ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 54, nº 1 (1991), p. 43-57
<http://www.numdam.org/item?id=AIHPA_1991__54_1_43_0>

Ann. Inst. Henri Poincaré,

Vol. 54, n° 1, 1991, p. 43-57. *Physique théorique*

Invariant subspaces for the Schrödinger evolution group

by

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Dedicated to Professor Shigetake Matsuura on his sixtieth birthday

ABSTRACT. — The formation of dispersion with finite velocity of quantum states is described in detail. To be more specific, we prove the invariance of the domains $D(|x|^m) \cap D(|p|^m)$, $m \in \mathbb{N}$, and of their topologies under the Schrödinger evolution group $\{e^{-itH}\}$, where we denote by x and p the position and momentum operator, respectively. Moreover, we give a characterization of invariant subspaces under unitary groups in a rather general setting.

RÉSUMÉ. — Nous analysons la formation de dispersion de vitesse finie des états quantiques. Plus précisément nous prouvons l'invariance des domaines $D(|x|^m) \cap D(|p|^m)$, $m \in \mathbb{N}$, et de leur topologie par le groupe d'évolution de Schrödinger $\{e^{-itH}\}$ où x et p sont les opérateurs de

(*) Partially supported by Grant-in Aid for Scientific Research, Ministry of Education.

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position et d'impulsion respectivement. Nous donnons de plus une caractérisation des sous espaces invariants par des groupes unitaires dans un cadre plus général.

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1. INTRODUCTION

In this paper we prove that the Schrödinger evolution group preserves the regularity and decay properties in the scale of weighted Sobolev spaces.

Let $H = H_0 + V$ be a Schrödinger operator in the Hilbert space $L^2 = L^2(\mathbb{R}^n)$, where $H_0 = -(1/2)\Delta$ is the free Hamiltonian and V is a H_0-bounded symmetric operator of multiplication with relative bound less than one, so that by the Kato-Rellich theorem H is self-adjoint in L^2 with domain $D(H) = D(H_0)$. We consider the Schrödinger evolution group $\{e^{-itH}\}$ in the scale of weighted Sobolev spaces $\mathscr{H}_m = H^{m,0} \cap H^{0,m}$, $m \in \mathbb{N} \cup \{0\}$, where

$$H^{r,s} = \{\psi \in \mathscr{S}';\ \|\psi\|_{r,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{r/2}\psi\| < \infty\}, \qquad r, s \in \mathbb{R}$$

and $\|\,.\,\|$ denotes the L^2-norm. $\mathscr{H}_m$ is a Hilbert space with the norm $|||\,.\,|||_m$, given by $|||\psi|||_m^2 = \|\psi\|_{m,0}^2 + \|\psi\|_{0,m}^2$. The free Schrödinger evolution group $\{e^{-itH_0}\}$ leaves $\mathscr{H}_m$ invariant since the Fourier transform is an isometry on $\mathscr{H}_m$ and the multiplication operator by $\exp(-i(t/2)|\xi|^2)$ preserves $\mathscr{H}_m$. We now state our main results:

THEOREM 1. – *Let $m \in \mathbb{N} \cup \{0\}$. Suppose that*

(H_m) $D(|H|^{m/2}) = H^{m,0}$

holds when $m \geqq 3$. Then:

(1) *$\mathscr{H}_m$ and $H^{m,0}$ are invariant under e^{-itH} for any $t \in \mathbb{R}$.*

(2) *The map $(t, \varphi) \mapsto e^{-itH}\varphi$ is continuous from $\mathbb{R} \times \mathscr{H}_m$ to $\mathscr{H}_m$ and from $\mathbb{R} \times H^{m,0}$ to $H^{m,0}$.*

(3) *e^{-itH} has the estimates*

$$\|e^{-itH}\varphi\|_{m,0} \leqq C(m)\|\varphi\|_{m,0}, \quad (t, \varphi) \in \mathbb{R} \times H^{m,0}, \tag{1.1}$$

$$\left.\begin{array}{c}\|e^{-itH}\varphi\|_{0,m} \leqq C(m)(\|\varphi\|_{0,m} + |t|^m \|\varphi\|_{m,0}),\\ (t, \varphi) \in \mathbb{R} \times \mathscr{H}_m,\end{array}\right\} \tag{1.2}$$

where $C(m)$ is independent of t and φ. In particular,

$$|||e^{-itH}\varphi|||_m \leqq \tilde{C}(m)(1+|t|^m)|||\varphi|||_m, \qquad (t, \varphi) \in \mathbb{R} \times \mathscr{H}_m. \tag{1.3}$$

(4) *For any* $\alpha\in(\mathbb{N}\cup\{0\})^n$ *with* $|\alpha|\leqq m$ *and any* $\varphi\in\mathcal{H}_m$, *the map* $\mathbb{R}\ni t\mapsto e^{-it\mathrm{H}}x^\alpha e^{-it\mathrm{H}}\varphi\in\mathrm{L}^2$ *is continuously differentiable and*

$$\frac{d}{dt}(e^{it\mathrm{H}}x^\alpha e^{-it\mathrm{H}}\varphi)=-ie^{it\mathrm{H}}((1/2)(\Delta x^\alpha)+(\nabla x^\alpha).\nabla)e^{-it\mathrm{H}}\varphi.$$

THEOREM 2. – *Suppose that* (H_m) *holds for all* $m\geqq 3$. *Then:*

(1) *For any* $(t,\varphi)\in\mathbb{R}\times\mathcal{S}$, $e^{-it\mathrm{H}}\varphi\in\mathcal{S}$ *and the map* $\mathbb{R}\ni t\mapsto e^{-it\mathrm{H}}\varphi\in\mathcal{S}$ *is* C^∞.

(2) *The map* $\mathbb{R}\times\mathcal{S}\ni(t,\varphi)\mapsto e^{-it\mathrm{H}}\varphi\in\mathcal{S}$ *is continuous.*

The estimate (1.2) is optimal with respect to the growth rate in time. In fact we have:

THEOREM 3. – *Let* $m\in\mathbb{N}$. *If* $\varphi\in\mathcal{H}_m$, *then*

$$\lim_{|t|\to\infty}\sum_{|\alpha|=m}\|e^{it\mathrm{H}_0}(x/t)^\alpha e^{-it\mathrm{H}_0}\varphi-(-i\partial)^\alpha\varphi\|=0.$$

In particular,

$$\lim_{|t|\to\infty}|t|^{-m}\|e^{-it\mathrm{H}_0}\varphi\|_{0,m}=\|(-\Delta)^{m/2}\varphi\|.$$

Theorems 1-2 describe the formation of dispersion with finite velocity of quantum dynamics. In other words, quantum states are well localized. Of course, as was noted by Hunziker [9], the description of localization in terms of supports of wavefunctions is in vain. The results of Hayashi-Ozawa [7], Masuda [11] and Ozawa [12] will explain this kind of uselessness.

