ANNALES DE L'I. H. P., SECTION A

C. GÉRARD

The Mourre estimate for regular dispersive systems

Annales de l'I. H. P., section A, tome 54, nº 1 (1991), p. 59-88
<http://www.numdam.org/item?id=AIHPA_1991__54_1_59_0>

Ann. Inst. Henri Poincaré,
Vol. 54, n° 1, 1991, p. 59-88. *Physique théorique*

The Mourre estimate for regular dispersive systems

by

C. GÉRARD
Centre de Mathématiques U.R.A. 169,
École Polytechnique, 91128 Palaiseau Cedex, France

ABSTRACT. — Dispersive N-body Hamiltonians arise when one replaces the non relativistic kinetic energy for N particles $\sum_{1}^{N} -\frac{1}{2m_i}\triangle_{x_i}$ by a general kinetic term $\omega(D_x)$ for a real fuction $\omega(\xi)$. We describe in this work a class of dispersive Hamiltonians which we call *regular* for which we prove a Mourre estimate outside a closed and countable set of energies called *thresholds* which can be explicitly described. The regularity condition depends only on the kinetic term $\omega(\xi)$ and on the familly of coincidence planes X_a which describe the N particle structure of the potential energy term. As a consequence of the Mourre estimate we establish absence of singular continuous spectrum and H-smoothness of $\langle x \rangle^{-s}$, for s bigger than 1/2, which is a basic tool in scattering theory. We then prove some results on scattering theory for short range interactions. We obtain existence of wave operators, orthogonality of channels and asymptotic completeness of wave operators in the two cluster region.

RÉSUMÉ. — Les Hamiltoniens à N corps dispersifs sont obtenus en remplaçant le terme d'énergie cinétique non relativiste pour N particules $\sum_{1}^{N} -\frac{1}{2m_i}\triangle_{x_i}$ par un terme cinétique général $\omega(D_x)$ pour une fonction réelle $\omega(\xi)$. Nous décrivons dans ce travail une classe de Hamiltoniens dispersifs que nous appellons *réguliers* pour lesquels nous établissons une estimation de Mourre en dehors d'un ensemble fermé et dénombrable de niveaux d'énergie appelés *seuils*. La condition de régularité ne dépend que du terme cinétique $\omega(\xi)$ et de la famille de plans de coincidence X_a qui décrit la structure à N corps du terme d'énergie potentielle. Comme conséquence de l'estimation de Mourre nous montrons l'absence de spectre singulier continu et la H-régularité de $\langle x \rangle^{-s}$ pour $s > 1/2$ qui est un outil de base dans la théorie du scattering. Nous montrons ensuite quelques résultats sur la théorie du scattering pour des interactions à courte portée. Nous obtenons l'existence des opérateurs d'onde, l'orthogonalité des canaux et la complétude asymptotique dans la région à deux corps.

1. INTRODUCTION

We describe in this paper a class of N-body dispersive systems for which a Mourre estimate can be proved outside a closed and countable set of points, called the *threshold set*. This threshold set can be described in terms of eigenvalues of some suitably defined dispersive sybsystems. To describe more in details this work, let us start by considering the standard N body Hamiltonians.

The basic Hamiltonian describing N non relativistic interacting particles is the following:

$$H = \sum_{i=1}^{N} -\frac{1}{2m_i} \triangle_{x_i} + \sum_{i<j} V_{ij}(x_i - x_j).$$

Here $x_i \in \mathbf{R}^3$ and $m_i > 0$ are the position and mass of particle number i.

The spectral and scattering theory of this type of Hamiltonians have been the subject of an vast amount of study since the fifties. Concerning the spectral theory, an important step was taken when Perry-Sigal-Simon [PSS] succeeded in applying to H the abstract commutator method of Mourre [M], generalizing earlier work of Mourre for $N=3$. This proof was greatly simplified afterwards by Froese-Herbst [F-H]. As a consequence of the Mourre estimate, one gets absence of singular continuous spectrum, some results about embedded eigenvalues, and local H-smoothness of the operator $\langle x \rangle^{-s}$, for $s > 1/2$, outside a set of energies called *thresholds*, which are the eigenvalues of some subhamiltonians. The H-smoothness of $\langle x \rangle^{-s}$ for $s > 1/2$, is an important tool in the proof by Sigal-Soffer [S-S] of asymptotic completeness for short range systems (*i.e.* with $V_{ij}(x)$ decaying faster than $\langle x \rangle^{-1-\varepsilon}$, $\varepsilon > 0$). It allows for example to control lower order terms in commutators of H with phase space operators (*see* [S-S]).

Dispersive systems arise when one changes the kinetic energy in H. Let us give two examples. In relativistic quantum mechanics, one can replace the operator $-\frac{1}{2m_i}\Delta_{x_i}$ by $(-\Delta_{x_i}+2m_i^2)^{1/2}$. Then

$$H=\sum_{i=1}^{N}(-\Delta_{x_i}+m_i^2)^{1/2}+\sum_{i<j}V_{ij}(x_i-x_j)$$

is a Hamiltonian where the kinetic energy is treated relativistically (but not the interaction term).

Another example (which seems more difficult to treat) comes from solid state physics. Consider N electrons in a perfect crystal, where one uses a Peierls effective Hamiltonian to describe the interaction of each electron with the crystal lattice. Then if one considers N electrons belonging to the same band of the one electron problem, one gets (at least on a formal level) the following Hamiltonian:

$$H=\sum_{1}^{N}E(D_{x_i})+\sum_{i<j}\frac{e^2}{|x_i-x_j|}$$

Here $E(\theta)$ is the Bloch eigenvalue of the band, which is Γ^* periodic, if Γ^* is the dual lattice of the crystal. Other examples where the kinetic energy is similar arise in the scattering of spinwaves for the Heisenberg model.

Let us first give the definition of a general dispersive system (*see* [De1], [De2]). We consider a finite dimensional vector space X, with a familly $\{X_a\}_{a\in A}$ of vector subspaces of X. We endow X with a norm and with a Lebesgue measure to be able to define Fourier transforms of functions on X.

– The kinetic energy term is defined by a real valued function $\omega(\xi)$ on X^*.

– The interaction term is equal to:

$$V(x)=\sum_{a\in A}V_a(x)$$

where V_a is a real function such that $V_a(x+y)=V_a(x)$, $\forall y\in X_a$. For the moment we may assume that $V_a(x)$ is a bounded function. Precise conditions will be given in Section I. We ask that if one restricts V_a to a subspace supplementary to X_a, then V_a tends to 0 at infinity.

On the familly $\{X_a\}_{a\in A}$ one puts a partial ordering (*see* [Ag]), by putting: $b\subset a$ if $X_a\subset X_b$. One asks that:

$$X_{a_{\max}}\stackrel{\text{def}}{=}\bigcap_{a\in A}X_a=\{0\}$$

This means that the system has no global translation invariance. For $a \in \mathrm{A}$, one denotes by $\sharp a$ the maximal number of a_i such that:

$$a = a_1 \subsetneqq a_2 \ldots \subsetneqq a_n = a_{\max}.$$

Finally if $\sharp a_{\min} = \mathrm{N}$, where $\mathrm{X}_{a_{\min}} \stackrel{\mathrm{def}}{=} \mathrm{X}$, one defines a dispersive N-body Hamiltonian to be:

$$\mathrm{H} = \omega(\mathrm{D}_x) + \sum_{a \in \mathrm{A}} \mathrm{V}_a(x)$$

Standard N-body Hamiltonians are obtained when the function $\omega(\xi)$ is a positive definite quadratic form $q(\xi)$ on X^*. It is important to notice that this quadratic form adds a lot of structure to the problem. First of all, one gets by duality a quadractic form $\tilde{q}$ on X. The Jacobi coordinates used in the three-body problem are just orthogonal coordinates for $\tilde{q}$. This allows to define a good notion of subhamiltonians H^a for $a \in \mathrm{A}$, which are the restriction of H to X^a, where $\mathrm{X}^a = \mathrm{X}_a^\perp$. Moreover $q(\xi)$ gives a intrinsic way to identify X and X^*, which is important in defining the propagation set of Sigal-Soffer [S-S]. All this structure is lost in general dispersive systems. For example there is no *a priori* way to fix a space X^a supplementary to X_a, or to define subhamiltonians. From these considerations, it seems very clear that, except for two-body systems, general dispersive systems can be very different from standard ones.

Derezinski [De1], introduced a set of hypotheses which imply that a dispersive N-body system satisfy a Mourre estimate (*see* [M], [CFKS]) on a given energy interval $\triangle$. He introduces a quadratic form on X to define supplementary spaces X^a and subhamiltonians. This also induces a decomposition $\mathrm{X}^* = \mathrm{X}^{a*} \oplus \mathrm{X}_a^*$, and allows to define subhamiltonians $\mathrm{H}^a(\xi_a) = \omega(\mathrm{D}_{x^a}, \xi_a) + \sum_{b \subset a} \mathrm{V}_b(x_a)$. His hypotheses are actually independent of the quadratic form but depend on some properties of the ξ_a dependent eigenvalues of $\mathrm{H}^a(\xi_a)$, like differentiability and positivity of certain derivatives of them, which seem very difficult to check on examples.

In this work we introduce a class of dispersive systems which we call *regular*. The hypotheses we make depend only on the familly of spaces $\{\mathrm{X}_a\}_{a \in \mathrm{A}}$, and on the function $\omega(\xi)$. They roughly state existence of "good coordinates" (*i.e.* good choices of spaces supplementary to the X_a), in which one can study the relative motion of clusters for a given cluster decomposition. These conditions are stated in Section 2. They have the advantage that one can check them quite easily on examples. However contrary to the hypotheses of Derezinski, they are not intrinsic in the sense that they depend on the choice of a familly of projections.

We give two classes of examples of regular dispersive systems. The first class is obtained from a standard N-body Hamiltonian by replacing $q(\xi)$ by $f(q(\xi))$ for a function $f(t)$ with $f'(t) \geqq \mathrm{C}_0 > 0$.

The second class is the Hamiltonian of three relativistic particles with same mass, interacting with an exterior field and/or by pair interactions:

$$H=\sum_{i=1}^{3}(-\Delta_{x_i}+2m^2)^{1/2}+\sum_{i<j}V_{ij}(x_i-x_j)+\sum_{i=1}^{3}V_i(x_i).$$

For a regular dispersive system, there is a good notion of subhamiltonians, and a good notion of *thresholds*. In Section 3, we prove that a Mourre estimate holds for regular dispersive systems, outside the threshold set, which is closed and countable (*see* Theorems 3.5, 3.6). The conjuguate operator is the generator of dilations $\frac{1}{2}(\langle x, D_x\rangle + \langle D_x, x\rangle)$ on X.

As a consequence of the Mourre estimate, we get absence of singular continuous spectrum, discreteness of eigenvalues outside the thresholds and local H smoothness of $\langle x\rangle^{-1/2-\varepsilon}$, $\forall\varepsilon>0$, outside the thresholds. In Section 4, we apply the Mourre estimate to prove some basic results on scattering theory. Under additional dynamical conditions on $\omega(\xi)$, and some implicit conditions on the spectral projections of subsystems, we prove existence of wave operators for short range interactions and orthogonality of channels. Using the Mourre estimate we prove asymptotic completeness in the *two cluster region.*

Note added in proof: after completion of this work we received a preprint by V. Iftimie where conditions for the existence of wave operators for dispersive Hamiltonians are given. These conditions seem to be more in the spirit of those of Derezinski [De1].

The plan of the paper is the following:

Section 2. Regular dispersive Hamiltonians.

Section 3. Proof of the Mourre estimate.

Section 4. Some results on scattering theory for dispersive systems.

Appendix A.

Appendix B.

2. REGULAR DISPERSIVE HAMILTONIANS

In this section, we introduce the class of dispersive N-body systems we are going to consider, and we give two non trivial examples of them. The first set of hypotheses on the kinetic energy function $\omega(\xi)$ are standard symbol properties used for example in the Weyl calculus of pseudodifferential operators. We refer to the book of Hörmander [Hö] for more details.

HYPOTHESES A:

(Ai) $\omega(\xi)$ is positive and tends to $+\infty$ when ξ tends to ∞.

(Aii) there exist a slowly varying metric $g_\xi(\delta\xi)$ (*see* [Hö], Def. 18.4.1) such that:

$$\omega \in S(\omega+1, g_\xi(\delta\xi))$$

This means that:

$$|D_\xi^k \omega(\xi, t_1, \ldots, t_k)| \leqq c_k(\omega(\xi)+1)\Pi_1^k g_\xi(t_i)^{1/2}, \qquad \forall k \in \mathbf{N}$$

(Aiii) ω is g_ξ continuous and σ, g_ξ temperate (*see* [Hö], Defs. 18.42 and 18.5.1). This means in our case that:

$$\exists c,\ C>0 \text{ such that: } \forall \xi,\ \eta \in X^*:$$

$$g_\xi(\xi-\eta) \leqq c \quad \Rightarrow \quad C^{-1} \leqq \frac{\omega(\xi)}{\omega(\eta)} \leqq C$$

$$\omega(\xi) \leqq C\,\omega(\eta)(1+|\xi-\eta|)^N$$

(Aiv) $g_\xi(\delta\xi) \leqq |\delta\xi|^2$ and $g_\xi(\xi) \leqq 1$.

We introduce now the crucial set of hypotheses on $\omega(\xi)$, which are needed to prove the Mourre estimate. These hypotheses correspond to the implicit hypotheses C_1, C_2 in the work of Derezinski [De1], and have the advantage to be much easier to check on actual examples. They intuitively mean that for each cluster decomposition, if one suppresses the intercluster interaction, then some clusters will eventually more away from the others under the classical motion.

