_0^t e^{is(x^2-z^2)}\hat{V}(x-z)K_{n-1}(s, z, y)\,ds\right|dy + \int |W_t(x^2-z^2)\hat{V}(x-z)K_{n-1}(t, z, y)|\,dy \leqq C\sup_{\substack{z\in\mathbf{R}\\ s\in[0,t]}}\int |K_{n-1}(s, z, y)|\,dy \leqq C',$$

and the Lemma follows.

Our induction procedure for bounding the differences $K_{n, V_k}-K_{n, V}$ will start at $n=2$, so that we need the following.

LEMMA 2. — *For V_k, $V \in \tilde{\mathscr{D}}$, if $V_k \to V$ boundedly, then*

$$\lim_{k\to\infty}\sup_x \int_{-\infty}^{\infty} |K_{2, V_k}(x, y)-K_{2, V}(x, y)|\,dy = 0.$$

Proof. — From the definition of K_2, the integral in the statement of the Theorem equals

$$\int\left|\int\int_0^t e^{is(x^2-z^2)}(\hat{V}_k(x-z)K_{1, V_k}(s, z, y)-\hat{V}(x-z)K_{1, V}(s, z, y))\,ds\,dz\right|dy$$
$$\leqq \int\left\{\left|\int(\hat{V}_k(x-z)-\hat{V}(x-z))\int_0^t e^{is(x^2-z^2)}W_s(z^2-y^2)\,ds\,\hat{V}(z-y)\,dz\right| + \left|\int \hat{V}_k(x-z)\int_0^t e^{is(x^2-z^2)}(\hat{V}_k(z-y)-\hat{V}(z-y))W_s(z^2-y^2)\,ds\,dz\right|\right\}dy. \qquad (3)$$

Note that the two terms on the right of (3) are almost identical, the main difference being that $x-z$ in $\hat{V}_k(x-z)-\hat{V}(x-z)$ in the first term is replaced by $z-y$ in the second term. The first term in (3), after integration by parts with respect to *ds*, is bounded by a constant times

$$\iint |\hat{V}_k(x-z)-\hat{V}(x-z)|\,|W_t(x^2-z^2)|\,|W_t(z^2-y^2)|\,dz\,dy$$
$$+\iint |\hat{V}_k(x-z)-\hat{V}(x-z)|\left|\int_0^t e^{is(z^2-y^2)}W_s(x^2-z^2)\,ds\right|dz\,dy. \quad (4)$$

The supremum over x of the first term in (4) approaches zero as $k\to\infty$, for integrating first with respect to dy, it is bounded by $C\int |\hat{V}_k(x-z)-\hat{V}(x-z)|\,|W_t(x^2-z^2)|\,dz$, and the supremum over x of this latter quantity approaches zero, by our Basic Lemma. The constant above is $C=\sup_z \int |M_t(z^2-y^2)|\,dy$, which is finite. Integrating the second term of (4) with respect to *ds*, we obtain a bound of

$$\iint \frac{|\hat{V}_k(x-z)-\hat{V}(x-z)|}{|x^2-z^2|}|W_t(x^2-y^2)-W_t(z^2-y^2)|\,dy\,dz$$
$$\leqq C\iint \frac{|\hat{V}_k(x-z)-\hat{V}(x-z)|}{|x^2-z^2|+1}\,dz,$$

by a slight modification of (2), with $\bar{K}_1(x, y, z)$ replaced by $K_1(x, y)$ (and the same proof). By the Corollary, the supremum over x of the last integral approaches zero as $k\to\infty$.

To consider the second term in (3), set $J=\{z\,|\,|x-z|<1 \text{ or } |x+z|<1\}$, and integrate with respect to z first over J. We have (changing the order of integration),

$$\sup_x \int_0^t \int_J \int |\hat{V}_k(z-y)-\hat{V}(z-y)|\,|W_s(z^2-y^2|\,dy\,dz\,ds$$
$$\leqq \sup_x \int_0^t \int_J \sup_z \int |\hat{V}_k(z-y)-\hat{V}(z-y)|\,|W_s(z^2-y^2|\,dy\,dz\,ds$$
$$\leqq C\int_0^t \sup_z \int |\hat{V}_k(z-y)-\hat{V}(z-y)|\,|W_s(z^2-y^2)|\,dy\,ds.$$

Now by our Basic Lemma, the integrand with respect to *ds* approaches zero pointwise as $k\to\infty$, and is bounded as a function of s. Hence by the dominated convergence theorem, the last integral approaches zero as $k\to\infty$.