There is a large literature on the problem of invariant domains for $e^{-it\mathrm{H}}$ ([3], [4], [6], [9], [13], [14], [18]). Hunziker [9] showed that for any $m\in\mathbb{N}\cup\{0\}$, $\mathrm{D}_m=\bigcap_{j+|\alpha|\leqq m}\mathrm{D}(x^\alpha\mathrm{H}^j)$ is invariant under $e^{-it\mathrm{H}}$ *without assuming* (H_m) for $m\geqq 3$. Moreover, in [9] it is shown that part (2) of Theorem 2 holds if all derivatives of V are bounded and continuous. The space D_m, however, does not always fit into a detailed description of the regularity preservation property of $e^{-it\mathrm{H}}$. For example, if

$\mathrm{V}(x)=-(n-1)/2|x|$, $\varphi(x)=e^{-|x|}$ for $n\geqq 3$, then $e^{-it\mathrm{H}}\varphi=e^{it/2}\varphi\in\bigcap_{m\geqq 0}\mathrm{D}_m$

while $e^{-it\mathrm{H}}\varphi\notin\mathrm{H}^{n/2+1,0}$, $t\in\mathbb{R}$. Radin and Simon [14] obtained part (1) and (1.3) of Theorem 1 in the case $m\leqq 2$, where the condition (H_1) is guaranteed by the Heinz-Kato theorem [16]. In addition, they showed some examples which illustrate how local singularities in V cause the breakdown of invariance in the case $m>2$. This leads to the observation that the assumption (H_m) controls local singularities in V. The problem then arises what conditions on V ensure (H_m). A sufficient condition is given by:

THEOREM 4. — *Let* $m \in \mathbb{N}$. *When* $m \geqq 3$, *suppose that* $\partial^\alpha V$ *is bounded from* $H^{1+|\alpha|,0}$ *to* L^2 *for all* $1 \leqq |\alpha| \leqq m-2$. *Then* $D(|H|^{m/2}) = H^{m,0}$.

Theorem 4 improves the previous results of Arai [2], Ozawa [13] and Wilcox [18]. As a simple application of Theorems 1, 4 and an inequality of Herbst ([8], Theorem 2.5), we have:

THEOREM 5. — *Let* $V(x) = \sum_{j=1}^{k} \lambda_j |x|^{-\mu_j}$, $\lambda_j \in \mathbb{R}$, $\mu_j > 0$, $k \in \mathbb{N}$, *and let* $\mu = \max_{1 \leqq j \leqq k} \mu_j$. *The assumptions in Theorem* 1 *are satisfied in the following cases:*

(1) $0 < \mu < n/2\,(n \leqq 4)$, $0 < \mu < 2\,(n \geqq 5)$, *when* $m \leqq 2$.

(2) $0 < \mu < n/2 - 1\,(n = 3,\ 4)$, $0 < \mu \leqq 1\,(n \geqq 5)$, *when* $m = 3$.

(3) $0 < \mu \leqq 1\,(n \geqq 2m-1)$, *when* $m \geqq 4$.

The contents of the paper are as follows. In Section 2 we prove Theorems 1-3. The proof of Theorem 1 uses a differential inequality for $\| e^{-itH} \varphi \|_{0,m}^2$ ([3], [4], [14]) and an integral representation for $x^\alpha e^{-itH} \varphi$ [9]. For this purpose we approximate the weight functions by rapidly decreasing functions [9] and the initial data by the resolvent of H. The regularization by the resolvent has the advantage that it commutes with e^{-itH}, which enables us to obtain *a priori* estimates without regularizing the potential V. In Section 3 we prove Theorem 4 by expanding $(H_0 + V)^m$ out. Section 4 is devoted to a characterization of invariant subspaces for e^{-itH} in terms of the resolvent estimates for H. This will be done in a rather general setting by making use of the Hille-Yosida theorem. Related results have been obtained by Schonbeck [15].

Throughout the paper we use the following notations. For $s \in \mathbb{R}$ $[s]$ denotes the largest integer $\leqq s$; $[.,.]$ denotes the commutator; $\partial_t = \partial/\partial t$; ∂_j denotes the distributional derivative with respect to the j-th coordinate; $\nabla = (\partial_1, \ldots, \partial_n)$; $\Delta = \sum_{j=1}^{n} \partial_j^2$; for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we set

$$|\alpha| = \sum_{j=1}^{n} \alpha_j, \qquad \alpha! = \sum_{j=1}^{n} \alpha_j!,$$

$$\binom{\alpha}{\beta} = \alpha!/\beta!(\beta-\alpha)! \qquad (\beta \leqq \alpha),$$

$$\partial^\alpha = \prod_{j=1}^{n} \partial_j^{\alpha_j}, \qquad x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j},$$

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ \partial^0 = x^0 = 1;$$

$\mathscr{F}$ denotes the Fourier transform according to the normalization $(\mathscr{F}\psi)(\xi)=(2\pi)^{-n/2}\int \exp(-ix.\xi)\psi(x)\,dx$; $\mathscr{S}$ denotes the Fréchet space of rapidly decreasing functions from $\mathbb{R}^n$ to $\mathbb{C}$; $\mathscr{S}'$ denotes the dual of $\mathscr{S}$; L^2 denotes the Lebesgue space $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)\otimes\mathbb{C}^n$, with the norm denoted by $\|.\|$; $(.,.)$ denotes the L^2-scalar product and various anti-dualities; $C(I; B)$ denotes the Fréchet space of continuous functions from an interval $I\subset\mathbb{R}$ to a Banach space B; $C^k(I; B)$, $k\in\mathbb{N}$, denotes the space of k-times continuously differentiable functions from I to B; $\mathscr{L}(B)$ denotes the Banach space of bounded operators in B.

Different constants might be denoted by the same letter C, and if necessary, by $C(*, \ldots, *)$ in order to indicate the dependence on the quantities appearing in parentheses. The summation over an empty set is understood to be zero. A function, its value at a point, and the multiplication operator by that function might be denoted by the same symbol when this causes no confusion.

2. PROOF OF THEOREMS 1-3

We start with some fundamental lemmas. For $\varepsilon\neq 0$ and $s\in\mathbb{R}$, ζ_ε and ω^s denote the functions on $\mathbb{R}^n$ given respectively by $\zeta_\varepsilon(x)=\exp(-|\varepsilon x|^2)$ and $\omega^s(x)=(1+|x|^2)^{s/2}$. For $\lambda\in\mathbb{R}\setminus\{0\}$, we set $R_\lambda=(H+i\lambda)^{-1}$.