HYPOTHESES B. – There exist a familly π_a for $a \in A$ of projections: $X \to X_a$ such that:

(Bi) if $b \subset a$ then $\pi_b \pi_a = \pi_a \pi_b = \pi_a$.

One then denotes by π^a the projection $\mathbf{1}-\pi_a$, by X^a the vector space $\operatorname{Ker} \pi_a$ and by ${}^t\pi_a: X_a^* \to X^*$, ${}^t\pi^a: X^{a*} \to X^*$ the dual applications.

(Bii) $\forall a \in A$, $\forall \xi^a \in X^{a*}$, $\forall \xi_a \in X_a^*$ one has:

$$\langle \nabla_\xi \omega({}^t\pi^a \xi^a + {}^t\pi_a \xi_a),\ {}^t\pi_a \xi_a \rangle > 0 \qquad \text{for} \quad \xi_a \neq 0.$$

Condition (Bi) corresponds to the fact that if $b \subset a$ then $X_a \subset X_b$ (this is implied by $\pi_b \pi_a = \pi_a$) and also $X^b \subset X^a$ (this is implied by $\pi_a \pi_b = \pi_a$). Then $\pi_b(X^a) \subset X^a$ and we denote this subspace by X_b^a. We put $\pi_b^a = \pi_b \pi^a: X \to X_b^a$ and ${}^t\pi_b^a: X_b^{a*} \to X^*$ the dual application. Then we ask that:

(Biii) $\forall a \in A$, $\forall b \in A$ with $b \subset a$, one has:

$$\langle \nabla_\xi \omega({}^t\pi^b \xi^b + {}^t\pi_b^a \xi_b^a + {}^t\pi_a \xi_a),\ {}^t\pi_b^a \xi_b^a \rangle \geqq 0$$

We now state some auxiliary hypotheses which do not have a direct dynamical meaning:

HYPOTHESES C:

(Ci) $\inf_{\xi^a \in X^{a*}} (\omega({}^t\pi^a \xi^a + {}^t\pi_a \xi_a) - \omega({}^t\pi^a \xi^a))$ tends to $+\infty$ when ξ_a tends to ∞.

(Cii) $\exists N \in \mathbb{N}$ such that:

$$\forall a \in A \; g_\xi ({}^t\pi^a \xi^a) (\omega ({}^t\pi^a \xi^a + {}^t\pi_a \xi_a) + 1)^{-N}$$

is bounded.

(Ciii) $X^a \cap \bigcap_{b \subsetneq a} X_b = \{0\}$.

Finally let us state the hypotheses on the potentials. Recall from the Introduction that the potential $V(x)$ can be written as a sum: $V(x) = \sum_{a \in A} V_a(x)$, where V_a is a multiplicative operator which commutes with the translation operators $\mathscr{U}_\tau : u(x) \mapsto u(x-\tau)$, for all $\tau \in X_a$. One can then write $V_a(x) = V_a(\pi^a x)$.

HYPOTHESES D:

(Di) $\langle x^a \rangle^\mu V_a(x^a)$ is $\omega({}^t\pi^a D_{x^a})$ bounded with relative bound 0, for some $\mu > 0$. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$ for a given norm $|\ |$ on X.

(Dii) $\langle x^a, \nabla_{x^a} V^a(x^a) \rangle$ is bounded from

$$\{u \in L^2(X^a) \mid \omega({}^t\pi^a D_{x^a}) u \in L^2(X^a)\}$$

into

$$\{u \in L^2(X^a) \mid (\omega({}^t\pi^a D_{x^a}) + 1)^{-1/2} u \in L^2(X^a)\} \overset{\text{def}}{=} H_{-1}$$

(Diii) $\langle x^a, \nabla_{x^a} \langle x^a, \nabla_{x^a} V^a(x^a) \rangle \rangle$ is bounded from

$$\{u \in L^2(X^a) \mid \omega({}^t\pi^a D_{x^a}) u \in L^2(X^a)\} \quad \text{into} \quad L^2(X^a)$$

Let us now make some comments on these hypotheses.

– One could weaken (Bii) by asking strict positivity for $\xi_a \neq 0$ and $\xi_a \neq \xi_{a_i}$, $i = 1, \ldots, n$. Then the results of Section 3 still hold if one adds to the threshold set the eigenvalues of $H^a(\xi_{a_i})\, i = 1, \ldots, n$. (*See* Section 3 for the notation).

– Condition (Ci) ensures that for a given cluster decomposition a on an energy level λ, the relative momentum of the clusters is bounded.

– Condition (Cii) is of a technical nature and condition (Ciii) is used to construct suitable partitions of unity on X^a (*see* Lemma 3.1).

– Condition (Di) implies that $H = \omega(D_x) + V(x)$ is selfadjoint with domain $D(H) = \{u \in L^2(X) \mid \omega(D_x) u \in L^2(X)\}$. This follows from showing that $V_a(\pi^a x)$ is $\omega(D_x)$ bounded with relative bound 0, which in turn follows easily from hypotheses A and writing $\omega(D_x)$ as the direct integral:

$$\int_{X_a^*}^{\oplus} \omega({}^t\pi^a D_{x^a} + {}^t\pi_a \xi_a)\, d\xi_a$$

– Conditions (Dii) and (Diii) are the standard hypotheses on multicommutators needed to apply Mourre's theory. We have assumed for simplicity that $V_a(x)$ is a multiplication operator. To treat non multiplicative potentials it suffices to look in the proofs where commutators between V_a and multiplication operators arise and to write down the necessary hypotheses on V_a to control these terms. We do not give the details.

Example 1. – Consider a function $g \in C^\infty(\bar{\mathbf{R}}^+)$ such that:

$$\left|\frac{d^\alpha g}{dt^\alpha}\right| \leqq C_\alpha (g(t)+1)\langle t\rangle^{-\alpha}.$$
$$g'(t) \geqq c_0 > 0 \text{ on } \bar{\mathbf{R}}^+.$$

We take on X^* a quadratic form $q(\xi)$ which we denote by ξ^2 for simplicity. For $a \in A$, we denote by π_a the orthogonal projection on X_a for the dual quadratic form $\tilde{q}$ of q. Then it is an easy exercice to check that hypotheses A, B, C hold for $\omega(\xi) = g(\xi^2)$, for any set of subspaces $\{X_a\}_{a \in A}$.

Example 2. – We consider a model of three relativistic particles with pair interactions and/or in an exterior field. We assume that the particles have all the same mass which can be fixed to one. Then the Hamiltonian is of the form:

$$H = \sum_{i=1}^{3} (1 + D_{x_i}^2)^{1/2} + \sum_{i<j} V_{ij}(x_i - x_j) + \sum_{i=1}^{3} V_i(x_i)$$

The set of subspaces $\{X_a\}$ is described in appendix A. Then in case of *exterior field*, H is a regular dispersive system. In case of *no exterior field*, H commutes with translations of the whole system. If we restrict it to the space $\{x \in \mathbf{R}^9 \mid x_1 + x_2 + x_3 = 0\}$ and to particles with total momentum $D_{x_1} + D_{x_2} + D_{x_3} = 0$ (this corresponds to separate the motion of the "center of mass") then H becomes a regular dispersive system. The proof of these properties is given in appendix A.

3. PROOF OF THE MOURRE ESTIMATE

In this Section we prove that for regular dispersive systems, the Mourre estimate holds outside a set of points which can be described explicitely (*see* Def. 3.4) in terms of some subhamiltonians and called the *threshold set* of H.

Let us first introduce some standard notations. For $a \in A$, we denote by H_a the Hamiltonian:

$$\omega(D_x) + \sum_{b \subset a} V_b(x) \stackrel{\text{def}}{=} \omega(D_x) + V^a(x)$$

We denote by I_a the potential $H - H_a$.

Since H_a commutes with the translations U_τ for $\tau \in X_a$, we can write H_a as a direct integral:

$$H_a = \int_{X_a^*}^{\oplus} H^a(\xi_a)\, d\xi_a \tag{1}$$

Here $H^a(\xi_a) = \omega({}^t\pi^a D_{x^a} + {}^t\pi_a \xi_a) + V^a(x^a)$ with domain

$$D(H^a) = \{ u \in L^2(X^a) \mid \omega({}^t\pi^a D_{x^a}) u \in L^2(X^a) \}.$$

Here we consider V^a as a function on X^a by restriction. The facts that $H^a(\xi_a)$ is self-adjoint with domain $D(H^a)$ and that $(H^a(\xi_a) + i)^{-1}$ is weakly measurable follow easily from (Aiii), (Aiv) and (Di).

One also easily checks that the domain of H_a (written as a direct integral in (1)) (*see* the Definition in [R-S]) coincide with D(H). We will denote by H_0 the operator $H_{a_{\min}}$ with domain D(H).

As a first step we state without proof a lemma about existence of suitable partitions of unity. These partitions of unity are a standard tool in the geometric method in scattering theory (*see* for example [F-H], [S-S], [C-F-K-S]). The important hypothesis for their existence is condition (Ciii).

LEMMA 3.1. — *For any* $a \in A$ *there exist for* $b \in A$, $b \subset a$, $\#_a b = 2$, C^∞ *functions* $J_a^b(x^a)$ *supported in* $U_a^b = \{ x \in X^a \mid |\pi^d x| \geqq \delta |x|, \forall d \subset a, d \not\subset b \}$, *homogeneous of degree* 0 *in* $\{ x \in X^a \mid |x| \geqq 1 \}$, *and a* C_0^∞ *function* $J_a^a(x^a)$ *supported in* $\{ x \in X^a \mid |x| \leqq 2 \}$ *such that*: $\sum_{b \subset a,\, \#_a b \leqq 2} J_a^b(x^a)^2 = 1.$

We will denote by A^a the generator of dilations in X^a, which is the Weyl quantization of the function $\langle x^a, \xi^a \rangle$ on $T^* X^a$, and put $A = A^{a_{\max}}$. It is not difficult to check that A^a is selfadjoint with domain

$$D(A^a) = \{ u \in L^2(X^a) \mid A^a u \in L^2(X^a) \}.$$

We now check that A satisfies the technical hypotheses needed to apply Mourre's method.

LEMMA 3.2. — (i) $D(A) \cap D(H)$ *is dense in* D(H).
(ii) $[H, iA]$ *extends as a bounded operator from* D(H) *into* H_{-1}.
(iii) $[H_0, iA]$ *extends as a bounded operator from* D(H) *into* $L^2(X)$.
(iv) $[[H, iA], iA]$ *extends as a bounded operator from* D(H) *into* $L^2(X)$.

Proof. — The proof uses hypotheses A and D and is left to the reader. □

Finally let us state one more Lemma, which will be useful in the sequel. We denote by $f \in C_0^\infty(\mathbf{R})$ a cutoff function and by $J^a(x)$ a function in a partition of unity on X constructed in Lemma 3.1.

LEMMA 3.3. — (i) $[f(H_b), J^a]$ *is compact.*
(ii) $(f(H_a) - f(H))\, J^a$ *is compact.*

(iii) $\xi_a \mapsto f(\mathrm{H}^a(\xi_a))$ *is continuous for the norm topology.*

(iv) $\xi_a \mapsto f(\mathrm{H}^a(\xi_a))[\mathrm{H}^a(\xi_a), i\mathrm{A}^a] f(\mathrm{H}^a(\xi_a))$ *is continuous for the norm topology.*

(v) $\exists\, \mathrm{R}>0$ *such that* $f(\mathrm{H}^a(\xi_a))=0$ *for* $|\xi_a| \geqq \mathrm{R}$.

Proof. – We will use the Stone-Weierstrass gavotte (*see* [C.F.K.S]) and assume that $f(\lambda) = (\lambda - z)^{-1}$ for $z \in \mathbf{C} \setminus \mathbf{R}$. To prove (i) we compute: $[(\mathrm{H}_b - z)^{-1}, \mathrm{J}^a] = (\mathrm{H}_b - z)^{-1} [\mathrm{H}_0, \mathrm{J}^a] (\mathrm{H}_b - z)^{-1}$. $[\mathrm{H}_0, \mathrm{J}^a]$ is a pseudodifferential operator with principal symbol bounded by $\mathrm{C}(\omega(\xi)+1)\langle x \rangle^{-1}$, which shows that $[(\mathrm{H}_b - z)^{-1}, \mathrm{J}^a]$ is compact.

To prove (ii) we use the first resolvent formula:

$$((\mathrm{H}_a - z)^{-1} - (\mathrm{H} - z)^{-1}) \mathrm{J}^a = (\mathrm{H} - z)^{-1} \mathrm{I}_a (\mathrm{H}_a - z)^{-1} \mathrm{J}_a = (\mathrm{H} - z)^{-1} \mathrm{I}_a \mathrm{J}^a (\mathrm{H}_a - z)^{-1} + \mathrm{K}$$

where K is compact by (i). Then (ii) follows from the fact that $\mathrm{I}_a \mathrm{J}^a = 0(\langle x \rangle^{-\mu})$.

