ANNALES DE L'I. H. P., SECTION A

DANIEL ALTSCHULER
CLAUDE ITZYKSON

Remarks on integration over Lie algebras

Annales de l'I. H. P., section A, tome 54, nº 1 (1991), p. 1-8
<http://www.numdam.org/item?id=AIHPA_1991__54_1_1_0>

Ann. Inst. Henri Poincaré,

Vol. 54, n° 1, 1991, p. 1-8. *Physique théorique*

Remarks on integration over Lie algebras

by

Daniel ALTSCHULER and Claude ITZYKSON

Service de Physique Théorique (*)
C.E.N.-Saclay, 91191 Gif-sur-Yvette Cedex, France

ABSTRACT. — We comment on certain problems of integration over Lie algebras and groups which arise naturally in the study of matrix models. In particular we give a new simple derivation of a classical formula due to Harish-Chandra, which was rediscovered by physicists as a mean of solving models with two matrices or more.

RÉSUMÉ. — Nous commentons certains problèmes d'intégration sur les algèbres et les groupes de Lie qui apparaissent naturellement dans l'étude des modèles de matrices. En particulier nous donnons une nouvelle dérivation simple d'une formule classique due à Harish-Chandra, qui a été redécouverte par les physiciens comme moyen de résoudre les modèles à deux matrices ou plus.

Matrix models have attracted the attention of physicists for a variety of reasons. Originally they were statistical models proposed and studied by Wigner and others for the spectroscopy of heavy nuclei [1]. Later particle physicists used them in the investigation of the large N limit of QCD, also called planar limit [2] as the only Feynman graphs which survive it are those that can be drawn on a plane. But soon they became

(*) Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Énergie Atomique.

a subject of interest of their own as it was discovered in [2] that the perturbative expansion is really of topological nature, if one views a graph as the spine of a simplicial decomposition of a surface. The contribution of a graph is proportional to N^{χ} where χ is the Euler characteristic of the simplicial decomposition. In this way surfaces of arbitrary χ participate in the perturbative expansion. Powerful techniques were developed ([3], [4], [5]) to sum the contributions of all graphs of a given topology.

This then led to the conclusion that these models could be of some help in understanding quantum gravity in two dimensions [6]. Two-dimensional quantum gravity is a theory of fluctuating random surfaces. A convenient way of regularizing it is to restrict the dynamical variables to a large but finite number of points on a surface considered as the vertices of a simplicial decomposition. A remarkable double limit scheme of the matrix models was found [7], such that they exhibit a scaling behavior for all χ. This is achieved by simultaneously sending N to infinity and carefully tuning the coupling constants. The resulting exponents agree with those which were obtained [8] directly in the continuum.

At this point one should also mention related work by mathematicians [9] who obtained information on the topology of the moduli space of surfaces using matrix models computations. The idea is that a suitable family of simplicial decompositions of surfaces gives rise to a cell decomposition of Teichmüller space. Then one reduces the computation of the (virtual) Euler characteristic of moduli space to a Feynman graph counting problem, which is solved by finding the appropriate generating function, which turns out to be the partition function of a matrix model.

The models are defined by the partition function:

$$Z=\int \prod_{a} d\mathrm{M}_a \exp\left[-\sum_{a} \mathrm{Tr}\, \mathrm{V}(\mathrm{M}_a)+\sum_{a,b} \beta_{ab} \mathrm{Tr}(\mathrm{M}_a \mathrm{M}_b)\right] \tag{1}$$

where the M_a, $1 \leqq a \leqq r$ are $\mathrm{N} \times \mathrm{N}$ matrices, which are supposed to be hermitian at first, and the integration is over $\mathrm{V} \times \mathrm{V} \times \ldots \times \mathrm{V}$ (r factors) where V is the real vector space of such matrices. The matrix of couplings β_{ab} is taken to be of the form:

$$\beta_{ab}=\beta \Gamma_{ab} \tag{2}$$

with Γ_{ab} the incidence matrix of a graph Γ. We assume that the potential V(M) is even:

$$\mathrm{V}(\mathrm{M})=\frac{1}{2}\mathrm{M}^2+\sum_{k \geqq 2} \frac{\beta_k}{\mathrm{N}^{k-1}} \mathrm{M}^{2k} \tag{3}$$

