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Localisation for the spin J-boson Hamiltonian

by

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ABSTRACT. — We investigate the phase diagram of the ground state for a spin J coupled linearly to a Bose field. We prove, under suitable infrared conditions, that there exists a critical coupling strength, $\alpha_c(J)$, above which the left-right symmetry of the system is broken: the spin becomes localized. We establish lower and upper bounds on $\alpha_c(J)$. In particular, they imply that $\alpha_c(J = \infty)$ agrees with the critical coupling strength of the semiclassical theory.

RÉSUMÉ. — Nous étudions le diagramme de phase de l'état fondamental pour un spin J couplé linéairement à un champ de Bosons. Nous montrons que sous des conditions infrarouges appropriées, il existe une valeur critique $\alpha_c(J)$ de l'amplitude du couplage au dessus de laquelle la symétrie droite-gauche du système est brisée : le spin devient localisé. Nous donnons des bornes supérieures et inférieures pour $\alpha_c(J)$. Elles impliquent en particulier que $\alpha_c(J = \infty)$ coïncide avec la valeur critique de la théorie semi classique.
1. INTRODUCTION

The spin-boson Hamiltonian models a spin 1/2 coupled to a bosonic field. It is the prototypical example of a dissipative quantum system. We refer to [1] for a recent review. The coupling between the spin and the environment may be so strong that the ground state of the system becomes twofold degenerate with a broken left-right symmetry. This phenomenon is necessarily associated with the generation of an infinite number of infrared bosons ([3], [4]). From a quantum mechanical point of view a natural question to ask is what happens if the spin 1/2 is replaced by a spin J. For large J one can use the semiclassical theory ([2], [13]). How does then the quantum regime (small J) link up with the semiclassical regime?

The spin J-boson Hamiltonian reads

\[
H = -\frac{c}{J} S^x \otimes 1 + 1 \otimes \int dk \omega(k) a^*(k) a(k) + \frac{\sqrt{\alpha}}{J} S^z \otimes \int dk \lambda(k) [a^*(k) + a(k)] - \frac{h}{J} S^z \otimes 1. \tag{1}
\]

Here \( S = (S^x, S^y, S^z) \) are the spin J matrices with \([S^x, S^y] = i S^x \) plus cyclic permutations and \( S \cdot S = J(J+1) \). \( \{ a(k), a^*(k) \mid k \in \mathbb{R}^d \} \) are annihilation and creation operators in momentum space of a d-dimensional Bose field, \([a(k), a^*(k')] = \delta(k-k') \). Since dimension plays no particular role, we set \( d=1 \) for simplicity. Our results hold for any dimension. \( \omega(k) \geq 0 \) is the dispersion relation of the Bose field and \( \lambda(k) = \lambda(k)^* \) are the couplings. For convenience we require

\[
\int dk \lambda(k)^2 < \infty. \tag{2}
\]

\( \alpha \geq 0 \) is the coupling parameter. We normalize it by setting

\[
\int dk \frac{\lambda(k)^2}{\omega(k)} = \frac{1}{2}. \tag{3}
\]

The integral in (3) has to be finite in order to ensure that \( H \) is bounded from below.

For \( h=0 \), \( H \) is invariant under the discrete symmetry, \( \tau \), defined by

\[
\tau a(k) = -a(k), \quad \tau a^*(k) = -a^*(k),
\tau S^x = S^x, \quad \tau S^y = -S^y, \quad \tau S^z = -S^z. \tag{4}
\]

Clearly \( \tau^2 = 1 \). We want to understand under what conditions this left-right symmetry is spontaneously broken in the ground state. We approach
the problem by means of an order parameter (other, equivalent, possibilities are discussed in [3], [4]), denoted by $m^*$, which may be defined through the following limit procedure: We confine the Bose field to a finite box, $\Lambda$, in physical space and impose periodic boundary conditions. Moreover we introduce a ultraviolet-cutoff $|k| \leq k_{\text{max}}$. The $k$-integrals in (1) become then finite sums over a momentum lattice, denoted by $K$. The Hamiltonian with these cutoffs has a unique ground state, denoted by $\Psi_{K, h}$. The order parameter is given by

$$m^*: = \lim_{h \to 0} \lim_{K \to \mathbb{R}} \langle \Psi_{K, h} | \frac{1}{J} \sum_{\mathbf{z}} | \Psi_{K, h} \rangle.$$  

The order of limits is essential. It is part of our proof that these limits exist. If $m^*=0$, then $H$ has a unique ground state. The $\tau$ symmetry is unbroken. If $m^*>0$, the $\tau$-symmetry is spontaneously broken and $H$ has a twofold degenerate ground state. In the following $\epsilon$ will be kept fixed and we investigate how $m^*$ depends on $\alpha$ and $J$. Actually, $m^*$ is increasing in $\alpha$. This allows us to define a critical coupling strength, $\alpha_c(J)$, by

$$m^* = 0 \quad \text{for} \quad \alpha < \alpha_c(J),$$

$$m^* > 0 \quad \text{for} \quad \alpha > \alpha_c(J).$$  

The two extreme cases, $J=1/2$ and $J=\infty$, are well understood. For the spin $1/2$ case the central quantity is the effective potential

$$W(t) = \int dk \, \lambda(k) \, e^{-\alpha(k)|t|}$$

(note that $W(t)$ is bounded because of (2) and $\int dt \, W(t) = 1$ by (3)). If

$$\lim_{t \to \infty} t^2 W(t) = 0,$$  

then $m^* = 0$ and hence $\alpha_c(1/2) = \infty$. On the other hand, if the limit in (8) is strictly positive (or infinite), then $\alpha_c(1/2) < \infty$. For sufficiently strong coupling the $\tau$-symmetry is broken. At $\alpha = \alpha_c(1/2)$, $m^*$ either vanishes or not, depending on details ([3], [4]).

On the other hand, for large $J$ we can use the result of Lieb [2] who proves that in the limit $J \to \infty$, $\frac{1}{J} \mathbf{S}$ becomes a classical variable and the
partition function for the Hamiltonian (1) converges to the partition function for the semiclassical Hamiltonian

\[ H_{sc} = -\varepsilon \cos(\varphi) \sin(\theta) + \int dk \omega(k) a^*(k) a(k) + \sqrt{\alpha} \cos(\theta) \int dk \lambda(k) [a^*(k) + a(k)] - h \cos(\theta), \]

where \(0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi\). Since now \(x\)- and \(z\)-component of the spin commute, the ground state is easily determined. Computing \(m^*\) through the limit \(h \to 0\), one obtains \(\alpha_c(\infty) = \varepsilon\), independent of the large \(t\) decay of the effective potential \(W(t)\), cf. Appendix.