LEMMA 2.1 (Hunziker [9; Lemma 2]). — *Let* $m\in\mathbb{N}$. *If* $u\in C(\mathbb{R}; L^2)$, *then* $\varepsilon^m\omega^m\zeta_\varepsilon u\to 0$, $\zeta_\varepsilon u\to u$ *in* $C(\mathbb{R}; L^2)$ *as* $\varepsilon\to 0$.

LEMMA 2.2. — *Let* $m\in\mathbb{N}$. *If* $\psi\in\mathscr{H}_m$, *then* $\psi\in\bigcap_{j=0}^{m} H^{j,m-j}$ *and for* $0\leqq j\leqq m$,

$$\sum_{|\alpha|=j}\|\partial^\alpha\psi\|_{0,m-j}\leqq C(j,m)\sum_{|\beta|=m}\|\partial^\beta\psi\|^{j/m}\|\psi\|_{0,m}^{1-j/m}. \tag{2.1}$$

Proof. — It suffices to prove (2.1) for $\psi\in C_0^\infty$, since C_0^∞ is dense in $\mathscr{H}_m$ and the norm $|||\psi|||_{k,s}=\|\psi\|_{0,s}+\sum_{|\alpha|=k}\|\partial^\alpha\psi\|_{0,s}$ is an equivalent norm on $H^{k,s}$ if $k\in\mathbb{N}$, $s\in\mathbb{R}$ (*see* Triebel [17], Theorems 1, 3 and 4). The L.H.S. of (2.1) is estimated by $C\sum_{|\alpha|=j}\|\partial^\alpha\psi\|+C\sum_{|\alpha|=j}\||x|^{m-j}\partial^\alpha\psi\|$. By Hölder's inequality,

$$\sum_{|\alpha|=j} \|\partial^\alpha \psi\| = \sum_{|\alpha|=j} \|\xi^\alpha \mathscr{F} \psi\|$$
$$\leq C \| |\xi|^m \mathscr{F} \psi \|^{j/m} \| \mathscr{F} \psi \|^{1-j/m}$$
$$= C \left(\sum_{|\beta|=m} \frac{m!}{\beta!} \| \xi^\beta \mathscr{F} \psi \|^2 \right)^{j/2m} \|\psi\|^{1-j/m}$$
$$\leq C \sum_{|\beta|=m} \|\partial^\beta \psi\|^{j/m} \|\psi\|^{1-j/m}.$$

By an interpolation inequality of Lin [10],

$$\sum_{|\alpha|=j} \| |x|^{m-j} \partial^\alpha \psi \| \leq C \sum_{|\beta|=m} \|\partial^\beta \psi\|^{j/m} \| |x|^m \psi \|^{1-j/m}.$$

Collecting these estimates, we obtain (2.1).

Q.E.D.

LEMMA 2.3. – *Let* $m \in \mathbb{N} \cup \{0\}$ *and* $|\alpha| \leq 1$. *Then:*

(1) *For any* $|\lambda| \geq 1$, $\partial^\alpha R_\lambda \in \mathscr{L}(H^{0,m})$ *and*

$$\sum_{|\alpha| \leq 1} \sup_{|\lambda| \geq 1} |\lambda|^{1-|\alpha|/2} \|\partial^\alpha R_\lambda\|_{\mathscr{L}(H^{0,m})} \leq C(m).$$

Moreover, $i\lambda R_\lambda \to 1$ *strongly in* $\mathscr{L}(H^{0,m})$ *as* $|\lambda| \to \infty$.

(2) *Suppose in addition that* (H_m) *holds when* $m \geq 3$. *Then for any* $|\lambda| \geq 1$, $R_\lambda \in \mathscr{L}(H^{m,0})$ *and* $\sup_{|\lambda| \geq 1} |\lambda| \|R_\lambda\|_{\mathscr{L}(H^{m,0})} \leq C(m)$. *Moreover,* $i\lambda R_\lambda \to 1$ *strongly in* $\mathscr{L}(H^{m,0})$ *as* $|\lambda| \to \infty$.

Remark. – Related results have been obtained by Amrein, Cibils and Sinha [1], Lemmas 1-3.

Proof of Lemma 2.3. – (1) The proof uses induction on m. For $m=0$, it suffices to consider the case $|\alpha|=1$. Let $\psi \in L^2$. Since $D(|H|^{1/2}) = H^{1,0}$, we obtain by the closed graph theorem and the moment inequality [16]

$$\|\partial^\alpha R_\lambda \psi\| \leq C \| |H|^{1/2} R_\lambda \psi \| + C \|R_\lambda \psi\|$$
$$\leq C \|HR_\lambda \psi\|^{1/2} \|R_\lambda \psi\|^{1/2}$$
$$+ C|\lambda|^{-1} \|\psi\| \leq C(|\lambda|^{-1/2} + |\lambda|^{-1}) \|\psi\|,$$

as required. Let $m \geq 1$ and assume that part (1) holds for all $j \leq m-1$. We have for $\psi \in H^{0,m}$.

$$\zeta_\varepsilon \omega^m R_\lambda \psi = R_\lambda \zeta_\varepsilon \omega^m \psi + R_\lambda [H, \zeta_\varepsilon \omega^m] R_\lambda \psi$$
$$= R_\lambda \zeta_\varepsilon \omega^m \psi - R_\lambda \zeta_\varepsilon f_{m,\varepsilon} R_\lambda \psi - R_\lambda \zeta_\varepsilon g_{m,\varepsilon} x \cdot \nabla R_\lambda \psi, \quad (2.2)$$

where

$$f_{m,\varepsilon}(x) = (m/2)(n\omega^2(x) + (m-2)|x|^2)\omega^{m-4}(x)$$
$$-((n-2|\varepsilon x|^2)\varepsilon^2 \omega^2(x) + 2m|\varepsilon x|^2)\omega^{m-2}(x),$$
$$g_{m,\varepsilon}(x) = (m - 2\varepsilon^2 \omega^2(x))\omega^{m-2}(x).$$

By the induction hypothesis and Lemma 2.1, the R.H.S. of (2.2) converges to $R_\lambda \omega^2 \psi - R_\lambda f_m R_\lambda \psi - R_\lambda g_m x.\nabla R_\lambda \psi$ in L^2 as $\varepsilon \to 0$, where $f_m(x) = (m/2)(n\omega^2(x) + (m-2)|x|^2)\omega^{m-4}(x)$, $g_m(x) = m\omega^{m-2}(x)$. It follows from the closedness of the multiplication operator ω^m that $R_\lambda \psi \in H^{0,m}$ and that

$$\omega^m R_\lambda \psi = R_\lambda \omega^m \psi - R_\lambda f_m R_\lambda \psi - R_\lambda g_m x.\nabla R_\lambda \psi. \tag{2.3}$$