Now for the part over $\mathbf{R}\diagdown\mathrm{J}$, we integrate the second term in (3) by parts with respect to ds. Our bound is then

$$\int\int_{\mathbf{R}\diagdown\mathrm{J}} |\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|\,|\mathrm{W}_t(x^2-z^2)|\,.\,|\mathrm{W}_t(z^2-y^2)|\,dz\,dy$$
$$+\int\int_{\mathbf{R}\diagdown\mathrm{J}} \frac{|\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|}{|x^2-z^2|}|\mathrm{W}_t(x^2-y^2)-\mathrm{W}_t(z^2-y^2)|\,dz\,dy. \quad (5)$$

Considering the first term, we have

$$\sup_x \int_{\mathbf{R}\diagdown\mathrm{J}}\int |\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|\,|\mathrm{W}_t(x^2-z^2)|\,|\mathrm{W}_t(z^2-y^2)|\,dy\,dz$$
$$=\sup_x \int_{\mathbf{R}\diagdown\mathrm{J}} |\mathrm{W}_t(x^2-z^2)| \int |\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|\,|\mathrm{W}_t(z^2-y^2)|\,dy\,dz.$$

The inner integral approaches zero as $k\to\infty$ uniformly in z, by the Basic Lemma. So given $\varepsilon>0$, choose k so large that the inner integral is less than ε. Since $\sup_x \int |\mathrm{W}_t(x^2-z^2)|\,dz<\infty$, the supremum approaches zero as $k\to\infty$. For the second term in (5), we have

$$\sup_x \int\int_{\mathbf{R}\diagdown\mathrm{J}} \frac{|\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|}{|x^2-z^2|}|\mathrm{W}_t(x^2-y^2)-\mathrm{W}_t(z^2-y^2)|\,dz\,dy. \quad (6)$$

We repeat the argument in our proof of the Basic Lemma, choosing a sequence $\{x_k\}$ so that (6) is less than a constant multiple of

$$\int\int_{\mathbf{R}\diagdown\mathrm{J}} \frac{|\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|}{|x_k^2-z^2|}|\mathrm{W}_t(x_k^2-y^2)-\mathrm{W}_t(z^2-y^2)|\,dz\,dy$$
$$\leqq \mathrm{C}\int\int_{\mathbf{R}\diagdown\mathrm{J}} \frac{|\hat{\mathrm{V}}_k(z-y)-\hat{\mathrm{V}}(z-y)|}{|x_k^2-z^2|+1}|\mathrm{W}_t(x_k^2-y^2)-\mathrm{W}_t(z^2-y^2)|\,dz\,dy. \quad (7)$$

(Note that J now depends on k.) Assuming that (7) does not approach zero as $k\to\infty$, we obtain a subsequence $\{x_{k_i}\}$ as before, for which (7) is bounded away from zero. If $|x_{k_i}|\leqq a$ for some $a>0$ and all i, then (6) is bounded by

$$\mathrm{C}\int\int \frac{|\hat{\mathrm{V}}_{k_i}(z-y)-\hat{\mathrm{V}}(z-y)|}{z^2+1}|\mathrm{W}_t(x_{k_i}^2-y^2)-\mathrm{W}_t(z^2-y^2)|\,dy\,dz. \quad (8)$$

Now

$$\int |\hat{\mathrm{V}}_{k_i}(z-y)-\hat{\mathrm{V}}(z-y)|\,|\mathrm{W}_t(x_{k_i}^2-y^2)-\mathrm{W}_t(z^2-y^2)|\,dy \underset{i\to\infty}{\to} 0$$

because the integrand approaches zero pointwise ($|W_t(x_{k_i}^2 - y^2)|$ is bounded), and is bounded by $\frac{C_1}{y^2+1} + \frac{C_2}{|z^2-y^2|+1}$. Thus the dz-integrand approaches zero pointwise, and is bounded by $\frac{C}{z^2+1}$, since

$$\sup_z \int |W_t(z^2-y^2)|\, dy < \infty.$$

Again by the dominated convergence theorem, (8) approaches zero as $k \to \infty$. In the event a sub-subsequence $|x_{k_{i_j}}| \to \infty$, (6) is bounded by

$$C \int_{\mathbf{R} \setminus \mathrm{J}} \frac{1}{|x_{k_{i_j}}^2 - z^2|}\, dz \to 0 \quad \text{as } k_{i_j} \to \infty,$$

as observed in the proof of the Basic Lemma. These contradictions establish Lemma 2.