The problem posed is the behaviour of \(\alpha_c(J)\) inbetween these two extreme cases. To our own surprise, the spin J-boson Hamiltonian interpolates in the simplest possible way: For \(h \neq 0\) and general \(W(t)\), we have

\[ \lim_{J \to \infty} m(J, h) = m_{sc}(h), \]

where \(m(J, h)\) is defined in (5) but without the limit \(h \to 0\) and \(m_{sc}(J, h)\) is the corresponding quantity obtained from the semiclassical Hamiltonian (9). If \(\alpha > \varepsilon\), then \(m_{sc}(h)\) has a jump discontinuity at \(h = 0\). For \(h = 0\) and if a decay condition slightly faster than in (8) holds, then \(\alpha_c(J) = \infty\) for every \(J\). On the other hand if

\[ \lim_{t \to \infty} t^2 W(t) > 0, \]

then \(\alpha_c(J) < \infty\). Presumably \(\alpha_c(J)\) is decreasing in \(J\). We will prove the bounds

\[ \varepsilon = \alpha_c(\infty) \leq \alpha_c(J) \leq \alpha_+(J) < \infty. \]

If the limit in (11) is infinite, then

\[ \lim_{J \to \infty} \alpha_+(J) = \varepsilon. \]

We expect this property to hold whenever (11) is satisfied. In the following figure we present a schematic phasediagram.

The technique to prove results as (12), (13) is similar to the spin 1/2 case with one extra twist however. For spin 1/2 one exploits a mapping to a ferromagnetic one-dimensional continuum Ising model (spin \(\sigma(t) = \pm 1/2\) with pair potential \(\alpha W(t)\). \(m^*\) becomes then the usual order parameter of spontaneous magnetisation. If the pair potential decays sufficiently slowly, then the Ising model orders and \(m^* > 0\). It turns out...
that in the corresponding mapping for the spin $J$ model the spin magnitude $J$ introduces an extra dimension. The continuum model now consists of $2J$ coupled Ising-lines. Let $\sigma_j(t)$ be the spin configuration in the $j$-th line, $1 \leq j \leq 2J$, $\sigma_j(t) = \pm 1/2$. The energy of the spin configuration in the two dimensional volume $[-\beta/2, \beta/2] \times \{1, 2, \ldots, 2J\}$ is then

$$\frac{1}{J} \sum_{i,j=1}^{2J} \int_{-\beta/2}^{\beta/2} dt \int_{-\beta/2}^{\beta/2} ds \alpha JW(J|t-s|) \sigma_i(t) \sigma_j(s).$$

(14)

In the $t$-direction the strength of the potential decreases, whereas in the $J$-direction the coupling is independent of the location of the pair of spins. As it should be, the total energy is extensive, i.e. proportional to $\beta J$.

The energy (14) has two mechanisms for ordering. If $W(t)$ decays slowly and if $\alpha$ is sufficiently large, then the spin system orders in the $t$-direction for fixed $J$. On the other hand, for fixed $\beta$, as $J \to \infty$ the energy (14) is of mean field type and the system must have a mean field phase transition. Note that as $J \to \infty$, $JW(J|t-s|)$ converges to $\delta(t-s)$ and, a priori, it is not quite obvious how the two mechanisms combine.

To give a short outline of the remainder of the paper: In Section 2 we establish the mapping between the spin $J$ boson Hamiltonian and the just mentioned system of $2J$ coupled Ising lines. In particular, we relate the order parameter $m^*$ to the spontaneous magnetisation. In Section 3 we
prove a lower and in Section 4 an upper bound on the critical coupling strength $\alpha_c(J)$.

2. ORDER PARAMETER

AND FUNCTIONAL INTEGRAL REPRESENTATION

To define the order parameter we first have to introduce a cutoff Hamiltonian, $H_K$. Let $\Lambda \subset \mathbb{R}$ be an interval of length $|\Lambda|$, the physical volume. We impose periodic boundary conditions. Let $K$ be the set of modes in $\Lambda$ with ultraviolet-cutoff $|k| \leq k_{\text{max}}$ (if necessary, zero modes are also removed from $K$). Then the cutoff Hamiltonian is given by

$$H_K = -\frac{\varepsilon}{J} S^z \otimes 1 + 1 \otimes \sum_{k \in K} \omega_k a_k^* a_k + \frac{\sqrt{2}}{J} S^z \otimes \sum_{k \in K} \lambda_k (a_k^* + a_k) - \frac{\hbar}{J} S^z \otimes 1,$$

(15)

with a suitable choice of $\omega_k$ and $\lambda_k$, cf. the proof of Proposition 2. \{ $a_k, a_k^* \mid k \in K$ \} constitute a representation of the CCR. Since $\vert K \vert < \infty$, this representation is equivalent to the Schrödinger representation. Therefore $H_K$ can be regarded as a linear operator on $\mathcal{H}_K = \mathbb{C}^{2J+1} \otimes \mathcal{F}_K^S$, where $\mathcal{F}_K^S \cong L^2(\mathbb{R}, d\lambda)^\nu \vert K \vert$ is the symmetric $\vert K \vert$-particle Fock space. Here $\nu \in \mathbb{N}$, $N \in \mathbb{N}$, denotes $N$-fold symmetric tensor product.

$H_K$ is a finite particle Hamiltonian generating a positivity improving one parameter semigroup, $e^{-\beta H_K}$, and thus $H_K$ has a unique ground state $\Psi_{K,h} \in \mathcal{H}_K$. We define the order parameter by

$$m(h) := \lim_{K \to \infty} \langle \Psi_{K,h} \vert \frac{1}{J} S^z \vert \Psi_{K,h} \rangle,$$

(16)

$$m^* := \lim_{h \to 0} m(h).$$

(17)

We will prove below that the sequence in (16) is monotone increasing and that $m(h)$ decreases monotonically to $m^*$.