Now we come to the matrix integration problems we shall discuss in this paper. Let us first consider the case $r=1$, $\beta_{ab}=0$. We observe that due to invariance of the trace, Tr V(M) only depends on the eigenvalues λ_i of M. If f(M) is an invariant function, $f(\mathrm{UMU}^{-1}) = f(\mathrm{M})$ for any unitary matrix U,

$$\int d\mathrm{M}\, f(\mathrm{M}) = \frac{(2\pi)^{\mathrm{N}(\mathrm{N}-1)/2}}{\prod\limits_{1 \leqq p \leqq \mathrm{N}} p!} \int \prod_{i=1}^{\mathrm{N}} d\lambda_i\, \Delta(\Lambda)^2 f(\Lambda) \tag{4}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_\mathrm{N})$ and

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \tag{5}$$

Applying (4) to $f(\mathrm{M}) = \exp[-\mathrm{Tr}\, \mathrm{V}(\mathrm{M})]$, one finds that Z is given up to a numerical factor by:

$$\mathrm{Z}_\mathrm{N} = \int \prod_{i=1}^{\mathrm{N}} d\mu(\lambda_i)\, \Delta(\Lambda)^2 \tag{6}$$

with

$$d\mu(\lambda) = d\lambda\, e^{-\mathrm{V}(\lambda)} \tag{7}$$

$$\mathrm{V}(\lambda) = \frac{1}{2}\lambda^2 + \sum_{k \geqq 2} \frac{\beta_k}{\mathrm{N}^{k-1}} \lambda^{2k} \tag{8}$$

Introduce the monic orthogonal polynomials $\mathrm{P}_n(\lambda)$ with respect to the measure $d\mu(\lambda)$, where n is the degree. Using the fact that $\Delta(\lambda)$ is the Vandermonde determinant, one shows that [4]

$$\mathrm{Z}_\mathrm{N} = \mathrm{N}! \prod_{n=0}^{\mathrm{N}-1} h_n \tag{9}$$

where h_n is the norm of $\mathrm{P}_n(\lambda)$:

$$h_n = \int d\mu(\lambda)\, \mathrm{P}_n(\lambda)^2 \tag{10}$$

The second integration problem occurs when one tries to evaluate (1) for two matrices or more. Let us consider two matrices, say M_1 and M_2, and suppose that $\beta_{12} = \beta \neq 0$. As before, one would like first of all to integrate over the angular variables to deal with a problem involving only the eigenvalues $\lambda_{1,i}$ and $\lambda_{2,i}$, so that one must compute the integral:

$$\mathrm{I}(\mathrm{M}_1, \mathrm{M}_2; \beta) = \int d\mathrm{U} \exp \beta\, \mathrm{Tr}\, (\mathrm{M}_1 \mathrm{U} \mathrm{M}_2 \mathrm{U}^{-1}) \tag{11}$$

where dU is the normalized Haar measure on the unitary group, $\int d\mathrm{U} = 1$. Since dU is both left and right-invariant, one has $\mathrm{I}(\mathrm{M}_1, \mathrm{M}_2; \beta) = \mathrm{I}(\Lambda_1, \Lambda_2; \beta)$, where Λ_1 and Λ_2 are the diagonal matrices of eigenvalues of M_1 and M_2. In [4] this integral was found to be:

$$I(\Lambda_1, \Lambda_2; \beta) = \beta^{-N(N-1)/2} \prod_{p=1}^{N-1} p! \, \frac{\det(\exp \beta \lambda_{1,i} \lambda_{2,j})}{\Delta(\Lambda_1) \Delta(\Lambda_2)} \tag{12}$$

