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Asymptotic and analytic properties of resonance functions

by

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ABSTRACT. – We consider the Schrödinger operator $-\Delta+V$ in $L^2(\mathbb{R}^n)$, $n \geqq 3$, where V is a dilation-analytic, short-range potential in the angle S_α. We prove that a resonance function $f(r, .)$ corresponding to the resonance k_0^2 is the restriction to $\mathbb{R}^+$ of an analytic, $L^2(S^{n-1})$-valued function $f(z, .)$ on S_α and establish asymptotics of $f(z, .)$ and $f'(z, .)$ for $z \to \infty$ in S_α.

RÉSUMÉ. – On considère l'opérateur de Schrödinger $-\Delta+V$ dans $L^2(\mathbb{R}^n)$, $n \geqq 3$, où V est un potentiel à courte portée, analytique par dilatation dans l'angle S_α. On montre qu'une fonction de résonnance $f(r, .)$ correspondant à la résonnance k_0^2 est la restriction à $\mathbb{R}^+$ d'une fonction $f(z, .)$ à valeur dans $L^2(S^{n-1})$, analytique dans S_α, et on établit le comportement asymptotique de $f(z, .)$ et $f'(z, .)$ lorsque $z \to \infty$ dans S_α.

1. INTRODUCTION

We consider the Schrödinger operator $-\Delta+V$ in $L^2(\mathbb{R}^n)$, $n \geqq 3$, where V is a short-range, dilation-analytic potential in an angle S_α. A resonance k_0^2

appears as a discrete eigenvalue of the complex-dilated Hamiltonian, a pole of the S-matrix and a pole of the analytically continued resolvent, acting from an exponentially weighted space to its dual. The latter property provides a natural definition of resonance functions associated with k_0^2 as solutions of $(-\Delta+V-k_0^2)f=0$ lying in the range of the residue at k_0^2 of the continued resolvent. This definition was introduced in [5] for the more general potential V + W, where W is exponentially decreasing. In the purely dilation-analytic case considered here, there is another natural candidate for resonance functions, namely the square-integrable eigenfunctions associated with k_0^2 and the complex-dilated Hamiltonian [2].

We prove as our main result (Theorem 2.1) that the resonance functions of [5] and [2] are simply the restrictions of analytic $L^2(S^{n-1})$-valued functions $f(z, .)$ on S_α to $\mathbb{R}^+$ and to rays $e^{i\varphi}\mathbb{R}^+$ with $\varphi > -\operatorname{Arg} k_0$, respectively.

Moreover, we establish the precise asymptotic behavior of any such function f and its derivative f':

$$\begin{aligned} f(z, .) &= e^{ik_0 z} z^{(1-n)/2} g(z), \\ g(z) &= \tau + O(|z|^{-\varepsilon}), \\ g'(z) &= O(|z|^{-1-\varepsilon}) \end{aligned} \tag{1.1}$$

for $z \to \infty$ in S_α, where $\tau \in L^2(S^{n-1})$.

Basic to our approach are various results from [5] including formulas of the stationary scattering theory, applicable to non-symmetric potentials (in our case complex-dilated potentials), and an extended limiting absorption principle. These ideas, summarized in Section 3, go into the proof of the analyticity property and approximate asymptotics of f, formulated in Lemma 2.3 and proved in Section 3. In order to establish the asymptotic behavior (1.1), we use this result combined with a recent *a priori* estimate by Agmon [1], quoted in Lemma 2.6, and a convexity result proved as Lemma 2.7.

The result (1.1) and the proof of Lemma 2.3 provide new insight into the dilation-analytic method in the two-body, short-range case. However, we would like to emphasize that we consider it as very likely that the square-integrable eigenfunctions of any two-body complex-dilated Hamiltonian (*i.e.*, including Hamiltonians with dilation-analytic long-range potentials) should have the above analyticity property. We just do not know how to prove this. The asymptotics of (1.1) should be modified accordingly, *i.e.*, $e^{ik_0 z}$ should be replaced by $e^{i\lambda(k_0, z)}$ where $\lambda(k_0, r)$ is a solution (or approximate solution) of the eikonal equation. If V is radial and analytic, the above conjectures can be proved by Jost-function-techniques.