We want to express $m(h)$ as an expectation value with respect to a stochastic process on the time interval $[-\beta/2, \beta/2]$ taking values in $\{-J, \ldots, J\}$. For this purpose we construct first the measure generated by $\exp(t S^z/J)$. Here and in what follows we will work in the $S^z$-basis. In this basis the ground state of $S^z$ is given by

$$\Omega_0 (m) = \frac{1}{2^J (J+m)^{1/2}} > 0, \quad -J \leq m \leq J.$$
Let $\Gamma^\beta$ be the set of piecewise constant paths on $[-\beta/2, \beta/2]$ taking values in $\{-J, \ldots, J\}$. Let $S(.)$ be a path in $\Gamma^\beta$ with jumps at $-\beta/2 < t_1 < \ldots < t_n < \beta/2$ and with the value $S(t) = m_i \in \{-J, \ldots, J\}$ for $t_i \leq t < t_{i+1}$, $0 \leq i \leq n$, $t_0 = -\beta/2$, $t_{n+1} = \beta/2$. We assign to $S(.)$ the weight

$$\Omega_0 (m_0) \Omega_0 (m_n) \langle m_0 | e^{\int_j S^x} | m_1 \rangle \times \ldots \times \langle m_{n-1} | e^{\int_j S^x} | m_n \rangle dt_1 \ldots dt_n,$$

(19)

where

$$\langle m | S^x | m' \rangle = \sqrt{J(J+1) - m(m+1)} \delta_{m,m'+1} + \sqrt{J(J+1) - m(m-1)} \delta_{m,m'-1}$$

are the matrix elements of $S^x$ in the $S^z$-basis. The so defined (unnormalized) measure on $\Gamma^\beta$ is denoted by $d\mu^\beta (S)$.

Let us define an action functional by

$$A_j (S) = -\frac{\alpha}{2J^2} \int_{-\beta/2}^{\beta/2} dt \int_{-\beta/2}^{\beta/2} ds \, W_K (t-s) S(t) S(s) - \frac{h}{J} \int_{-\beta/2}^{\beta/2} dt \, S(t),$$

(20)

where

$$W_K (t) = \frac{2\pi}{|A|} \sum_{k \in K} \lambda_k^2 e^{-\omega_k |t|}.$$

(21)

This is a Riemann sum with limit

$$W(t) = \lim_{K \to R} W_K (t) = \int dk \, \lambda (k)^2 e^{-\omega (k) |t|},$$

(22)

compare with (7). Expectation values with respect to the normalized measure $\frac{1}{Z} \exp \left[ -A_j (S) \right] d\mu^\beta (S)$ are denoted by $\langle . \rangle_1 (\beta, K)$.

**Proposition 1.** Let $\Psi_{K, h}$ be the ground state of $H_K$. Then

$$\langle \Psi_{K, h} | \int_j S^z | \Psi_{K, h} \rangle = \lim_{J \to \infty} \langle \frac{1}{J} S(0) \rangle_1 (\beta, K).$$

**Proof.** Let $H_K^0$ be the Hamiltonian (15) with $\alpha = h = 0$. This is the Hamiltonian of a spin $J$ and $|K|$ independent harmonic oscillators. Its ground state, $\Phi_K$, is the product of $\Omega_0$ and $|K|$ harmonic oscillator ground
states. Since $s\lim_{\beta \to \infty} \exp[-\beta (H_K - E_K, 0)] = \text{Pr}_{\Psi_{K,h}}$, the orthogonal projection on $\Psi_{K,h}$, and since $\langle \Phi_K | \Psi_{K,h} \rangle > 0$ by positivity, we have

$$
\lim_{\beta \to \infty} \frac{1}{\| e^{-\beta H_K} \Phi_K \|^2} \langle \Phi_K \big| e^{-\beta H_K} \int J S_z e^{-\beta H_K} | \Phi_K \rangle = \langle \Psi_{K,h} \big| \frac{1}{J} S_z | \Psi_{K,h} \rangle. \tag{23}
$$

$\langle \Phi_K \big| e^{-\beta H_K} \int J S_z e^{-\beta H_K} | \Phi_K \rangle$ can be rewritten as a functional integral. The free process is a product of $du^\beta(S)$ and $|K|$ independent Ornstein-Uhlenbeck processes. The action is given by

$$
\sqrt{\alpha} \int_{-\beta/2}^{\beta/2} dt S(t) \sum_{k \in K} \lambda_k q_k(t) - \frac{\hbar}{J} \int_{-\beta/2}^{\beta/2} dt S(t), \tag{24}
$$

where the $q_k(.)$ are Ornstein-Uhlenbeck paths on the time interval $[-\beta/2, \beta/2]$. The bosonic degrees of freedom can be integrated out, compare with [3, 5]. The net result is

$$
\frac{1}{\| e^{-\beta H_K} \Phi_K \|^2} \langle \Phi_K \big| e^{-\beta H_K} \int J S_z e^{-\beta H_K} | \Phi_K \rangle = \langle \frac{1}{J} S(0) \rangle_J (\beta, K) \tag{25}. \quad \square
$$

It turns out that the limit $J \to \infty$ can be better controlled in a system of $2J$ coupled Ising lines, which we introduce next. As an additional bonus this system makes it easy to prove correlation inequalities. The $2J$ coupled Ising lines can be viewed as a quantum version of Griffiths' method of analogue systems, [6].

For $1 \leq j \leq 2J$ let $\sigma_j(.)$ be a piecewise constant path on $[-\beta/2, \beta/2]$ with values $\pm 1/2$. By $d\sigma^\beta(\sigma_j)$ we denote $d\mu^\beta(S)$ for $J = 1/2$. In particular, if $\sigma_j(.)$ flips at $-\beta/2 < t_1 < \ldots < t_n < \beta/2$, its weight is $\left(\frac{\varepsilon}{J}\right)^n dt_1 \ldots dt_n$, independent of the initial and final values of $\sigma_j(.)$.

---

(1) Note that due to our boundary conditions expectation values are taken in the harmonic oscillator ground states rather than over thermal states as in [5] or [3], compare with equation (5.47) in [5].
**Lemma 1.** Let \( S(t) := \sum_{j=1}^{2J} \sigma_j(t) \). The weight of \( S(t) \) under \( \prod_{j=1}^{2J} d\nu^\beta(\sigma_j) \) equals \( d\mu^\beta(S) \).

**Proof.** Let \( S(t) \) take values \( m_i \) in the intervals \([t_i, t_{i+1})\), \( 0 \leq i \leq n \), \( t_0 = -\beta/2 \), \( t_{n+1} = \beta/2 \). Its weight under \( \prod_{j=1}^{2J} d\nu^\beta(\sigma_j) \) is of the form

\[
\sum_{\sigma} u(m_0) p(m_0, m_1) \cdots p(m_{n-1}, m_n),
\]

where \( u(m_0) \) is the number of ways \( m_0 \) can be realized and \( p(m, m') \) is the number of ways \( m' \) can be obtained given \( m \), weighted by \( \varepsilon/2J \) (the factor \( 1/2 \) is the proper normalisation).