In this paper we will study the generalisation of (4) and (11) when one replaces the unitary group by a compact simple Lie group G and the antihermitian matrices iM by the Lie algebra $\mathcal{G}$ of G. For that we have to fix some notations first. Let $\mathcal{G}^C$ be the complexification of $\mathcal{G}$. Let Ad be the adjoint representation of G on $\mathcal{G}$. Let $\langle \, , \, \rangle$ be an invariant form on $\mathcal{G}^C$ which is positive definite on $i\mathcal{G}$. Let dg denote the normalized Haar measure on G. Choose a Cartan subalgebra $\mathcal{H}^C$ of $\mathcal{G}^C$, let $\mathcal{H} = \mathcal{H}^C \cap \mathcal{G}$, and choose a set of positive roots $\Sigma_+ \subset i\mathcal{H}$. We identify $\mathcal{H}^C$ with its dual by means of $\langle \, , \, \rangle$. To each $\alpha \in \Sigma_+$ we associate $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. These are called coroots. The coroot lattice $Q^\vee$ is the lattice generated by the coroots. Let P be the weight lattice, which is the dual of $Q^\vee$. Let W be the Weyl group and $m_1, m_2, \ldots, m_l$ the exponents of W, with $l = \dim \mathcal{H}$ the rank of $\mathcal{G}$.

We introduce the analog of (5). For $h \in \mathcal{H}^C$ it is the polynomial:

$$\Delta(h) = \prod_{\alpha \in \Sigma_+} \langle \alpha, h \rangle. \tag{13}$$

It is the infinitesimal version of Weyl's denominator,

$$\sigma(h) = e^{i \langle \rho, h \rangle} \prod_{\alpha \in \Sigma_+} (1 - e^{-i \langle \alpha, h \rangle}) \tag{14}$$

where as usual ρ denotes

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha. \tag{15}$$

The following generalisation of (4) is easily proved:

$$\int_{\mathcal{G}} dx\, f(x) = \text{Const} \int_{\mathcal{H}} dh\, \Delta(h)^2 f(h) \tag{16}$$

where $f(x)$ is any invariant function on $\mathcal{G}$, *i.e.* $f(\mathrm{Ad}(g)\,x) = f(x)$ for any $g \in G$. There is a similar formula for class functions $f(g)$ on G:

$$\int_G dg\, f(g) = \text{Const} \int_H dt\, \sigma^2(i \ln t)\, f(t) \tag{17}$$

with $H = \exp \mathcal{H}$ the Cartan subgroup.

Formula (16) enables us to solve one-matrix models on any classical Lie algebra $\mathcal{G}$. One finds

$$Z_{2l} = l!\, h_0 h_2 \ldots h_{2l-2} \tag{18}$$

for $\mathcal{G}=\mathrm{so}(2l)$ and

$$Z_{2l-1}=l!\,h_1 h_3 \ldots h_{2l-1} \tag{19}$$

for $\mathcal{G}=\mathrm{so}(2l+1)$. Here Z_N is defined as in (6) but Δ given by (13). The partition function for $\mathcal{G}=\mathrm{sp}(2l)$ is the same as (19) up to a numerical factor. To derive these results one uses the assumption that the potential (3) is even, and the explicit structure of roots systems [10]. One could use (17) to solve the one-matrix models on *groups*, as was done for the unitary group in [11].

