ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 54, nº 1 (1991), p. 89-113
<http://www.numdam.org/item?id=AIHPA_1991__54_1_89_0>

Ann. Inst. Henri Poincaré,

Vol. 54, n° 1, 1991, p. 89-113. *Physique théorique*

The Bethe lattice spin glass at zero temperature

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ABSTRACT. — We prove the existence of a stable solution of the renormalized fixed point equation for the distribution of the single-site magnetization in the Bethe lattice spin glass at zero temperature. The proof is computer assisted.

RÉSUMÉ. — Nous prouvons l'existence d'une solution stable de l'équation de point fixe de renormalisation pour la distribution de la magnétisation d'un spin dans le modèle de verre de spin sur l'arbre de Bethe à température nulle. La preuve s'appuie sur des calculs numériques exacts.

1. INTRODUCTION

The spin glass on the Bethe lattice was studied intensively last years in works ([1]-[4]). The Hamiltonian of the model is

$$H = - \sum_{\langle i, j \rangle} J_{ij} \sigma_i \sigma_j$$

where $J_{ij} = \pm 1$ with probability 1/2 for any edge $\langle i, j \rangle$ independently. The analysis of the model in [1]-[4] was based on the study of the distribution of the single-site magnetization $M = \langle \sigma_i \rangle$ at the end point of the half-space' Bethe lattice. As the interaction J_{ij} is random the magnetization M is a random variable. It satisfies the fixed point equation

$$M \stackrel{d}{=} F(M) \qquad (1.1)$$

where $\stackrel{d}{=}$ means the equality of the distributions and

$$F(M) = \frac{p_1 M_1 + p_2 M_2}{1 + p_1 p_2 M_1 M_2} \qquad (1.2)$$

where p_1, p_2, M_1, M_2 are independent real random variables,

$$M_1 \stackrel{d}{=} M_2 \stackrel{d}{=} M$$

and $p_1 \stackrel{d}{=} p_2 \stackrel{d}{=} \tanh(J_{ij}/T)$, $T>0$ is the temperature, *i.e.* $p_i = \pm\tanh(1/T)$ with probability 1/2, $i=1, 2$. The distribution of M is symmetric and it is stable in the sense that the iterations

$$M^{(n+1)} \stackrel{d}{=} F(M^{(n)})$$

converge to M in a neighborhood of M.

In [2] it was proved that the trivial solution $M=0$ (*i.e.* $M=0$ with probability 1) is stable for $0<p<p_c$ where $p=\tanh(1/T)$ and $p_c=1/\sqrt{2}$. For $p>p_c$ the trivial solution is unstable and in [2] it was proved that for $p=p_c+\varepsilon$, where $\varepsilon>0$ is sufficiently small there exists a stable non-trivial solution M of equation (1.1). The distribution of M is absolutely continuous with respect to the Lebesgue measure in that case and its density is close to a Gaussian density.

In the present paper we shall prove the existence of a stable solution M of a *renormalized* version of equation (1.1) at $T=0$, the so-called Kwon-Thouless equation (*see* [4]). The structure of the paper is as following. In Sect. 2 we give the renormalized version of equation (1.1) at $T=0$ and formulate our main result. In Sect. 3 the proof of the main result is presented and in Sect. 4 we discuss briefly some unstable solutions of the renormalized equation.

2. RENORMALIZED FIXED POINT EQUATION AND FORMULATION OF THE MAIN THEOREM

Multiplying equation (1.1) by $p_0=\tanh(J_{ij}/T)$ and denoting $p_0 M$ by X we get the equation

$$X \stackrel{d}{=} \frac{p_0 (X_1 + X_2)}{1 + X_1 X_2}.$$

Multiplying now this equation by sgn p_0 and taking into account that X is a symmetric random variable (sgn $p_0 X \overset{d}{=} X$) we get

$$X \overset{d}{=} \frac{p(X_1+X_2)}{1+X_1 X_2} \tag{2.1}$$

where $p=|p_0|=\tanh(1/T)$. Note that $|M|=|\langle \sigma_i \rangle| \leqq 1$ which implies $|X|=|p_0 M| \leqq |p_0|=p$.

Consider the solutions of equation (2.1) for $T=0$, *i.e.* for $p=1$. In that case (2.1) implies

$$X \overset{d}{=} \frac{X_1+X_2}{1+X_1 X_2}. \tag{2.2}$$

Let $X=\tanh Y$. Therefore

$$\tanh Y \overset{d}{=} \frac{\tanh Y_1 + \tanh Y_2}{1+\tanh Y_1 \tanh Y_2} = \tanh(Y_1+Y_2)$$

and

$$Y \overset{d}{=} Y_1 + Y_2.$$

It is known (*see* [5]) that this equation has a unique solution $Y=0$ and it is unstable: the iterations $Y^{(n+1)}=Y_1^{(n)}+Y_2^{(n)}$ goes to infinity as $n \to \infty$ if $Y^{(0)} \not\equiv 0$. This leads to the conclusion that equation (2.2) does not give seemingly a good description of the model at $T=0$. Therefore we shall try to renormalize equation (2.1) in such a way that it would have a non-trivial solution at $T=0$.

Substituting $X=\tanh Y$ into equation (2.1) we get

$$Y \overset{d}{=} \operatorname{arcth}(p \tanh(Y_1+Y_2)) \tag{2.3}$$

where arcth is the inverse function to tanh. Since $|\tanh x|<1$,

$$\sup_{Y_1,\, Y_2} |\operatorname{arcth}(p \tanh(Y_1+Y_2))| < \operatorname{arcth} p,$$

and $|Y|<\operatorname{arcth} p$. Denote

$$Z = Y/\operatorname{arcth} p. \tag{2.4}$$

Then we get by (2.3) that

$$Z \overset{d}{=} f_p(Z_1+Z_2) \tag{2.5}$$

where

$$f_p(x) = \operatorname{arcth}(p \tanh(x \operatorname{arcth} p))/\operatorname{arcth} p. \tag{2.6}$$

In figure 1 graphs of $f_p(x)$ are given for $p=0.9$, 0.99, 0.999, 0.9999. They show that

$$\lim_{p\to 1} f_p(x)=f(x)=\begin{array}{ll} -1, & x\leqq -1, \\ x, & -1\leqq x\leqq 1, \\ 1, & 1\leqq x \end{array} \tag{2.7}$$

(of course the last relation is easy established analytically). Thus for $p=1$ (2.5) is reduced to

$$Z\overset{d}{=}f(Z_1+Z_2). \tag{2.8}$$

For further use we introduce the non-linear operator

$$F:\quad Z\to f(Z_1+Z_2) \tag{2.9}$$

in the space of random variables. Here $Z_1\overset{d}{=}Z$, $Z_2\overset{d}{=}Z$ and Z_1, Z_2 are independent. Then (2.8) is rewritten as

$$Z\overset{d}{=}F(Z). \tag{2.10}$$

The main result of this paper is the following theorem.

THEOREM 2.1. – *Equation* (2.10) *has a stable solution* Z_*.

Numerically it was constructed and studied in [4]. Its density and the distribution function are shown in Figure 2. The density $\rho_*(x)$ can be written in the form

$$\rho_*(x)=v_*\delta(x)+q_*\delta(x-1)+q_*\delta(x+1)+t_*(x), \tag{2.11}$$

where $\delta(x)$ is the Dirac δ-function, $t_*(x)\in L^2([-1, 1])$, and $t_*(-x)=t_*(x)$. To describe the stability properties of the solution Z_* we introduce a general class P of symmetric real random variables Z whose densities are written in the form

$$\rho(x)=v\,\delta(x)+q\,\delta(x-1)+q\,\delta(x+1)+t(x), \tag{2.12}$$

where $t(x)\in L^2(R^1)$, $t(-x)=t(x)$, $t(x)=0$ if $|x|>1$. By the normalization condition $1=\int_0^1 \rho(x)\,dx=v+2q+2\int_0^1 t(x)\,dx$, so

$$v=1-2q-2\int_0^1 t(x)\,dx. \tag{2.13}$$

Define a distance in P by

$$d(Z, Z')=(|q-q'|^2+\|t(x)-t'(x)\|_2^2)^{1/2} \tag{2.14}$$

where $\|t(x)\|_2=\left(\int_0^1 t^2(x)\,dx\right)^{1/2}$. For simplicity we shall denote the set of probability densities of the form (2.12) also by P.