(iii) is easy and left to the reader. Using (iii), we remark that it suffices in order to prove (iv) to show that

$$\xi_a \mapsto (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}} [\mathrm{H}^a(\xi_a), i\mathrm{A}^a] (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}}$$

is norm continuous for some N big enough. Then we write:

$$\begin{aligned}(\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}} [\mathrm{H}^a(\xi_a), i\mathrm{A}^a] (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}} &= (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}} (\langle \nabla_\xi \omega({}^t\pi^a \mathrm{D}_{x^a} + {}^t\pi_a \xi_a), {}^t\pi^a \mathrm{D}_{x^a} \rangle \\ &\quad - \langle x^a, \nabla_{x^a} \mathrm{V}^a(x^a) \rangle) (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}}.\end{aligned}$$

By (Dii) $\langle x^a, \nabla_{x^a} \mathrm{V}^a(x^a) \rangle$ is $\omega({}^t\pi^a \mathrm{D}_{x^a})$ bounded from which it follows that $(\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}} \langle x^a, \nabla_{x^a} v^a(x^a) \rangle (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}}$ is norm continuous using (iii). We then remark that

$$\xi_a \mapsto \langle \nabla_\xi \omega({}^t\pi^a \xi^a + {}^t\pi_a \xi_a), {}^t\pi^a \xi^a \rangle (\omega({}^t\pi^a \xi^a) + 1)^{-\mathrm{N}}$$

is continuous in the $\mathrm{L}^\infty(\mathrm{X}^{a*})$ topology by (Aii), (Cii), which proves that

$$\xi_a \mapsto (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}} \langle \nabla_\xi \omega({}^t\pi^a \mathrm{D}_{x^a} + {}^t\pi_a \xi_a), {}^t\pi^a \mathrm{D}_{x^a} \rangle (\mathrm{H}^a(\xi_a) - z)^{-\mathrm{N}}$$

is norm continuous.

Finally (v) is an immediate consequence of (Ci) and of the fact that $\mathrm{H}^a(0)$ is bounded from below. □

We will now define the set of points where the Mourre estimate fails.

DEFINITION 3.4. – *For* $a \in \mathrm{A}$, *the set of a-thresholds denoted by* τ_a *is the union of the eigenvalues of* $\mathrm{H}^b(0)$ *for* $b \subsetneq a$.

We will denote by τ the set $\tau_{a_{\max}}$ and by σ_a for $a \in \mathrm{A}$ the set of eigenvalues of $\mathrm{H}^a(0)$. We now state the main result of this Section.

THEOREM 3.5. – *Let* λ *an energy level such that* $\lambda \notin \tau$. *Then there exist* $\mathrm{C}_0 > 0$, Δ *neighborhood of* λ, K *compact operator such that*:

$$\mathrm{E}_\Delta(\mathrm{H}) [\mathrm{H}, i\mathrm{A}] \mathrm{E}_\Delta(\mathrm{H}) \geqq c_0 \mathrm{E}_\Delta(\mathrm{H}) + \mathrm{K}.$$

Proof. – Our proof will be inspired by the beautiful proof of Froese-Herbst [F-H] of the Mourre estimate for standar N-body systems, which uses an induction on the number of particles. Let us first describe our induction hypothesis. We consider the set of $a \in A$ with $N(a)=j$, $j<N$, and we fix some compact sets U_a in X_a^*. Then we assume the following:

(H.j 1) $\forall \varepsilon>0$, $\forall \lambda \in R$, $\exists \triangle$ neighborhood of λ, $\exists K^a(\xi_a)$ compact operator such that:

$$\forall \xi_a \in U_a, \quad E_\triangle(H^a(\xi_a))[H^a(\xi_a), iA^a]E_\triangle(H^a(\xi_a)) \geqq -\varepsilon E_\triangle(H^a(\xi_a)) + K^a(\xi_a). \quad (2)$$

(H.j 2) If $\lambda \notin \tau_a$, there exist $c_0>0$, V_a neighborhood of 0 in X_a^*, $\triangle$ neighborhood of λ, and $K^a(\xi_a)$ compact operator such that:

$$\forall \xi_a \in V_a, \quad E_\triangle(H^a(\xi_a))[H^a(\xi_a), iA^a]E_\triangle(H^a(\xi_a)) \geqq c_0 E_\triangle(H^a(\xi_a)) + K^a(\xi_a). \quad (3)$$

The proof is divided in two steps:

Step 1. – In Step 1, we prove that (H.j) imply the following properties:

($\tilde{\text{H}}$.j 1) $\forall \varepsilon>0$, $\forall \lambda \in \mathbf{R}$, $\exists \triangle$ neighborhood of λ, such that:

$$\forall \xi_a \in U_a, \quad E_\triangle(H^a(\xi_a))[H^a(\xi_a), iA^a]E_\triangle(H^a(\xi_a)) \geqq -\varepsilon E_\triangle(H^a(\xi_a)) \quad (4)$$

($\tilde{\text{H}}$.j 2) if $\lambda \notin \tau_a \cup \sigma_a$, there exist $c_0>0$, V_a neighborhood of 0 in X_a^*, $\triangle$ neighborhood of λ such that:

$$\forall \xi_a \in V_a, \quad E_\triangle(H^a(\xi_a))[H^a(\xi_a), iA^a]E_\triangle(H^a(\xi_a)) \geqq c_0 E_\triangle(H^a(\xi_a)). \quad (5)$$

Let us fix some $\xi_{a0} \in U_a$. If $\lambda \notin \sigma_{pp}(H^a(\xi_{a0}))$, then $E_\triangle(H^a(\xi_{a0}))$ converges strongly to 0 when $\triangle$ tends to $\{\lambda\}$. Hence $K(\xi_{a0})E_\triangle(H^a(\xi_{a0}))$ tends to 0 in norm when $\triangle$ tends to $\{\lambda\}$. Starting from (2) with $\varepsilon/2$, we can find some neighborhood $\triangle_1$ of λ such that:

$$E_{\triangle_1}(H^a(\xi_{a0}))[H^a(\xi_{a0}), iA^a]E_{\triangle_1}(H^a(\xi_{a0})) \geqq -\varepsilon E_{\triangle_1}(H^a(\xi_{a0})). \quad (6)$$

If $\lambda \in \sigma_{pp}(H^a(\xi_{a0}))$, then we can use the argument of Froese-Herbst (*see* for example [F.H], [C.F.K.S, Thm 4.21]) and apply the virial theorem to $H^a(\xi_{a0})$ to get a neighborhood $\triangle_1$ of λ such that (6) holds. If we compose (6) to the left and right by $\chi(H^a(\xi_{a0}))$ for a cutoff function $\chi \in C_0^\infty(\triangle_1)$, $\chi=1$ on $\triangle_2 \subset \triangle_1$, and use the norm continuity of: $\xi_a \mapsto \chi(H^a(\xi_a))$ and of: $\xi_a \mapsto \chi(H^a(\xi_a))[H^a(\xi_a), iA^a]\chi(H^a(\xi_a))$ (*see* Lemma 3.3), we get: $\forall \varepsilon>0$, $\forall \lambda \in \mathbf{R}$, $\exists V_a$ neighborhood of ξ_{a0}, $\exists \triangle_3$ neighborhood of λ such that:

$$\forall \xi_a \in V_a, \quad E_{\triangle_3}(H^a(\xi_a))[H^a(\xi_a), iA^a]E_\triangle(H^a(\xi_a)) \geqq -\varepsilon E_{\triangle_3}(H^a(\xi_a)). \quad (7)$$

By the compactness of U_a, we can find some finite covering of U_a made of neighborhoods V_i of some points ξ_{a_i} and some smaller $\triangle_4$ such that (4) holds with $\triangle = \triangle_4$. Let us prove now (5). Since $\lambda \notin \sigma^a = \sigma_{pp}(H^a(0))$, by the arguments above (5) holds for $\xi_a=0$, for some neighborhood $\triangle$ of λ. Then the continuity argument above proves that there exist a neighborhood V_a of 0 in X_a^* such that (5) holds.

Step 2. – In Step 2, we prove that ($\tilde{\mathrm{H}}$.j) implies (H.j+1). Let us denote by:

$$\mathbf{1}=\sum_{\#_a^b\leq 2} \mathrm{J}_a^b(x^a)^2$$

a partition of unity constructed in Lemma 3.1. Then we have:

$$\begin{aligned}\mathrm{I}\overset{\text{def}}{=}\chi(\mathrm{H}^a(\xi_a))[\mathrm{H}^a(\xi_a), i\mathrm{A}^a]\chi(\mathrm{H}^a(\xi_a))\\ =\sum_b \mathrm{J}_a^{b2}\chi(\mathrm{H}^a(\xi_a))[\mathrm{H}^a(\xi_a), i\mathrm{A}^a]\chi(\mathrm{H}^a(\xi_a))\\ \sum_b \mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))\mathrm{J}_a^b[\mathrm{H}^a(\xi_a), i\mathrm{A}^a]\chi(\mathrm{H}^a(\xi_a))+\mathrm{K}_1(\xi_a)\end{aligned}$$

where $\mathrm{K}_1(\xi_a)$ is compact by Lemma (3.3 i).

$$\mathrm{I}=\sum_b \mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))[\mathrm{H}^a(\xi_a), i\mathrm{A}^a]\mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))+\mathrm{K}_2(\xi_a).$$

Here $\mathrm{K}_2(\xi_a)$ is again compact since the commutator between $[\mathrm{H}^a(\xi_a), i\mathrm{A}^a]$ and J_a^b is a pseudodifferential operator with principal symbol $\mathrm{D}_\xi^2\omega(\xi)({}^t\pi^a\xi^a, \nabla_{x^a}\mathrm{J}_a^b(x^a))$, which is compact after composition by $\chi(\mathrm{H}^a(\xi_a))$, using (Cii). Using again Lemma 3.3 (i) we get:

$$\mathrm{I}=\sum_b \mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))[\mathrm{H}^a(\xi_a), i\mathrm{A}^a]\chi(\mathrm{H}^a(\xi_a))\mathrm{J}_a^b+\mathrm{K}_3(\xi_a)\tag{8}$$

where $\mathrm{K}_3(\xi_a)$ is compact. We also have:

$$\mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))=\mathrm{J}_a^b\int_{\mathrm{X}_b^{a*}}^{\oplus}\chi(\mathrm{H}^b(\xi_b))\,d\xi_b^a+\mathrm{K}_4(\xi_a)\tag{9}$$

where $\mathrm{K}_4(\xi_a)$ is compact. This follows from Lemma 3.3 (ii) and from writing:

$$\omega({}^t\pi^a\mathrm{D}_{x^a}+{}^t\pi_a\xi_a)+\mathrm{V}^b(x^b)$$

as:

$$\int_{\mathrm{X}_b^{a*}}^{\oplus}\omega({}^t\pi^b\mathrm{D}_{x^b}+{}^t\pi_b^a\xi_b^a+{}^t\pi_a\xi_a)+\mathrm{V}^b(x^b)\,d\xi_b^a.$$

Finally we get:

$$\begin{aligned}\mathrm{I}=\sum_b \mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))\bigg(\int_{\mathrm{X}_b^{a*}}^{\oplus}[\mathrm{H}^b(\xi_b), i\mathrm{A}^b]\\ +\langle\nabla_\xi\omega({}^t\pi^b\mathrm{D}_{x^b}+{}^t\pi_b^a\xi_b^a+{}^t\pi_a\xi_a), {}^t\pi_b^a\xi_b^a\rangle\,d\xi_b^a\bigg)\\ \times\chi(\mathrm{H}^a(\xi_a))\mathrm{J}_a^b+\mathrm{K}(\xi_a)\end{aligned}\tag{10}$$

where $\mathrm{K}(\xi_a)$ is compact. To check this it suffices to notice that:

$$\mathrm{J}_a^b\chi(\mathrm{H}^a(\xi_a))\langle x^a, \nabla_{x^a}\mathrm{I}_b^a(x^a)\rangle\chi(\mathrm{H}^a(\xi_a))\mathrm{J}_b^a$$

is compact, where

$$I_b^a(x^a)=\sum_{d\subset a,\, d\not\subset b} V_d(x^a).$$

Combining (9), (10), we get:

$$I=\sum_b J_a^b\left(\int_{X_b^{a*}}^{\oplus}\chi(H^b(\xi_b))([H^b(\xi_b),\, iA^b]+T_a^b)\,\chi(H^b(\xi_b))\,d\xi_b^a\right)J_a^b+K(\xi_a),\qquad(11)$$

where $K(\xi_a)$ is compact, and T_a^b is equal to:

$$\langle \nabla_\xi\,\omega({}^t\pi^b D_{x^b}+{}^t\pi_b^a\xi_b^a+{}^t\pi_a\xi_a),\ {}^t\pi_b^a\xi_b^a\rangle.$$

By Lemma 3.3 (v) we see that if ξ_a stays in U_a and if $|\xi_b^a|\gg 1$, then $\chi(H^b(\xi_b))=0$. We use here the fact X_b^* is isomorphic to $X_a^*\oplus X_b^{a*}$. Hence in (11), we integrate only on a compact set U_b^a in X_b^{a*}. Using the induction hypotheses ($\tilde{H}$.j) for $H^b(\xi_b)$, and condition (Biii), we get that $\forall\varepsilon>0$, $\forall\lambda>0$, $\exists\,\triangle$ neighborhood of λ such that:

$$\forall\xi_b\in U_b,\quad E_\triangle(H^b(\xi_b)([H^b(\xi_b),\, iA^b]+T_a^b)\,E_\triangle(H^b(\xi_b))\geqq-\varepsilon\,E_\triangle(H^b(\xi_b))$$

Hence by (11) we get by taking χ with small enough support:

$$\chi(H^a(\xi_a))\,[H^a(\xi_a),\, iA^a]\,\chi(H^a(\xi_a))\geqq-\varepsilon\sum_b J_a^b\int_{X_b^{a*}}^{\oplus}\chi^2(H^b(\xi_b))\,d\xi_b^a\,J_a^b+K(\xi_a)$$