Therefore, again by the induction hypothesis we obtain

$$\| R_\lambda \psi \|_{0,m} \leqq C(|\lambda|^{-1} + |\lambda|^{-2} + |\lambda|^{-3/2}) \| \psi \|_{0,m},$$
$$\| i\lambda R_\lambda \psi - \psi \|_{0,m} \leqq \| (i\lambda R_\lambda - 1)\omega^m \psi \| + C(|\lambda|^{-1} + |\lambda|^{-1/2}) \| \psi \|_{0,m} \to 0$$

as $|\lambda| \to \infty$. Noting that every term on the R.H.S. of (2.3) is in $H^{1,0}$, we have $\omega^m R_\lambda \psi \in H^{1,0}$ so that for $|\alpha| = 1$, $\partial^\alpha(\omega^m R_\lambda \psi) - (\partial^\alpha \omega^m) R_\lambda \psi \in L^2$, *i.e.*, $\partial^\alpha R_\lambda \psi \in H^{0,m}$. Consequently,

$$\sum_{|\alpha|=1} \| \partial^\alpha R_\lambda \psi \|_{0,m} \leqq \sum_{|\alpha|=1} (\| \partial^\alpha R_\lambda \omega^m \psi \| + \| \partial^\alpha R_\lambda f_m R_\lambda \psi \|$$
$$+ \| \partial^\alpha R_\lambda g_m x.\nabla R_\lambda \psi \| + \| (\partial^\alpha \omega^m) R_\lambda \psi \|)$$
$$\leqq C(|\lambda|^{-1/2} + |\lambda|^{-3/2} + |\lambda|^{-1}) \| \psi \|_{0,m}.$$

This proves part (1).

(2) Since $D(|H|^{m/2}) = H^{m,0}$, we obtain for $\psi \in H^{m,0}$

$$\| R_\lambda \psi \|_{m,0} \leqq C \| (|H|^{m/2} + 1) R_\lambda \psi \|$$
$$= C \| R_\lambda (|H|^{m/2} + 1) \psi \| \leqq C |\lambda|^{-1} \| \psi \|_{m,0},$$
$$\| (i\lambda R_\lambda - 1)\psi \|_{m,0} \leqq C \| (i\lambda R_\lambda - 1)(|H|^{m/2} + 1)\psi \| \to 0 \text{ as } |\lambda| \to \infty.$$

Q.E.D.

Proof of Theorem 1. — From (H_m), the commutativity on $D(|H|^{m/2})$ of e^{-itH} and $|H|^{m/2}$, and the unitary in L^2 of e^{-itH}, we see that e^{-itH} leaves $H^{m,0}$ invariant and has the estimate (1.1) and that the map $\mathbb{R} \times H^{m,0} \ni (t, \varphi) \mapsto e^{-itH}\varphi \in H^{m,0}$ is continuous. From now on we use these facts without particular comments. Parts (1)-(3) will follow if we can show that

$(1)_m$ $e^{-itH}(\mathscr{H}_m) \subset \mathscr{H}_m$, $t \in \mathbb{R}$;

$(2)_m$ the map $\mathbb{R} \times \mathscr{H}_m \ni (t, \varphi) \mapsto e^{-itH}\varphi \in H^{0,m}$ is continuous;

$(3)_m$ $\| e^{-itH}\varphi \|_{0,m} \leqq C(m)(\| \varphi \|_{0,m} + |t|^m \| \varphi \|_{m,0})$, $(t, \varphi) \in \mathbb{R} \times \mathscr{H}_m$.

Since (H_m) implies (H_j) for all $j \leqq m$ by the Heinz-Kato theorem [16], we use induction on m in order to prove that (H_m) implies $(1)_m - (3)_m$. For $m = 0$ we have nothing to prove. Let $m \geqq 1$ and assume that our claim holds for $m-1$. We proceed to the case m. For $\psi \in (H^{\max(m,2),0} \cap H^{0,m-1}) \setminus \{0\}$, we set $v(t) = e^{-itH}\psi$, $t \in \mathbb{R}$. Then $\zeta_\varepsilon \omega^m v \in C^1(\mathbb{R}; L^2) \cap C(\mathbb{R}; H^{2,0})$, $i\frac{d}{dt}\zeta_\varepsilon \omega^m v = \zeta_\varepsilon \omega^m H v$ and moreover,

$$\frac{d}{dt} \| (\zeta_\varepsilon \omega^m + i) v(t) \|^2$$

$$
\begin{aligned}
&= 2\,\mathcal{R}e\left(\frac{d}{dt}\zeta_\varepsilon\,\omega^m v(t),\ \zeta_\varepsilon\,\omega^m v(t)\right)\\
&= 2\,\mathcal{I}m\,(\zeta_\varepsilon\,\omega^m \mathrm{H}_0\, v(t),\ \zeta_\varepsilon\,\omega^m v(t))\\
&= 2\,\mathcal{I}m\,([\zeta_\varepsilon\,\omega^m,\ \mathrm{H}_0]\, v(t),\ \zeta_\varepsilon\,\omega^m v(t))\\
&= 2\,m\,\mathcal{I}m\,(\omega^{m-2}\, x\,.\,\nabla(\zeta_\varepsilon\, v(t)),\ \zeta_\varepsilon\,\omega^m v(t))\\
&\qquad -4\,\varepsilon^2\,\mathcal{I}m\,(\zeta_\varepsilon\,\omega^m x\,.\,\nabla v(t),\ \zeta_\varepsilon\,\omega^m v(t)).
\end{aligned}
$$

By Lemma 2.2, the R.H.S. of the last equality is estimated by

$$
\begin{aligned}
&\mathrm{C}\,\|\,\omega^m\zeta_\varepsilon\, v(t)\,\|^{2-1/m}\sum_{|\alpha|=m}\|\,\partial^\alpha(\zeta_\varepsilon\, v(t))\,\|^{1/m}\\
&\qquad +\mathrm{C}\,\|\,\varepsilon^2\zeta_\varepsilon\,\omega^m x\,.\,\nabla v(t)\,\|\,\|\,\zeta_\varepsilon\,\omega^m v(t)\,\|\\
&\leqq \mathrm{C}\,\|\,(\zeta_\varepsilon\,\omega^m + i)\, v(t)\,\|^{2-1/m}\sum_{|\alpha|=m}\|\,\partial^\alpha(\zeta_\varepsilon\, v(t))\,\|^{1/m}\\
&\qquad +\mathrm{C}\,\|\,\varepsilon^2\zeta_\varepsilon\,\omega^m x\,.\,\nabla v(t)\,\|\,\|\,(\zeta_\varepsilon\,\omega^m + i)\, v(t)\,\|.
\end{aligned}
$$