We have

\[
u(m) = \frac{1}{2^{2J}} \left( \frac{2J}{J+m} \right) = \Omega_0(m)^2 \text{ and}
\]

\[
p(m, m') = \begin{cases}
\frac{\varepsilon}{J} (J-m) & \text{if } m' = m + 1 \\
\frac{\varepsilon}{J} (J+m) & \text{if } m' = m - 1 \\
0 & \text{else}.
\end{cases}
\]

Comparing with (19) the claim follows from

\[
\Omega_0(m) p(m, m') \Omega_0(m')^{-1} = \langle m | \frac{\varepsilon}{J} S^x | m' \rangle. \quad \square
\]

As a Consequence of Lemma 1 we have

\[
\int d\mu^\beta(S) f(S) = \int \left( \prod_{j=1}^{2J} d\nu^\beta(\sigma_j) \right) f(\sigma_1 + \ldots + \sigma_{2J})
\]

for any (bounded) function \( f \) on \( \Gamma^\beta \).

The \( 2J \) coupled Ising lines have \( \prod_{j=1}^{2J} d\nu^\beta(\sigma_j) \) as free measure and in terms of the \( \sigma_j \) the action (20) reads

\[
A(\sigma) = -\frac{\alpha}{2J^2} \int_{-\beta/2}^{\beta/2} dt \int_{-\beta/2}^{\beta/2} ds \mathcal{W}_K(t-s) \sum_{i,j=1}^{2J} \sigma_i(t) \sigma_j(s)
\]

\[
-\frac{\hbar}{J} \int_{-\beta/2}^{\beta/2} dt \sum_{j=1}^{2J} \sigma_j(t),
\]

where we use \( \sigma \) as a short hand for \((\sigma_1, \ldots, \sigma_{2J})\). Expectations with respect to the normalized measure \( \frac{1}{Z} \exp \left[ -A(\sigma) \right] \prod_{j=1}^{2J} d\nu^\beta(\sigma_j) \) are denoted by \( \langle . \rangle (\beta, K) \).
The functional (27) is explicitly ferromagnetic. Also each $dv^\beta(\sigma)$ can be approximated by discrete Ising spin chains with ferromagnetic interactions, see [3]. Therefore the $2J$ coupled Ising lines is a ferromagnetic spin model.

**Proposition 2.** — The limits (16) and (17) exist and $m^*$ agrees with the spontaneous magnetisation of the $2J$ coupled Ising lines. Furthermore the limits $\beta \to \infty$ and $K \to \mathbb{R}$ commute,

$$m(h) = \lim_{K \to \mathbb{R}} \lim_{\beta \to \infty} \left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K)$$

$$= \lim_{\beta \to \infty} \lim_{K \to \mathbb{R}} \left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K). \quad (28)$$

**Proof.** — By Proposition 1 and Lemma 1,

$$\langle \Psi_{K, h} \left| \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K) = \lim_{\beta \to \infty} \left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K).$$

Let us first prove that $\left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K)$ increases monotonically as $K \to \mathbb{R}$ for all $\beta > 0$.

We choose the discretisation of $\omega(k)$ and $\lambda(k)$ such that $W_k(t)$ approximates $W(t)$ monotonically from below for all $t \in \mathbb{R}$. Let $k_1, k_2$ be in the closed interval of length $2\pi/|\Lambda|$ with center at $k$ such that $\omega(k_1) \geq \omega(k')$ and $|\lambda(k_2)| \leq |\lambda(k')|$ for all $k'$ in the corresponding interval. Let $\omega_k = \omega(k_1)$ and $\lambda_k = \lambda(k_2)$ for all $k \in \mathbb{K}$. Then $\lambda_k e^{-\omega_k |t|} \leq \lambda(k') e^{-\omega(k') |t|}$ for all $t$. Since (21) is a Riemann sum approximating the integral (22), this choice amounts in approximating the integral monotonically from below as $K \to \mathbb{R}$ for all $t \in \mathbb{R}$. By Griffiths' second inequality, the same monotonicity property holds then for

$$\left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K)$$

for all $\beta > 0$. Therefore $\left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K)$ is monotone increasing in $K$ also in the limit $\beta \to \infty$ and $m(h)$ is well defined.

The limits $K \to \mathbb{R}$ and $\beta \to \infty$ commute since $\left\langle \frac{1}{J} \sum_{j=1}^{2J} \sigma_j(0) \right\rangle (\beta, K)$ increases monotonically with $\beta$ for all $K$ because each Ising line has free boundary conditions at $t = \pm \beta/2$.

Again by Griffiths' second inequality, $m(h)$ decreases with $h$. Therefore, $m^* = \lim_{h \to 0} m(h)$ is well defined. It is known that this $m^*$ agrees with the spontaneous magnetisation defined by taking the infinite volume limit with "+" boundary conditions ([3], [4]). 

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LOCALISATION FOR THE SPIN

We have shown that ground state expectations of the spin J-boson Hamiltonian agree with the expectations for the 2J coupled Ising lines. Because they are ferromagnetic, the "infinite volume" limit $\beta \to \infty$ and the limit $K \to \mathbb{R}$ exist. From now on we study the 2J coupled Ising lines and adapt our notation accordingly. $\langle \cdot \rangle (\sigma)$ denotes infinite volume expectations, where the brackets indicate the coupling parameter.

Finally we note that $A(\sigma)$ originates in the Euclidean action of a Hamiltonian. Integrating out the bosonic degrees of freedom in a system of 2J independent spins coupled linearly to a harmonic lattice yields the effective action $A(\sigma)$.

3. LOWER BOUNDS ON THE CRITICAL COUPLING

Let us differentiate the pair correlation for $h=0$ with respect to $\sigma$. Using the Lebowitz inequality we obtain for the infinite volume expectations

$$
\frac{d}{d\sigma} \langle \sigma_j(0) \sigma_k(t) \rangle (\sigma) = \frac{1}{2J^2} \int ds \int ds' W(s-s') \sum_{l,n=1}^{2J} [\langle \sigma_j(0) \sigma_k(t) \sigma_l(s) \sigma_n(s') \rangle (\sigma) - \langle \sigma_j(0) \sigma_k(t) \rangle (\sigma) \langle \sigma_l(s) \sigma_n(s') \rangle (\sigma)]
$$

$$
\leq \frac{1}{J^2} \int ds \int ds' W(s-s') \sum_{l,n=1}^{2J} \langle \sigma_j(0) \sigma_l(s) \rangle (\sigma) \langle \sigma_k(t) \sigma_n(s') \rangle (\sigma)
$$

for $1 \leq j, k \leq 2J$. $\langle \sigma_j(0) \sigma_k(t) \rangle (\sigma)$ is bounded by the solution of the differential equation corresponding to (29) with initial condition

$$
\langle \sigma_j(0) \sigma_k(t) \rangle (\sigma=0) = \int d\nu (\sigma_j) \sigma_j(0) \sigma_j(t) \delta_{jk} = \frac{1}{4} e^{-|t|/\lambda} \delta_{jk} \quad ([3],[8]).
$$