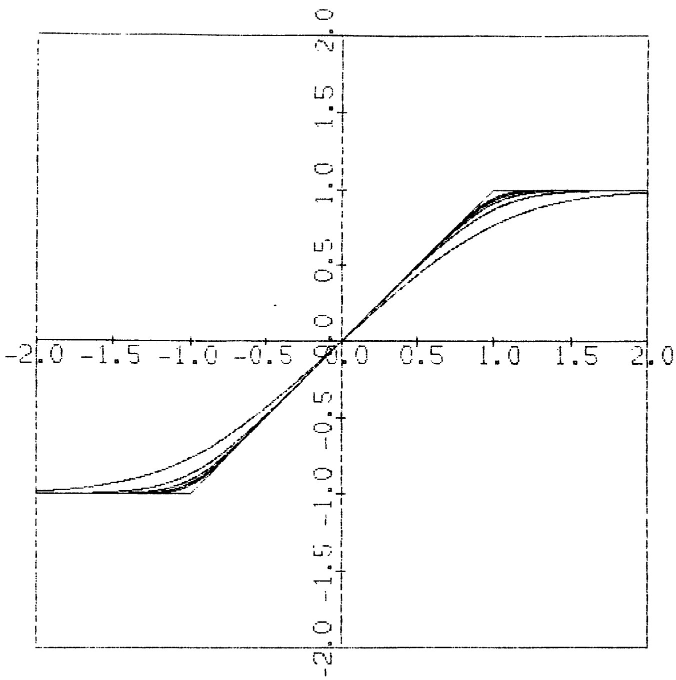

Fig. 1. – Graphs of the functions $f_p(x)$ [*see* (2.6)] for $p=0.9.$, 0.99, 0.999, 0.9999, 0.99999.

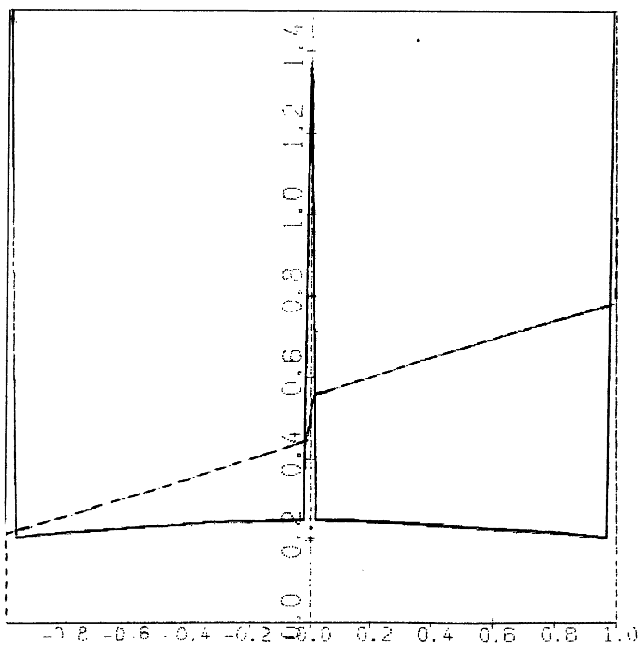

Fig. 2. – The density (solid line) and the distribution function (dash line of the stable solution of equation (2.8)

We introduce also a real Hilbert space H of generalized functions $\rho(x)$ on $\mathbf{R}^1$ of the form (2.12) with the same conditions $t(x) \in L^2(\mathbf{R}^1)$, $t(-x)=t(x)$, $t(x)=0$ if $|x| \geqq 1$, and such that $\int_{\infty}^{\infty} \rho(x)\,dx=0$, *i. e.*

$$v=-2q-2\int_0^1 t(x)\,dx. \qquad (2.15)$$

The space H is isomorphic to the space of pairs $(q, t(x))$ where $q \in \mathbf{R}^1$ and $t(x) \in L^2([0, 1])$ and we shall often identify the element $\rho(x) \in H$ of the form (2.12) satisfying condition (2.15) with the pair $(q, t(x))$. The norm in H is defined as

$$\|(q, t(x))\| = \left(q^2 + \int_0^1 t^2(x)\,dx\right)^{1/2}. \qquad (2.16)$$

It is noteworthy that by (2.12)-(2.14) any element $\rho(x) \in P$ can be written as $\rho(x)=\delta(x)+\rho_0(x)$, where $\rho_0(x) \in H$ and $d(Z, Z')=\|\rho_0-\rho'_0\|$. We introduce also an orthonormal basis in H. Let

$$p_k(x)=\sqrt{2k+1}\,P_k(2x-1),$$

where

$$P_k(x)=\frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$$

is the k-th Legendre polynomial, $k=0, 1, 2, \ldots$ The functions $p_k(x)$ form an orthonormal basis in $L^2([0, 1])$. It enables to write any element $\rho=(q, t(x)) \in H$ as a vector $(q, t_0, t_1, t_2, \ldots)$, where $t(x)=\sum_{k=0}^{\infty} t_k p_k(x)$. The vectors $e=(1, 0, 0, \ldots)$, $e_0=(0, 1, 0, \ldots)$, $e_1=(0, 0, 1, \ldots)$, ... form an orthonormal basis in H.

Consider now a random variable $Z^0 \in P$ whose density is

$$\rho^0(x)=v^0\,\delta(x)+q^0\,\delta(x-1)+q^0\,\delta(x+1)+\sum_{k=0}^{3} t_k^0 p_k(|x|) \qquad (2.17)$$

with $v^0=1-2q^0-2t_0^0$ and

$$\begin{aligned} &q^0=0.218{,}423{,}930{,}8;\ t_0^0=0.228{,}160{,}639{,}3;\\ &t_1^0=-0.012{,}506{,}828{,}5;\ t_2^0=-0.002{,}015{,}912{,}5;\\ &t_3^0=0.000{,}071{,}737{,}9. \end{aligned} \qquad (2.18)$$

We shall verify that Z^0 is an approximate solution of equation (2.10), namely

$$d(F(Z^0), Z^0)<2.10^{-6}. \qquad (2.19)$$

In the proof of Theorem 1 we shall show that there exists a solution Z_* of equation (2.10) which lies in a small neighborhood of Z^0,

$$d(Z_*, Z^0) < 10^{-5}, \tag{2.2}$$

and which possesses the following stability property:

$$d(F(Z), Z_*) \leqq 0.796\, d(Z, Z_*) \tag{2.21}$$

if $d(Z, Z^0) \leqq 5.10^{-4}$. The last property ensures the local stability of Z_* in P.

Numerical simulations indicate that Z_* is seemingly globally stable in P but we can't prove it rigorously. As concerns some unstable solutions of equation (2.10) see a discussion in Sect. 4 below.

Numerical simulations show also that for any $1/\sqrt{2} = p_c < p < 1$ there is a unique stable non-trivial solution $Z_*^{(p)}$ of the equation (2.5) and

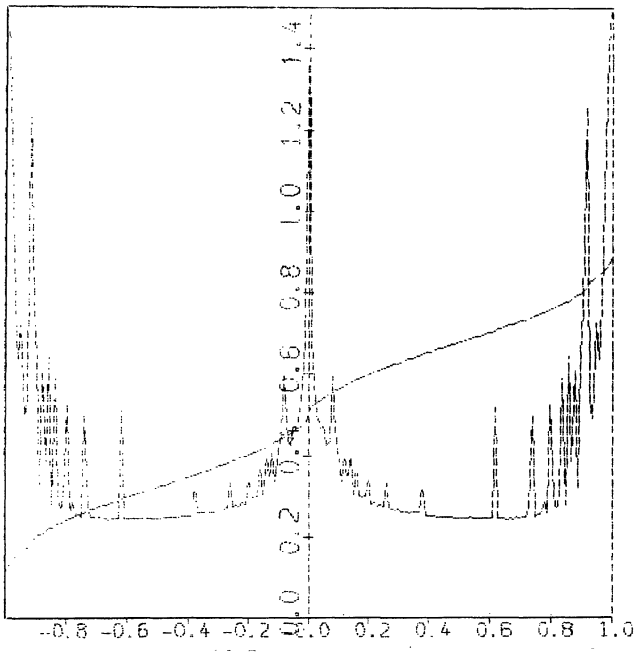

Fig. 3. – Solution $Z_*^{(p)}$ for $p = 0.9995$.

weak $-\lim_{p \to 1} Z_*^{(p)} = Z_*$. In Figure 3 one can see the solution $Z_*^{(p)}$, obtained with the help of computer for $p = 1 - 5.10^{-4}$. One can note that $Z_*^{(p)}$ has a lot of peaks if p is close to 1, so the convergence $Z_*^{(p)} \underset{p \to 1}{\to} Z_*$ is only in the weak sense. We hope in the future to extend the existence theorem to p's sufficiently close to 1.