On the other hand, the method of this paper is generalizable to exterior-dilation-analytic, short-range potentials, treated for example in [6] by a more general complex-distortion method. However, this seems to involve a number of technically somewhat complicated "error estimates". Consequently, we restrict ourselves to the dilation-analytic case which allows a relatively simple treatment while illustrating clearly the basic ideas.

Apart from the fact that it makes the dilation-analytic theory more transparent, we are also interested in (1.1) because it implies exponential decay in time of resonance states defined as suitably cut-off resonance functions, as proved in [8].

We conclude with a discussion of the generalized eigenfunctions, establishing their analyticity and certain asymptotic properties (Theorem 4.1).

2. THE MAIN RESULT

We introduce the weighted L^2-spaces $L^2_{\delta, b} = L^2_{\delta, b}(\mathbb{R}^n)$ for $\delta, b \in \mathbb{R}$ by

$$L^2_{\delta, b} = \left\{ f \,\middle|\, \|f\|^2_{\delta, b} = \int_{\mathbb{R}^n} |f(x)|^2 (1+r^2)^{\delta} e^{2br}\, dx < \infty \right\},$$

where $x \in \mathbb{R}^n$, $r = |x|$. We shall always assume that the dimension $n \geqq 3$. The weighted Sobolev spaces $H^2_{\delta, b} = H^2_{\delta, b}(\mathbb{R}^n)$ are defined by

$$H^2_{\delta, b} = \{ f \,|\, \|f\|^2_{2, \delta, b} = \sum_{|\alpha| \leqq 2} \|D^{\alpha} f\|^2_{\delta, b} < \infty \},$$

where $D^{\alpha} = \prod_j \partial_j^{\alpha_j}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum_j \alpha_j$. We set $L^2_{\delta} = L^2_{\delta, 0}$, $H^2_{\delta} = H^2_{\delta, 0}$, and

$$h = L^2(S^{n-1}), \qquad S^{n-1} = \{ x \in \mathbb{R}^n \,|\, |x| = 1 \}.$$

We identify $L^2_{\text{loc}}(\mathbb{R}^n)$ with $L^2_{\text{loc}}(\mathbb{R}^+, h; r^{(n-1)/2}\, dr)$, setting $f(x) = f(r, \omega)$, $\omega = \dfrac{x}{|x|}$.

We introduce the following subsets of the complex plane $\mathbb{C}$,

$$\mathbb{C}^{+(-)} = \{ k \in \mathbb{C} \,|\, \operatorname{Im} k \underset{(<)}{>} 0 \}, \ \tilde{\mathbb{C}}^+ = \overline{\mathbb{C}^+} \setminus \{0\}.$$

$\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $\mathscr{C}(\mathscr{H}_1, \mathscr{H}_2)$ denote the spaces of bounded and compact operators from $\mathscr{H}_1$ into $\mathscr{H}_2$, respectively, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces.

For any closed, linear operator A in a Hilbert space $\mathscr{H}$, we denote by $\mathscr{D}(A)$, $\mathscr{R}(A)$, and $\mathscr{N}(A)$ the domain, the range, and the null space of A, respectively.

The free Hamiltonian H_0 in L^2 is defined for $u \in \mathscr{D}(H_0) = H^2$ by $H_0 u = -\Delta u$ with resolvent $R_0(k) = (H_0 - k^2)^{-1} \in \mathscr{B}(L^2)$ for $k \in \mathbb{C}^+$.

Let $\{U(\rho)\}_{\rho \in \mathbb{R}^+}$ be the dilation group on L^2 defined by

$$(U(\rho) f)(x) = \rho^{n/2} f(\rho x).$$

For $\alpha' > 0$, $S_{\alpha'} = \{\rho e^{i\varphi} \mid \rho > 0, |\varphi| < \alpha'\}$.

Condition on the Potential V. – We assume that V is a symmetric, H_0-compact operator in L^2, such that for some $\alpha \in (0, \pi/2)$, $\delta_1, \delta_2 > 1/2$, $0 < b_1 \leqq \infty$, the following holds.

(A.1) $V(\rho) = U(\rho) V U^{-1}(\rho)$ has a $\mathscr{C}(H^2_{-\delta_1}, L^2_{\delta_2})$-valued analytic extension $V(z)$ to S_α.