The formula which is really interesting to generalise is (11). The answer is given as follows. Let $x_1, x_2 \in i\mathcal{H}$. Then

$$\int_G dg \exp \beta \langle x_1, \mathrm{Ad}(g)\, x_2 \rangle = \beta^{-(\dim G - l)/2}\, \Delta(\rho) \frac{\sum_{w \in W} \varepsilon(w)\, e^{\beta \langle x_1, w(x_2) \rangle}}{\Delta(x_1)\, \Delta(x_2)} \tag{20}$$

where $\varepsilon(w)=(-1)^{l(w)}$, $l(w)$ = length of w expressed as a product of reflections. Notice that

$$\Delta(\rho)=\prod_{\alpha \in \Sigma_+} \frac{|\alpha|^2}{2} \prod_{i=1}^{l} m_i! \tag{21}$$

and that (20) is unchanged if we simultaneously multiply $\langle\ ,\ \rangle$ by a factor a and divide β by a. Therefore we can evaluate (21) by setting $|\alpha_L|^2=2$,

$$\Delta(\rho)=\left(\frac{|\alpha_S|^2}{|\alpha_L|^2}\right)^{\#\text{short roots}} \prod_{i=1}^{l} m_i! \tag{22}$$

and put it back into (20) without having to choose a normalisation for $\langle\ ,\ \rangle$. (Here α_L and α_S denote long and short roots.)

Before embarking on our short proof of (20), we would like to make a few remarks. Formula (20) is due to Harish-Chandra [12] who proved it by studying the invariant differential operators on G. He never used Weyl's character formula in his proof, in fact (20) is an ingredient in the derivation of the character formula based on harmonic analysis [13]. Our proof uses the character formula, which can be established independently in a purely algebraic way, together with elementary harmonic analysis on G. As already mentioned before, in the particular case of unitary groups the formula was rediscovered by one of us in collaboration with J.-B. Zuber [4]. For a recent application in another context in physics see [15].

In more recent mathematical works, the l.h.s. of (20) is often expressed as the integral over an orbit of the coadjoint representation. It is possible to express the characters themselves as orbital integrals [14]. One can also use the fact that the orbits in the coadjoint representation have a symplectic structure to prove (20) more directly: one computes the saddle-point approximation to the integral, the presence of the symplectic structure then implies that the approximation is in fact an exact computation. *See* [16] and the references in [15].

Now we come to the proof of (20). We note that the integrands is proportional to the Heat Kernel on $i\mathscr{G}$,

$$\exp\left(-\frac{1}{2t}|x_1-\mathrm{Ad}(g)x_2|^2\right) \tag{23}$$

with $\beta=1/t$. For short times t, it doesn't matter if we consider instead the Heat Kernel on G, $\mathrm{K}(g_1 g g_2^{-1} g^{-1}, t)$ with $g_j=\exp ix_j$, $j=1,2$. It turns out that the latter integral is easy to compute. The evaluation of its short-time limit then furnishes the result. Thus we need the expression for the Heat Kernel on G, $\mathrm{K}(g, t)$. The set of all irreducible unitary representations of G is

$$\mathrm{P}_+=\mathrm{P}\cap\{x\in i\mathscr{H}\,|\,\langle x, \alpha\rangle\geqq 0,\ \alpha\in\Sigma_+\}. \tag{24}$$

More precisely, P_+ is the set of highest weights of these representations. Denote by $\chi_\lambda(g)$ and d_λ the character and the dimension of the representation corresponding to $\lambda\in\mathrm{P}_+$. Then

$$\mathrm{K}(g, t)=\sum_{\lambda\in\mathrm{P}_+} d_\lambda\,\chi_\lambda(g)\,e^{-c_\lambda t/2} \tag{25}$$

where

$$c_\lambda=|\lambda+\rho|^2-|\rho|^2. \tag{26}$$

Let us compute

$$\mathrm{F}(g_1, g_2, t)=\int_{\mathrm{G}} dg\,\mathrm{K}(g_1 g g_2^{-1} g^{-1}, t)=\sum_{\lambda\in\mathrm{P}_+}\chi_\lambda(g_1)\,\chi_\lambda(g_2^{-1})\,e^{-c_\lambda t/2} \tag{27}$$