Since $\|\,(\zeta_\varepsilon\,\omega^m + i)\, v(t)\,\| \geqq \|\, v(t)\,\| = \|\,\psi\,\| > 0$, we obtain

$$
\begin{aligned}
&\left|\frac{d}{dt}\|\,(\zeta_\varepsilon\,\omega^m + i)\, v(t)\,\|^{1/m}\right|\\
&\qquad \leqq \mathrm{C}\sum_{|\alpha|=m}\|\,\partial^\alpha(\zeta_\varepsilon\, v(t))\,\|^{1/m}\\
&\qquad\qquad +\mathrm{C}\,\|\,\varepsilon^2\zeta_\varepsilon\,\omega^m x\,.\,\nabla v(t)\,\|\,\|\,\psi\,\|^{1/m-1}. \qquad (2.4)
\end{aligned}
$$

Now, for $\varphi \in \mathcal{H}_m \setminus \{0\}$, we set $u(t) = e^{-it\mathrm{H}}\varphi$, $t \in \mathbb{R}$. By the induction hypothesis, $u \in \mathrm{C}(\mathbb{R};\ \mathrm{H}^{0,\,m-1} \cap \mathrm{H}^{m,\,0})$. It follows from Lemma 2.3 that for $|\lambda| \geqq 1$, $i\lambda\,\mathrm{R}_\lambda\,\varphi \in (\mathrm{H}^{\max(m,\,2),\,0} \cap \mathrm{H}^{0,\,m}) \setminus \{0\}$ and furthermore,

$$
\omega^{m-2}\, x\,.\,\nabla\, i\lambda\,\mathrm{R}_\lambda\, u \in \mathrm{C}(\mathbb{R};\ \mathrm{L}^2), \qquad (2.5)
$$

$$
\sup_{|\lambda|\geqq 1}\|\, i\lambda\,\mathrm{R}_\lambda\,\varphi\,\|_{m,\,0} \leqq \mathrm{C}\,\|\,\varphi\,\|_{m,\,0}, \qquad (2.6)
$$

$$
\sup_{|\lambda|\geqq 1}\|\, i\lambda\,\mathrm{R}_\lambda\,\varphi\,\|_{0,\,m} \leqq \mathrm{C}\,\|\,\varphi\,\|_{0,\,m}, \qquad (2.7)
$$

$$
i\lambda\,\mathrm{R}_\lambda\,\varphi \to \varphi \ \text{ in } \mathcal{H}_m \ \text{ as } |\lambda| \to \infty. \qquad (2.8)
$$

Since R_λ and $e^{-it\mathrm{H}}$ commute, $i\lambda\,\mathrm{R}_\lambda\, u \in \mathrm{C}(\mathbb{R};\ \mathrm{H}^{m,\,0})$ and

$$
\sup_{t\in\mathbb{R}}\|\, i\lambda\,\mathrm{R}_\lambda\, u(t) - u(t)\,\| = \|\, i\lambda\,\mathrm{R}_\lambda\,\varphi - \varphi\,\| \to 0 \text{ as } |\lambda| \to \infty.
$$

Integrating (2.4) with ψ replaced by $i\lambda\,\mathrm{R}_\lambda\,\varphi$, we obtain

$$
\begin{aligned}
&\|\,(\zeta_\varepsilon\,\omega^m + i)\, i\lambda\,\mathrm{R}_\lambda\, u(t)\,\|^{1/m} \leqq \|\,(\zeta_\varepsilon\,\omega^m + i)\, i\lambda\,\mathrm{R}_\lambda\,\varphi\,\|^{1/m}\\
&\qquad +\mathrm{C}\left|\int_0^t \sum_{|\alpha|=m}\|\,\partial^\alpha(\zeta_\varepsilon\, i\lambda\,\mathrm{R}_\lambda\, u(s))\,\|^{1/m}\, ds\right| + \mathrm{C}\,\|\, i\lambda\,\mathrm{R}_\lambda\,\varphi\,\|^{1/m-1}\\
&\qquad\qquad \times\left|\int_0^t \sum_{|\alpha|=m}\|\,\varepsilon^2\zeta_\varepsilon\,\omega^m x\,.\,\nabla\, i\lambda\,\mathrm{R}_\lambda\, u(s)\,\|\, ds\right|, \qquad t \in \mathbb{R}. \qquad (2.9)
\end{aligned}
$$

By Lemma 2.1 and (2.5), the R.H.S. of (2.9) converges to

$$\|(\omega^m+i)\,i\lambda\,R_\lambda\,\varphi\|^{1/m}+C\left|\int_0^t \sum_{|\alpha|=m}\|\partial^\alpha(i\lambda\,R_\lambda\,u(s))\|^{1/m}\,ds\right| \tag{2.10}$$

as $\varepsilon\to 0$. By Fatou's lemma, $(\omega^m+i)\,i\lambda\,R_\lambda\,u(t)\in L^2$ and $\|(\omega^m+i)\,i\lambda\,R_\lambda\,u(t)\|^{1/m}$ is estimated by (2.10). Now we use (2.6), (2.7) and the commutativity of R_λ and e^{-itH} to obtain

$$\begin{aligned}\|(\omega^m+i)\,i\lambda\,R_\lambda\,u(t)\| &\leqq C\,\|(\omega^m+i)\,i\lambda\,R_\lambda\,\varphi\| \\ &\quad + C\left|\int_0^t \sum_{|\alpha|=m}\|\partial^\alpha(i\lambda\,R_\lambda\,u(s))\|^{1/m}\,ds\right|^m \\ &\leqq C\,\|\varphi\|_{0,m}+C\,|t|^m\,\|\varphi\|_{m,0}, \qquad t\in\mathbb{R},\end{aligned} \tag{2.11}$$

where C is independent of $|\lambda|\geqq 1$. In the same way as above,

$$\begin{aligned}\|(\omega^m+i)\,(i\lambda\,R_\lambda\,u(t)-i\mu\,R_\mu\,u(t))\| &\leqq C\,\|i\lambda\,R_\lambda\,\varphi-i\mu\,R_\mu\,\varphi\|_{0,m} \\ &\quad + C\,|t|^m\,\|i\lambda\,R_\lambda\,\varphi-i\mu\,R_\mu\,\varphi\|_{m,0}, \qquad t\in\mathbb{R},\end{aligned} \tag{2.12}$$