(A.2) On the domain $S_\alpha \times \{k \mid |k| < b_1\}$,

$$V(z, k) = e^{-izkr} z^2 V(z) e^{izkr}$$

defines a $\mathscr{C}(H^2_{-\delta_1}, L^2_{\delta_2})$-valued, jointly analytic function of z and k.

Remark. – The joint analyticity in (A.2) follows from local boundedness of $\|V(z, k)\|$ in $\mathscr{B}(H^2_{-\delta_1}, L^2_{\delta_2})$. If V is multiplicative, so is $V(z)$ and (A.2) with $b_1 = \infty$ follows from (A.1).

The Hamiltonian $H = H_0 + V$ is self-adjoint on $\mathscr{D}(H) = H^2$, and associated with H is a self-adjoint, analytic family of type A, $H(z)$, given by

$$H(z) = z^{-2} H_0 + V(z), \qquad z \in S_\alpha.$$

Under the assumptions (A.1) and (A.2), it is known from [5] that for any $b_2 \in (0, b_1)$, the resolvent $R(k) = (H - k^2)^{-1}$ continues meromorphically as a $\mathscr{B}(L^2_{0, b_2}, H^2_{0, -b_2})$-valued function from $\mathbb{C}^+$ across $\mathbb{R}^+$ to

$$S_\alpha^{b_2} := \{k \in S_\alpha \mid -b_2 < \operatorname{Im} k < 0\}$$

This continuation will be denoted $\tilde{R}(k)$. [For an explicit construction, *see* (3.4) in Section 3.] The set of poles of $\tilde{R}(k)$ in $S_\alpha^{b_2}$ will be denoted by $\mathscr{R}^{b_2}$. We also introduce the set of dilation-analytic resonances $\mathscr{R}$ defined by

$$\mathscr{R} = \bigcup_{0 < \varphi < \alpha} \mathscr{R}(z)$$

where for $0 < \operatorname{Arg} z = \varphi < \alpha$,

$$\mathscr{R}(z) = \{k \mid 0 > \operatorname{Arg} k > -\varphi, k^2 \in \sigma(H(z))\}.$$

By considering suitable expectation values of the resolvent, one can easily prove that

$$\mathscr{R}^{b_2} = \mathscr{R} \cap \{k \mid \operatorname{Im} k > -b_2\}.$$

Another characterization of resonances is provided by the following theorem proved in [5] (in fact, for more general potentials).

THEOREM 2.0. – *The S-matrix* $S(k)$ *for the pair* (H_0, H) *extends meromorphically to* S_α. *The extension* $\tilde{S}(k)$ *has no poles in* $S_\alpha \cap \mathbb{C}^+$ [i.e., $\tilde{S}$

$^{-1}(k)$ *is analytic in* $S_\alpha \cap \mathbb{C}^-$]. *The sets of poles of* $\tilde{S}(k)$ *and* $\tilde{R}(k)$ *in* $S_\alpha^{b_2}$ *coincide* ($=\mathscr{R}^{b_2}$) *and are of the same order.*

We recall the notion of resonance functions as introduced in the more general setting of [5]. Let $k_0 \in \mathscr{R}^{b_2}$ be given. The space $\mathscr{F}$ of resonance functions at k_0 is defined by

$$\mathscr{F} = \mathscr{F}(k_0) = \{ f \in \mathscr{R}(P) \mid (H^{-b_2} - k_0^2) f = 0 \},$$

where P denotes the (finite-rank) residue of $\tilde{R}(k)$ at k_0, and H^{-b_2} is the extension of H to $L^2_{0,\,-b_2}$ with $\mathscr{D}(H^{-b_2}) = H^2_{0,\,-b_2}$. We suppress b_2 in the notation for the simple reason that $\mathscr{F}$ and $\mathscr{R}(P)$ are obviously independent of $b_2 > -\operatorname{Im} k_0$. It is known that $\mathscr{F} = \mathscr{R}(P)$ if and only if $\tilde{R}(k)$ has a simple pole at k_0.

Our main result is a follows.