Thus we have

$$
\langle \sigma_j(0) \sigma_k(t) \rangle (\sigma) \leq \frac{1}{\sqrt{2\pi J}} \int d\omega e^{\text{int} \sum_{l=1}^{2J} e^{i\lambda l (j-k)/J} \frac{\hat{G}(\omega)}{1 - (4\pi/J) \delta_{\rho} x \hat{W}(\omega) \hat{G}(\omega)}},
$$

(30)

where $\hat{W}(\omega)$ and $\hat{G}(\omega)$ are the Fourier transforms of $W(t)$ and $\frac{1}{4} e^{-|t|/\lambda}$, respectively. (30) is valid as long as $1 > (4\pi/J) x \hat{W}(\omega) \hat{G}(\omega)$ for all $\omega$. Since
\( \hat{W}(\omega) \) and \( \hat{G}(\omega) \) take their maximum at \( \omega = 0 \), this means
\[
1 > \frac{4 \pi}{J} \hat{W}(0) \hat{G}(0) = \frac{\alpha}{\varepsilon}.
\] (31)

(Note that \( \hat{W}(0) = \int dt W(t) = 1 \) by (3).) As in [3] and [8] we thus arrive at the mean field bound

**PROPOSITION 3.** - If \( \alpha < \varepsilon \), then \( m^* = 0 \).

If the interaction decays faster than \( t^{-2} \) for \( t \to \infty \), we can use the energy-entropy argument of [3] and [7] to prove

**PROPOSITION 4.** - Let \( \int dt W(t) < \infty \). Then \( m^* = 0 \) for all \( \varepsilon > 0 \), \( \alpha \geq 0 \) and all \( J \).

### 4. UPPER BOUNDS ON THE CRITICAL COUPLING

We state the main result of our investigation.

**THEOREM 1.** - Let \( \lim_{t \to \infty} t^2 W(t) > 0 \). Then for any \( J \geq 1/2 \) there exists a \( \alpha_+(J) \) such that
\[
\varepsilon \leq \alpha_+(J) \leq \alpha_+(J) < \infty.
\] (32)

Furthermore, if \( \lim_{|t| \to \infty} t^2 W(t) = \infty \), then
\[
\lim_{J \to \infty} \alpha_+(J) = \varepsilon.
\] (33)

The bound \( \varepsilon \leq \alpha_+(J) \) is an obvious consequence of Proposition 3.

Our proof of \( \alpha_+(J) < \infty \) and (33) is divided into two steps. We first partition the system into blocks of length \( \delta \) and decouple the free measure (this yields a lower bound on \( m^* \)). The magnetisations per block form then a standard spin model over the one dimensional lattice. Applying Wells' inequality, its magnetisation is bounded below by the magnetisation of a \( \pm 1 \) Ising spin system—a well understood model [12]. To obtain useful bounds we have to control the a priori distribution of the magnetisation in a single block, in particular its behavior for large \( J \). This is carried through in step two. The crucial point there is that for sufficiently large coupling the single block has a mean field phase transition as \( J \to \infty \). Therefore the single site measure cannot concentrate at zero as \( J \to \infty \).

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STEP 1. — We change the time-scale in the action (27) by setting $t' = t/J$.

Let $\beta' = \beta/J$, then the free process, $\prod_{j=1}^{2J} dv^j(\sigma_j)$, refers to paths on the time interval $[-\beta'/2, \beta'/2]$ and the action is given by

$$A(\sigma) = -\frac{\alpha}{2} \int_{-\beta'/2}^{\beta'/2} dt \int_{-\beta'/2}^{\beta'/2} ds \mathcal{W}(J|t-s) \times \sum_{i,j=1}^{2J} \sigma_i(t) \sigma_j(s) - h \int_{-\beta'/2}^{\beta'/2} dt \sum_{j=1}^{2J} \sigma_j(t). \quad (34)$$

With the new scale the action is explicitly extensive, i.e. proportional to $\beta'J$. (34) has a mean field interaction in the "spatial" direction, $\{-J, \ldots, J\}$. In the time direction, $[-\beta'/2, \beta'/2]$, the interaction strength, $\mathcal{W}(J|t|)$, becomes strong and short-ranged as $J$ increases with total (integrated) strength independent of $J$.

We partition the interval $[-\beta'/2, \beta'/2]$ into intervals of length $\delta$, independent of $J$. For notational convenience we set $\beta' = N\delta$ with $N \in \mathbb{N}$. For $-N \leq l, n \leq N$ let

$$\mathcal{W}(l,s) = \mathcal{W}(n-l) = \min \left\{ \mathcal{W}(t-s) \left| \left( l - \frac{1}{2} \right) \delta \leq s \leq \left( l + \frac{1}{2} \right) \delta \right. \right\}. \quad (35)$$

Then $\mathcal{W}(t-s) \geq \mathcal{W}(l,s)$. As in Section 2 let $S(l) = \sum_{j=1}^{2J} \sigma_j(t)$ and define the magnetisation per volume in the block $l$ by

$$M_l = \frac{1}{\delta J} \int_{(l-1/2)\delta}^{(l+1/2)\delta} dt S(t). \quad (36)$$

Clearly, $|M_l| \leq 1$.

By $d\rho_j(M_l)$ we denote the distribution of $M_l$ under

$$\frac{1}{Z} \exp \left[ \frac{\alpha}{2} \int_{-\delta/2}^{\delta/2} dt \int_{-\delta/2}^{\delta/2} ds \mathcal{W}(J|t-s|) S(t) S(s) \right] d\mu^\delta(S). \quad (37)$$

Here $d\mu^\delta(S)$ is the measure on $\Gamma^\delta$ generated by $\exp(\epsilon \delta S^\tau)$ with free boundary conditions as defined in Section 2 and $Z$ is the normalisation constant. If obvious from the context we will suppress the $J$ dependence of $d\rho_j$. Let $\langle \cdot \rangle (\alpha)$ denote expectations with respect to the normalized
measure
\[
\frac{1}{Z} \exp \left[ \frac{\alpha}{2} \sum_{l \neq n}^N \tilde{W}(J|n-l|) M_l M_n + h \delta J \sum_{l=-N}^N M_l \prod_{l=-N}^N d\phi(M_l). \right]
\] (38)

Since compared to \( \langle . \rangle(\alpha) \) ferromagnetic interactions have been decreased, \( m^* \geq \lim_{h \to 0} \langle M_0 \rangle(\alpha) \).

To control the width of the single site measure in the limit \( J \to \infty \) we use the following property.