where C is independent of $|\lambda|$, $|\mu|\geqq 1$. By (2.8), (2.11), (2.12) and the closedness of the multiplication operator ω^m+i, we obtain $(1)_m$ and $(3)_m$. Since part (4) with $(3)_m$ gives $(2)_m$, we prove part (4), following Hunziker [9]. Let $|\alpha|\leqq m$. By integrating the Heisenberg type equation $\dfrac{d}{dt}(e^{itH}\zeta_\varepsilon x^\alpha u(t))=ie^{itH}[H,\,\zeta_\varepsilon x^\alpha]\,u(t)$,

$$\begin{aligned}\zeta_\varepsilon x^\alpha u(t) &= e^{-itH}\zeta_\varepsilon x^\alpha\varphi - i\int_0^t e^{-i(t-s)H}((1/2)(\Delta x^\alpha) \\ &\quad -(n+2|\alpha|)\,\varepsilon^2 x^\alpha+2\,\varepsilon^4 x^\alpha|x|^2)\,\zeta_\varepsilon\,u(s)\,ds \\ &\quad - i\int_0^t e^{-i(t-s)H}\zeta_\varepsilon((\nabla x^\alpha)-(2\,\varepsilon^2 x^\alpha)\,x).\nabla u(s)\,ds.\end{aligned} \tag{2.13}$$

Since we already know $(3)_m$ and $u\in C(\mathbb{R};\,H^{m,0}\cap H^{0,m-1})$, we see from Lemma 2.2 that $(\nabla x^\alpha).\nabla u\in C(\mathbb{R};\,L^2)$ and that (2.13) converges to

$$x^\alpha u(t)=e^{-itH}x^\alpha\varphi-i\int_0^t e^{-i(t-s)H}((1/2)(\Delta x^\alpha)+(\nabla x^\alpha).\nabla)\,u(s)\,ds$$

in L^2 as $\varepsilon\to 0$. Part (4) then follows by the standard argument.

Q.E.D.

Proof of Theorem 2. — It follows from Sobolev's lemma and Lemma 2.2 that $\{|||.|||_m;\ m\in\mathbb{N}\cup\{0\}\}$ constitutes a fundamental system of seminorms on $\mathscr{S}$, and hence part (2) follows from Theorem 1. It remains to prove that $\mathbb{R}\ni t\mapsto e^{-itH}\varphi\in\mathscr{S}$ is C^∞ for any $\varphi\in\mathscr{S}$. In view of part (2), this is equivalent to showing that $H^k\varphi\in\mathscr{S}$ for any $k\in\mathbb{N}$. By assumption, for any $m\in\mathbb{N}$,

$$\|H^k\varphi\|_{2m,0}\leqq C\|H^{m+k}\varphi\|+C\|H^k\varphi\|\leqq C\|\varphi\|_{2(k+m),0}$$

so that $H^k\varphi\in\bigcap_{l\geqq 0}H^{l,0}$. Therefore we are reduced to proving that $H^k\varphi\in\bigcap_{m\geqq 0}H^{0,m}$ for any $k\in\mathbb{N}$. To this end we first prove by induction on k that for any $m\geqq 1$ and $\psi\in\mathscr{H}_{2k+m-2}$

$$\sum_{\substack{j+l\leqq k\\ j\geqq 0,\, l\geqq 1}}\|H^j[H^l,\omega^m]\psi\|\leqq C|||\psi|||_{2k+m-2}. \tag{2.14}$$

Let $k=1$. We have $[H,\omega^m]\psi=-(1/2)(\Delta\omega^m)\psi-(\nabla\omega^m).\nabla\psi$. By Lemma 2.2, $\|[H,\omega^m]\psi\|\leqq C|||H\psi|||_m$, as required. Let $k\geqq 1$ and assume that (2.14) holds. We proceed to the case $k+1$. We use the formula

$$[H^{\tilde{l}},\omega^m]=H^{\tilde{l}-1}[H,\omega^m]+\sum_{\tilde{j}=0}^{\tilde{l}-2}H^{\tilde{j}}[H^{\tilde{l}-1-\tilde{j}},\omega^m]H. \tag{2.15}$$

Now let $j+l\leqq k$. By (2.15) and the induction hypothesis,

$$\begin{aligned}\|H^{j+1}[H^l,\omega^m]\psi\|&\leqq\|H^{l+j}[H,\omega^m]\psi\|\\&\quad+\sum_{\tilde{j}=0}^{l-2}\|H^{j+1+\tilde{j}}[H^{l-1-\tilde{j}},\omega^m]H\psi\|\\&\leqq C\|[H,\omega^m]\psi\|_{2(l+j),0}+C|||H\psi|||_{2k+m-2}\leqq C|||\psi|||_{2k+m},\end{aligned}$$

where we have used (H_{l+j}), Lemma 2.2, and Hunziker's lemma [9], Lemma 1. Similarly,

$$\|H^j[H^{l+1},\omega^m]\psi\|\leqq C|||\psi|||_{2k+m}.$$

Therefore (2.14) holds for any $k\in\mathbb{N}$.

We now prove that $H^k\varphi\in\bigcap_{m\geqq 0}H^{0,m}$ for any $k\in\mathbb{N}$. By (2.14) and Lemma 2.2,

$$\begin{aligned}\|\omega^m H^k\psi\|&\leqq[H^k,\omega^m]\psi\|+\|H^k\omega^m\psi\|\\&\leqq C|||\psi|||_{2k+m-2}+C\|\omega^m\psi\|_{2k,0}\leqq C|||\psi|||_{2k+m},\end{aligned}$$

as desired.

Q.E.D.