We come now to our main result.

THEOREM 1. – *If* $V_k \to V$ *pointwise boundedly, then* $-\Delta + V_k \to -\Delta + V$ *in the sense of norm group convergence.*

Proof. – We will show that for all $n \geq 1$, V_k, $V \in \mathscr{D}$,

$$\lim_{k \to \infty} \sup_x \int_{-\infty}^{\infty} |K_{n, V_k}(x, y) - K_{n, V}(x, y)|\, dy = 0; \tag{9}$$

by the remarks preceding Lemma 1, this is sufficient. We have already established the result for $n=1$ and 2, by the Basic Lemma and Lemma 2, respectively.

The above supremum is bounded by

$$\begin{aligned} &\sup_x \int\int |\hat{V}_k(x-z) - \hat{V}(x-z)| \cdot \left| \int_0^t e^{is(x^2-z^2)} K_{n-1, V}(s, z, y)\, ds \right| dz\, dy \\ &\qquad + \sup_x \int\int |\hat{V}_k(x-z)| \\ &\qquad \times \left| \int_0^t e^{is(x^2-z^2)} [K_{n-1, V_k}(s, z, y) - K_{n-1, V}(s, z, y)]\, ds \right| dz\, dy. \end{aligned} \tag{10}$$

We shall refer to these two terms as (10.1) and (10.2). Integrating (10.1) by parts with respect to ds, we get a bound of

$$\sup_x \iint |\hat{V}_k(x-z)-\hat{V}(x-z)| . |W_t(x^2-z^2)| |K_{n-1,V}(t, z, y)| \, dy \, dz$$
$$+\sup_x \iint \frac{|\hat{V}_k(x-z)-\hat{V}(x-z)|}{|x^2-z^2|} . |\bar{K}_{n-1}(t, x, y, z)-K_{n-1}(t, z, y)| \, dy \, dz;$$

we call these (10.1 A) and (10.1 B), respectively.

The term (10.1 A) is bounded by

$$C \sup_x \int |\hat{V}_k(x-z)-\hat{V}(x-z)| . |W_t(x^2-z^2)| \, dz,$$

which approaches zero as $k \to \infty$, by the Basic Lemma. We use here the fact that $\sup_z \int |K_{n-1}(z, y)| \, dy < \infty$ [*see* (1)]. For (10.1 B), use Lemma 1 to obtain a bound of

$$C \sup_x \int \frac{|\hat{V}_k(x-z)-\hat{V}(x-z)|}{|x^2-z^2|+1} \, dz \to 0,$$

as $k \to \infty$, by the Corollary to our Basic Lemma. Now, term (10.2) is bounded by

$$C \sup_x \iint \left| \int_0^t e^{is(x^2-z^2)} [K_{n-1,V_k}(s, z, y)-K_{n-1,V}(s, z, y)] \, ds \right| dz \, dy. \quad (11)$$

We will prove by induction that this term approaches zero as $k \to \infty$. For $n=2$, this was shown in the course of proving Lemma 2. Indeed, for $n=2$, (11) is bounded in exactly the same way as the second term on the right side of (3). So we assume the result holds with n replaced by $n-1$ [that is, with K_{n-2} in the integral (11)], and show it then holds for n (that is, with K_{n-1} in the integral). We first note that this inductive hypothesis implies that

$$\lim_{k \to \infty} \sup_x \int |K_{n-1,V_k}(x, y)-K_{n-1,V}(x, y)| \, dy = 0, \quad (12)$$

for we already know that (10.1) $\underset{k \to \infty}{\to}$ 0 for all n. [This explains our choice of inductive hypothesis, rather than the more obvious (12).]