THEOREM 2.1. – *Let* $k_0 \in \mathscr{R} \cap \{ k \mid |k| < b_1 \}$ *be given. Then*

(*a*) *any* $f \in \mathscr{F}(k_0)$ *has an analytic extension to* S_α *as an h-valued function with the following properties. There exist* $\tau \in h$ *and for any* $\alpha' \in (0, \alpha)$ *an* $\varepsilon > 0$ *such that the asymptotic formulas*

$$f(z, .) = e^{ik_0 z} z^{(1-n)/2} g(z)$$

with

$$g(z) = \tau + O(|z|^{-\varepsilon})$$
$$\frac{d}{dz} g(z) = O(|z|^{-1-\varepsilon})$$

hold for $z \to \infty$, *uniformly in* $S_{\alpha'}$.

(*b*) *Any* $\tau \in h$ *obtained in accordance with* (*a*) *is in* $\mathscr{N}(\tilde{S}^{-1}(k_0))$, *and the correspondance* $\mathscr{F} \ni f \to \tau \in \mathscr{N}(\tilde{S}^{-1}(k_0))$ *is an isomorphism.*

(*c*) *The space* $\mathscr{F}_z$ *obtained from* $\mathscr{F}$ *by z-rotating* $f \in \mathscr{F}$ *in accordance with* (*a*), i. e.,

$$f(x) = f(r, .) \to f(rz, \omega) = f_z(x),$$

coincides with $\mathscr{N}(H(z) - k_0^2)$, *provided* $\operatorname{Arg} z k_0 > 0$.

Remark 2.2. – (*a*) We expect that ε in (*a*) can be chosen independent of α', and, in fact, that the statement holds with $\varepsilon = \min\left(\delta_2 - \frac{1}{2}, 1\right)$, where δ_2 is given in (A.2), but we do not know how to prove this. Our proof, however, gives more information than stated above. In particular, we can show that the asymptotics of (*a*) hold for $z = r \to +\infty$ (*i. e.*, for the resonance function) with any $\varepsilon < \min\left(\delta_2 - \frac{1}{2}, \frac{1}{2}\right) \alpha/(\alpha - \operatorname{Arg} k_0)$.

(*b*) If the order m of the resonance pole k_0 is larger than one, it is also possible to obtain information on $\mathscr{R}(P) \setminus \mathscr{F}(k_0)$. For instance, one can prove that there exists $f \in \mathscr{R}(\mathrm{P})$, such that $f(r, .)$ extends to S_α as an analytic, h-valued function and

$$f(z, .) = z^{m-n/2-1/2} e^{ik_0 z} (\tau + O(|z|^{-\varepsilon})), \qquad \tau \neq 0.$$

We shall not elaborate this here.

We proceed to the proof of Theorem 2.1 using the following closely related result, which will be proved in Section 3.

An analytic, $\mathrm{H}^2_\delta(\mathrm{L}^2_\delta, \mathrm{L}^1)$-valued function $\chi(z)$ on S_α is said to be dilation-analytic if

$$\mathrm{U}(\rho)\chi(z) = \chi(\rho z), \qquad \rho > 0, \quad z \in \mathrm{S}_\alpha.$$

LEMMA 2.3. – *Let* $k_0 \in \mathscr{R} \cap \{k \mid |k| < b_1\}$, $\delta > \frac{1}{2}$ *and* $f \in \mathscr{F}$ *be given. Then there exists a dilation-analytic* $\mathrm{H}^2_{-\delta}$*-valued function* $\chi(z)$ *on* S_α *such that*

$$f(r, .) = e^{ik_0 r} \chi(1)(r, .), \qquad r > 0,$$

and for any $z \in \mathrm{S}_\alpha$ *with* $\operatorname{Arg} z k_0 > 0$,

$$\psi(z)(r, .) = e^{ik_0 rz} \chi(z)(r, .) \in \mathscr{N}(\mathrm{H}(z) - k_0^2).$$

The map $\mathscr{F} \ni f \to \psi(z) \in \mathscr{N}(\mathrm{H}(z) - k_0^2)$ *is onto.*

Under the assumption of Lemma 2.3, we obtain by the statement of the lemma that $f(r, .)$ has an analytic extension $f(z, .)$ to S_α given by the relation

$$e^{ik_0 rz} \chi(z)(r, .) = z^{n/2} f(rz, .).$$

Notice that functions in $\mathrm{H}^2_{-\delta}$ have restrictions to spheres.