**Proposition 5.** — For each \( \alpha > \varepsilon \) there exists a \( v > 0 \), independent of \( J \), and a \( \delta_1 > 0 \) such that for all \( \delta > \delta_1 \)
\[
\int d\phi(M_0) M_0^2 \geq v^2.
\] (39)

This proposition will be proved in step two.

Let \( \langle . \rangle_1(\alpha') \) denote expectations with respect to the normalized Ising measure
\[
\frac{1}{Z} \exp \left[ \frac{\alpha'}{2} \sum_{l \neq n}^N \tilde{W}(J|n-l|) M_l M_n + h' \sum_{l=-N}^N M_l \prod_{l=-N}^N \frac{1}{2} (\delta_{l-1}(M_l) + \delta_1(M_l)). \right]
\] (40)

We apply Wells' inequality \([3, 9]\) to (38). By Proposition 5 there exists then a \( 0 < u \leq v \) independent of \( J \), such that
\[
\langle M_0 \rangle_{\phi}(\alpha) \geq \langle M_0 \rangle_1(\alpha J^2 \delta^2 u^2).
\] (41)

The phase diagram of the Ising model (40) for \( N \to \infty \), equivalent \( \beta' \to \infty \), with coupling \( \alpha' = \alpha J^2 \delta^2 u^2 \) is discussed in \([12]\). If \( \lim_{t \to \infty} t^2 W(t) > 0 \), then the Ising model orders provided \( \alpha' \), equivalently \( \alpha \), is large enough. This proves (32). Let us chose an arbitrary \( \alpha > \varepsilon \) and let \( \lim_{t \to \infty} t^2 W(t) = \infty \). Then the nearest neighbor coupling, \( J^2 W(Jt) \), diverges as \( J \to \infty \). Furthermore, for \( J \) sufficiently large,
\[
\lim_{n \to \infty} n^2 \alpha \delta^2 J^2 u^2 \tilde{W}(Jn) > 1.
\] (42)

Therefore, \( \varepsilon < \alpha_+ (J) < \alpha \) provided \( J \) is large enough. \( \square \)

**Step 2 (Proof of Proposition 5).** — We have to investigate the single block measures \( d\phi_i \) in the limit \( J \to \infty \). Substituting \( J W(Jt) \) by \( \delta(t) \) (which
gives a negligible error) we obtain the mean field problem
\[ \frac{1}{Z} \exp \left[ \frac{\alpha}{2J} \sum_{i \neq j}^{2J} \int_{-\delta/2}^{\delta/2} dt \, \sigma_i(t) \sigma_j(t) \right] \prod_{j=1}^{2J} d\nu^\delta(\sigma_j). \]  
(43)

In more familiar cases the single site space consists only of two points, say ±1. Here we must deal with the \textit{a priori} measure \( d\nu^\delta \). Fortunately such general mean field systems have been studied before. In [10] the single site space is a bounded volume in \( \mathbb{R}^d \) equipped with the Lebesgue measure. The proof in [10] has to be modified only slightly in order to apply to (43). Before doing so let us explain the main result of [10].

Let \( \rho \) be a bounded density relative to \( d\nu^\delta, 0 \leq \rho \leq a \), with normalisation
\[ \int d\nu^\delta(\sigma) \rho(\sigma) = 1. \]  
For such a "state" \( \rho \) we define the energy
\[ E(\rho) = \alpha \int d\nu^\delta(\sigma) \int d\nu^\delta(\sigma') \rho(\sigma) \rho(\sigma') \int_{-\delta/2}^{\delta/2} dt \, \sigma(t) \sigma'(t), \]  
(44)

the entropy
\[ S(\rho) = -\int d\nu^\delta(\sigma) \rho(\sigma) \ln \rho(\sigma), \]  
(45)

and the free energy
\[ F(\rho) = E(\rho) - S(\rho). \]  
(46)

\( F(\rho) \) is bounded from below. Let \( \mathcal{M}_f \) be the set of \( \rho \)'s minimizing \( F \).

For each \( \rho \) we can build the product measure
\[ d\nu_\rho = \prod_{j=1}^{\infty} \rho(\sigma_j) d\nu^\delta(\sigma_j). \]  
(47)

Now let us choose a subsequence \( J \to \infty \) such that \( \varphi_j \) converges weakly to \( \overline{\varphi} \). Since \( \overline{\varphi} \) must be permutation invariant, the theorem of Hewitt and Savage ensures that \( \overline{\varphi} \) can be decomposed into product measures as
\[ \overline{\varphi} = \int \psi(dp; \overline{\varphi}) \nu_\rho. \]  
(48)

The main result of [10] is that the decomposition measure, \( \psi(dp; \overline{\varphi}) \), is concentrated on \( \mathcal{M}_f \). In particular, along the chosen subsequence,
\[ \lim_{J \to \infty} \int d\varphi_j(M_0) M_0^2 = \int_{\mathcal{M}_f} \psi(dp; \overline{\varphi}) \int d\nu^\delta(\sigma) \rho(\sigma) \left[ \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} dt \, \sigma(t) \right]^2. \]  
(49)

Thus the proof of Proposition 5 is accomplished by studying the minima of the free energy functional (46).

Let us now introduce some notation. We write \( \Gamma^\delta_{1/2} \) for \( \Gamma^\delta \) if \( J = 1/2 \). Let \( \mathcal{S} \) be the set of all probability measures on \( (\Gamma^\delta_{1/2})^\mathbb{N} \) which are
invariant under permutations. This means, \( \mu \in \mathcal{S} \) if \( \mu(A_1 \times \ldots \times A_n) = \mu(A_{\pi_1} \times \ldots \times A_{\pi_n}) \) for any measurable sets \( A_1, \ldots, A_n \subseteq \Gamma^1/2 \), all \( n \in \mathbb{N} \) and all permutations \( \pi \) of \( \{1, \ldots, n\} \). Let \( \mathcal{S}_a \subseteq \mathcal{S} \) be the set of all permutation invariant measures \( d\mu \) on \( (\Gamma^1/2)^n \) such that there exist densities \( f_k(\sigma_1, \ldots, \sigma_k) \), bounded above by \( a^k \) for some \( a > 0 \), and satisfying

\[
d\mu_k(\sigma_1, \ldots, \sigma_k) = d\mu_k(\Gamma^1/2)^k = f_k(\sigma_1, \ldots, \sigma_k) \, d\nu^\delta(\sigma_1) \ldots d\nu^\delta(\sigma_k). \quad (50)
\]

**Lemma 2.** The sequence of measures \( d\varphi_j \), has weak limit points in \( \mathcal{S}_a \) as \( J \to \infty \). Each limit point, \( \varphi_\infty \), can be decomposed into extremal measures such that the decomposition measure is concentrated on \( \mathcal{M}_f \). If \( \psi(d\rho; \varphi) \)

denotes the decomposition measure, then \( \varphi = \int_{\mathcal{M}_f} \psi(d\rho; \varphi) \, \nu_\rho \).