We do the dz-integration over J and $\mathbf{R}\diagdown J$, as in Lemma 2. Changing the order of integration in (11), we have the bound

$$C \sup_x \int_0^t \int_J \int | K_{n-1, V_k}(s, z, y) - K_{n-1, V}(s, z, y) | \, dy \, dz \, ds$$
$$\leqq C \int_0^t \sup_z \int | K_{n-1, V_k}(s, z, y) - K_{n-1, V}(s, z, y) | \, dy \, ds.$$

By (12), the integrand approaches zero as $k \to \infty$, for almost every s. It is also bounded as a function of s, so by dominated convergence, the entire integral on the right approaches zero as $k \to \infty$. Now, on $\mathbf{R}\diagdown J$, we integrate by parts with respect to ds in (11), yielding

$$\sup_x \int_{\mathbf{R}\diagdown J} \int | W_t(x^2 - z^2) | . | K_{n-1, V_k}(t, z, y) - K_{n-1, V}(t, z, y) | \, dy \, dz$$
$$+ \sup_x \int_{\mathbf{R}\diagdown J} \int \frac{1}{|x^2 - z^2|} | \bar{K}_{n-1, V_k}(t, x, y, z) - \bar{K}_{n-1, V}(t, x, y, z) | \, dy \, dz$$
$$+ \sup_x \int_{\mathbf{R}\diagdown J} \int \frac{1}{|x^2 - z^2|} | K_{n-1, V_k}(t, z, y) - K_{n-1, V}(t, z, y) | \, dy \, dz,$$

which we denote (11.1), (11.2), and (11.3), respectively.

Now (11.1) and (11.3) approach zero as $k \to \infty$, by (12). Term (11.2) is bounded by

$$\sup_x \int_{\mathbf{R}\diagdown J} \int \int \frac{| \hat{V}_k(z-w) - \hat{V}(z-w) |}{|x^2 - z^2|} \left| \int_0^t e^{is(x^2 - w^2)} K_{n-2, V}(s, w, y) \, ds \right| dw \, dy \, dz$$
$$+ \sup_x \int_{\mathbf{R}\diagdown J} \int \int \frac{| \hat{V}_k(z-w) |}{|x^2 - z^2|}$$
$$\times \left| \int_0^t e^{is(x^2 - w^2)} [K_{n-2, V_k}(s, w, y) - K_{n-2, V}(s, w, y)] \, ds \right| dw \, dy \, dz,$$

which we denote (11.2 A) and (11.2 B). The term (11.2 B) approaches zero by the inductive hypothesis. Indeed, given $\varepsilon > 0$, choose k so large that the dz-integrand is less than $\frac{\varepsilon}{|x^2 - z^2|}$. Since

$$\sup_x \int_{\mathbf{R}\diagdown J} \frac{1}{|x^2 - z^2|} dz < \infty,$$

we are done. As for (11.2A), we argue as in the proof of Lemma 2. Choose $\{x_k\}$ so that (11.2A) is less than

$$C\int_{\mathbf{R}\setminus J}\int\int\frac{|\hat{V}_k(z-w)-\hat{V}(z-w)|}{|x_k^2-z^2|+1} \times\left|\int_0^t e^{is(x_k^2-w^2)}K_{n-2,v}(s,w,y)\,ds\right|dw\,dy\,dz. \tag{13}$$

If (13) does not approach zero, obtain a subsequence $\{x_{k_i}\}$ for which it is bounded away from zero. If $|x_{k_i}|\leqq a$ for all i, then (13) is bounded by

$$C\int\int\int\frac{|\hat{V}_{k_i}(z-w)-\hat{V}(z-w)|}{z^2+1}\left|\int_0^t e^{is(x_{k_i}^2-w^2)}K_{n-2,v}(s,w,y)\,ds\right|dw\,dy\,dz.$$

Now the dz-integrand is bounded by $\dfrac{C}{z^2+1}$, and approaches zero pointwise a.e. Indeed,

$$\begin{aligned}\int\int|\hat{V}_{k_i}(z-w)-\hat{V}(z-w)|.&\left|\int_0^t e^{is(x_{k_i}^2-w^2)}K_{n-2,v}(s,w,y)\,ds\right|dw\,dy\\ \leqq\int\int&|\hat{V}_{k_i}(z-w)-\hat{V}(z-w)|.|W_t(x_{k_i}^2-w^2)|.|K_{n-2,v}(t,w,y)|\,dw\,dy\\ +\int\int&\frac{|\hat{V}_{k_i}(z-w)-\hat{V}(z-w)|}{|x_{k_i}^2-w^2|}|\bar{K}_{n-2,v}(x_{k_i},y,w)-K_{n-2,v}(w,y)|\,dw\,dy,\end{aligned}$$

after integrating by parts with respect to ds. Using (1) and Lemma 1, both terms are bounded by

$$C\int\frac{|\hat{V}_{k_i}(z-w)-\hat{V}(z-w)|}{w^2+1}dw\to 0$$

as $k_i\to\infty$.