We define an h-valued function $g(z)$ by

$$g(z) = z^{(n-1)/2} e^{-ik_0 z} f(z, .), \qquad z \in \mathrm{S}_\alpha.$$

We want to prove

LEMMA 2.4. – *There exists* $\tau \in h$ *such that for any* $\alpha' \in (0, \alpha)$ *and some* $\varepsilon = \varepsilon(\alpha') > 0$, *the asymptotics*

$$g(z) = \tau + O(|z|^{-\varepsilon})$$
$$\frac{d}{dz} g(z) = O(|z|^{-1-\varepsilon})$$

hold for $z \to \infty$ *uniformly in* $\mathrm{S}_{\alpha'}$.

Given Lemmas 2.3 and 2.4, Theorem 2.1 (*a*) and (*c*) follow. The statement (*b*) follows from (*a*) and [5], Theorems 2.4 and 2.5.

In order to prove Lemma 2.4, we will need three lemmas. The first one follows from a standard application of Cauchy's integral formula.

LEMMA 2.5. — *Let* $g_1(z)$ *be an analytic h-valued function on* S_α. *Suppose there exists* $\delta \in \mathbb{R}$ *such that for any* $\alpha' \in (0, \alpha)$

$$\sup_{z \in S_{\alpha'}} |z| \int_0^\infty \| g_1(rz) \|_h^2 \, |1+zr|^{-2\delta} \, dr < \infty.$$

Then for any $\alpha' \in (0, \alpha)$, $m = 0, 1, 2, \ldots$, *there exists* $K > 0$ *such that*

$$\left\| \frac{d^m}{dz^m} g_1(z) \right\| \leqq K |z|^{-1/2-m+\delta}$$

for all $z \in S_{\alpha'} \cap \{ z' \mid |z'| \geqq 1 \}$.

Proof (sketch). — Let $z \in S_\alpha$ be given and set for $r > 0$,

$$S_r(z) = \{ z' \in \mathbb{C} \mid |z - z'| = r \}.$$

Then for ε small enough and $\varepsilon |z| < r < 2 \varepsilon |z|$,

$$\left\| \frac{d^m}{dz^m} g_1(z) \right\| \leqq K_m \int_{S_r(z)} \| g_1(z') \| \, |z'-z|^{-m-1} \, |dz'|.$$

By integration

$$\left\| \frac{d^m}{dz^m} g_1(z) \right\| \leqq \frac{K_m}{(\varepsilon |z|)^{m+2}} \int_{\varepsilon |z|}^{2 \varepsilon |z|} dr \int_{S_r(z)} \| g_1(z') \| \, |dz'|.$$

We conclude by applying the Cauchy-Schwarz inequality. ■

LEMMA 2.6. — *Let* $k \in \mathbb{C}^+$, $b' < \operatorname{Im} k$ *and* $\delta' > \frac{1}{2}$ *be given. Suppose* $u \in H^2_{0,-b'}$ *satisfies*

$$(-\Delta - k^2) u \in L^2_{\delta', \operatorname{Im} k}.$$

Then, with $\delta = \min(\delta', 1)$,

$$\int_0^\infty \left\| \frac{d}{dr} \left\{ e^{-ikr} r^{(n-1)/2} u(r, .) \right\} \right\|_h^2 r^{2\delta} \, dr < \infty.$$

This lemma follows from the *a priori* estimate given by Agmon [1], Theorem 2.2. Incidentally, this estimate is stronger in some sense (for k bounded away from the reals) than the extended limiting absorption principle stated in Lemma 3.2 of Section 3. However, it is singular as k approaches $\mathbb{R} \setminus \{0\}$.