Using (9) and the arguments in the beginning of Step 2, we get (2) for a. It remains to check (3). Since $\lambda\notin\tau_a$, we can apply (5) to each H^b with $b\subset a$, $b\neq a$. So there exist $c_0>0$, $\triangle$ neighborhood of λ and some $0<\varepsilon_1\ll 1$ such that if $|\xi_a|\leqq\varepsilon_1$, $|\xi_b^a|\leqq\varepsilon_1$, then:

$$\chi(H^b(\xi_b))\,[H^b(\xi_b),\, iA^b]\,\chi(H^b(\xi_b))\geqq c_0\,\chi^2(H^b(\xi_b)).$$

Here χ is a cutoff function with support in $\triangle$. On the other hand, condition (Bii) and the fact that ${}^t\pi_b\xi_b={}^t\pi_a\xi_a+{}^t\pi_b^a\xi_b^a$ gives:

$$\langle \nabla_\xi\,\omega({}^t\pi^b\xi^b+{}^t\pi_b^a\xi_b^a),\ {}^t\pi_b^a\xi_b^a\rangle>0\qquad\text{for}\quad \xi_b^a\neq 0$$

Hence there exist $c_1>0$, $\varepsilon_2>0$ such that, if $\xi_b^a\in U_b^a$, $|\xi_b^a|\geqq\varepsilon_1$, $|\xi_a|\leqq\varepsilon_2$ one has:

$$\langle \nabla_\xi\,\omega({}^t\pi^b\xi^b+{}^t\pi_b^a\xi_b^a+{}^t\pi_a\xi_a),\ {}^t\pi_b^a\xi_b^a\rangle\geqq c_1>0.$$

If we take $2\varepsilon_0\leqq\inf(c_0, c_1)$, and $\triangle_1\subset\triangle$ such that (4) holds for $\triangle_1$ and ε_0, we get:

$$\chi(H^b(\xi_b))([H^b(\xi_b),\, iA^b]+T_a^b)\,\chi(H^b(\xi_b))\geqq\frac{1}{2}\inf(c_0, c_1)\,\chi^2(H^b(\xi_b))\qquad(12)$$

for $|\xi_a|\leqq\inf(\varepsilon_1, \varepsilon_2)$, $\xi_b^a\in U_b^a$, and χ a cutoff function supported in $\triangle_1$. Then we can use the same arguments as in the proof of (2) and we get

(3) for a. To complete the proof of Theorem 3.5, it just remains to start the induction. For $N(a)=1$, *i.e.* $a=a_{\min}$, the operator $H^a(\xi_a)$ is the operator of multiplication by $\omega(\xi_a)$ on $\mathbf{C}$, and $A^a=0$. Then (4) is obviously satisfied. To check (5) we note that if $\lambda\notin\sigma_{a_{\min}}$, then $\lambda\neq\omega(0)$. Then there exist a neighborhood V_a of 0 in $X_a^*=X^*$, a neighborhood $\triangle$ of λ such that $E_\triangle(H^a(\xi_a))=0$ for $\xi_a\in V_a$. Hence (5) is satisfied. Then the Theorem follows by noticing that $H=H^{a_{\max}}(0)$. This completes the proof of the Theorem. □

We will now state some consequences of Theorem 3.5.

THEOREM 3.6. — (i) *The singular spectrum of a regular dispersive Hamiltonian* H *is empty.*

(ii) *the eigenvalues of* H *can accumulate (with multiplicity) only at the threshold set of* H.

(iii) *if* $\lambda_0\notin\tau\cup\sigma_{pp}(H)$, *there exist a neighborhood* $\triangle$ *of* λ_0 *such that:*

$$\limsup_{\varepsilon\to 0^+}\|\langle x\rangle^{-s}E_\triangle(H)(H-\lambda\pm i\varepsilon)^{-1}\langle x\rangle^{-s}\|<+\infty\quad\text{for any } s>1/2.$$

(iv) $\langle x\rangle^{-s}$ *for* $s>1/2$ *is locally* H*-smooth at an energy* $\lambda_0\notin\tau\cup\sigma_{pp}(H)$.

Proof. — (i) and (ii) follow from Theorem 3.5 and Lemma 3.2 by the abstract Mourre's method (*see* [M], [C.F.K.S]), and the fact (*see* comments below) that τ is countable. To prove (iii), it suffices to prove that $(A+i)\langle x\rangle^{-1}E_\triangle(H)$ is bounded. By (Di), this is equivalent to prove that $(A+i)\langle x\rangle^{-1}\chi(H_0)$ is bounded for a cutoff function $\chi\in C_0^\infty(\mathbf{R})$. This is an easy consequence of the pseudodifferential calculus and of the fact $\omega(\xi)$ tends to $+\infty$ when $|\xi|$ tends to $+\infty$. Finally (iv) follows from (iii) (*see* [R.S]). □

Let us now make some comments on this result. (i) and (ii) show that the behavior of the spectrum of H is quite similar to that of a standard N-body Hamiltonian. In particular one can easily verify that for $a\neq a_{\max}$, $H^a(0)$ is again a regular dispersive N-body Hamiltonian on $L^2(X^a)$. Hence the eigenvalues of $H^a(0)$ can accumulate only at the eigenvalues of $H^b(0)$ for $b\subsetneq a$, the last accumulation point in this hierarchy being the "N-body threshold" $\omega(0)$. In particular the threshold set τ is closed and countable. Property (iv) for standard N-body Hamiltonians is an important tool in the proof of asymptotic completeness by Sigal-Soffer [S-S]. It allows to control all lower order terms when one constructs positive commutators to estimate the propagation set. In particular (iv) was used as an implicit hypothesis by Derezinski [De2] in his study of H-smoothness for dispersive N-body systems.

4. SOME RESULTS ON SCATTERING THEORY FOR DISPERSIVE SYSTEMS

In this Section we give some basic results on scattering theory for short range N-body dispersive systems. We introduce a set of dynamical conditions on $\omega(\xi)$ which together with some implicit conditions on the spectral projections on the point spectrum of subsystems are sufficient to ensure existence of channel wave operators and orthogonality of channels. These conditions strengthen conditions B in the following sense. Condition B (ii) intuitively means that *some* clusters separate from the others under the classical motion. To have existence of wave operators one needs to be sure that *all* clusters move away from the others. This is ensured by condition (Ei) below. Finally we apply the Mourre estimate of Section 3 to prove asymptotic completeness of wave operators in the *two cluster region.*

Let us now describe the conditions on $\omega(\xi)$. For $a, b \in \mathscr{A}$, $b \not\subset a$, we know that $\pi^b X_a \neq \{0\}$ since $X_a \not\subset X_b$. Let us denote by $X_{ab} \subset X_a$ the space $\operatorname{Ker} \pi^b|_{X_a}$. If we choose a projection $\pi_{ab}: X_a \to X_{ab}$, we denote by X_a^b the space $\operatorname{Ker} \pi_{ab}$, by $\tilde{\pi}_a^b: X_a \to X_a^b$ the projection $1 - \pi_{ab}$ and by ${}^t\pi_{ab}: X_{ab}^* \to X_a^*$, ${}^t\tilde{\pi}_a^b: X_a^{b*} \to X_a^*$ the dual projections. We sometimes identify ${}^t\tilde{\pi}_a^b$ and ${}^t\pi_{ab}$ with ${}^t\pi_a^t \tilde{\pi}_a^b$ and ${}^t\pi_a^t \pi_{ab}$. We ask that $\tilde{\pi}_a^b$ can be choosen to satisfy:

(Ei) $\langle \nabla_\xi \omega({}^t\pi^a \xi^a + {}^t\pi_{ab} \xi_{ab} + {}^t\tilde{\pi}_a^b \xi_a^b, {}^t\tilde{\pi}_a^b \xi_a^b \rangle > 0$ for $\xi_a^b \neq 0$.

It is straightforward to check that the dispersive system of Example 1 in Sect. 2 satisfies Hypotheses E, if one takes $\tilde{\pi}_a^b$ to be orthogonal projection on X_a^b. The Hamiltonian considered in Example 2 in the case *without* exterior field satisfies (E). Indeed we can easily check that for any b, a with $b \not\subset a$, $a \neq a_{\min}$, one has $X_{ab} = 0$, so (Ei) follows from (Bii). If $a = a_{\min}$, then one has $X_{ab} = X_b$, and (Ei) is also satisfied [*see* Appendix A, (A.4)]. In case with exterior field, there exist clusters a, b with $b \not\subset a$ such that (Ei) is not satisfied. Let us now define the wave operators. For $a \in A$, we denote by P_a the direct integral:

$$P_a = \int_{X_a^*}^{\oplus} E_{pp}(H^a(\xi_a))\, d\xi_a \tag{13}$$

The fact that $\xi_a \mapsto E_{pp}(H^a(\xi_a))$ is weakly measurable is well known (*see* for example [C.F.K.S., Proof of Thm. 9.4]). To prove existence of the wave operators, we need an extra assumption on the decay and regularity properties of the map $\xi_a \to E_{pp}(H^a(\xi_a))$. We ask that the following condition hold:

(Eii) the map $\xi_a \in X_a^* \mapsto E_{pp}(H^a(\xi_a)) \langle x^a \rangle^{s_0}$ belongs to the space $C_{\text{loc}}^{s_0}(X_a^*, \mathscr{L}(L^2(X^a)))$ for some $s_0 > 1$. Here $C_{\text{loc}}^{s_0}$ denotes the usual local Hölder space of exponent s_0.

We define ([1]) the wave operator $\Omega_a^\pm$ to be:

$$\Omega_a^\pm = s.\lim_{t\to\pm\infty} e^{it\mathrm{H}} e^{-it\mathrm{H}_a} \mathrm{P}_a. \tag{14}$$

The result of this Section is the following theorem:

THEOREM 4.1. – *Assume that* A, B, C, D, E *are satisfied with* $\mu > 1$. *Then:*

(i) *the wave operators* $\Omega_a^\pm$ *exist.*

(ii) *for* $a \neq b$, $\operatorname{Im}\Omega_a^\pm$ *is orthogonal to* $\operatorname{Im}\Omega_b^\pm$.

Proof. – To prove (i) we will use the Cook argument. We take u in a dense subset of $\mathrm{L}^2(\mathrm{X})$ and we will prove that $\lim_{t\to\pm\infty} e^{it\mathrm{H}} e^{-it\mathrm{H}_a} \mathrm{P}_a u$ exist. By the Cook argument, it suffices to show that

$$\| \mathrm{I}_a(x) e^{-it\mathrm{H}_a} \mathrm{P}_a u \| \quad \text{is in } \mathrm{L}^1(\mathbf{R}).$$

Here $\mathrm{I}_a(x) = \sum_{b\subset a} \mathrm{V}_b(x^b)$ and let us consider one term $\| \mathrm{V}_b(x^b) e^{-it\mathrm{H}_a} \mathrm{P}_a u \|$. We can write:

$$\mathrm{H}_a = \int_{\mathrm{X}_a^{b*}}^{\oplus} \mathrm{H}^a(\xi_a^b)\, d\xi_a^b. \tag{15}$$

Let us take u such that $\mathrm{E}_\triangle(\mathrm{H}_a) u = u$ for some interval $\triangle \subset \mathbf{R}$,

$$\hat{u}(x^a, x_{ab}, \xi_a^b) = 0, \quad \text{for } |\xi_a^b| \leqq \varepsilon_0, \qquad \text{and} \qquad \langle x_b^a \rangle^s u \in \mathrm{L}^2(\mathrm{X}), \quad \forall s \geqq 0.$$

(Here $\hat{u}$ is the Fourier transform in x_a^b). It is easy to see that the set of u satisfying these conditions for any $b \not\subset a$ and for some ε_0 is dense in $\mathrm{L}^2(\mathrm{X})$. We take then a cutoff function $\chi(\xi_a^b)$ supported in $|\xi_a^b| \geqq \varepsilon_0/4$, equal to 1 in $|\xi_a^b| \geqq \varepsilon_0/2$. Then we have $\mathrm{H}_a u = \chi(\mathrm{D}_{x_a^b}) \mathrm{H}_a u$ for u as above. Let us denote by A_a^b the Weyl quantization of $\langle x_a^b, \xi_a^b \rangle \chi_1(\xi_a)$, where $\chi_1 \in \mathrm{C}_0^\infty(\mathrm{X}_a^*)$ is a cutoff function equal to 1 in $|\xi_a| \leqq \mathrm{R}$. We introduce $\chi_1(\xi_a)$ to avoid having to consider what happens for large ξ_a. We have:

$$\begin{aligned} &\mathrm{E}_\triangle(\mathrm{H}_a) \chi(\mathrm{D}_{x_a^b}) [\mathrm{H}_a, i\mathrm{A}_a^b] \chi(\mathrm{D}_{x_a^b}) \mathrm{E}_\triangle(\mathrm{H}_a) \\ &\quad = \int_{\mathrm{X}_a^{b*}}^{\oplus} \mathrm{E}_\triangle(\mathrm{H}^a({}^t\pi_{ab} \mathrm{D}_{x_{ab}} + {}^t\pi_a^b \xi_a^b)) \chi_1({}^t\pi_{ab} \mathrm{D}_{x_{ab}}^t + {}^t\pi_a^b \xi_a^b) \chi(\xi_a^b) \\ &\qquad\qquad \mathrm{T}_a^b \chi(\xi_a^b) \mathrm{E}_\triangle(\mathrm{H}^a({}^t\pi_{ab} \mathrm{D}_{x_{ab}} + {}^t\pi_a^b \xi_a^b))\, d\xi_a^b \end{aligned} \tag{16}$$

where:

$$\mathrm{T}_a^b = \langle \nabla_\xi \omega({}^t\pi^a \mathrm{D}_{x^a} + {}^t\pi_{ab} \mathrm{D}_{x_{ab}} + {}^t\pi_a^b \xi_a^b), {}^t\pi_a^b \xi_a^b \rangle$$