Finally, if $|x_{k_{i_j}}|\to\infty$ for some sub-subsequence, note that along this subsequence (13) is bounded by $C\int_{\mathbf{R}\setminus J}\dfrac{1}{|x_{k_{i_j}}^2-z^2|+1}dz \underset{j\to\infty}{\to} 0$, using the proof that 10.1 in (10) is bounded. This again contradicts the assumption that (13) does not approach zero, proving (9). From the general remarks preceding Lemma 1, we thus obtain convergence of the corresponding unitary groups.

3. NORM RESOLVENT CONVERGENCE AND INFINITELY MANY POINT INTERACTIONS

The Fourier transform approach leads us to a simple proof of norm resolvent convergence that is independent of Theorem 1 (*cf.* [1], Theorem III.2.2). Indeed, if $V_n \to V$ boundedly, then standard Sobolev estimates imply that $V_n \to V$ in norm as operators from H^1 to H^{-1} (*see* Proposition 1 below). Consequently, $H_0 + V_n \to H_0 + V$ in the norm resolvent sense (*cf.* [10], Theorem VIII.25 (*c*)), and so the approximation by means of local scaled short-range interactions obtained in [1], Theorems I.3.2.3 and II.2.2.2, follow immediately. Note that if $V \in \tilde{\mathscr{D}}$, then the quadratic form $V(\varphi, \psi) = \langle \hat{V} * \hat{\varphi}, \hat{\psi} \rangle$ defined on H^1 gives rise in the usual way to an operator in $L(H^1, H^{-1})$; we write $V(\varphi, \psi) = \langle V\varphi, \psi \rangle$.

PROPOSITION 1. — *If* V_k, $V \in \tilde{\mathscr{D}}$, *and* $V_k \to V$ *boundedly, then*

$$\sup_{\substack{\varphi,\, \psi \in H^1 \\ \varphi,\, \psi \neq 0}} \frac{|\langle (V_k - V)\varphi, \psi \rangle|}{\|\varphi\|_1 \|\psi\|_1} \to 0$$

as $k \to \infty$.

Proof. — For $\varphi, \psi \in H^1$, and $\varepsilon > 0$, we have

$$|\langle (V_k - V)\varphi, \psi \rangle| = \left| \int\!\!\int (\hat{V}_k(x-y) - \hat{V}(x-y))\, \hat{\varphi}(y)\, dy\, \overline{\hat{\psi}(x)}\, dx \right|$$
$$\leqq \left(\int\!\!\int \frac{|\hat{V}_k(x-y) - \hat{V}(x-y)|^2}{(1+y^2)(1+x^2)}\, dx\, dy \right)^{1/2} \|\varphi\|_1 \|\psi\|_1$$

$< \varepsilon \|\varphi\|_1 \|\psi\|_1$ (with $\|\cdot\|_1$ denoting Sobolev H^1 norm), for n sufficiently large, by the dominated convergence theorem.

We now consider the case of infinitely many point interactions. More generally, we recall that the method of [6] enables us to define $-\Delta + \sum_{i=1}^{\infty} \alpha_i V_i$ as a self-adjoint operator, where $\{V_i\} \subset \tilde{\mathscr{D}}$, $\{\alpha_i\} \in l^1$, and $|\hat{V}_i|$ are uniformly bounded. (The case $V_i = \delta_{x_i}$ yields infinitely many point interactions. Note that while we must assume $\{\alpha_i\} \in l^1$, in order to control the behavior of the Fourier transforms, we need not assume that the singularities of the potential are separated by a minimum positive distance.) With $\{V_i\}$ as described above, and (formally) $V = \sum_{i=1}^{\infty} \alpha_i V_i$, we have as an immediate consequence of Theorem 1 and Proposition 1:

PROPOSITION 2. — *Let* $V_N = \sum_{i=1}^{N} \alpha_i V_i$. *Then* $-\Delta + V_N \to -\Delta + V$ *in both the norm group and norm resolvent sense.*

Comparing this with [1], Theorem III.2.1, we see that the condition on the coupling constants precludes the possibility of gaps in the essential spectrum that occur, for example, in the case of periodic potentials. Thus, in the case of infinitely many δ-interactions considered here, we have $\sigma_{ess}(-\Delta + V) = [0, \infty)$.