**Proof.** We first prove that the sequence of measures \( \varphi_j \) has weak limit points in \( \mathcal{S} \). We cannot adopt the argument of [10] since \( \Gamma^1/2 \) is not compact. Instead we apply results of [11], chapter 4, in particular Proposition 4.7 and Example 1. We have to check that for all \( 1 \leq j \leq 2J \)

\[
\left| \sum_{i=1}^{2J} \left( \int_{-\delta/2}^{\delta/2} ds J W(J \, | \, t-s \, |) \sigma_i(t) \sigma_j(s) \right) \right| \leq \frac{1}{2} \int_{-\delta/2}^{\delta/2} dt \left( |\sigma_i(t)| + |\sigma_j(t)| \right). \quad (51)
\]

is bounded uniformly in \( J \). This is obvious since (51) is bounded by \( \delta/2 \) (in the terminology of [11] this means that the interaction is absolutely summable).

The Lipschitz continuity used in [10] is replaced by

\[
\left| \int_{-\delta/2}^{\delta/2} dt \int_{-\delta/2}^{\delta/2} ds J W(J \, | \, t-s \, |) \sigma_i(t) \sigma_j(s) \right| \leq \frac{1}{2} \int_{-\delta/2}^{\delta/2} dt \left( |\sigma_i(t)| + |\sigma_j(t)| \right). \quad (52)
\]

Here we have used that \( xy - x'y' = \frac{1}{2} (x + x')(y - y') + \frac{1}{2} (x - x')(y + y') \).

For \( k \leq 2J \) we set

\[
f_k^2(\sigma_1, \ldots, \sigma_k) = \frac{1}{Z} \int d\nu^\delta(\sigma_{k+1}) \ldots \int d\nu^\delta(\sigma_{2J}) e^{-\Lambda(\sigma)} , \quad (53)
\]

where \( Z \) is the normalisation constant. Let \( Z_0 = \int d\nu^\delta(\sigma) \). Then we have

\[
0 \leq f_k^2(\sigma_1, \ldots, \sigma_k) \leq \left( \frac{e}{Z_0} \right)^k . \quad (54)
\]
This replaces the corresponding estimate (2.6) in [10]. Furthermore there exist constants $C$, $a > 0$, independent of $J$ and $k$, such that for all $k \leq 2J$

$$|f^J_k(\sigma_1, \ldots, \sigma_k) - f^J_k(\sigma'_1, \ldots, \sigma'_k)| \leq C a^k \sum_{j=1}^{\delta/2} dt |\sigma_j(t) - \sigma'_j(t)| \leq \frac{1}{2} C a^k k\delta. \quad (55)$$

This replaces Lemma 2 in [10].

Along the given subsequence, $f^J_k$ converges weakly to a limit $f_k$ which is the marginal of $\bar{\varphi}$ on the sites $\{1, \ldots, k\}$. The main technical tool in [10] is to make sure that also the entropy of $f^J_k$ converges to the entropy of $f_k$. For this weak convergence is not enough. In [10] the uniform Lipschitz continuity of the densities $f^J_k$ was used. This is substituted here by (55). By the theorem of Arzela-Ascoli it implies the existence of pointwise convergent subsequences of $f^J_k$ as $J \to \infty$ on compact sets. Since by weak convergence the limit is unique, $f^J_k \to f_k$ almost surely. Because of (54) this implies the convergence of entropies. The energy of the "state" $\rho$ is given by (44) since $JW(Jt) \to \delta(t)$ as $J \to \infty$. The remainder of the proof is identical to [10].

Let us write $\langle \cdot \rangle_{\rho}$ for expectations with respect to the measure $\rho(\sigma) d\nu(\sigma)$ and let $m(t) = \langle \sigma(t) \rangle_{\rho}$. $\rho$ is a stationary point of the free energy functional $F(\rho)$ iff

$$\rho(\sigma) = \frac{\exp\left[2\alpha \int_{-\delta/2}^{\delta/2} dt \sigma(t) m(t) \right]}{\int d\nu(\sigma) \exp\left[2\alpha \int_{-\delta/2}^{\delta/2} dt \sigma(t) m(t) \right]}. \quad (56)$$

Clearly, the weak coupling solution is $\rho_0 = Z_0^{-1} = \left[\int d\nu(\sigma)\right]^{-1}$ with $m(t) = 0$ for all $t$. To prove Proposition 5 we have to show that there are absolute minima of $F(\rho)$ with $m(t) \neq 0$.

**Lemma 3.** For $\alpha < \epsilon$ there exists a $\delta_0 > 0$ such that for all $\delta > \delta_0$ $\rho_0$ is the unique minimum of $F(\rho)$.

For $\alpha > \epsilon$ there exists a $\delta_1 > 0$ such that for all $\delta > \delta_1$ $\rho_0$ is an unstable stationary point of $F(\rho)$.

**Proof.** By inserting (56) into $F(\rho)$ we obtain the functional

$$\tilde{F}(m(\cdot)) = \alpha \int_{-\delta/2}^{\delta/2} dt m(t)^2 - \ln \int d\nu(\sigma) \exp\left[2\alpha \int_{-\delta/2}^{\delta/2} dt \sigma(t) m(t) \right]. \quad (57)$$

Since the stationary points of $F(\rho)$ and $\tilde{F}(m(\cdot))$ with $m(t) = \langle \sigma(t) \rangle_{\rho}$ agree, we only have to investigate the absolute minima of $\tilde{F}(m(\cdot))$. 

The quadratic variation of $\bar{F}$ with respect to $m(.)$ at $m(.) = 0$ is given by

$$\frac{\delta^2 \bar{F}}{\delta m(t) \delta m(s)} \bigg|_{m(.) = 0} = 2\alpha [\delta(t-s) - 2\alpha \langle \sigma(t) \sigma(s) \rangle_{\rho_0}].$$  \hspace{1cm} (58)

For $J = 1/2$ we have $\Omega_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and thus

$$\langle \sigma(t) \sigma(s) \rangle_{\rho_0} = \langle \Omega_0 | \sigma^2 e^{-\varepsilon|t-s|} | \sigma^2 \rangle_{\Omega_0} = \frac{1}{4} e^{-\varepsilon|t-s|}.$$

The Fourier coefficients of $\delta(t) - \frac{\alpha}{2} e^{-\varepsilon|t|}$, $-\delta/2 \leq t \leq \delta/2$, are given by

$$\omega_n^2 + \varepsilon (\varepsilon - \alpha) + (-1)^{n+1} \frac{\varepsilon \alpha}{\omega_n^2 + \varepsilon^2} e^{-\delta \alpha/2}$$  \hspace{1cm} (59)

with $\omega_n = \pi n / \delta$, $n \in \mathbb{Z}$. (59) is positive for all $\omega_n$ if $\alpha < \varepsilon$ and $\delta$ is large enough. This implies stability.