It is easy to see that if $\mathrm{E}_\triangle(\mathrm{H}_a) \neq 0$, then $\chi_1(\xi_a) = 1$, provided we take R large enough. Indeed this follows from (Ci) and (Di). Using now (Ei), we

---

([1]) This definition was suggested to us by J. Derezinski.

get that:

$$\langle \nabla_{\xi}\,\omega\,({}^t\pi^a\,D_{x^a}+{}^t\pi_{ab}\,D_{x_{ab}}+{}^t\pi_a^b\,\xi_a^b),\ {}^t\pi_a^b\,\xi_a^b\rangle\,\chi^2(\xi_a^b)\geqq c_1\,\chi^2(\xi_a^b)\qquad \text{for}\quad c_1>0.$$

Hence H_a satisfies the following estimate:

$$E_{\triangle}(H_a)\,\chi(D_{x_a^b})\,[H_a,\,i\,A_a^b]\,\chi(D_{x_a^b})\,E_{\triangle}(H_a)\geqq c_1\,(E_{\triangle}(H_a)\,\chi(D_{x_a^b}))^2. \tag{17}$$

This can be considered as a refined version of the Mourre estimate. Indeed the Mourre estimate is a positivity condition on commutators which holds locally in energy with respect to a given Hamiltonian H. In our case H_a and $[H_a,\,i\,A_a^b]$ commute with all operators $\chi(D_{x_a^b})$. The estimate (17) is a version of the Mourre estimate which is local not only in energy but also in $D_{x_a^b}$. In Prop. 4.2, we will prove that one can extend the local time decay estimates of Jensen-Mourre-Perry [J.M.P] to cover the case where only (17) is satisfied. Of course the time decay estimates will hold only locally in $D_{x_a^b}$. The next estimate is proved in Proposition 4.2 below:

$$\|\langle x_a^b\rangle^{-s}\,e^{-it\,H_a}\,E_{\triangle}(H_a)\,\chi(D_{x_a^b})\,\langle x_a^b\rangle^{-s}\|\leqq C\,(1+|t|^{-s'})\;\forall\,0<s'<s \tag{18}$$

Recall that u is such that:

$$\langle x_a^b\rangle^s\,u\in L^2(X),\qquad \forall\,s>1 \tag{19}$$

Let us choose some number $1<s<s_0$. Then we get:

$$\begin{aligned}\|V_b(x^b)\,e^{-it\,H_a}\,P_a\,u\| &= \|V_b(x^b)\,E_{\triangle}(H_a)\,\langle x^b\rangle^{\mu}\,\langle x^b\rangle^{-\mu}\,\chi_1(D_{x_a})\,P_a\,e^{-it\,H_a}\,\chi(D_{x_a^b})\|\\ &= \|V_b(x^b)\,E_{\triangle}(H_a)\,\langle x^b\rangle^{\mu}\,\langle x^b\rangle^{-\mu}\,\langle x^a\rangle^{-s}\,\langle x_a^b\rangle^{s}\,\tilde{P}_a\,\langle x_a^b\rangle^{-s}\\ &\qquad\qquad e^{-it\,H_a}\,\chi(D_{x_a^b})\,u\|\end{aligned}$$

where $\tilde{P}_a$ is equal to $\langle x^a\rangle^s\,\langle x_a^b\rangle^{-s}\,P_a\,\chi_1(D_{x_a})\,\langle x_a^b\rangle^s$. By (Di) the term $V_b(x^b)\,E_{\triangle}(H_a)\,\langle x^b\rangle^{\mu}$ is bounded. Using also (Eii), one proves that $\tilde{P}_a$ is bounded (*see* Appendix B). Finally $\langle x^b\rangle^{-\mu}\,\langle x^a\rangle^{-s}\,\langle x_a^b\rangle^s$ is bounded since $\langle x_a^b\rangle\leqq C\,\langle \pi^b\,x_a\rangle$ and since we can always take s such that $1<s<\mu$. So we get:

$$\begin{aligned}\|V_b(x^b)\,e^{-it\,H_a}\,P_a\,u\| &\leqq C\,\|\langle x_a^b\rangle^{-s}\,e^{-it\,H_a}\,\chi(D_{x_a^b})\,u\|\\ &\leqq C\,\|\langle x_a^b\rangle^{-s}\,e^{-it\,H_a}\,E_{\triangle}(H_a)\,\chi(D_{x_a^b})\,\langle x_a^b\rangle^{-s}\,\langle x_a^b\rangle^{s}\,u\|\\ &\leqq C\,(1+|t|)^{-s'},\qquad \forall\,0<s'<s,\quad \text{by (18).}\end{aligned}$$

Since $s>1$, we get that $\|V_b(x^b)\,e^{-it\,H_a}\,u\|$ is in $L^1(\mathbf{R})$, for any $b\not\subset a$, which prove the existence of the wave operators $\Omega_a^{\pm}$. Let us now prove (ii): it suffices to show that for each u, v in a dense subset of $L^2(X)$, and for a sequence $t_n\to\pm\infty$,

$$I_n=\langle e^{-it_n}\,H_a\,P_a\,u,\ e^{-it_n\,H_b}\,P_b\,v\rangle$$

tends to 0 when n tends to $+\infty$. We have $a\neq b$ so we can assume that $a\not\subset b$. Then:

$$\mathrm{I}_n=\langle \mathbf{1}_{\{|x^a|\leqq \mathrm{R}_n\}}e^{-it_n\mathrm{H}_a}\mathrm{P}_a u,\ e^{-it_n\mathrm{H}_b}\mathrm{P}_b v\rangle + \langle \mathbf{1}_{\{|x^a|\geqq \mathrm{R}_n\}}e^{-it_n\mathrm{H}_a}\mathrm{P}_a u,\ e^{-it_n\mathrm{H}_b}\mathrm{P}_b v\rangle=\mathrm{I}_{n,1}+\mathrm{I}_{n,2}.$$

We have

$$\|\mathbf{1}_{\{|x^a|\geqq \mathrm{R}_n\}}e^{-it_n\mathrm{H}_a}\mathrm{P}_a u\|^2 = \int_{\mathrm{X}_a^*}\|\mathbf{1}_{\{|x^a|\geqq \mathrm{R}_n\}}e^{-it_n\mathrm{H}^a(\xi_a)}\mathrm{E}_{pp}(\mathrm{H}^a(\xi_a))\,\hat{u}(.,\xi_a)\|^2_{\mathrm{L}^2(\mathrm{X}^a)}\,d\xi_a = \int_{\mathrm{X}_a^*} f_n(\xi_a)\,d\xi_a.$$

The sequence $f_n(\xi_a)$ converges simply to 0 if R_n is a sequence tending to $+\infty$, and $0\leqq f_n(\xi_a)\leqq \|\hat{u}(.,\xi_a)\|^2_{\mathrm{L}^2(\mathrm{X}^a)}$, hence by Lebesgue dominated convergence Theorem:

$$\lim_{n\to+\infty}\|\mathbf{1}_{\{|x^a|\geqq \mathrm{R}_n\}_{n\to+\infty}}e^{-it_n\mathrm{H}_a}\mathrm{P}_a u\|=0.$$

So $\mathrm{I}_{n,2}$ tends to 0 when n tends to $+\infty$. Let us consider now $\mathrm{I}_{n,1}$,

$$|\mathrm{I}_{n,1}|\leqq \|\mathbf{1}_{\{x^a|\leqq \mathrm{R}_n\}}e^{-it_n\mathrm{H}_b}\mathrm{P}_b v\|\leqq c\,\mathrm{R}_n^s\|\langle x^a\rangle^{-s}e^{-it_n\mathrm{H}_b}\mathrm{P}_b v\|$$

As in the proof of (i), we can choose v with Fourier transform in ξ_b^a supported away for $\xi_b^a=0$. By the same arguments, we get for such v:

$$|\mathrm{I}_{n,1}|\leqq \mathrm{C}\,\mathrm{R}_n^s(1+|t_n|)^{-s}.$$

Taking $\mathrm{R}_n=t_n^{1/2}$ yields that $\lim_{n\to+\infty}\mathrm{I}_{n,1}$, which proves (ii). This completes the proof of the Theorem. □

Let us now state the Proposition used in the proof of Theorem 4.1. Since we do not aim for generality, we will not prove the most general abstract result in this direction. Let us just remark that our method could probably be used to prove other existence results for wave operators in cases where the stationary phase technique cannot be applied.

PROPOSITION 4.2. – *Let $\chi(\xi_a^b)$ a cutoff function supported where the function χ in* (17) *is equal to* 1. *Then for any s, $s'0<s'<s$, there exists $c>0$ such that the following estimates hold:*

$$\|\langle x_a^b\rangle^{-s}\chi(\mathrm{D}_{x_a^b})e^{-it\mathrm{H}_a}\mathrm{E}_\Delta(\mathrm{H}_a)\langle x_a^b\rangle^{-s}\|\leqq c(1+|t|)^{-s'},\qquad t\in\mathbf{R}.\quad(20)$$

Proof. – We will follow closely the proof of [J.M.P., Thm 2.2] and use similar notations. We will also sometimes refer to the book [C.F.K.S]. To simplify notations, let us put $\mathrm{H}_a=\mathrm{H}$, $\mathrm{A}_a^b=\mathrm{A}$ and $\chi_0=\chi_0(\mathrm{D}_{x_a^b})$ for a cutoff function $\chi_0(\xi_a^b)$ supported in a region where the estimate (17) holds. We will also denote by F the operator $f(\mathrm{H}_a)$ for a smooth cutoff function f

supported in $\triangle$, and by D the operator $(A^2+1)^{1/2}$. We will denote by $\| \quad \|_{0,2}$ the operator norm from $L^2(X)$ into $D(H)$. The proof will be divided in several steps.

Step 1. – We put

$$A_1=[H, A], \qquad B_1=\chi_0 A_1 \chi_0,$$

and for $\varepsilon>0$, $\operatorname{Im} z>\operatorname{Re} z\in\triangle$, we consider $G_z^M(\varepsilon)=(H-z+\varepsilon F B_1 F)^{-1}$. For later use, note that χ_0 commutes with $G_z^M(\varepsilon)$ and with F. One knows that $G_z^M(\varepsilon)$ exists by the standard argument of [M] and the positivity of $F B_1 F$. We will sometimes drop the subscript z in $G_z^M(\varepsilon)$ for simplicity of notations. Let us prove the following estimates:

$$\| F \chi_0^2 G^M(\varepsilon) \|_{0,2} \leqq C \varepsilon^{-1} \tag{21}$$

$$\| (1-F) G^M(\varepsilon) \|_{0,2} \leqq C \tag{22}$$

$$\| \chi_0^2 G^M(\varepsilon) D \|_{0,2} \leqq C \varepsilon^{-1/2} \tag{23}$$

By the same arguments as in the proof of Lemma 4.14 in [C.F.K.S.], we get:

$$\| F \chi_0^2 G^M(\varepsilon) \varphi \| \leqq C \varepsilon^{-1/2} | \langle \chi_0 \varphi, G^M(\varepsilon) \chi_0 \varphi \rangle |^{1/2} \tag{24}$$

Then one has:

$$(1-F) G^M(\varepsilon) = (1-F) G^M(0) (1+\varepsilon F B_1 F G^M(\varepsilon)) \tag{25}$$

Since $(1-F) G^M(0)$ is bounded from $L^2(X)$ into $D(H)$, and $F[H, A]$ is bounded, (22) will follow from (21). To prove (21), we write:

$$\begin{aligned} \| \chi_0^2 G^M(\varepsilon) \| + 1 &\leqq \| F \chi_0^2 G^M(\varepsilon) \| + \| (1-F) \chi_0^2 G^M(\varepsilon) \| + 1 \\ &\leqq C \varepsilon^{-1/2} \| \chi_0^2 G^M(\varepsilon) \|^{1/2} + C \varepsilon \| \chi_0^2 G^M(\varepsilon) \| + 1 \quad \text{by (24) (25)} \end{aligned}$$

From this it follows directly that $\| \chi_0^2 G^M(\varepsilon) \| \leqq C \varepsilon^{-1}$. So we have proven (21), (22). To prove (23), we introduce $F^M(\varepsilon)=D \chi_0 G^M(\varepsilon) \chi_0 D$. From (24) with $\varphi=D\psi$, we get:

$$\| F \chi_0^2 G^M(\varepsilon) D \psi \| \leqq C \varepsilon^{-1/2} | \langle \psi, D \chi_0 G^M(\varepsilon) \chi_0 D \psi \rangle |^{1/2}$$

Hence:

$$\| F \chi_0^2 G^M(\varepsilon) D \| \leqq C \varepsilon^{-1/2} \| F^M(\varepsilon) \|^{1/2}. \tag{26}$$

So (23) will follow from the fact that $\| F^M(\varepsilon) \|$ is bounded. This will be a consequence of Mourre's differential inequality technique. Indeed one has:

$$\dot{F}^M(\varepsilon) = -D \chi_0 G^M(\varepsilon) \chi_0 F [H, A] F \chi_0 G^M(\varepsilon) \chi_0 D = Q_1+Q_2+Q_3$$

where:

$$Q_1 = -D \chi_0 G^M(\varepsilon) \chi_0 (1\text{-}F) [H, A] (1-F) \chi_0 G^M(\varepsilon) \chi_0 D$$

$$\begin{aligned} Q_2 = &-D \chi_0 G^M(\varepsilon) \chi_0 F [H, A] (1-F) \chi_0 G^M(\varepsilon) \chi_0 D \\ &-D \chi_0 G^M(\varepsilon) \chi_0 (1-F) [H, A] F \chi_0 G^M(\varepsilon) \chi_0 D \end{aligned}$$

$$Q_3 = -D \chi_0 G^M(\varepsilon) \chi_0 [H, A] \chi_0 G^M(\varepsilon) \chi_0 D$$