3. PROOF OF THE MAIN THEOREM

To prove Theorem 2.1 we shall consider Galerkin's approximations of equation (2.10) and construct an approximate solution of this equation and a neighborhood of this approximate solution where the map $Z \to F(Z)$ is contracting. It will be a computer-assisted proof in the sense that we use the computer in estimating various quantities.

Proof of Theorem 2.1. – N-modes Galerkin's approximations. First we write the map F in terms of distribution densities. If $\rho(x)$ is a probability density (with respect to dx) on $\mathbf{R}^1$, define

$$g(\rho)(x)=\chi_{(-1,\,1)}(x)\,\rho(x)+\int_{-\infty}^{-1}\rho(y)\,dy\,\delta(x+1)+\int_{1}^{\infty}\rho(y)\,dy\,\delta(x-1) \quad (3.1)$$

where $\chi_{(-1,\,1)}(x)$ is the characteristic function of the interval $(-1, 1)$. The meaning of the operator $\rho\to g(\rho)$ is that g "rakes up" the mass of the measure $\rho(x)\,dx$ on the half-line $(-\infty, -1]$ to the point -1 and that on the half-line $[1, \infty)$ to the point 1. It is clear that g is extended to a linear operator in the space of probability distributions on $\mathbf{R}^1$.

Now the map F can be described in the following way. Let $\rho(x)$ be a probability distribution density of a random variable Z. Then

$$F:\ \rho(x)\to g(\rho*\rho)(x). \quad (3.2)$$

In fact, the convolution $\rho*\rho$ corresponds to the sum of random variables Z_1+Z_2 in (2.9) and g corresponds to the function f in (2.9). To specialize F for $\rho\in P$ let us give the following definition. Define for $t(x)\in L^2([0, 1])$ the function

$$\hat{t}(x)=\begin{cases} t(x), & x\in[0, 1],\\ 0 & \text{otherwise,}\end{cases}$$

in $L^2(\mathbf{R}^1)$ and for $t(x),\ s(x)\in L^2([0, 1])$ the function

$$A(t, s)(x)=(\hat{t}(x)+\hat{t}(-x))*(\hat{s}(x)+\hat{s}(-x))\,|_{[0,\,1]}, \quad (3.4)$$

where $|_{[0,\,1]}$ is the restriction onto the segment [0, 1], and

$$a(t, s)=\int_1^2 \hat{t}(x)*\hat{s}(x)\,dx. \quad (3.5)$$

Then for $\rho\in P$, $\rho(x)=v\,\delta(x)+q\,\delta(x+1)+q\,\delta(x-1)+t(x)$, we have:

$$F(\rho)(x)=\rho'(x)=g(\rho*\rho)(x)$$
$$=v'\,\delta(x)+q'\,\delta(x+1)+q'\,\delta(x-1)+t'(x) \quad (3.6)$$

where

$$v' = v^2 + 2q^2,$$
$$q' = 2vq + q^2 + 2q\int_0^1 t(x)\,dx + a(t, t), \qquad (3.7)$$
$$t'(x) = 2vt(x) + 2qt(1-x) + \mathrm{A}(t, t)(x), \qquad 0 \leqq x \leqq 1.$$

Substituting $v = 1 - 2q - 2\int_0^1 t(x)\,dx$, we get the reduced system:

$$q' = 2q - 3q^2 - 2q\int_0^1 t(x)\,dx + a(t, t),$$
$$t'(x) = \left(2 - 4q - 4\int_0^1 t(x)\,dx\right) t(x) \qquad (3.8)$$
$$+ 2qt(1-x) + \mathrm{A}(t, t)(x),$$

$0 \leqq x \leqq 1$. Expanding $t(x)$ and $t'(x)$ in the basis $\{p_k(x), k \geqq 0\}$ we get

$$q' = 2q - 3q^2 - 2qt_0 + \sum_{m,\,n=0}^{\infty} a_{mn} t_m t_n,$$
$$t'_k = (2 - 4q - 4t_0 + 2q(-1)^k) \,.\, t_k + \sum_{m,\,n=0}^{\infty} a_{kmn} t_m t_n, \qquad (3.9)$$

$k = 0, 1, 2, \ldots$, where

$$a_{mn} = a(p_m, p_n),$$
$$a_{kmn} = \int_0^1 p_k(x)\, \mathrm{A}(p_m, p_n)(x)\,dx. \qquad (3.10)$$

Now we present some formulae for the coefficients a_{mn}, a_{kmn}. An extended exposition of these formulae with their proofs can be found in the Appendix.

General properties.

$$a_{kmn} = a_{knm} = a_{nmk} = \ldots$$
$$a_{mn} = a_{nm} \qquad (3.11)$$
$$a_{kmn} = 0 \quad \text{if} \quad k > n + m + 1$$
$$a_{mn} = 0 \quad \text{if} \quad m > n + 1. \qquad (3.12)$$

Particular values

$$a_{00} = 1/2, \qquad a_{mm} = 0, \qquad m \geqq 1,$$
$$a_{m,\,m+1} = \frac{(-1)^m}{2\sqrt{(2m+1)(2m+3)}}, \qquad m \geqq 0. \qquad (3.13)$$

$$a_{000}=3/2, \qquad a_{0\,mm}=0, \qquad m\geqq 1,$$
$$a_{0\,m,\,m+1}=\frac{(-1)^{m+1}}{2\sqrt{(2m+1)(2m+3)}}, \qquad m\geqq 0. \tag{3.14}$$
$$a_{1\,mm}=-\frac{(2+(-1)^{m+1})\sqrt{3}}{(2m-1)(2m+3)}, \qquad m\geqq 0,$$
$$a_{1\,m,\,m+1}=\frac{(-1)^{m+1}\sqrt{3}}{2\sqrt{(2m+1)(2m+3)}}, \qquad m\geqq 0, \tag{3.15}$$
$$a_{1\,m,\,m+2}=\frac{(2+(-1)^{m+1})\sqrt{3}}{2(2m+3)\sqrt{(2m+1)(2m+5)}}, \qquad m\geqq 0.$$

We now return to equation (3.9). In the N-modes Galerkin approximation we cut the last system to

$$\begin{aligned} q' &= 2q-3q^2-2qt_0\sum_{m,\,n=0}^{N-1} a_{mn}t_m t_n, \\ t'_k &= (2-4q-4t_0+2q(-1)^k)\,t_k+\sum_{m,\,n=0}^{N-1} a_{kmn}t_m t_n, \end{aligned} \tag{3.16}$$

$k=0, 1, \ldots, N-1$. Define the space

$$P_N=\Big\{\rho(x)=(1-2q-2t_0)\,\delta(x) + q\,\delta(x-1)+q\,\delta(x+1)+\sum_{k=0}^{N-1} t_k p_k(|x|)\Big\}, \tag{3.17}$$

$P_N\subset P$ and the operator

$$F_N:\quad \rho(x)\to\rho'(x),$$

where $\rho'(x)$ is determined by equations (3.16), $F_N: P_N\to P_N$. Then for $\rho\in P_N$,

$$F(\rho)-F_N(\rho)=\sum_{k=N}^{2N-1}\ \sum_{m,\,n=0}^{N-1} a_{kmn}t_m t_n. \tag{3.18}$$

This formula enables to estimate the error $\|F(\rho)-F_N(\rho)\|$.