LEMMA 2.7. — *Suppose for some* $t \in \mathbb{R}$, $\eta(z)$ *is a dilation-analytic* L^2_{-t}*-valued function*, $|\operatorname{Arg} z| < \alpha$. *Define for any* $\theta \in]-\alpha, \alpha[$,

$$f(\theta) = \inf \{ s \mid \eta(e^{i\theta}) \in L^2_{-s} \}.$$

Then one of the following alternatives holds:

(*a*) $f(\theta)=-\infty$ *for all* $\theta\in]-\alpha,\alpha[$;

(*b*) $f(\theta)\in]-\infty, t]$ *for all* $\theta\in]-\alpha,\alpha[$ *and* f *is convex.*

Proof. – Suppose $f(\theta)>-\infty$ for all $\theta\in]-\alpha,\alpha[$. Then the assumed upper boundedness, together with a result of Jensen [7], implies convexity provided we can verify the inequality

$$\left.\begin{array}{c} f(\theta-\kappa)+f(\theta+\kappa)\geqq 2f(\theta) \\ \text{for all } \theta, \kappa \text{ such that } \theta-\kappa, \theta+\kappa\in]-\alpha,\alpha[. \end{array}\right\} \quad (2.1)$$

Suppose (2.1) is false. Then we can find θ, κ, s_1, $s_2\in\mathbb{R}$ such that

$$\theta-\kappa, \theta+\kappa\in]-\alpha,\alpha[, \qquad s_1+s_2<2f(\theta),$$

$\eta(e^{i(\theta-\kappa)})\in L^2_{-s_1}$, and $\eta(e^{i(\theta+\kappa)})\in L^2_{-s_2}$.

We introduce the dilation-analytic L^1-valued function

$$g_\varepsilon(z)=\eta(z)\overline{\eta(\bar{z}e^{i2e})}(1+zr\varepsilon)^{s_1+s_2-2t}(1+zr)^{-s_1-s_2}$$

defined on $\{z\in S_\alpha \mid |\operatorname{Arg} z-\theta|<\kappa\}$, $\varepsilon>0$.

By Hadamard's three line theorem and our assumptions, the L^1-norm of $g_\varepsilon(e^{i\theta})$ is uniformly bounded as $\varepsilon\downarrow 0$. This implies that $f(\theta)\leqq(s_1+s_2)/2$, a contradiction.

If $f(\theta)=-\infty$ for some $\theta\in]-\alpha,\alpha[$, we can use the above idea (interpolation) to establish that $f(\theta)\equiv-\infty$. ■

Proof of Lemma 2.4. – Since $g(rz)=z^{-1/2}r^{(n-1)/2}\chi(z)(r,.)$, Lemma 2.3 implies that the condition of Lemma 2.5 holds for $g_1(z)=g(z)$ and any $\delta>\frac{1}{2}$. We define for $r>0$ and $z\in S_\alpha$

$$\chi_1(z)(r,.)=z^{1/2}r^{(1-n)/2}g'(rz,.), \qquad g'=\frac{d}{dz}g.$$

By the statement of Lemma 2.5 (for $m=1$), we obtain that $\chi_1(z)$ is a dilation-analytic, $L^2_{1/2+\varepsilon}$-valued function on S_α for any $\varepsilon>0$.

For $z\in S_\alpha$ with $\operatorname{Arg} zk_0>0$, we have by (A.2) and Lemma 2.3 that $\psi(z)(r,.)=e^{ik_0rz}\chi(z)(r,.)$ satisfies

$$(-\Delta-(zk_0)^2)\psi(z)=-z^2V(z)\psi(z)\in L^2_{\delta_2, \operatorname{Im} zk_0}.$$

Hence, by Lemma 2.6, with $\delta=\min(\delta_2, 1)$,

$$\chi_1(z)\in L^2_\delta \text{ for } \operatorname{Arg} zk_0>0.$$

With the notation of Lemma 2.7, we have proved for $\eta(z)=\chi_1(z)$ that $f(\theta)\leqq-\frac{1}{2}$ for $\theta\in(-\alpha, -\operatorname{Arg} k_0)$ and $f(\theta)\leqq-\min(\delta_2, 1)$ for $\theta\in[-\operatorname{Arg} k_0), \alpha)$. It follows from the convexity assured by the lemma that

$f(\theta) < -\frac{1}{2}$ for all $\theta \in (-\alpha, \alpha)$. Thus, for any $\alpha'' \in (0, \alpha)$, there exists $\varepsilon > 0$ such that $f(\theta) < -\frac{1}{2} - \varepsilon$ for $\theta \in (-\alpha'', \alpha'')$. From this it follows readily that $\chi_1(z)$ is a dilation-analytic, $L^2_{1/2+\varepsilon}$-valued function on $S_{\alpha''}$. Now we apply Lemma 2.5 [with $g_1(z) = g'(z)$ and $m = 0$] to obtain for any $\alpha' \in (0, \alpha'')$ that

$$|g'(z)| \leqq K |z|^{-1-\varepsilon}, \qquad z \in S_{\alpha'} \cap \{ z' \,|\, |z'| \geqq 1 \}.$$

This estimate holds for arbitrarily chosen $\alpha' \in (0, \alpha)$, since we can pick $\alpha'' \in (\alpha', \alpha)$ and use the above argument.