To obtain this result, substitute (25) in (27). It reduces to

$$d_\lambda\int_{\mathrm{G}} dg\,\chi_\lambda(g_1 g g_2^{-1} g^{-1})=\chi_\lambda(g_1)\,\chi_\lambda(g_2^{-1}) \tag{28}$$

which follows from the orthogonality relations for matrix elements (Peter-Weyl theorem). Now recall the Weyl character formula:

$$\chi_\lambda(e^{ix})=\frac{\nu_{\lambda+\rho}(x)}{\sigma(x)} \tag{29}$$

where $x\in i\mathscr{H}$ and

$$\nu_\lambda(x)=\sum_{w\in\mathrm{W}}\varepsilon(w)\,e^{i\langle w(\lambda), x\rangle}. \tag{30}$$

We want to express (27) in terms of theta functions. We have

$$\sum_{\lambda \in P_+} e^{-c_\lambda t/2} \nu_{\lambda+\rho}(x_1)\, \nu_{\lambda+\rho}(-x_2) = e^{|\rho|^2 t/2} \sum_{w \in W} \varepsilon(w) \sum_{\lambda \in P} e^{i\langle \lambda, x_1 - w(x_2)\rangle} e^{-|\lambda|^2 t/2} \quad (31)$$

since $\nu_\lambda(x) = 0$ if the stabilizer of λ in W is non-trivial. With the Poisson summation formula we arrive at

$$F(g_1, g_2, t) = \frac{e^{|\rho|^2 t/2} v^{1/2} (2\pi/t)^{l/2}}{\sigma(x_1)\sigma(-x_2)} \sum_{w \in W} \varepsilon(w) \sum_{\beta \in 2\pi Q^\vee} e^{-|x_1 - w(x_2) + \beta|^2/2t} \quad (32)$$

where v is the index of $Q^\vee$ in P. This formula was also derived in [17] for the study of the orbital theory of affine algebras. Now we take a short time limit:

$$\lim_{\varepsilon \to 0} \varepsilon^{\dim G} F(e^{i\varepsilon x_1}, e^{i\varepsilon x_2}, \varepsilon^2 t) = f(x_1, x_2, t) \quad (33)$$

All terms in (32) with $\beta \neq 0$ become negligible and we get:

$$f(x_1, x_2, t) = v^{1/2} \left(\frac{2\pi}{t}\right)^{l/2} \frac{\sum_{w \in W} \varepsilon(w) e^{-|x_1 - w(x_2)|^2/2t}}{\Delta(x_1)\Delta(x_2)} \quad (34)$$

To complete the proof we have to work out the precise relation between $K(g, t)$ and the Heat Kernel on $i\mathcal{G}$. As before we express K in terms of theta functions:

$$\begin{aligned} K(e^{ix}, t) &= \lim_{\eta \to 0} \sum_{\lambda \in P_+} \chi_\lambda(e^{ix}) \chi_\lambda(e^{-i\varepsilon\eta\rho}) e^{-c_\lambda t/2} \\ &= \lim_{\eta \to 0} \frac{e^{|\rho|^2 t/2} v^{1/2} (2\pi/t)^{l/2}}{\sigma(x)\sigma(-\varepsilon\eta\rho)} \sum_{w \in W} \varepsilon(w) \sum_{\beta \in 2\pi Q^\vee} e^{-|x - \varepsilon\eta w(\rho) + \beta|^2/2t} \end{aligned} \quad (35)$$

Hence

$$\lim_{\varepsilon \to 0} \varepsilon^{\dim G} K(e^{i\varepsilon x}, \varepsilon^2 t) = \lim_{\eta \to 0} \frac{v^{1/2} (2\pi/t)^{l/2}}{\Delta(x)\Delta(\eta\rho)} \sum_{w \in W} \varepsilon(w) e^{-|x - \eta w(\rho)|^2/2t} \quad (36)$$

Thus we get

$$\lim_{\varepsilon \to 0} \varepsilon^{\dim G} K(e^{i\varepsilon x}, \varepsilon^2 t) = \frac{v^{1/2} (2\pi/t)^{l/2}}{t^{(\dim G - l)/2} \Delta(\rho)} e^{-|x|^2/2t} \quad (37)$$

using the denominator formula $\nu_\rho(-i\eta x/t) = \sigma(-i\eta x/t)$. Comparing (34) and (37) we arrive at (20).