Let $\rho^0\in P_N$ be an approximate solution of the equation $F_N(\rho)=\rho$. Then we can consider ρ^0 also as an approximate solution of the original equation $F(\rho)=\rho$. The error is estimated as follows,

$$\begin{aligned} \|F(\rho^0)-\rho^0\| &\leqq \|F_N(\rho^0)-\rho^0\|+\|F(\rho^0)-F_N(\rho^0)\| \\ &= \|F_N(\rho^0)-\rho^0\|+\Big\|\sum_{k=N}^{2N-1}\ \sum_{m,\,n=0}^{N-1} a_{kmn}t_m^0 t_n^0 p_k(x)\Big\| \\ &= \|F_N(\rho^0)-\rho^0\|+\Big(\sum_{k=N}^{2N-1}\Big(\sum_{m,\,n=0}^{N-1} a_{kmn}t_m^0 t_n^0\Big)^2\Big)^{1/2}. \end{aligned} \tag{3.19}$$

An approximate solution of the equation $F_N(\rho)=\rho$ can be found with the help of computer. One starts with an initial vector $\rho_0=(q_0, t_{00}, t_{01}, \ldots, t_{0,N-1})$ and by iterating equations (3.16) one converges to an approximate solution ρ^0. In such a way for $N=4$ we have calculated $\rho^0(x)$, defined in (2.17), (2.18). One can verify with the help of a computer that

$$\|F_4(\rho^0)-\rho^0\|<10^{-9}$$

and

$$\|F(\rho^0)-F_4(\rho^0)\|=\left(\sum_{k=4}^{7}\left(\sum_{m,n=0}^{3} a_{kmn}t_m^0 t_n^0\right)^2\right)^{1/2}<1.9.10^{-6}$$

which imply

$$\|F(\rho^0)-\rho^0\|<2.10^{-6}. \tag{3.20}$$

To prove the contractive property of the map F in a neighborhood of the point $\rho^0\in P$ we shall establish that the differential of this map is contracting in a neighborhood of ρ^0.

Estimation of the differential. – The map $F: \rho\to\rho'=g(\rho*\rho)$ is quadratic in ρ and its matrix form is given by equations (3.9). The differential D_ρ of the map F at the point $\rho\in P$ is

$$D_\rho:\quad \sigma\to 2g(\rho*\sigma).$$

We prove now that D_ρ is a linear bounded operator in H. Let $\|D_\rho\|=\sup_{\|\sigma\|=1}\|D_\rho\sigma\|$.

LEMMA 3.1. – *If* $t(x), s(x)\in L^2([0, 1])$, *then*

$$\|A(t, s)(x)\|_2^2+[a(t, s)]^2\leqq 4\|t\|_2^2\|s\|_2^2. \tag{3.21}$$

If $\rho=(q, t(x))\in H$ *and* $\sigma=(r, s(x))\in H$ *then*

$$\|g(\rho*\sigma)\|\leqq(8+4\sqrt{2})\|\rho\|\|\sigma\|. \tag{3.22}$$

If $\rho\in P$, $\rho(x)=\delta(x)+\rho_1(x)$, $\rho_1\in H$, *then*

$$\|D_\rho\|\leqq 2+(16+8\sqrt{2})\|\rho_1\|. \tag{3.23}$$

Proof. – Recall a general inequality

$$\|f*g\|_{L^2(R^1)}\leqq\|f\|_{L^1(R^1)}\|g\|_{L^2(R^1)}.$$

Moreover

$$\|f\|_{L^1(R^1)}\leqq|\operatorname{supp} f(x)|^{1/2}\|f\|_{L^2(R^1)},$$

where $|\operatorname{supp} f(X)|$ is the length of the support of $f(x)$. Now let

$$w(x)=(\hat{t}(x)+\hat{t}(-x))*(\hat{s}(x)+\hat{s}(-x)).$$

Then $A(t, s)(x) = w(x)|_{[0, 1]}$ and $a(t, s) = \int_1^2 w(x)\,dx$. We have: $w(-x) = w(x)$ and

$$\int_0^1 [A(t, s)(x)]^2\,dx + [a(t, s)]^2 = \int w^2(x)\,dx + \left(\int_1^2 w(x)\,dx\right)^2$$
$$\leqq \int_0^2 w^2(x)\,dx = 1/2 \int_{-2}^2 w^2(x)\,dx$$
$$\leqq 1/2 \| \hat{t}(x) + \hat{t}(-x) \|_{L^1(\mathbf{R}^1)}^2 \| \hat{s}(x) + \hat{s}(-x) \|_{L^2(\mathbf{R}^1)}^2$$
$$= 4 \| t(x) \|_{L^1([0, 1])}^2 \| s(x) \|_{L^2([0, 1]}^2 \leqq 4 \| t \|_2^2 \| s \|_2^2.$$

The inequality (3.21) is proved.

If $\rho, \sigma \in H$, $\rho = (q, t(x))$, $\sigma = (r, s(x))$ then $g(\rho * \sigma) = \rho' = (r', s'(x))$, where

$$r' = -3qr - qs_0 - t_0 r + a(t, s),$$
$$s'(x) = -2t(x)r - 2t(x)s_0 - 2qs(x)$$
$$-2t_0 s(x) + t(1-x)r + qs(1-x) + A(t, s)(x)$$

where $t_0 = \int_0^1 t(x)\,dx$, $s_0 = \int_0^1 s(x)\,dx$. Note that $|t_0| \leqq \| t \|_2$, $|s_0| \leqq \| s \|_2$.

We now estimate $\| s'(x) \|_2$:

$$\| s' \|_2 \leqq 2 \| t \|_2 |r| + 2 \| t \|_2 \| s \|_2 + 2|q| \| s \|_2 + 2 \| t \|_2 \| s \|_2 + \| t \|_2 r$$
$$+ |q| \| s \|_2 + \| A(t, s) \|_2 = 3 \| t \|_2 r + 3|q| \| s \|_2 + 4 \| t \|_2 \| s \|_2 + \| A(t, s) \|_2.$$

Similarly,

$$|r'| \leqq |q| |r| + |q| \| s \|_2 + \| t \|_2 |r| + |a(t, s)|,$$

hence

$$\| g(\rho * \sigma) \| = (\| s' \|_2^2 + |r'|^2)^{1/2}$$
$$\leqq \| s' \|_2 + |r'| \leqq 3|q| |r| + 4 \| t \|_2 |r|$$
$$+ 4|q| \| s \|_2 + 4 \| r \|_2 \| s \|_2 + |a(t, s)| + \| A(t, s) \|_2.$$

By (3.21),

$$|a(t, s)| + \| A(t, s) \|_2 \leqq [2((a(t, s)^2 + (\| A(t, s) \|_2)^2)]^{1/2} \leqq 2\sqrt{2} \| t \|_2 \| s \|_2,$$

so

$$\| g(\rho * \sigma) \| \leqq 3|q| |r| + 4 \| t \|_2 |r| + 4|q| \| s \|_2 + (4 + 2\sqrt{2}) \| t \|_2 \| s \|_2$$
$$\leqq (4 + 2\sqrt{2})(|q| + \| t \|_2)(|r| + \| s \|_2) \leqq (8 + 4\sqrt{2})(|q|^2 + \| t \|_2^2)^{1/2}$$
$$\times (|r|^2 + \| s \|_2^2)^{1/2} = (8 + 4\sqrt{2}) \| \rho \| \| \sigma \|.$$

The estimate (3.22) is proved.

Let $\rho(x) \in P$, $\rho(x) = \delta(x) + \rho_1(x)$, $\rho_1(x) \in H$. Then

$$\| D_\rho \sigma \| = \| 2g(\rho * \sigma) \| = \| 2g(\delta + \rho_1) * \sigma) \|$$
$$= \| 2\sigma + 2g(\rho_1 * \sigma) \| \leqq 2 \| \sigma \| + 2 \| g(\rho_1 * \sigma) \|.$$

By (3.22) $\|g(\rho*\sigma)\| \leqq (8+4\sqrt{2})\|\rho_1\|\|\sigma\|$, hence

$$\|D_\rho\| \leqq 2+(16+8\sqrt{2})\|\rho_1\|.$$

Lemma 3.1 is proved.