To approximate the Hamiltonian $-\Delta + \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ by scaled short-range interactions, let $\{W_i\}$ be real-valued functions such that $|W_i| \leqq W$, $W \in L^1(\mathbf{R}^1)$, and for $\varepsilon > 0$, define $W_{i,\varepsilon}(x) = \frac{1}{\varepsilon} W_i\left(\frac{x - x_i}{\varepsilon}\right)$. Then

$$\sum_{i=1}^{\infty} \alpha_i W_{i,\varepsilon} \underset{\varepsilon \to 0}{\to} \sum_{i=1}^{\infty} \alpha_i \int W_i(x)\, dx\, \delta_{x_i}$$

in the bounded sense, and so the corresponding Hamiltonians converge in both the norm group and norm resolvent sense.

4. A COUNTEREXAMPLE

It is natural to ask whether our result can be strengthened. Bounded convergence may be an unnecessarily strong requirement; it does, after all, imply norm convergence of V_k as operators from H^1 to H^{-1}. We will show here the two natural weakenings of our hypotheses lead to failure of our theorem. The two weakenings (in the following example) involve more general convergence for potentials, *i.e.*, weak form convergence and convergence in the space of tempered distributions.

In a well-known paper (*cf.* [4], p. 359), Friedman gives an example in three dimensions to show that, in general, weak convergence of forms need not imply even strong group convergence. Our example shows this holds as well in one dimension. We remark that Friedman's example shows that Theorem 1 does not hold in three dimensions.

Let $\{V_k\} \subset \tilde{\mathcal{D}}$ satisfy $\hat{V}_k(x) = \sin kx\, e^{-x^2}$. Then for each k, V_k is bounded as a multiplication operator from H^1 to H^{-1}, and

$$\langle V_k \varphi, \psi \rangle = \langle \hat{V}_k * \hat{\varphi}, \hat{\psi} \rangle$$

for all φ, $\psi \in H^1$. Moreover,

$$|\langle V_k \varphi, \psi \rangle| = \left| \int \hat{V}_k(x-y)\,\hat{\varphi}(y)\,\overline{\hat{\psi}(x)}\,dy\,dx \right| \underset{k \to \infty}{\to} 0, \tag{14}$$

by the two dimensional version of the Riemann-Lebesgue lemma and the fact that $\hat{\varphi}(y)\hat{\psi}(x) \in L^1$. In fact (14) shows that $V_k \to 0$ in the strong operator topology of $L(H^1, H^{-1})$, by taking suprema over ψ of unit norm. It is also easy to show that $V_k \to 0$ in the topology of tempered distributions $\mathscr{S}'$. We will show that, however, $e^{it(H_0+V_k)}$ fails to converge to e^{itH_0} in norm, with $H_0 = -\Delta$ in $L^2(\mathbf{R}^1)$. Thus these two weaker forms of convergence do not suffice to guarantee norm group convergence of the corresponding Hamiltonians. That is, the mapping from potentials to Hamiltonians in the norm group topology is discontinuous from either of these two topologies, while it is of course continuous from the topology of pointwise bounded convergence.

To show that $e^{it(H_0+V_k)}$ fails to converge to e^{itH_0} in norm, it suffices to show $U_{V_k}(t, 0) = e^{-itH_0} e^{it(H_0+V_k)}$ fails to converge to $U_0(t, 0)$. Note that if $V \equiv 0$, we have $U_V(t, 0) = U_0(t, 0) = I$, the identity. We will show that $\| U_{V_k}(t, 0) - I \|$ fails to converge to 0. Since the first term in the expansion of $U_{V_k}(t, 0)$ is I, we need to show that the remainder of the expansion fails to vanish in norm as $k \to \infty$. Because multiplication by V_k is a bounded self-adjoint operator on $L^2(\mathbf{R}^1)$, $U_{V_k}(t, 0)$ is given by a norm-convergent time-ordered expansion whose first-order term $K_{1,k}$ is (unitarily equivalent under Fourier transformation to) an integral operator with kernel $W_t(x^2-y^2)\hat{V}_k(x-y)$.