Q_1 is bounded since $(1-F)G^M(\varepsilon)(H+i)$ is bounded. To estimate Q_2 it suffices to estimate $\|\chi_0^2 G^M(\varepsilon) D\|$, which is bounded by

$$C\varepsilon^{-1/2}\|F^M(\varepsilon)\|^{1/2} \quad \text{by (26).}$$

Let us consider now Q_3. One can write as in [C.F.K.S., Lemma 4.15], $Q_3 = Q_4 + Q_5$, where:

$$Q_4 = D\chi_0^2 G^M(\varepsilon)[H - z + \varepsilon F B_1 F, A] G^M(\varepsilon)\chi_0^2 D$$
$$Q_5 = -\varepsilon D\chi_0^2 G^M(\varepsilon)[F B_1 F, A] G^M(\varepsilon)\chi_0^2 D$$

Q_4 is bounded by $C\|D\chi_0^2 G^M(\varepsilon)\|\|A\chi_0^2 D\|$ by expanding the commutator. Since χ_0 is smooth, $A\chi_0^2 D$ is bounded by an easy application of pseudodifferential calculus. So we get by (26), (22):

$$\|Q_4\| \leqq C(\varepsilon^{-1/2}\|F^M(\varepsilon)\|^{1/2} + 1)$$

Q_5 is bounded by:

$$C\varepsilon(1 + \|D\chi_0^2 G^M(\varepsilon)\|) \leqq C\varepsilon(1 + \varepsilon^{-1}\|F^M(\varepsilon)\|)$$

since $[F B_1 F, A]$ is bounded. Finally we get the inequality:

$$\|\dot{F}^M(\varepsilon)\| \leqq C(1 + \varepsilon^{-1/2}\|F^M(\varepsilon)\|^{1/2} + \|F^M(\varepsilon)\|). \tag{27}$$

From this the fact that $\|F^M(\varepsilon)\|$ is uniformly bounded follows by the argument of Mourre (*see* for example [C.F.K.S., Prop. 4.11]).

Step 2. – In Step 2, we introduce $B_j = \chi_0[B_{j-1}, A]\chi_0$, for $j \geqq 2$ and put:

$$C_n(\varepsilon) = \sum_1^n \frac{\varepsilon^j}{j!} B_j$$

We will prove that $G(\varepsilon) \overset{\text{def}}{=} (H - z + C_n(\varepsilon))^{-1}$ existes as a bounded operator and satisfies (21), (22), (23). This can be done by following the proof of [J.M.P., Lemma 3.1]. Let us indicate the principal steps. One first defines:

$$\begin{aligned} G^0(\varepsilon) = {} & G^M(\varepsilon) - G^M(\varepsilon)\chi_0(1-F) \\ & \times (1 + \varepsilon A_1\chi_0 F G^M(\varepsilon)\chi_0(1-F))^{-1}\varepsilon A_1\chi_0 G^M(\varepsilon) \\ & = G^M(\varepsilon) - G^M(\varepsilon)(1-F) \\ & \times (1 + \varepsilon A_1\chi_0 F G^M(\varepsilon)\chi_0(1-F))^{-1}\varepsilon A_1\chi_0^2 G^M(\varepsilon) \end{aligned}$$

$G^0(\varepsilon)$ exists for $0 < \varepsilon \ll 1$, since $\|A_1\chi_0 F G^M(\varepsilon)\chi_0(1-F)\| \leqq C$ by (22). Then it is not difficult to check that $G^0(\varepsilon)$ satisfies (21), (22), (23) and is the inverse of $(H - z + \varepsilon B_1 F)$. Then one defines:

$$\begin{aligned} G^1(\varepsilon) = {} & G^0(\varepsilon) - G^0(\varepsilon)\varepsilon\chi_0 A_1 \\ & (1 + \varepsilon\chi_0(1-F)G^0(\varepsilon)\chi_0 A_1)^{-1}\chi_0(1-F)G^0(\varepsilon). \end{aligned}$$

One uses here that $\|(1-F)G^0(\varepsilon)\chi_0 A_1\|$ is bounded by a constant by (22) for $G^0(\varepsilon)$. Again we check easily that $G^1(\varepsilon)$ satisfies (21), (22), (23) and is the inverse of $(H - z + \varepsilon B_1)$, which proves (21), (22), (23) for $n=1$. For

$n \geqq 2$, we remark that:

$$(C_n(\varepsilon) - \varepsilon B_1) G^1(\varepsilon) = \sum_2^n \frac{\varepsilon^j}{j!} \chi_0 [B_{j-1}, A] \chi_0 G^1(\varepsilon) = \sum_2^n \frac{\varepsilon^j}{j!} [B_{j-1}, A] \chi_0^2 G^1(\varepsilon)$$

By (21), (22) for $G^1(\varepsilon)$, we get that $\| (C_n(\varepsilon) - \varepsilon B_1) G^1(\varepsilon) \| \leqq C\varepsilon$. So for $0 < \varepsilon \ll 1$, we can define:

$$G_z(\varepsilon) = G^1(\varepsilon) - G^1(\varepsilon)(1 + (C_n(\varepsilon) - \varepsilon B_1) G^1(\varepsilon))^{-1} (C_n(\varepsilon) - \varepsilon B_1) G^1(\varepsilon).$$

Then $G_z(\varepsilon)$ is the inverse of $(H - z + \varepsilon B_1)$ and satisfies:

$$\| \chi_0^2 G_z(\varepsilon) \|_{0,2} \leqq C \varepsilon^{-1} \tag{28}$$

$$\| \chi_0^2 G_z(\varepsilon) D \|_{0,2} \leqq C \varepsilon^{-1/2}. \tag{29}$$

Let us now prove that $G_z(\varepsilon)$ is norm differentiable and satisfies on $D(A)$:

$$\dot{G}_z(\varepsilon) = \chi_0 [G_z(\varepsilon), A] \chi_0 + \frac{\varepsilon^n}{n!} \chi_0 G_z(\varepsilon) [B_n, A] G_z(\varepsilon) \chi_0 \tag{30}$$

To give a meaning to (30), let us first check that $\chi_0 G_z(\varepsilon) A \chi_0$ and $\chi_0 A G_z(\varepsilon) \chi_0$ are bounded on $D(A)$ and that $\chi_0 [G_z(\varepsilon), A] \chi_0$ defined on $D(A)$ extends to $L^2(X)$ as a bounded operator. Since

$$[A, \chi_0] = \langle \nabla_\xi \chi_0, \xi \rangle (D_{x_a^b})$$

is bounded, $\chi_0 G_z(\varepsilon) A \chi_0$ is bounded on $D(A)$. Also

$$\chi_0 A G_z(\varepsilon) \chi_0 = \chi_0 G_z(\varepsilon) A \chi_0 + \chi_0 G_z(\varepsilon) [H + C_n(\varepsilon), A] G_z(\varepsilon) \chi_0,$$

which proves that $\chi_0 A G_z(\varepsilon) \chi_0$ is bounded on $D(A)$ and that $\chi_0 [G_z(\varepsilon), A] \chi_0$ extends to $L^2(X)$ as a bounded operator. The verification of (30) can now be done as in [J.M.P., Lemma 3.1].

Step 3. – In Step 3 we will consider for $n \geqq 2$, $s > n - 1/2$, $\operatorname{Im} z > 0$, $\varepsilon > 0$, operators of the form $F_{1,2}(\varepsilon) = D^s \chi_1 G_z(\varepsilon)^n \chi_2 D^s$. Here $\chi_i = \chi_i(D_{x_a^b})$ where $\chi_i(\xi_a^b)$ is a cutoff function such that $\chi_i \chi_0 = \chi_i$. We will prove that $F_{1,2}(\varepsilon)$ has a norm limit when $\varepsilon \to 0$ by adapting Mourre's differential inequality technique (*see* [J.M.P., Lemma 3.3]). However there is a somewhat subtle point here: in [J.M.P.] one has $\chi_i = \mathbf{1}$ and one uses a differential inequality satisfied by $F(\varepsilon)$ to prove by induction a weaker and weaker singularity of $F(\varepsilon)$ on $\varepsilon = 0$ to get the desired result. In our case, due to the insertion of the cutoffs χ_i one has to use different χ_i at each step of the induction. Let us describe this more in details. One has:

$$\begin{aligned} \dot{F}_{1,2}(\varepsilon) &= D^s \chi_1 \dot{G}_z(\varepsilon)^n \chi_2 D^s \\ &= D^s \chi_1 \sum_0^{n-1} G_z(\varepsilon)^j \dot{G}_z(\varepsilon) G_z(\varepsilon)^{n-j-1} \chi_2 D^s \\ &= D^s \chi_1 \sum_0^{n-1} G_z(\varepsilon)^j \chi_0 [G_z(\varepsilon), A] \chi_0 G_z(\varepsilon)^{n-j-1} \chi_2 D^s \end{aligned}$$

$$+\frac{\varepsilon^n}{n!}\sum_0^{n-1} G_z(\varepsilon)^j \chi_0 [B_n, A]\chi_0 G_z(\varepsilon)^{n-j}\chi_2 D^s$$
$$= D^s\chi_1 [G_z(\varepsilon)^n, A]\chi_2 D^s$$
$$+\frac{\varepsilon^n}{n!}\sum_0^{n-1} G_z(\varepsilon)^j [B_n, A] G_z(\varepsilon)^{n-j}\chi_2 D^s$$
$$= \mathrm{I}(\varepsilon) + \mathrm{II}(\varepsilon)$$

We estimate $\mathrm{II}(\varepsilon)$ by:

$$C\varepsilon^n \| D^s\chi_1 G_\varepsilon\| \sum_0^{n-1} \| G_z(\varepsilon)^j \chi_3 (H+i)\| \times \|(H+i)^{-1}[B_n, A](H+i)^{-1}\| \times \| G_z(\varepsilon)^{n-j-1}\chi_3(H+i)\| \, \|\chi_2 G_z(\varepsilon) D^s\|$$

Here χ_3 is a cutoff function such that $\chi_0\chi_3 = \chi_3$, $\chi_i\chi_3 = \chi_i$. Hence:

$$\mathrm{II}(\varepsilon) \leqq C\varepsilon^n \varepsilon^{-1/2}\varepsilon^{-j}\varepsilon^{j-n+1}\varepsilon^{-1/2} \leqq C$$

If we expand the commutator in $\mathrm{I}(\varepsilon)$, we get:

$$\mathrm{I}(\varepsilon) = D^s\chi_1 G_z^n(\varepsilon) A\chi_2 D^s - D^s\chi_1 A G_z^n(\varepsilon)\chi_2 D^s$$

Let us just consider the first term. One has:

$$D^s\chi_1 G_z^n(\varepsilon) A\chi_2 D^s = D^s\chi_1 G_z^n(\varepsilon)\chi_2 A D^s + D^s\chi_1 G_z^n(\varepsilon)[A, \chi_2] D^s = \mathrm{III}(\varepsilon) + \mathrm{IV}(\varepsilon)$$

We have that:

$$\|\mathrm{III}(\varepsilon)\| \leqq C \| D^s\chi_1 G_z^n(\varepsilon)\chi_2 D^{s-1}\|$$

Using the same interpolation argument as in [J.M.P., Thm 2.2], we get:

$$\|\mathrm{III}(\varepsilon)\| \leqq C\| D^s\chi_1 G_z^n(\varepsilon)\chi_2 D^s\|^{1-1/s} \| D^s\chi_1 G_z^n(\varepsilon)\chi_2\|^{1/s}.$$

By (21), (22), (23) one gets:

$$\|\mathrm{III}(\varepsilon)\| \leqq C\| F_{1,2}(\varepsilon)\|^{1-1/s}\varepsilon^{-(n-1/2)/s}$$

Finally we have:

$$\|\mathrm{IV}(\varepsilon)\| \leqq \| D^s\chi_1 G_z^n(\varepsilon)\chi_3 D^s\| \, \| D^{-s}[\chi_2, A] D^s\|$$

if $\chi_3\chi_2 = \chi_2$. Using again pseudodifferential calculus and interpolation, one sees that $D^{-s}[A, \chi_2] D^s$ is bounded. Putting all these estimates together, we get:

$$\|\dot{F}_{1,2}(\varepsilon)\| \leqq C(\| F_{1,2}(\varepsilon)\|^{1-1/s}\varepsilon^{-(n-1/2)/s} + \| F_{1,3}(\varepsilon)\| + \| F_{2,3}(\varepsilon)\| + 1) \quad (31)$$

Also (21), (22), (23) give:

$$\| F_{1,2}(\varepsilon)\| + \| F_{1,3}(\varepsilon)\| + \| F_{2,3}(\varepsilon)\| \leqq C\varepsilon^{-(n-1)}. \quad (32)$$

Inserting (32) into (31), we get:

$$\|\dot{F}_{1,2}(\varepsilon)\| \leqq C(\|F_{1,2}(\varepsilon)\|^{\alpha}\varepsilon^{-\beta}+\varepsilon^{-\delta})$$
$$\|F_{1,2}(\varepsilon)\| \leqq C\varepsilon^{-\gamma}$$

where $\gamma=\delta=(n-1)$, $\alpha=1-1/s$, $\beta=(n-1/2)/s$. The argument of [J.M.P., Lemma 3.3] gives:

$$\|F_{1,2}(\varepsilon)\| \leqq (\varepsilon^{-\alpha\delta-\beta+1}+\varepsilon^{-\delta+1}). \tag{33}$$

This is better since $\alpha<1$, $\beta<1$. Since the estimate (33) also holds for $\|F_{1,3}(\varepsilon)\|$, $\|F_{2,3}(\varepsilon)\|$, we can use it instead of (32) and insert it into (31). Integrating again, we get a better decay. After a finite number of steps, we get that:

$$\|D^s\chi_1(H-z)^{-1}\chi_2 D^s\| \leqq C$$

Then by the same arguments as in [J.M.P., Theorem 2.2] one gets that for $s>n-1/2$, $\lambda\in\triangle$, the norm limits $\lim_{\varepsilon\to 0} D^s\chi(H-\lambda\pm\varepsilon)^{-n}D^s$ exists and are equal to $\left(\frac{d}{d\lambda}\right)^{n-1} D^s\chi(H-\lambda\pm\varepsilon)^{-1}D^s$, if χ is a cutoff function supported where $\chi_0=1$. Using then that $\langle x_a^b\rangle^s D^s E_\triangle(H)$ is bounded, one gets that $\langle x_a^b\rangle^s E_\triangle(H)\chi(D_{x_a^b})\frac{dE}{d\lambda}\langle x_a^b\rangle^s$ is C^n in norm for $s>n+1/2$. By integration and interpolation (*see* [J.M.P., Theorem 4.2]) one gets finally the estimate (20). This completes the proof of the Proposition. □

As an application of the local H-smoothness of $\langle x\rangle^{-s}$, $s>1/2$, we will prove now the asymptotic completeness of wave operators in the two-cluster region. This is the exact analogue of asymptotic completeness below the three-body threshold for standard N-body systems (*see* [C], [S], [E]).