Proof of Theorem 3. – Let $\varphi\in\mathscr{H}_m$. By making use of the Fourier transform and the Hermite polynomials, we have for $|\alpha|=m$,

$$\begin{aligned}&\mathscr{F}(e^{itH_0}(x/t)^\alpha e^{-itH_0}\varphi-(-i\partial)^\alpha\varphi)\\&=(i/t)^m(\exp(i(t/2)|\xi|^2)\partial^\alpha(\exp(-i(t/2)|\xi|^2)-(-it\xi)^\alpha)\mathscr{F}\varphi\\&=(i/t)^m\Bigg(\sum_{\beta\leqq\alpha}\sum_{\gamma\leqq[\beta/2]}\binom{\alpha}{\beta}\frac{\beta!(-1)^{|\beta+\gamma|}}{\gamma!(\beta-2\gamma)!}2^{-|\gamma|}(it)^{|\beta-\gamma|}\xi^{\beta-2\gamma}\partial^{\alpha-\beta}\mathscr{F}\varphi\end{aligned}$$

$$+\sum_{0\neq\gamma\leqq[\alpha/2]}\frac{\alpha!(-1)^{|\alpha+\gamma|}}{\gamma!(\alpha-2\gamma)!}2^{-|\gamma|}(it)^{|\alpha-\gamma|}\xi^{\alpha-2\gamma}\mathcal{F}\varphi\Big),$$

where $[\beta/2]=([\beta_1/2], \ldots, [\beta_n/2])$. By Lemma 2.2, the R.H.S. of the last equality converges to zero in L^2 as $|t|\to\infty$. This implies the first equality in the theorem. The second equality follows from the first one since

$$\|e^{-itH_0}\varphi\|_{0,m}^2=\sum_{j=1}^{m}\sum_{|\alpha|=j}\binom{m}{j}\frac{j!}{\alpha!}\|x^\alpha e^{-itH_0}\varphi\|^2$$

$$=\sum_{j=1}^{m}\sum_{|\alpha|=j}\binom{m}{j}\frac{j!}{\alpha!}\|e^{itH_0}x^\alpha e^{-itH_0}\varphi\|^2.$$

Q.E.D.

3. PROOF OF THEOREM 4

It is enough to consider the case $m\geqq 3$. Let $k=[m/2]$. By the assumption made on the derivatives of V, V^l is bounded from $H^{2l,0}$ to L^2 for all $l\leqq k-1$, and moreover, $\prod_{h=1}^{l}\partial^{\alpha_h}V$ is bounded from $H^{l+|\alpha_1+\ldots+\alpha_l|,0}$ to L^2 whenever $1\leqq l\leqq k-1$, $1\leqq|\alpha_1+\ldots+\alpha_l|\leqq 2(k-l)$. Let $\psi\in\mathcal{S}$. We have by induction that for all $j=1, \ldots, k$, $H^j\psi$ is in $D(H)=H^{2,0}$ and

$$H^j\psi=\sum_{l=0}^{j}\binom{j}{l}V^lH_0^{j-l}\psi+\sum_{l=1}^{j-1}$$

$$\times\sum_{\substack{|\beta|\leqq 2(j-l)-1\\|\alpha_1+\ldots+\alpha_l+\beta|=2(j-l)}}C(j,l,\{\alpha_h\},\beta)\Big(\sum_{h=1}^{l}\partial^{\alpha_h}V\Big)\partial^\beta\psi,\quad(3.1)$$

where every term on the R.H.S. is in L^2 by the preceding remarks. This proves $\mathcal{S}\subset D(H^k)$ and

$$\||H|^k\psi\|=\|H^k\psi\|\leqq C\|\psi\|_{2k,0},\qquad\psi\in\mathcal{S}.\qquad(3.2)$$

If $m=2k+1$, again by the above remarks every term on the R.H.S. of (3.1) with $j=k$ is in $H^{1,0}$ and

$$\|H^k\psi\|_{1,0}\leqq C\|\psi\|_{2k+1,0},\qquad\psi\in\mathcal{S},$$

which when combined with the fact $D(|H|^{1/2})=H^{1,0}$, shows

$$\||H|^{m/2}\psi\|=\||H|^{1/2}H^k\psi\|$$

$$\leqq C\|H^k\psi\|_{1,0}\leqq C\|\psi\|_{m,0},\qquad\psi\in\mathcal{S}.\quad(3.3)$$

The inclusion $D(|H|^{m/2})\supset H^{m,0}$ then follows from (3.2) and (3.3), since $\mathcal{S}$ is dense in $H^{m,0}$ and $|H|^{m/2}$ is closed. We now prove the reverse inclusion

$D(|H|^{m/2}) \subset H^{m,0}$ by induction. Let $m \geqq 3$ and assume that $D(|H|^{(m-1)/2}) \subset H^{m-1,0}$. Let $\psi \in D(|H|^{m/2})$. By the induction hypothesis, $\psi \in H^{m-1,0}$. In order to prove that $\psi \in H^{m,0}$, we distinguish between the following two cases:

(1) $m=2k+1$, $k \geqq 1$. (2) $m=2k$, $k \geqq 2$.

(1) When $m=2k+1$, it is sufficient to prove that $\partial^\alpha \psi \in D(H^k)$ for all $|\alpha|=1$. This will follow if we can show that

$$\sum_{|\alpha|=1} |(\partial^\alpha \psi, H^k \varphi)| \leqq C(\| |H|^{m/2} \psi \| + \|\psi\|) \|\varphi\|, \qquad \varphi \in D(H^k). \tag{3.4}$$

We approximate ψ by a sequence $\{\psi_j\}$ in $\mathscr{S}$ such that $\psi_j \to \psi$ in $H^{m-1,0}$ as $j \to \infty$. Consequently, $H^k \psi_j \to H^k \psi$ in L^2 as $j \to \infty$. By (3.1) with $j=k$,

$$\begin{aligned}(\partial^\alpha \psi_j, H^k \varphi) &= ([H^k, \partial^\alpha] \psi_j, \varphi) + (\partial^\alpha H^k \psi_j, \varphi) \\ &= \sum_{l=1}^{k} \binom{k}{l} ([V^l, \partial^\alpha] H_0^{k-l} \psi_j, \varphi) - (H^k \psi_j, \partial^\alpha \varphi) \\ &+ \sum_{l=1}^{k-1} \sum_{\substack{|\beta| \leqq 2(k-l)-1 \\ |\alpha_1 + \ldots + \alpha_l + \beta| = 2(k-l)}} C(k, l, \{\alpha_h\}, \beta) \left(\left[\prod_{h=1}^{l} \partial^{\alpha_h} V, \partial^\alpha \right] \partial^\beta \psi_j, \varphi \right).\end{aligned}$$

In the same way as before, we obtain

$$\| [V^l, \partial^\alpha] H_0^{k-l} \psi_j \| \leqq C \| H_0^{k-l} \psi_j \|_{2l, 0} \leqq C \| \psi_j \|_{2k, 0},$$

$$\left\| \left[\sum_{h=1}^{l} \partial^{\alpha_h} V, \partial^\alpha \right] \partial^\beta \psi_j \right\| \leqq C \| \partial^\beta \psi_j \|_{l+|\alpha_1 + \ldots + \alpha_l|+1, 0} \leqq C \| \psi_j \|_{2k, 0},$$

and therefore

$$|(\partial^\alpha \psi_j, H^k \varphi)| \leqq C \| \psi_j \|_{2k, 0} \| \varphi \| + |(H^k \psi_j, \partial^\alpha \varphi)| \tag{3.5}$$