The inequality (3.23) gives a rough estimate of the differential D_ρ. Now we shall oftain a much better estimate of $\|D_\rho\|$ for ρ lying in a neighborhood of the approximate fixed point ρ^0. To that end we shall use the matrix form of the differential D_ρ. It can be obtained by differentiating equations (3.9). We get $\sigma'=D_\rho\sigma=(r', s'(x))$, where

$$\begin{aligned} r' &= (2-6q-2t_0)r-2qs_0+2\sum_{m,\,n=0}^{\infty} a_{mn}t_m s_n, \\ s'_k &= (-4+2(-1)^k)t_k r-4t_k s_0 \\ &\quad +(2-4q-4t_0+2q(-1)^k)s_k+2\sum_{m,\,n=0}^{\infty} a_{kmn}t_m s_n, \end{aligned} \tag{3.24}$$

$k=0, 1, 2, \ldots$ One can rewrite these equations in matrix notations as

$$\begin{aligned} r' &= d_{11}r+\sum_{n=0}^{\infty} d_{1,\,n+2}s_n, \\ s'_k &= d_{k+2,\,1}r+\sum_{n=0}^{\infty} d_{k+2,\,n+2}s_n, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} d_{11} &= 2-6q-2t_0, \\ d_{12} &= -2q+2\sum_{m=0}^{\infty} a_{m0}t_m, \\ d_{1,\,n+2} &= 2\sum_{m=0}^{\infty} a_{mn}t_m, \qquad n\geqq 1, \\ d_{k+2,\,1} &= (-4+2(-1)^k)t_k, \qquad k\geqq 0, \\ d_{22} &= 2-2q-8t_0+2\sum_{m=0}^{\infty} a_{0m0}t_m, \\ d_{k+2,\,2} &= -4t_k+2\sum_{m=0}^{\infty} a_{km0}t_m, \qquad k\geqq 1 \\ d_{k+2,\,k+2} &= 2-4q-4t_0+2q(-1)^k+2\sum_{m=0}^{\infty} a_{kmn}t_m,\ k\geqq 1, \\ d_{k+2,\,n+2} &= 2\sum_{m=0}^{\infty} a_{kmn}t_m, \qquad k, n\geqq 1, \qquad k\neq n. \end{aligned} \tag{3.26}$$

First we estimate the differential D_ρ at $\rho=\rho^0$. Recall that $t_m^0=0$ for $m\geqq 4$. Decompose the matrix $D_{\rho^0}=(d_{ij})_{i,\ j=1}^{i,\ j=\infty}$ into four blocks:

$$D_{\rho^0}=\begin{pmatrix} D^{(11)} & D^{(12)} \\ D^{(21)} & D^{(22)} \end{pmatrix},$$

where

$$D^{(11)}=(d_{ij})_{i,\ j=1}^{i,\ j=3}, \qquad D^{(12)}=(d_{ij})_{i=1,\ j=4}^{i=3,\ j=\infty},$$
$$D^{(21)}=(d_{ij})_{i=4,\ j=1}^{i=\infty,\ j=3}, \qquad D^{(22)}=(d_{ij})_{i,\ j=4}^{\infty}.$$

We now estimate the quantities $\| D^{(ij)} \|$.

Estimation of $\| D^{(11)} \|$, $\| D^{(12)} \|$, $\| D^{(21)} \|$. Numerical calculations on the computer give

$$D^{(11)}=\begin{vmatrix} 0.233{,}135{,}136{,}4 & -0.212{,}297{,}632{,}7 & 0.132{,}249{,}112{,}9 \\ -0.456{,}321{,}278{,}6 & 0.429{,}569{,}762{,}6 & -0.132{,}249{,}112{,}9 \\ 0.075{,}040{,}970{,}9 & -0.082{,}221{,}799{,}0 & -0.198{,}076{,}462{,}2 \end{vmatrix}$$

where the error in each matrix element does not exceed 10^{-9}. It enables to prove that

$$\| D^{(11)} \| < 0.727. \qquad (3.27)$$

Namely one can verify with the help of a computer that all determinants of the matrix $0.727^2\,E-(D^{(11)})^*\,D^{(11)}$ are positive, so

$$0.727^2\,E>(D^{(11)})^*\,D^{(11)},$$

hence (3.27) holds.

Next one can estimate with the help of a computer that

$$\left(\sum_{j=4}^{7}\sum_{i=1}^{3} d_{ij}^2\right)^{1/2}<0.054,$$
$$\left(\sum_{i=4}^{7}\sum_{j=1}^{3} d_{ij}^2\right)^{1/2}<0.055.$$

As $d_{ij}=0$ if $|i-j|>4$ (because of $t_m=0$ for $m\geqq 4$), we get

$$\begin{cases} \| D^{(12)} \| \leqq 0.054, \\ \| D^{(21)} \| \leqq 0.055. \end{cases} \qquad (3.28)$$

Estimation of $\| D^{(22)} \|$. – Decompose $D^{(22)}$ as

$$D^{(22)}=D_0^{(22)}+D_1^{(22)},$$

where

$$D_0^{(22)}=((2-4\,q^0-4\,t_0^0+2\,q^0\,(-1)^k)\,\delta_{kn})_{k,\ n=3}^{\infty}$$

and

$$D_1^{(22)}=\left(2\sum_{m=0}^{3} a_{kmn}\,t_m^0\right)_{k,\ n=3}^{\infty}.$$

As $D_0^{(22)}$ is a diagonal matrix,

$$\|D_0^{(22)}\| \leqq \max_{\pm} |2-4q^0-4t_0^0 \pm 2q^0| = 2-2q^0-4t_0^0 < 0.650,6. \quad (3.29)$$

Next,

$$D_1^{(22)} = \sum_{m=0}^{3} t_m^0 D_m,$$

where $D_m = (2a_{kmn})_{k,\, n=3}^{\infty}$. We now estimate $\|D_m\|$. As t_0^0 is large (with respect to other t_m^0's) the most crucial point is an accurate estimation of $\|D_0\|$. By (3.11), (3.12), (3.14)

$$D_0 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{5.7}} & 0 & 0 & \vdots \\ -\frac{1}{\sqrt{5.7}} & 0 & \frac{1}{\sqrt{7.9}} & 0 & \vdots \\ 0 & \frac{1}{\sqrt{7.9}} & 0 & -\frac{1}{\sqrt{9.11}} & \vdots \\ 0 & 0 & -\frac{1}{\sqrt{9.11}} & 0 & \vdots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

LEMMA 3.2. – $\|D_0\| \leqq 1/4$.

Proof. – D_0 is a symmetric Jacobi matrix. We recall a general result (*see* [6]). Let

$$A = \begin{bmatrix} 0 & b_1 & 0 & 0 & \vdots \\ b_1 & 0 & b_2 & 0 & \vdots \\ 0 & b_2 & 0 & b_3 & \vdots \\ 0 & 0 & b_3 & 0 & \vdots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

be a symmetric Jacobi matrix. Then $\|A\| \leqq \lambda$ if (and only if) the recurrent sequence

$$\Delta_0 = 1, \qquad \Delta_1 = \lambda, \qquad \Delta_n = \lambda\,\Delta_{n-1} - b_{n-1}^2\,\Delta_{n-2}$$

is positive for all n (Δ_n is up to the sign the determinant of the characteristic matrix $n \times n$). Assume $|b_1| \geqq |b_2| \geqq |b_3| \geqq \dots$ Then we state that for any $\lambda \geqq 2|b_1|$ the sequence $d_n = \Delta_n / |b_1|^n$ is increasing and so $\Delta_n > 0$. In fact,

$$d_0 = 1 < 2 \leqq \frac{\lambda}{|b_1|} = d_1$$

and

$$d_n = \frac{\lambda}{|b_1|} d_{n-1} - \left(\frac{b_{n-1}}{b_1}\right)^2 d_{n-2} \geqq 2 d_{n-1} - d_{n-2} \geqq d_{n-1}$$

by induction. Thus we gut $\|A\| \leqq \lambda$ for any $\lambda \geqq 2|b_1|$, which implies $\|A\| \leqq 2|b_1|$. Similarly, if for some $k \geqq 1$

$$|\lambda| \geqq 2|b_k| \tag{3.30}$$

and $\Delta_j > 0$, $j=0, 1, \ldots, k$, and

$$\Delta_k \geqq \Delta_{k-1} |b_k| \tag{3.31}$$

then the sequence $d_n = \frac{\Delta_n}{|b_k|^n}$, $n \geqq k$, is increasing and $\Delta_n > 0$. In fact,

$$d_k = \frac{\Delta_k}{|b_k|^k} \geqq \frac{\Delta_{k-1}|b_k|}{|b_k|^k} = d_{k-1}$$

and

$$d_n = \frac{\lambda}{|b_k|} d_{n-1} - \left(\frac{b_{n-1}}{b_k}\right)^2 d_{n-2} \geqq 2 d_{n-1} - d_{n-2} \geqq d_{n-1}$$

for $n > k$ by induction. Thus (3.30), (3.31) imply $\|A\| \leqq \lambda$.