DEFINITION 4.3. — *A point* $\lambda\in\mathbf{R}$ *belongs to the two cluster region* (*TCR*) *if there exist a neighborhood* $\triangle$ *of* λ *such that* $E_\triangle(H_a)=0$, *for any* $a\in A$ *with* $\# a\geqq 3$.

In Exemple 1, any point in $]\inf\sigma_{ess}(H), \inf_{\# a>3}\sigma(H^a(0))[$ is in TCR. In Exemple 2, any point in $]\inf\sigma_{ess}(H), O[$ is in TCR. We will prove the following result:

THEOREM 4.4:

$$\mathscr{H}_c(H)\cap \operatorname{Im}(E_{TCR}(H))=\oplus_{\# a=2}\operatorname{Im}\Omega_a^{\pm}.$$

Proof. — We will only sketch the main steps of the proof, since it is a quite standard application of the Mourre estimate and of the geometric method. We will use some partitions of unity $\varphi_a(x)$, $a\in A$, $a\neq a_{max}$, having the following properties: $\varphi_a(x)$ is C^∞, homogeneous of degree 0 outside $\{x\in X\,|\,|x|\leqq 1\}$ and supported in:

$$\{x\in X\,|\,|\pi^a x|\leqq\varepsilon_a|x|\}\cap\{x\in X\,|\,\forall b,\ a\nsubseteq b,\ |\pi^b x|\geqq\varepsilon_b|x|\}$$

for some constants ε_a. Also one asks that $\sum_{a \neq a_{\max}} \varphi_a^2(x) = 1$ in

$$\{x \in X \mid |x| \geqq 2\}.$$

A proof of their existence can be found for example in [S-S, Sect 4]. Fix $\lambda \in \mathbf{R}$, and $\triangle$ a neighborhood of λ such that $E_\triangle(H_a) = 0$, $\forall a \in A$, $\sharp a \geqq 3$. Then it suffices to show that:

$$\forall u \in \mathscr{H}_c(H) \cap \operatorname{Im} E_\triangle(H),\ \forall \varepsilon > 0,\ \exists u_{a,\varepsilon}^{\pm} t\, L^2(x)$$

such that:

$$\lim_{t \to \pm\infty} \| e^{-itH} u - \sum_{\sharp a = 2} e^{-itH_a} P_a u_{a,\varepsilon}^{\pm} \| \leqq \varepsilon. \tag{34}$$

To prove that (34) implies Theorem 4.3, one has to use orthogonality of channels, Theorem 4.1 (ii). Let $\chi \in C_0^\infty(\mathbf{R})$ be a cutoff function supported in TCR such that $\chi = 1$ on $\triangle$. Using the arguments of Sigal-Soffer [S-S, Sects 2, 3], it suffices to prove that the Deift-Simon wave operators defined by:

$$W_b^{\pm} u = \lim_{t \to \pm\infty} e^{itH_b} \varphi_b^2 e^{-itH} \chi(H) u \tag{35}$$

exist for $u \in \mathscr{H}_c(H)$, $b \neq a_{\max}$, and that $W_b^{\pm} = 0$ if $\sharp b \geqq 3$.

From the proof of (35) will also follow that for $\sharp a = 2$, $b \nsubseteq a$

$$s \lim_{t \to \pm\infty} e^{itH_b} \varphi_b^2 e^{-itH_a} \chi(H_a)(1 - P_a) = 0$$

(remark that $\operatorname{Im}(1 - P_a) \subset \mathscr{H}_c(H_a)$) from which (34) follows easily. So it remains to prove (35). Since $u \in \mathscr{H}_c(H)$, we can replace u modulo an arbitrary small error by a finite sum $\sum_1^k u_i$, where $\chi_i(H) u_i = u_i$, and $\chi_i \in C_0^\infty(\mathbf{R})$ is a cutoff function supported in TCR and outside all thresholds and eigenvalues of both H and the H_a for $\sharp a = 2$. In fact one just has to use the fact that the eigenvalues of H can accumulate only at thresholds of H, *see* Theorem 3.6. So we may assume that $u = u_i$. It is also easy to see that:

$$\lim_{t \to \pm\infty} (1 - \tilde{\chi}_i(H_b)) e^{itH_b} \varphi_b^2 e^{-itH} u = 0 \tag{36}$$

where $\tilde{\chi}_i$ is a cutoff function such that $\tilde{\chi}_i \chi_i = \chi_i$. (use Lemma 3.3 (i), (ii), and the fact that $u \in \mathscr{H}_c(H)$). From now on, we drop the index i in χ_i, $\tilde{\chi}_i$. Again by Lemma 3.3, we get that:

$$\chi(H) = \sum_{a \neq a_{\max}} \tilde{\varphi}_a \chi(H_a) \tilde{\varphi}_a + K$$

where $\{\tilde{\varphi}_a\}$ is another partition of unity similar to $\{\varphi_a\}$, and K is a compact operator. Since χ is supported in TCR, we get:

$$\chi(\mathrm{H}) = \sum_{\# a=2} \tilde{\varphi}_a \chi(\mathrm{H}_a) \tilde{\varphi}_a + \mathrm{K} \tag{37}$$

To prove existence of the limit (35), we write using also (36):

$$\tilde{\chi}(\mathrm{H}_b) e^{it\mathrm{H}_b} \varphi_b^2 \chi(\mathrm{H}) e^{-it\mathrm{H}} u$$
$$= \sum_{\# a=2} \tilde{\chi}(\mathrm{H}_b) e^{it\mathrm{H}_b} \varphi_b^2 \tilde{\varphi}_a \chi(\mathrm{H}_a) \tilde{\varphi}_a e^{it\mathrm{H}} u + \tilde{\chi}(\mathrm{H}_b) e^{it\mathrm{H}_b} \varphi_b^2 \mathrm{K} e^{it\mathrm{H}} u. \tag{38}$$

One can choose the partition of unity $\tilde{\varphi}_a$ such that: $\varphi_b \tilde{\varphi}_a$ is supported near 0 if $\# a=2$, $b \neq a$. Since all terms in (38) containing H compact operators will give 0 in the time limit $t \to \pm\infty$, we get $\mathrm{W}_b^{\pm}=0$ for $\# b \geqq 3$. For $\# b=2$ one has:

$$\tilde{\chi}(\mathrm{H}_b) e^{it\mathrm{H}_b} \varphi_b^2 \chi(\mathrm{H}) e^{-it\mathrm{H}} u$$
$$= \tilde{\chi}(\mathrm{H}_b) e^{it\mathrm{H}_b} \varphi_b^2 \tilde{\varphi}_b \chi(\mathrm{H}_b) \tilde{\varphi}_b e^{-it\mathrm{H}} \chi(\mathrm{H}) u + o(1). \tag{39}$$

Denote by

$$\mathrm{I}(t) = \langle \varphi_b^2 \tilde{\varphi}_b \chi(\mathrm{H}_b) \tilde{\varphi}_b e^{-it\mathrm{H}} \chi(\mathrm{H}) u, \tilde{\chi}(\mathrm{H}_b) e^{-it\mathrm{H}_b} v \rangle,$$

for $v \in \mathrm{L}^2(\mathrm{X})$. We have:

$$\mathrm{I}(t) - \mathrm{I}(0) = \int_0^t \langle (\mathrm{H}_b \varphi_b^2 \tilde{\varphi}_b \chi(\mathrm{H}_b) \tilde{\varphi}_b - \varphi_b^2 \tilde{\varphi}_b$$
$$\chi(\mathrm{H}_b) \tilde{\varphi}_b \mathrm{H}) e^{-it\mathrm{H}} \chi(\mathrm{H}) u, \tilde{\chi}(\mathrm{H}_b) e^{-it\mathrm{H}_b} v \rangle \, dt \tag{40}$$

Now $\mathrm{H} = \mathrm{H}_b + \mathrm{I}_b(x)$, where $\langle x \rangle^{\mu/2} \chi(\mathrm{H}_b) \tilde{\varphi}_b \mathrm{I}_b \langle x \rangle^{\mu/2}$ is bounded by (Di). So in the right hand side of (40), the term containing

$$\langle \varphi_b^2 \tilde{\varphi}_b \chi(\mathrm{H}_b) \tilde{\varphi}_b \mathrm{I}_b e^{-it\mathrm{H}} \chi(\mathrm{H}) u, e^{-it\mathrm{H}_b} \tilde{\chi}(\mathrm{H}_b) v \rangle$$

as integrand is integrable in t, by Cauchy-Schwartz inequality and H (resp. H_b) smoothness of $\langle x \rangle^{-\mu/2}$ on support of χ (resp. support of $\tilde{\chi}$). The other term in the right hand side of (40) has the integrand:

$$[\mathrm{H}_b, \varphi_b^2 \tilde{\varphi}_b \chi(\mathrm{H}_b) \varphi_b] = [\mathrm{H}_b, \varphi_b^2 \tilde{\varphi}_b] \chi(\mathrm{H}_b) \varphi_b + \varphi_b^2 \tilde{\varphi}_b \chi(\mathrm{H}_b) [\mathrm{H}_b, \varphi_b] \tag{41}$$

An easy extension of Lemma 3.3 gives that:

$$\mathrm{R}_b \overset{\text{def}}{=} \chi(\mathrm{H}_b) - \sum_{c \subsetneqq b} \chi(\mathrm{H}_c) \psi_c$$

for some partition of unity $\{\psi_c\}$ similar to $\{\varphi_a\}$ (with the X_a for $a \not\subset b$ deleted) satisfies: $\mathrm{H}_b \mathrm{R}_b \langle x \rangle$ is bounded. So $\chi(\mathrm{H}_b) \varphi_b = \mathrm{R}_b \varphi_b$, since χ is supported in TCR and $\# b=2$.

Using now hypotheses (A) $[\mathrm{H}_b, \varphi_b^2 \tilde{\varphi}_b](\mathrm{H}_b + i)^{-1}$ is bounded. Hence the first term in the r.h.s. of (41) is of the form: $\langle x \rangle^{-1} \mathrm{K}_b \langle x \rangle^{-1}$, where K_b is bounded. The second term is similar, and both give integrable terms in (40), using again Cauchy-Schwartz inequality and H (resp. H_b) smoothness of $\langle x \rangle^{-1}$ on support of χ (resp. support of $\tilde{\chi}$). The proof of the existence of the limit (35) follows as in [S-S], which completes the proof of the theorem 4.4. □

APPENDIX A

In this appendix we prove that the Hamiltonian of 3 relativistic particles with same mass with pair interactions and in an exterior field is regular. By a change of scale we can fix the mass to be equal to 1 and the Hamiltonian is:

$$H = \sum_{i=1}^{3} (1 + D_{x_i}^2)^{1/2} + \sum_{i=1}^{3} V_i(x_i) + \sum_{i<j} V_{ij}(x_i - x_j).$$

The set of subspaces $\{X_a\}$ is the following: for

$$a = \{(i, j), k\}, \qquad X_a = \{x \in X \mid x_i = x_j\} \quad (\text{here } X = \mathbf{R}^9)$$
$$a = \{1, 2, 3\}, \qquad X_a = X$$
$$a = \{(1, 2, 3)\}, \qquad X_a = \{x \in X \mid x_1 = x_2 = x_3\}$$

Finally one has three X_a corresponding to the exterior field: for $a = i$, $X_a = \{x \in X \mid x_i = 0\}$. Let us first consider the case without exterior fields, *i.e.* X_1, X_2, X_3 are suppressed.

The case without exterior field:

As it stands our system is not dispersive system since $X_{a_{\max}} \neq \{0\}$ (in other words H commutes with translations of the whole system). To reduce the problem one has to fix the total momentum. One must be aware of the fact that there is no *a priori* way to do that.