Taking the limit $j \to \infty$ in (3.5), we have

$$|(\partial^\alpha \psi, H^k \varphi)| \leqq C \| \psi \|_{2k, 0} \| \varphi \| + |(H^k \psi, \partial^\alpha \varphi)|,$$

which yields (3.4) since

$$\begin{aligned}\| \psi \|_{2k} + \| \partial^\alpha H^k \psi \| \leqq C(\| |H|^k \psi \| + \| \psi \| \\ + \| |H|^{1/2} H^k \psi \|) \leqq C \| |H|^{m/2} \psi \| + C \| \psi \|.\end{aligned}$$

(2) When $m=2k$, it suffices to prove that $\Delta \psi \in D(H^{k-1})$. This will follow if we can show that

$$|(\Delta \psi, H^{k-1} \varphi)| \leqq C(\| |H|^{m/2} \psi \| + \| \psi \|) \| \varphi \|, \qquad \varphi \in D(H^{k-1}). \tag{3.6}$$

The proof of (3.6) is parallel to that of (3.4). We approximate ψ by a sequence $\{\psi_j\}$ in $\mathscr{S}$ such that $\psi_j \to \psi$ in $H^{m-1,0}$ as $j \to \infty$. Consequently, $H^{k-1} \psi_j \to H^{k-1} \psi$ in L^2 as $j \to \infty$. In the same way as before,

$$\| [H^{k-1}, \Delta] \psi_j \| \leqq C \| \psi_j \|_{2k-1, 0}$$

and therefore

$$|(\Delta \psi_j, H^{k-1} \varphi)| \leqq C \| \psi_j \|_{2k-1,0} \| \varphi \| + |(H^{k-1} \psi_j, \Delta\varphi)|,$$

which in turn implies (3.6).

Q.E.D.

Remark. — The argument given above shows that the inclusion $H^{m,0} \subset D(|H|^{m/2})$ follows from a weaker assumption that $\partial^\alpha V$ is bounded from $H^{2+|\alpha|,0}$ to L^2 for all $|\alpha| \leqq m-2$ because this implies that $\prod_{h=1}^{l} \partial^{\alpha_h} V$ is bounded from $H^{2l+|\alpha_1+\ldots+\alpha_l|,0}$ to L^2 whenever $l \leqq k-1$, $|\alpha_1+\ldots+\alpha_l| \leqq 2(k-l)$.

4. A CHARACTERIZATION OF INVARIANT SUBSPACES UNDER UNITARY GROUPS

Our purpose in this section is to prove the following:

THEOREM 6. — *Let* X *and* Y *be Hilbert spaces such that* Y *is densely and continuously embedded in* X. *Let* T *be a self-adjoint operator in* X. *Let* m, $M \in (0, \infty)$. *Then the following conditions are equivalent.*

(1) e^{-itH} *leaves* Y *invariant for any* $t \in \mathbb{R}$ *and has the estimate*

$$\| e^{-itT} \varphi \|_Y \leqq M(1+|t|^m) \| \varphi \|_Y, \qquad \varphi \in Y. \tag{4.1}$$

(2) $(T+i\lambda)^{-k}$ *leaves* Y *invariant for any* $k \in \mathbb{N}$ *and any* $\lambda \in \mathbb{R} \setminus \{0\}$ *and has the estimate*

$$\| (T+i\lambda)^{-k} \varphi \|_Y \leqq M(|\lambda|^{-k} + (\Gamma(m+k)/\Gamma(k)) |\lambda|^{-m-k}) \| \varphi \|_Y, \qquad \varphi \in Y, \tag{4.2}$$

where Γ *denotes the gamma function.*

Proof. — (1) $\Rightarrow$ (2): Let $\varphi \in Y$. For any $\lambda > 0$ and $k \in \mathbb{N}$ we have in X

$$(T \pm i\lambda)^{-k} \varphi = (1/(\pm i)^k \Gamma(k)) \int_0^\infty t^{k-1} e^{-t\lambda} e^{\pm itT} \varphi \, dt. \tag{4.3}$$

Since the map $\mathbb{R} \ni t \mapsto e^{-itT} \varphi \in X$ is continuous and statisfies (4.1), it follows that the map $\mathbb{R} \ni t \mapsto e^{-itT} \varphi \in Y$ is weakly continouous (*see*, Ginibre-Velo [5], Appendix 2), so that the maps $[0, \infty) \ni t \mapsto t^{k-1} e^{-t\lambda} e^{\pm itT} \varphi \in Y$ are strongly measurable and

$$\| t^{k-1} e^{-t\lambda} e^{\pm itT} \varphi \|_Y \leqq M t^{k-1} (1+t^m) e^{-t\lambda} \| \varphi \|_Y, \qquad t \geqq 0.$$

Therefore, by Bochner's theorem the integral in (4.3) converges in Y and the R.H.S. of (4.3) is estimated in Y by

$$M(\lambda^{-k} + (\Gamma(m+k)/\Gamma(k)) \lambda^{-m-k}) \| \varphi \|_Y, \qquad \lambda > 0,$$

since for any $\psi \in X$

$$|((T \pm i\lambda)^{-1} \varphi, \psi)_X| \leqq M (\lambda^{-k} + (\Gamma(m+k)/\Gamma(k)) \lambda^{-m-k}) \| \varphi \|_Y \| \psi \|_{Y^*}.$$

This implies part (2).

(2) $\Rightarrow$ (1): Let $\varphi \in Y$. By (4.2), we have for any $t>0$ and $k \in \mathbb{N}$

$$\| (1 \pm i(t/k) T)^{-k} \varphi \|_Y \leqq M (1 + (\Gamma(m+k)/\Gamma(k) k^m) t^m) \| \varphi \|_Y.$$

By Stirling's formula,

$$\limsup_{k \to \infty} \| (1 \pm i(t/k) T)^{-k} \varphi \|_Y \leqq M (1 + t^m) \| \varphi \|_Y.$$

On the other hand, $(1 \pm i(t/k)T)^{-k} \varphi \to e^{\mp itT} \varphi$ in X as $k \to \infty$. Therefore, $e^{\mp itT} \varphi \in Y$ and $(1 \pm i(t/k)T)^{-k} \varphi \to e^{\mp itT} \varphi$ weakly in Y as $k \to \infty$ (*see* Ginibre-Velo [5], Appendix 2). This implies part (1).

Q.E.D.

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(*Manuscript received January* 22, 1990)
(*in revised form April* 24, 1990.)