We apply the last inequality for $A = D_0$, $\lambda = \frac{1}{4}$, $k=3$. We have $\lambda = 1/4 > 2/\sqrt{9.11} = 2|b_3|$, hence (3.30) holds.

Next,

$$\Delta_0 = 1, \quad \Delta_1 = \lambda, \quad \Delta_2 = \lambda^2 - b_1^2, \quad \Delta_3 = \lambda(\lambda^2 - b_1^2) - \lambda b_2^2;$$

$$\begin{aligned}\Delta_3 - |b_3|\Delta_2 &= \lambda(\lambda^2 - b_1^2 - b_2^2) - |b_3|(\lambda^2 - b_1^2) \\ &= \frac{1}{4}\left(\frac{1}{16} - \frac{1}{35} - \frac{1}{63}\right) - \frac{1}{\sqrt{99}}\left(\frac{1}{16} - \frac{1}{35}\right) \\ &= \left(\frac{1}{4} - \frac{1}{\sqrt{99}}\right)\left(\frac{1}{16} - \frac{1}{35}\right) - \frac{1}{4.63} > 0,\end{aligned}$$

hence (3.31) holds. Thus $\|D_0\| < \lambda = 1/4$ and Lemma 3.2 is proved.

LEMMA 3.3. – $\|D_1\| < \sqrt{3}/2$.

Proof. – By (3.11), (3.12), (3.15) we have

$$D_1=\sqrt{3}\begin{bmatrix} -\frac{2}{3.7} & -\frac{1}{\sqrt{5.7}} & \frac{1}{7\sqrt{5.9}} & 0 & \vdots \\ -\frac{1}{\sqrt{5.7}} & -\frac{6}{5.9} & \frac{1}{\sqrt{7.9}} & \frac{3}{9\sqrt{7.11}} & \vdots \\ \frac{1}{7\sqrt{5.9}} & \frac{1}{\sqrt{7.9}} & -\frac{2}{7.11} & -\frac{1}{\sqrt{9.11}} & \vdots \\ 0 & \frac{3}{9\sqrt{7.11}} & -\frac{1}{\sqrt{9.11}} & -\frac{6}{9.13} & \vdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \end{bmatrix}$$

D_1 is a symmetric five-diagonal matrix. We exstimate the sum of squares of its elements. We have

$$\left(\frac{2}{3.7}\right)^2+\left(\frac{2}{7.11}\right)^2+\dots$$
$$=4\left[\frac{1}{7^2-3^2}\left(\frac{1}{3^2}-\frac{1}{7^2}\right)+\frac{1}{11^2-7^2}\left(\frac{1}{7^2}-\frac{1}{11^2}\right)+\dots\right]$$
$$<\frac{4}{7^2-3^2}\left[\left(\frac{1}{3^2}-\frac{1}{7^2}\right)+\left(\frac{1}{7^2}-\frac{1}{11^2}\right)+\dots\right]=\frac{4}{40.9}=\frac{1}{90};$$

similarly,

$$\left(\frac{6}{5.9}\right)^2+\left(\frac{6}{9.13}\right)^2+\dots<\frac{36}{(9^2-5^2)\,5^2}=\frac{9}{350},$$
$$2\left(\frac{1}{5.7}+\frac{1}{7.9}+\dots\right)=\frac{1}{5},$$
$$2\left(\frac{3^2}{9^2.7.11}+\frac{3^2}{13^2.11.15}+\dots\right)<\frac{2.9}{9^2(11-7)\,7}=\frac{1}{126}.$$

Thus the sum of squares of the elements of matrix the D_1 does not exceed

$$3\left(\frac{1}{90}+\frac{9}{350}+\frac{1}{5}+\frac{1}{490}+\frac{1}{126}\right)<\frac{3}{4},$$

so $\|D_1\|<\sqrt{3}/2$. Lemma 3.3 is proved.

LEMMA 3.4. – $\|D_m\|\leqq 4$ *for any* m.

Proof. – Let $P: L^2([0, 1])\to L^2([0, 1])$ be a projection defined in the basis $\{p_k(x), k\geqq 0\}$ by $P: (t_0, t_1, t_2, t_3, \dots)\to(0, 0, t_2, t_3, \dots)$.

Then for $\sigma=(q, t(x))$

$$D_m\sigma=(0, 2\,PA(p_m, P\,t)),$$

hence $\| D_m \sigma \| \leqq 2 \| A(p_m, Pt) \|_2$, and by lemma 3.1

$$\| D_m \sigma \| \leqq 2.2 \| p_m \|_2 \| Pt \|_2 \leqq 4 \| t \|_2 \leqq 4 \| \sigma \|,$$

hence $\| D_m \| \leqq 4$. Lemma 3.4 is proved.

We return to the estimation of $\| D^{(22)} \|$. Applying Lemmas 3.2-3.4 we get

$$\| D_1^{(22)} \| \leqq \sum_{m=0}^{3} | t_m^0 | \| D_m \| \leqq \frac{| t_0^0 |}{4} + \frac{\sqrt{3} | t_1^0 |}{2} + 4 | t_2^0 | + 4 | t_3^0 | \leqq 0.0763.$$

Adding the estimate (3.29) we get

$$\| D^{(22)} \| < 0.727. \tag{3.32}$$

Thus we have estimated in (3.27), (3.28) and (3.32) the norms of all the blocks $D^{(ij)}$, $i, j = 1, 2$, of the matrix D_{ρ^0}. Note now that

$$\| D_{\rho^0} \| \leqq \left\| \begin{pmatrix} D^{(11)} & 0 \\ 0 & D^{(22)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & D^{(12)} \\ D^{(21)} & 0 \end{pmatrix} \right\|$$
$$\leqq \max \{ \| D^{(11)} \|, \| D^{(22)} \| \} + \max \{ \| D^{(12)} \|, \| D^{(21)} \| \}.$$

Using the estimates of $\| D^{(ij)} \|$ we get

$$\| D_{\rho^0} \| \leqq 0.727 + 0.055 = 0.782. \tag{3.33}$$

This estimates the norm of the differential at $\rho = \rho^0$.

Let now $\rho = \rho^0 + \rho_1$, where $\rho_1 \in H$. Then

$$\| D_\rho \sigma \| = \| 2 (\rho * \sigma) \| = \| 2 g (\rho^0 * \sigma) + 2 g (\rho_1 * \sigma) \|$$
$$\leqq \| D_{\rho^0} (\sigma) \| + 2 \| g (\rho_1 * \sigma) \|.$$

By Lemma 3.1 $\| g (\rho_1 * \sigma) \| \leqq (8 + 4\sqrt{2}) \| \rho_1 \| \| \sigma \| \leqq 14 \| \rho_1 \| \| \sigma \|$, so we get estimate $\| D_\rho \sigma \| \leqq \| D_{\rho^0} \sigma \| + 28 \| \rho_1 \| \| \sigma \|$, or

$$\| D_\rho \| \leqq \| D_{\rho^0} \| + 28 \| \rho_1 \|. \tag{3.34}$$

Inequalities (3.33), (3.34) give the desired estimate of the differential D_ρ in a neighborhood of the approximate fixed point ρ^0. We return to the proof of Theorem 2.1.