Uniqueness follows by a contraction argument analogous to the one given in [10] in the proof of Theorem 3. We remark that nonuniqueness also contradicts Proposition 3 because the argument in step 1 would yield $m^* > 0$ for $\alpha < \varepsilon$.

If $\alpha > \varepsilon$, then (59) is negative for $|\omega_n|$ small enough and $\delta$ sufficiently large. From this we conclude that the quadratic variation of $\bar{F}$ at $m(.) = 0$ is not positive definite. \qed

**Proof of Proposition 5.** - Let $\alpha > \varepsilon$ and let $\bar{\phi}$ be any weak limit point of $\varphi_J$ as $J \to \infty$. Then, along the given subsequence, $\lim J \to \infty \varphi_J(M^2_0) = 0$ by (49). Suppose that $\bar{\phi}(M^2_0) = 0$. Then $\nu_p(M^2_0) = 0$ and hence $\nu_p(M_0) = 0$ for almost all $p \in \mathcal{M}_f$ which contradicts Lemma 3. Hence $\varphi_J(M^2_0)$ has to be bounded away from zero uniformly in $J$. \qed

The proof of Proposition 5 has the

**Corollary.** - For $\alpha > \varepsilon$ we have

$$\lim_{h \to 0} \lim_{J \to \infty} m(h) > 0,$$  \hspace{1cm} (60)

independent of the choice of $W(t)$. 

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APPENDIX

As an example we explain how to calculate ground state expectations of $\frac{1}{J} S^z$ in the semiclassical limit $J \to \infty$. We introduce the cutoff Hamiltonian corresponding to the semiclassical Hamiltonian (9),

$$H_{xc}^c = -e \cos \varphi \sin \theta + \sum_{k \in K} \omega_k a_k^* a_k + \sqrt{\alpha} \cos \theta \sum_{k \in K} \lambda_k (a_k^* + a_k) - h \cos \theta,$$

where we suppress the $K$ dependence of $H_{xc}^c$ in our notation. This Hamiltonian is defined on $\mathcal{H}_{K,xc} = S^2 \otimes \mathcal{F}_K$, where $S^2$ is the two sphere. Diagonalizing $H_{xc}^c$ one finds that its ground state energy for fixed $\theta$ and $\varphi$ is given by

$$g^- (\theta, \varphi) = e \cos \varphi \sin \theta - \frac{\alpha}{2} \cos^2 \theta - h \cos \theta.$$ (62)

By $H_{xc}^c$ we denote the Hamiltonian (61) with all terms except $\sum_{k \in K} \omega_k a_k^* a_k$ multiplied by $(J + 1)/J$. Its ground state energy for fixed $\theta$ and $\varphi$ is given by

$$g^+ (\theta, \varphi) = e \frac{J + 1}{J} \cos \varphi \sin \theta - \frac{\alpha}{2} \left( \frac{J + 1}{J} \right)^2 \cos^2 \theta - h \frac{J + 1}{J} \cos \theta.$$ (63)

Thus the ground state energies of $H_{xc}^c$ are determined by

$$e_j^\pm (h) = \min_{\theta, \varphi} g^\pm (\theta, \varphi).$$ (64)

Taking the limit $\beta \to \infty$ in equation (5.4) of [2] yields then the bounds

$$\frac{e^- (h + \eta) - e_j^+ (h)}{\eta} \leq \left( \frac{1}{J} S^z \right) \leq \frac{e^- (h) - e_j^+ (h - \eta)}{\eta}$$ (65)

for all $\eta \geq 0$. In (65) one can take the limit $K \to \mathbb{R}$. We have

$$\lim_{J \to \infty} g^+ (\theta, \varphi) = g^- (\theta, \varphi) \equiv g (\theta, \varphi).$$

Thus

$$e (h) : = \lim_{J \to \infty} e_j^+ (\theta, \varphi) = \min_{\theta, \varphi} g (\theta, \varphi).$$ (66)

Taking the limit $J \to \infty$ and then $\eta \searrow 0$ for the lower and $\eta \nearrow 0$ for the upper bound in (65) yields

$$\frac{d}{dh'} e (h') \bigg|_{h' = h^+} \leq \lim_{J \to \infty} \left( \frac{1}{J} S^z \right) \leq \frac{d}{dh'} e (h') \bigg|_{h' = h^+}.$$ (67)
If $h \neq 0$, then $g(\theta, \phi)$ has a unique minimum $(\theta_0(h), \phi_0(h))$ and (67) yields

$$
\lim_{j \to \infty} \langle \Psi_{K,h} | \frac{1}{j} S^z | \Psi_{K,h} \rangle = \cos \theta_0(h).
$$

(68)

Let $h = 0$. If $\alpha < \varepsilon$, then $g(\theta, \phi)$ has the unique minimum $(\theta_0, \phi_0) = \left( \frac{\pi}{2}, \frac{\pi}{2} \right)$ and

$$
\lim_{h \to 0} \lim_{j \to \infty} \langle \Psi_{K,h} | \frac{1}{j} S^z | \Psi_{K,h} \rangle = \cos \theta_0 = 0.
$$

(70)

If $\alpha > \varepsilon$, $g(\theta, \phi)$ has the two minima $(\theta_0^-, \phi_0^-) = \left( \pi - \arcsin \frac{\alpha}{\varepsilon}, 0 \right)$ and

$(\theta_0^+, \phi_0^+) = \left( \arcsin \frac{\alpha}{\varepsilon}, 0 \right)$. Then

$$
\lim_{h \to 0} \lim_{j \to \infty} \langle \Psi_{K,h} | \frac{1}{j} S^z | \Psi_{K,h} \rangle = \cos \theta_0^- = -\sqrt{1 - \left( \frac{\varepsilon}{\alpha} \right)^2},
$$

(71)

and

$$
\lim_{h \to 0} \lim_{j \to \infty} \langle \Psi_{K,h} | \frac{1}{j} S^z | \Psi_{K,h} \rangle = \cos \theta_0^+ = \sqrt{1 - \left( \frac{\varepsilon}{\alpha} \right)^2}.
$$

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