PROPOSITION 3. – *$\| K_{1,k}(t) \| \geq C\,t$, uniformly in k sufficiently large.*

Proof. – The L^2 operator norm of $K_{1,k}(t)$ is

$$\sup_{\|\varphi\| = \|\psi\| = 1} \left| \int\!\!\int W_t(x^2-y^2)\,\hat{V}_k(x-y)\,\psi(y)\,dy\,\overline{\varphi(x)}\,dx \right|.$$

Let $\varphi_k(x) = K_k e^{-x^2} \sin kx$, and $\psi_k(y) = M_k e^{-y^2} \cos ky$, where K_k and M_k are normalizing constants. Then

$$\begin{aligned} \| K_{1,k}(t) \| &\geq \left| \int\!\!\int W_t(x^2-y^2)\,\hat{V}_k(x-y)\,\psi_k(y)\,dy\,\overline{\varphi_k(x)}\,dx \right| \\ &= K_k M_k \left| \int\!\!\int W_t(x^2-y^2)\,e^{-(x-y)^2} \sin k(x-y) \right. \\ &\qquad \left. \times e^{-x^2} \sin kx\, e^{-y^2} \cos ky\,dx\,dy \right| \\ &= K_k M_k \left| \int\!\!\int l_t(x, y) \sin^2 kx \cos^2 ky\,dx\,dy - \frac{1}{4}\int \right. \\ &\qquad \left. \times \int l_t(x, y) \sin 2kx \cos 2ky\,dx\,dy \right|, \end{aligned} \tag{15}$$

where $l_t(x, y) = W_t(x^2 - y^2) e^{-2(x^2+y^2-xy)} \in L^1(\mathbf{R}^2)$. Now the second term in (15) approaches zero by the two-dimensional Riemann-Lebesgue lemma. For the remaining term, note that

$$K_k^{-2} = \int_{-\infty}^{\infty} e^{-2x^2} \sin^2 kx\, dx \to \frac{1}{2} \int_{-\infty}^{\infty} e^{-2x^2} dx,$$

using $\sin^2 kx = \dfrac{1 - \cos 2kx}{2}$ and the Riemann-Lebesgue lemma. Consequently, $K_k \geqq c > 0$, and similarly for M_k. Thus, the remaining term

$$K_k M_k \iint l_t(x, y) \sin^2 kx \cos^2 ky\, dx\, dy \underset{k \to \infty}{\to} C \iint l_t(x, y)\, dx\, dy$$

for a suitable $C > 0$. Note however that

$$\iint l_t(x, y)\, dx\, dy = t \iint \frac{e^{it(x^2-y^2)} - 1}{it(x^2 - y^2)} e^{-2(x^2+y^2-xy)}\, dx\, dy \geqq t\, G(t),$$

where

$$G(t) \underset{t \to 0}{\to} c = \iint e^{-2(x^2+y^2-xy)}\, dx\, dy,$$

by the dominated convergence theorem. This completes the proof of the proposition.

We begin the proof of the main result by noting that $|W_t(x^2 - y^2)| \leqq t$. Recall now that in the expansion $U_{V_k}(t, 0) - I = \sum_{n=1}^{\infty} i^n K_{n,k}(t)$ the operator $K_{n,k}(t) \equiv K_{n, V_k}(t)$ is a bounded integral operator with kernel $K_{n, V_k}(t, x, y) \equiv K_{n,k}(t, x, y)$. We next show that:

PROPOSITION 4. – *The sum* $U_{V_k}^{(2)}(t, 0) = \sum_{n=2}^{\infty} i^n K_{n,k}(t) = O(t^2)$ *in that* $t^{-2} U_{V_k}^{(2)}(t, 0)$ *remains bounded as an* L^2 *operator, uniformly in* k, *as* $t \to \infty$.

Proof. – By the remarks after (1), we can bound $K_{n,k}(t)$ by bounding $\sup_x \int |K_{n,k}(t, x, y)|\, dy$, using $K_n(t, 0, x, y) = K_n(0, t, y, x)$. For $n = 1$, we have

$$\sup_x \int |K_{1,k}(t, x, y)|\, dy = \sup_x \int |W_t(x^2 - y^2) \sin(k(x-y))\, e^{-(x-y)^2}|\, dy$$
$$\leqq \sup_x \int |W_t(x^2 - (y+x)^2)\, e^{-y^2}|\, dy \leqq \sqrt{\pi}\, t.$$

Thus $\| K_{1,k}(t) \| \leqq \sqrt{\pi}\, t$.