End of proof of Theorem 2.1. — Consider the neighborhood $U = \{ \rho = \rho^0 + \rho_1 \in P, \| \rho_1 \| \leqq 5.10^{-4} \}$ of the point $\rho^0 \in P$. Then by (3.33), (3.34) for $\rho \in U$,

$$\| D_\rho \| \leqq \| D_{\rho^0} \| + 28 \| \rho_1 \| \leqq 0.782 + 0.014 = 0.796.$$

Thus the differential D_ρ of the map F is a contractive operator for $\rho \in U$. It leads to the contractive property of the map F itself in U. Namely let

$\rho_0, \rho_1 \in U$. Define $\rho_t = (1-t)\rho_0 + t\rho_1$, $0 \leqq t \leqq 1$. Then

$$\begin{aligned}\| F(\rho_1) - F(\rho_0) \| &= \left\| \int_0^1 \frac{d}{dt} F(\rho_t)\, dt \right\| \\ &= \left\| \int_0^1 D_{\rho_t} \frac{d\rho_t}{dt}\, dt \right\| = \left\| \int_0^1 D_{\rho_t} (\rho_1 - \rho_0)\, dt \right\| \\ &\leqq \int_0^1 \| D_{\rho_t} \| \, \| \rho_1 - \rho_0 \| \, dt \leqq 0.796 \| \rho_1 - \rho_0 \|. \quad (3.35)\end{aligned}$$

We prove now that $F(U) \subset U$. We have for $\rho \in U$,

$$\begin{aligned}\| F(\rho) - \rho^0 \| &\leqq \| F(\rho) - F(\rho^0) \| + | F(\rho^0) - \rho^0 \| \\ &\leqq 0.796 \| \rho - \rho^0 \| + 2.10^{-6} \\ &\leqq 0.796\, 5.10^{-4} + 2.10^{-6} \leqq 5.10^{-4},\end{aligned}$$

so $F(\rho) \in U$, *i.e.* $F(U) \subset U$.

Thus we showed that $F: U \to U$ is a contractive map so there exists a fixed point $\rho_* \in U$, $F(\rho_*) = \rho_*$. As

$$\begin{aligned}0.796 \| \rho^0 - \rho_* \| &\geqq \| F(\rho^0) - F(\rho_*) \| = \| F(\rho^0) - \rho_* \| \\ &\geqq \| \rho^0 - \rho_* \| - \| F(\rho^0) - \rho^0 \| \geqq \| \rho^0 - \rho_* \| - 2.10^{-6},\end{aligned}$$

we get that

$$\| \rho^0 - \rho_* \| \leqq \frac{2.10^{-6}}{1 - 0.796} < 10^{-5}.$$

Moreover by (3.35)

$$\| F(\rho) - \rho_* \| = \| F(\rho) - F(\rho_*) \| \leqq 0.796 \| \rho - \rho_* \|$$

if $\rho \in U$. Theorem 2.1 is proved.

4. ON SOME UNSTABLE FIXED POINTS OF THE MAP $Z \to F(Z)$

Consider a finite lattice

$$L_n = \left\{ -1 + \frac{j}{n} \right\}_{j=0}^{2n}$$

and the set R_n of symmetric random variables taking values in L_n. The set R_n is invariant with respect to the operator F as $f(Z_1 + Z_2) \in L_n$ if Z_1, $Z_2 \in L_n$. It is natural to consider the question about the existence of stable fixed points of the operator F in the sets R_n. For instance, for $n = 1$, $L_1 = \{-1, 0, 1\}$ and the stable fixed point is

$$Z_*^{(1)} = -1, 0, 1 \text{ with probability } \frac{1}{3}.$$

A small refinement of the proof of Theorem 2.1 enables to prove the existence of a stable fixed point $Z_*^{(n)}$ in the set R_n for sufficiently large n. Moreover weak$-\lim_{n\to\infty} Z_*^{(n)} = Z_*$, the fixed point of Theorem 2.1. Note that $Z_*^{(n)}$ is unstable in the whole space of random variables on $[-1, 1]$ and even in R_{nk} for $k>1$. In this connection it is noteworthy also that although Z is stable in the metrics $d(Z, Z')$ it can't be stable in the topology of weak convergence in the whole space of random variables on $[-1, 1]$ as there exists a sequence $Z_*^{(n)}$ of fixed points which converges weakly to Z_* as $n\to\infty$.

APPENDIX
CALCULUS OF THE MATRIX ELEMENTS a_{kmn} AND a_{mn}

Let

$$p_k(x) = \begin{cases} \sqrt{2k+1}\, P_k(2x-1), & 0 \leqq x \leqq 1 \\ 0 & \text{otherwise} \end{cases} \tag{A 1}$$

where $P_k(x)$ is the k-th Legendre polynomial and $p_k^{\pm}(x) = p_k(\pm x)$. The by definition

$$a_{mn} = \int_1^2 p_m * p_n(x)\, dx, \tag{A 2}$$

$$a_{kmn} = \int_0^1 p_k(x)\, (p_m^+ + p_m^-) * (p_n^+ + p_n^-)(x)\, dx. \tag{A 3}$$

Let

$$b_{kmn} = (-1)^k \int_0^1 p_k(x)\, p_m * p_n(x)\, dx \tag{A 4}$$

and

$$c_{kmn} = \frac{\sqrt{2k+1}}{\sqrt{(2m+1)(2n+1)}} \int_0^1 p_k(x)\, p_m * p_n(x)\, dx. \tag{A 5}$$

We shall prove the following statements.

PROPOSITION A 1. — *For* c_{kmn} *the following recurrent relations hold:*

$$c_{kmn}=\frac{1}{4}\delta_{m0}\delta_{nk}+\frac{(-1)^n}{2(2m+1)}(\delta_{m+1,k}-\delta_{m-1,k})$$
$$+\frac{1}{2m+1}\sum_{j=0}^{\infty}(2n-4j-1)(c_{k,m+1,n-1-2j}-c_{k,m-1,n-1-2j}) \quad (A\,6)$$

where δ_{ij} *is the Kronecker symbol and it is assumed that* $c_{kmn}=0$ *if either* m *or* n *is negative.*

COROLLARY:

$$c_{000}=\frac{1}{2}, \qquad c_{mm0}=0, \qquad m\geqq 1;$$
$$c_{m+1,m0}=\frac{1}{2(2m+1)}, \qquad c_{m,m+1,0}=-\frac{1}{2(2m+3)}, \qquad m\geqq 0;$$
$$c_{mm1}=-\frac{1}{(2m-1)(2m-3)}, \qquad m\geqq 1; \quad (A\,7)$$
$$c_{m,m+1,1}=\frac{1}{2(m+3)}, \qquad m\geqq 1;$$
$$c_{m,m+2,1}=\frac{1}{2(2m+3)(2m+5)}, \qquad m\geqq 1.$$

PROPOSITION A 2. — $b_{kmn}=0$ *if* $k>m+n+1$.

PROPOSITION A 3:

$$b_{kmn}=b_{knm}=b_{nmk}=\ldots; \quad (A\,8)$$
$$a_{kmn}=((-1)^k+(-1)^m+(-1)^n)\,b_{kmn}$$
$$=(1+(-1)^{m+k}+(-1)^{n+k})\sqrt{\frac{2(m+1)(2n+1)}{2k+1}}\,c_{kmn}; \quad (A\,9)$$
$$c_{mn}=(-1)^{m+n}.b_{0mn},$$
$$a_{mn}=-a_{0mn}=-b_{0mn}, \qquad \text{if} \quad m+n>0. \quad (A\,10)$$

COROLLARY. — *Relations* (3.11)-(3.15) *hold.*

Now we turn to the proofs of the formulated statements.