ACKNOWLEDGEMENTS

We thank D. Bernard and J.-B. Zuber for discussions and for guiding us in the literature on this subject.

REFERENCES

[1] For extensive discussions and references *see* M. L. MEHTA, *Random Matrices and the Statistical Theory of Energy Levels*, Academic Press, New York, 1967.
[2] G. 'T HOOFT, *Nucl. Phys.*, Vol. **B72**, 1974, p. 461; *Nucl. Phys.*, Vol. **B75**, 1974, p. 461.
[3] E. BRÉZIN and C. ITZYKSON, G. PARISI and J.-B. ZUBER, *Comm. Math. Phys.*, Vol. **59**, 1978, p. 35.
[4] C. ITZYKSON and J.-B. ZUBER, *J. Math. Phys.*, Vol. **21**, 1980, p. 411.
[5] D. BESSIS, C. ITZYKSON and J.-B. ZUBER, *Adv. Applied Math.*, Vol. **1**, 1980, p. 109.
[6] F. DAVID, *Nucl. Phys.*, Vol. **B257**, 1985, p. 45; V. KAZAKOV, *Phys. Lett.*, Vol. **150B**, 1985, p. 282; V. KAZAKOV, I. KOSTOV and A. A. MIGDAL, *Phys. Lett.*, Vol. **B157**, 1985, p. 295.
[7] E. BRÉZIN and V. A. KAZAKOV, *Phys. Lett.*, Vol. **B236**, 1990, p. 144; M. DOUGLAS and S. SHENKER, Rutgers preprint, 1989; D. J. GROSS and A. A. MIGDAL, *Phys. Rev. Lett.*, Vol. **64**, 1990, p. 127.
[8] V. G. KNIZHNIK, A. M. POLYAKOV and A. B. ZAMOLODCHIKOV, *Mod. Phys. Lett.*, Vol. **A3**, 1988, p. 819; F. DAVID *Mod. Phys. Lett.*, Vol. **A3**, 1988, p. 1651; J. DISTLER and H. KAWAI, *Nucl. Phys.*, Vol. **B321**, 1989, p. 509.
[9] J. HARER and D. ZAGIER, *Invent. Math.*, Vol. **85**, 1986, p. 457; R. C. PENNER, *Bull. A.M.S.*, Vol. **15**, 1986, p. 73; *J. Diff. Geom.*, Vol. **27**, 1988, p. 35; *Comm. Math. Phys.*, Vol. **113**, 1987, p. 299.
[10] N. BOURBAKI, *Groupes et Algèbres de Lie*, chapitres 4, 5 et 6, Masson, Paris, 1981.
[11] D. J. GROSS and E. WITTEN, *Phys. Rev.*, Vol. **D21**, 1980, p. 446.
[12] HARISH-CHANDRA, *Am. J. Math.*, Vol. **79**, 1957, p. 87.
[13] S. HELGASON, *Groups and Geometric Analysis*, Academic Press, 1984.
[14] A. A. KIRILLOV, *Funct. Anal. Appl.*, Vol. **2**, 1968, p. 133.
[15] M. STONE, *Nucl. Phys.*, Vol. **314**, 1989, p. 557.
[16] J. J. DUISTERMAAT and G. J. HECKMAN, *Invent. Math.*, Vol. **69**, 1982, p. 259.
[17] I. B. FRENKEL, *Invent. Math.*, Vol. **77**, 1984, p. 301.

(*Manuscript received July* 17, 1990.)