We now show by induction that

$$\sup_x \int |K_{n,k}(t, x, y)|\, dy \leqq (\sqrt{\pi}\, t)^n. \tag{16}$$

Using the inductive definition of $K_{n,k}(t)$ we have

$$K_{n,k}(t, x, y) = \int_0^t \int e^{is(x^2 - w^2)}\, \hat{V}(x-w)\, K_{n-1}(s, w, y)\, dw\, ds,$$

so that

$$\sup_x \int |K_{n,k}(t, x, y)|\, dy \leqq \sup_x \int\int\int_0^t |\hat{V}(x-w)\, K_{n-1}(s, w, y)|\, ds\, dw\, dy$$
$$\leqq \sup_x \int\int_0^t |\hat{V}(x-w)| \sup_w \int |K_{n-1}(s, w, y)|\, dy\, ds\, dw$$
$$\leqq \sup_x \int\int_0^t |\hat{V}(x-w)|\, (\sqrt{\pi}\, s)^{n-1}\, ds\, dw$$
$$\leqq (\sqrt{\pi})^{n-1}\, t^n/n \sup_x \int |\hat{V}(x-w)|\, dw$$
$$\leqq (\sqrt{\pi})^{n-1}\, t^n/n \sup_x \int e^{-(x-w)^2}\, dw$$
$$\leqq (\sqrt{\pi})^n\, t^n/n \leqq (\sqrt{\pi}\, t)^n. \tag{17}$$

Thus by (17) and the remark at the beginning of the proof, we conclude $\| K_{n,k}(t) \| \leqq (C_2\, t)^n$ uniformly in k. Hence $\| U_{V_k}^{(2)}(t, 0) \| \leqq \sum_{n=2}^{\infty} (C_2\, t)^n = O(t^2)$ for t sufficiently small, uniformly in k, as desired.

Proposition 3, together with Proposition 4, shows that

$$\| U_{V_k}(t, 0) - U_0(t, 0) \| \geqq C\, t$$

for some constant C and t sufficiently small, uniformly in k, so that this difference does not go to 0, providing the desired counterexample. It is thus not sufficient for $V_k \to V$ in the topology of forms or of distributions for norm convergence of the corresponding Schrödinger operators to occur. We summarize this in:

THEOREM 2. — *The mapping of potentials* V *to operators* $-\Delta+V$ *is discontinuous from either the distributional topology or the weak* H^1 *form topology to the topology of norm group convergence.*

REFERENCES

[1] S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN and H. HOLDEN, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York, 1988.

[2] S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN and W. KIRSCH, On Point Interactions in One Dimension, *J. Operator Theory*, Vol. **12**, 1984, pp. 101-126.

[3] W. G. FARIS, *Self-Adjoint Operators*, Springer-Verlag, New York, 1975.

[4] C. N. FRIEDMAN, Perturbations of the Schrödinger Equation by Potentials with Small Support, *J. Funct. Anal.*, Vol. **10**, 1972, pp. 346-360.

[5] F. GESZTESY and H. HOLDEN, A New Class of Solvable Models in Quantum Mechanics Describing Point Interactions on the Line, *J. Phys.*, Vol. **A 20**, 1987, pp. 5157-5177.

[6] R. HUGHES and I. SEGAL, Singular Perturbations in the Interaction Representation, *J. Funct. Anal.*, Vol. **38**, 1980, pp. 71-98.

[7] T. KATO, *Perturbation Theory for Linear Operators*, Second Edition, Springer-Verlag, New York, 1980.

[8] M. REED and B. SIMON, Methods of Modern Mathematical Physics, Vol. I, *Funct. Anal.*, Academic Press, New York, 1972.

[9] M. REED and B. SIMON, Methods of Modern Mathematical Physics, Vol. II, *Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[10] I. E. SEGAL, Singular Perturbations of Semigroup Generators, Linear Operators and Approximation, *Proceedings, Conference in Oberwolfach*, 1971, pp. 54-68.

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