Proof of Proposition A 1. — We have for $0\leqq x\leqq 1$

$$p_m * p_n(x)=\sqrt{(2m+1)(2n+1)}\int_0^x \mathrm{P}_m(2x-2y-1)\,\mathrm{P}_n(2y-1)\,dy$$
$$=\frac{\sqrt{(2m+1)(2n+1)}}{2}\int_{-1}^{x}\mathrm{P}_m(x'-y'-1)\,\mathrm{P}_n(y')\,dy'$$

where $y' = 2y - 1$, $x' = 2x - 1$. Let

$$Q_{mn}(x) = \int_{-1}^{x} P_m(y - x + 1) P_n(y) \, dy. \tag{A 11}$$

Then as $P_m(-x) = (-1)^m P_m(x)$,

$$p_m * p_n(x) = (-1)^m \frac{\sqrt{(2m+1)(2n+1)}}{2} Q_{mn}(2x - 1). \tag{A 12}$$

Now, since

$$P_m(x) = \frac{1}{2m+1} (P'_{m+1}(x) - P'_{m-1}(x)), \tag{A 13}$$

we have

$$Q_{mn}(x) = \frac{1}{2m+1} \int_{-1}^{x} [P'_{m+1}(y - x + 1) - P'_{m-1}(y - x + 1)] P_n(y) \, dy.$$

Integrating by parts we get

$$Q_{mn}(x) = \frac{1}{2m+1} \Big\{ [P_{m+1}(1) - P_{m-1}(1)] P_n(-x) - [P_{m+1}(-x) - P_{m-1}(-x)] P_n(-1) - \int_{-1}^{x} [P_{m+1}(y - x + 1) - P_{m-1}(y - x + 1)] P'_n(y) \, dy \Big\}.$$

We have: $P_m(1) = 1$, and $P_{m+1}(1) - P_{m-1}(1) = \delta_{m0}$;

$$P_n(-1) = (-1)^n, \qquad P_{m\pm 1}(-x) = -(-1)^m P_{m\pm 1}(x);$$

$$P'_n(y) = (2n-1) P_{n-1}(y) + P'_{n-2}(y) = \ldots = \sum_{j=0}^{\infty} (2n - 4j - 1) P_{n-1-2j}(y),$$

and we get

$$Q_{mn}(x) = \frac{1}{2m+1} \Big\{ \delta_{m0} P_n(x) + (-1)^{m+n} [P_{m+1}(x) - P_{m-1}(x)] - \sum_{j=0}^{\infty} (2n - 4j - 1) [Q_{m+1,\, n-1-2j}(x) - Q_{m-1,\, n-1-2j}(x)] \Big\}. \tag{A 14}$$

Let $Q_{mn}(x) = \sum_{k=0}^{\infty} q_{kmn} P_k(x)$. Then the last equation means that

$$q_{kmn} = \frac{1}{2m+1} \Big\{ \delta_{m0} \delta_{nk} + (-1)^{m+n} (\delta_{m+1,\, k} - \delta_{m-1,\, k}) - \sum_{j=0}^{\infty} (2n - 4j - 1) (q_{k,\, m+1,\, n-1-2j} - q_{k,\, m-1,\, n-1-2j}) \Big\}. \tag{A 15}$$

By (A 12)

$$p_m * p_n(x) = (-1)^m \frac{\sqrt{(2m+1)(2n+1)}}{2} \sum_{k=0}^{\infty} q_{kmn} \mathrm{P}_k(2x-1)$$
$$= \frac{(-1)^m}{2} \sum_{k=0}^{\infty} \sqrt{\frac{(2m+1)(2n+1)}{2k+1}} q_{kmn} p_k(x),$$

therefore

$$c_{kmn} = \sqrt{\frac{2k+1}{(2m+1)(2n+1)}} \int p_k(x) p_m * p_n(x)\, dx = \frac{(-1)^m}{2} q_{kmn}.$$

Multiplying (A 15) by $(-1)^{m/2}$ we get

$$c_{kmn} = \frac{1}{2} \delta_{m\,0} \delta_{mk} + \frac{(-1)^n}{4m+2} (\delta_{m+1,\,k} - \delta_{m-1,\,k})$$
$$+ \frac{1}{2m+1} \sum_j (2n-4j-1)(c_{k,\,m+1,\,n-1-2j} - c_{k,\,m-1,\,n-1-2j}).$$

Proposition A 1 is proved.

Proof of Corollary. – For $n=0$ by (A 6)

$$c_{km\,0} = \frac{1}{2} \delta_{m\,0} \delta_{k\,0} + \frac{1}{2(m+1)} (\delta_{m+1,\,k} - \delta_{m-1,\,k})$$

which gives formulae (A 7) with $n=0$. Moreover by (A 6) $c_{km\,1}$ is expressed via $c_{k,\,m\pm 1,\,0}$. It enables to get all the other formulae in (A 7).

Proof of Proposition A 2. – By (A 11) $\mathrm{Q}_{mn}(x)$ is a polynomial of degree $m+n+1$, so by (A 12) $p_m * p_n(x)$ is also. Hence $p_m * p_n(x)$ is orthogonal to $p_k(x)$ if $k > m+n+1$. Proposition A 2 is proved.

Proof of Proposition A 3. – We have: $p_n(1-x) = (-1)^n p_n(x)$, therefore

$$b_{kmn} = (-1)^k \int_{-\infty}^{\infty} p_k(x) \int_{-\infty}^{\infty} p_m(x-y) p_n(y)\, dy\, dx$$
$$= (-1)^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_k(1-x) p_m(x-y) p_n(1-y)\, dy\, dx$$
$$= (-1)^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_k(x') p_m(y'-x') p_n(y')\, dy'\, dx'$$
$$= (-1)^n \int_{-\infty}^{\infty} p_n(y') \int_{-\infty}^{\infty} p_m(y'-x') p_k(x')\, dx'\, dy' = b_{nmk}.$$

Moreover, as $p_m * p_n = p_n * p_m$, $b_{kmn} = b_{knm}$. This proves (A 8).

Next, by (A 3)

$$a_{kmn} = \sum_{\pm,\ \pm} \int_0^1 p_k(x)\, p_m^{\pm} * p_n^{\pm}(x)\, dx.$$

Now,

$$\int_0^1 p_k(x)\, p_m^{+} * p_n^{+}(x)\, dx = (-1)^k b_{kmn},$$

$$\int_0^1 p_k(x)\, p_m^{+} * p_n^{-}(x)\, dx = \int_{-\infty}^{\infty} p_k(x) \int_{-\infty}^{\infty} p_m(x-y)\, p_n(-y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_k(x)_m (x-y'+1)\, p_n(1-y')\, dy'\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_k(x)\, p_m(y'-x)\, p_n(y')\, dy'\, dx$$

$$= (-1)^{m+n} \int_{-\infty}^{\infty} p_n(y')\, p_k * p_m(y')\, dy' = (-1)^m b_{nkm} = (-1)^m b_{kmn};$$

similarly,

$$\int_0^1 p_k(x)\, p_m^{-} * p_n^{+}(x)\, dx = (-1)^n b_{kmn}. \tag{A 16}$$

At last

$$\int_0^1 p_k(x)\, p_m^{-} * p_n^{-}(x) = 0.$$

Thus $a_{kmn} = (-1)^k b_{kmn} + (-1)^m b_{kmn} + (-1)^n b_{kmn}$. This proves (A 9).

Now,

$$a_{mn} = \int_1^2 \int_{-\infty}^{\infty} p_m(y)\, p_n(x-y)\, dy\, dx$$

$$= \int_0^1 \int_{-\infty}^{\infty} p_m(1+y')\, p_n(x'-y')\, dy'\, dx'$$

$$= (-1)^m \int_{-\infty}^{\infty} p_0(x') \int_{-\infty}^{\infty} p_m(y')\, p_n(x'-y')\, dy'\, dx'$$

$$= (-1)^m \int_{-\infty}^{\infty} p_0(x')\, p_m^{-} * p_n^{+}(x')\, dx',$$

or by (A 16)

$$a_{mn} = (-1)^{m+n} b_{0\,mn},$$

as stated. By (A 9)

$$a_{0\,mn} = (1 + (-1)^m + (-1)^n)\, b_{0\,mn}.$$

Note that by formulae (A 7)-(A 9) and Proposition A 2 $b_{0\,m} \neq 0$ only for $|m-n|=1$ if $m+n>0$. In such a case $(-1)^{m+n}=-1$ and $1+(-1)^m+(-1)^n=1$, hence $a_{mn}=b_{0\,mn}$ and $a_{0\,mn}=b_{0\,mn}$. Thus (A 10) and consequently Proposition A 3 are proved.

Proof of relations (3.11)−(3.15). − (3.11) follows from (A 8), (A 9). (3.12) is a consequence of Proposition A 2 and formula (A 10). Particular values (3.13)-(3.15) of the matrix elements a_{mn}, $a_{0\,mn}$ and $a_{1\,mn}$ follow from (A 7), (A 9), (A 10).

ACKNOWLEDGEMENTS

The author is grateful to J. T. Chayes and L. Chayes for useful discussions. The author thanks Italian C.N.R.-G.N.F.M. for the financial support given to his visit to the University of Rome "La Sapienza", where part of this work was done.

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(*Manuscript received May* 18, 1990.)