We consider the following matrix on $\mathbf{R}^9$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -21 \end{pmatrix}$$

and put ourselves in the coordinates $\tilde{x} = A x$. A is suited to the study of the cluster $\{1, (2, 3)\}$.

– For $a = \{(1, 2, 3)\} = a_{\max}$ we put: $\pi_{a_{\max}}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (\tilde{x}_1, 0, 0)$.

– For $a = \{(1, (2, 3)\}$ we put: $\pi_a(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (\tilde{x}_1, 0, \tilde{x}_3)$.

Since we have $\xi = {}^t A \tilde{\xi}$, H can be rewritten as:

$$(1 + (D_{\tilde{x}_1} + D_{\tilde{x}_2} + D_{\tilde{x}_3})^2)^{1/2} + (1 + (D_{\tilde{x}_3} - D_{\tilde{x}_2})^2)^{1/2} + (1 + 4 D_{\tilde{x}_3}^2)^{1/2} + V(\tilde{x}). \quad (A.1)$$

To separate the "center of mass" motion we restrict H to $L^2(X^{a_{\max}})$ where $X^{a_{\max}} = \operatorname{Ker} \pi_{a_{\max}}$, by replacing $D_{\tilde{x}_1} = D_{x_1} + D_{x_2} + D_{x_3}$ by 0 in (A.1). This amounts to fix the total momentum to be 0. Then one checks that $X^{a_{\max}} = \{ x \in \mathbf{R}^9 \mid x_1 + x_2 + x_3 = 0 \}$ is independent of the choice of cluster we made, so every thing will work similarly for clusters $\{(1, 2), 3\}$ and $\{(1, 3), 2\}$.

– To check (Bii) for $a = \{1, (2, 3)\}$ with π_a defined above we have to show that if $\omega = (1 + (\xi_2 + \xi_3)^2)^{1/2} + (1 + (\xi_3 - \xi_2)^2)^{1/2} + (1 + 4\xi_3^2)^{1/2}$, one has:

$$\left\langle \frac{\partial \omega}{\partial \xi_3}(\xi_2, \xi_3), \xi_3 \right\rangle > 0 \qquad \text{for} \quad \xi_3 \neq 0. \tag{A.2}$$

The condition (Biii) has to be checked only if $b = a_{\min} = \{1, 2, 3\}$, for which $\pi_{a_{\min}} = \mathbf{1}$, and $\pi_b^a = \pi^a$. Then one has to check that:

$$\left\langle \frac{\partial \omega}{\partial \xi_2}(\xi_2, \xi_3), \xi_2 \right\rangle \geqq 0 \tag{A.3}$$

Note that (A.3) implies (A.2). Let us check (A.3):

$$\left\langle \frac{\partial \omega}{\partial \xi_2}(\xi_2, \xi_3), \xi_2 \right\rangle = \frac{(\xi_2 + \xi_3, \xi_2)}{(1 + (\xi_2 + \xi_3)^2)^{1/2}} + \frac{(\xi_2 - \xi_3, \xi_2)}{(1 + (\xi_2 - \xi_3)^2)^{1/2}} = \mathrm{I}.$$

Let us set

$$x = \|\xi_2\|^2, \qquad y = \langle \xi_3, \xi_2 \rangle, \qquad z = \|\xi_3\|^2,$$
$$a = \|\xi_2 + \xi_3\|^2, \qquad b = \|\xi_2 - \xi_3\|^2.$$

Then

$$\mathrm{I} = (x(2 + 2x + 2z + 2(1+a)^{1/2}(1+b)^{1/2}) - 4y^2) \times (1+a)^{-1/2}(1+b)^{-1/2}(((1+a)^{1/2} + (1+b)^{1/2})^{-1}.$$

Let us check that $x + x^2 + zx + x(1+a)^{1/2}(1+b)^{1/2} - 2y^2 \geqq 0$. This is equivalent to:

$$\begin{aligned} & x(1+a)^{1/2}(1+b)^{1/2} \geqq 2y^2 - x - x^2 - xz \\ \Leftrightarrow \quad & x^2(1 + 2x + 2z + x^2 + 2zx + z^2 - 4y^2) \geqq x^2 + x^4 + x^2 z^2 + 4y^4 + 2x^3 \\ & \qquad + 2x^2 z - 4xy^2 + 2x^3 z - 4x^2 y^2 - 4xzy^2 \\ & \qquad \Leftrightarrow \quad 4xy^2 - 4y^4 + 4zxy^2 \geqq 0 \\ & \qquad \Leftrightarrow \quad \|\xi_2\|^2 - \langle \xi_3, \xi_2 \rangle^2 + \|\xi_2\|^2 \|\xi_3\|^2 \geqq 0 \end{aligned} \tag{A.4}$$

which is obvious. Hence (A.3) and (A.2) hold, which shows that the conditions B hold. For conditions A, C, we take the metric

$$g_\xi(\delta\xi) = \frac{\delta\xi_1^2}{\langle \xi_1 \rangle^2} + \frac{\delta\xi_2^2}{\langle \xi_2 \rangle^2} + \frac{\delta\xi_3^2}{\langle \xi_3 \rangle^2},$$

and restric it to $T^*(X^{a_{\max}})$. The verification of conditions A, C is easy and left to the reader. Let us now consider the case with exterior field.

The case with exterior field:

Then $X_{a_{\max}} = \{0\}$, which corresponds to the fact that H does not commute any more with translations of the system. Let us describe the set of projections we use. For $a=1, 2, 3$, $X_a = \{x \in X \mid x_a = 0\}$ we take π_a to be the projection on X_a with kernel $\{x \in X \mid x_i = 0, i \neq a\}$. For example if $a=1$, then we have to check that:

$$\left\langle \frac{\partial \omega}{\partial \xi_1}, \xi_1 \right\rangle > 0 \quad \text{for} \quad \xi_1 \neq 0 \tag{A.5}$$

$$\left\langle \frac{\partial \omega}{\partial \xi_2}, \xi_2 \right\rangle + \left\langle \frac{\partial \omega}{\partial \xi_3}, \xi_3 \right\rangle \geqq 0 \tag{A.6}$$

(A.5) and (A.6) are immediate. For $a = \{1, (2, 3)\}$ we take the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

and put ourselves in the coordinates $\tilde{x} = A x$. Then $X_a = \{\tilde{x}_2 = 0\}$, and we take $\pi_a(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (\tilde{x}_1, 0, \tilde{x}_3)$. The kinetic term can be rewritten as:

$$\omega(\tilde{\xi}) = (1 + (\tilde{\xi}_3 + \tilde{\xi}_2)^2)^{1/2} + (1 + (\tilde{\xi}_3 - \tilde{\xi}_2)^2)^{1/2} + (1 + \tilde{\xi}_1^2)^{1/2}.$$

From (A.4), we get that:

$$\left\langle \frac{\partial \omega}{\partial \tilde{\xi}_1}, \tilde{\xi}_1 \right\rangle + \left\langle \frac{\partial \omega}{\partial \tilde{\xi}_3}, \tilde{\xi}_3 \right\rangle > 0 \quad \text{for} \quad (\tilde{\xi}_1, \tilde{\xi}_3) \neq 0.$$

and: $\left\langle \frac{\partial \omega}{\partial \tilde{\xi}_2}, \tilde{\xi}_2 \right\rangle \geqq 0$. Hence conditions (B) hold for exterior field. Conditions (A) and (C) can be checked as in the preceeding case. □

APPENDIX B

In this appendix we will prove the boundedness of $\tilde{P}_a$ used in the proof of Thm. 4.1. Recall that $\tilde{P}_a$ is equal to $\|x^a\|^s \|x_a^b\|^{-s} P_a \chi_1(D_{x_a}) \|x_a^b\|^s$. If we conjuguate everything by the Fourier transform in x_a, we see that this property is equivalent to the following: on $L^2(X_a^*, L^2(X^a))$, one consider the following multiplier:

$$A: \quad u(\xi_a) : \mapsto \chi_1(\xi_a) \|x^a\|^s E_{pp}(H^a(\xi_a)) u(\xi_a).$$

Then $\tilde{P}_a$ is bounded if and only if A is bounded from the Sobolev space $H^{-s}(X_a^*, L^2(X^a))$ into itself, or equivalently the multiplier A^* of symbol $\chi_1(\xi_a) E_{pp}(H^a(\xi_a)) \|x^a\|^s$ is bounded on $H^s(X_a^*, L^2(X^a))$. Since $\chi_1(\xi_a)$ has compact support, and the function: $\xi_a : \mapsto E_{pp}(H^a(\xi_a)) \|x^a\|^s$ is in the

Hölder space $C^{s_0}(X_a^*, \mathscr{L}(L^2(X^a)))$, this will be a consequence of (a vector version of) the fact that the operator of multiplication by a compactly supported $C^{s_0}(R^n)$ function is bounded on $H^s(R^n)$ for all $0<s<s_0$. A simple way to prove this fact is by means of the Littlewood-Paley decomposition (*see* for example [Bo]). For completeness we will sketch the proof. In the sequel the variable x will denote the actual variable on R^n. Let us take functions $\varphi(\xi)$, $\psi(\xi)\in C_0^\infty(R^n)$ such that $\psi(\xi)=1$ for $|\xi|\leqq 1/2$, $\psi(\xi)=0$ for $|\xi|\geqq 1$, φ is supported in the shell $1/2\leqq|\xi|\leqq 2$ and:

$$\psi(\xi)+\sum_0^\infty \varphi(2^{-p}\xi)=1.$$

For $u\in S'(R^n)$ let us put $u_{-1}=\psi(D_x)u$, $u_p=\varphi(D_x)u$, and $S_p u=\sum_{-1}^{p-1} u_q$.

Then the spaces $H^s(R^n)$ and $C^\rho(R^n)$ are caracterized (with norm equivalence) by the properties:

$$u\in C^\rho \Leftrightarrow \|u_p\|_\infty\leqq C\,2^{-p\rho}c_p \quad \text{where} \quad c_p\in l^\infty$$
$$u\in H^s \Leftrightarrow \|u_p\|_2\leqq C\,2^{-ps}c_p \quad \text{where} \quad c_p\in l^2$$

Let us now write for $a\in C^{s_0}$, $u\in H^s$, the product au as:

$$au=\sum_{p\leqq q+N_0} a_p u_q+\sum_{p\geqq q-N_0} a_p u_q+\sum_{|p-q|<N_0} a_p u_q=\Sigma_1+\Sigma_2+\Sigma_3.$$

For N_0 big enough, the terms in Σ_1 have their Fourier transform supported in $\{(1/2-\varepsilon)2^q\leqq|\xi|\leqq(2+\varepsilon)2^q\}$ One can hence estimate $\|\Sigma_1\|_s^2$ by:

$$\sum_q 2^{2qs}\|u_q\|_0^2\|a\|_\infty^2.$$

Similarly $\|\Sigma_2\|_s^2$ can be estimated by:

$$\sum_p 2^{2ps}\|a_p\|_\infty^2\|u\|_0^2$$

which is bounded since $s<s_0$. Finally it is very easy to see that Σ_3 is also in H^s. □

REFERENCES

[Ag] S. AGMON, *Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations*, Princeton University Press, Princeton, 1982.

[Bo] J. M. BONY, Calcul symbolique et propagation des singularités pour les équations non linéaires, *Ann. Scient. E.N.S.*, Vol. **14**, 1981, p. 209-246.

[C] J. M. COMBES, Time Dependent Approach to Nonrelativistic Multichannel Scattering, *Nuevo Cimento*, Vol. **64 A**, 1969, p. 111-144.

[C.F.K.S.] H. L. CYCON, R. G. FROESE, W. KIRSCH and B. SIMON, *Schrödinger operators*, Texts and Monographs in Physics, Springer Verlag, 1987.

[De1] J. DEREZINSKI, *The Mourre Estimate for Dispersive* N-*Body Schrödinger Operators*, Preprint Virginia Polytechnic Institute, 1988.

[De2] J. DEREZINSKI, *Criteria for the Kato Smoothness with Respect to a Dispersive* N-*Body Schrödinger Operator*, Preprint, Virginia Polytechnic Institute, 1988.

[E] V. ENSS, Two Cluster Scattering of N Charged Particles, *Comm. Math. Phys.*, Vol. **65**, 1979, pp. 151-165.

[F-H] R. G. FROESE and I. HERBST, A New Proof of the Mourre Estimate, *Duke Math. J.*, Vol. **49**, 1982, pp. 1075-1085.

[Hö] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, Vol. **III**, Springer Verlag, 1985.

[R-S] M. REED and B. SIMON, *Methods of Modern Mathematical Physics*, Vol. **IV**, Analysis of operators, Academic Press, 1980.

[P-S-S] P. PERRY, I. M. SIGAL and B. SIMON, Spectral Analysis of N-Body Schrödinger Operators, *Ann. Math.*, Vol. **114**, 1981, pp. 519-567.

[S] B. SIMON, N-Body Scattering in the Two Cluster Region, *Comm. Math. Phys.*, Vol. **58**, 1978, pp. 205-210.

[M] E. MOURRE, Abscence of Singular Continuous Spectrum for Certain Self-Adjoint Operators, *Comm. Math. Phys.*, Vol. **78**, 1981, pp. 391-408.

[S-S] I. M. SIGAL and A. SOFFER, The N Particle Scattering Problem: Asymptotic Completeness for Short Range Systems, *Ann. Math.*, Vol. **126**, 1987, pp. 35-108.

(*Manuscript received March* 2, 1990.)