By integration we get for some $\tau \in h$,

$$g(z) = \tau + O(|z|^{-\varepsilon}). \quad \blacksquare$$

3. PROOF OF LEMMA 2.3

This section is devoted to the proof of the basic result, Lemma 2.3. The proof is based on the analytic theory developed in [5] for short-range potentials, not necessarily symmetric (nor local). We will apply this theory to the potential $z^2 V(z)$, $z \in S_\alpha$ fixed, and then examine the analyticity dependence on z. For simplicity, we will assume throughout this section that $b_1 = \infty$.

For $z \in S_\alpha$, we define

$$V_z = z^2 V(z)$$
$$H_z = z^2 H(z) = H_0 + V_z.$$

For $k \in \mathbb{C}^+$, $k^2 \notin \sigma(H_z)$, we set $R_z(k) = (H_z - k^2)^{-1}$.

The sets of singular points are given by

$$\Sigma_d(z) = \{ k \in \mathbb{C}^+ \,|\, k^2 \in \sigma(H_z) \}$$
$$\Sigma_r(z) = \{ k \in \mathbb{R} \setminus \{0\} \,|\, \mathcal{N}(1 + V_z R_0(k + i0)) \neq 0 \},$$

where

$$R_0(k + i0) = \lim_{\varepsilon \downarrow 0} R_0(k + i\varepsilon) \text{ in } \mathscr{B}(L^2_\delta, H^2_{-\delta}), \delta > \frac{1}{2},$$
$$\Sigma(z) = \Sigma_d(z) \cup \Sigma_r(z).$$

We have the following identification of the singular points.

LEMMA 3.1. — *(a) If* $\alpha > \operatorname{Arg} z > 0$, *then*

$$z^{-1} \Sigma_d(z) = \mathscr{R}(z) \cup \{ k > 0 \,|\, k^2 \in \sigma_p(H) \} \cup \{ i\beta \,|\, \beta > 0,\ -\beta^2 \in \sigma_d(H) \},$$

and

$$\Sigma_r(z) = z\mathscr{R} \cap \mathbb{R}^+.$$

(*b*) *If* $0 > \operatorname{Arg} z > -\alpha$, *then*

$$\Sigma(z) = -\overline{\Sigma(\bar{z})}.$$

(*c*) *For* $z \in \mathbb{R}^+$,

$$\Sigma_d(z) = z\{i\beta \mid \beta > 0,\ -\beta^2 \in \sigma_d(H)\}$$

and

$$\Sigma_r(z) = z\{k \in \mathbb{R} \setminus \{0\} \mid k^2 \in \sigma_p(\mathrm{H})\}.$$

where $\sigma_d(\mathrm{H})$ *denotes the discrete spectrum and* $\sigma_p(\mathrm{H})$ *the point spectrum of* H.

Proof. – The statements on $\Sigma_d(z)$ follow from standard dilation-analytic theory [2]. The statement on $\Sigma_r(z)$ for $z \in \mathbb{R}^+$ was proved in [5], Lemma 2.7. The proof goes along the line of the proof of the statements on $\Sigma_r(z)$ for z non-real, which in turn was given in [3]. ■

For $z \in \mathrm{S}_\alpha$, $k \in \tilde{\mathbb{C}}^+$, we define $\mathrm{V}_z^\pm(k) = e^{\mp ikr}\mathrm{V}_z e^{\pm ikr}$, which by (A.2) is in $\mathscr{C}(\mathrm{H}^2_{-\delta_1}, \mathrm{L}^2_{\delta_2})$. The following extended limiting absorption principle follows from [5] together with the analyticity in k guaranteed by (A.2).

LEMMA 3.2. – *Let* $\delta > 1/2$ *be given.*

(*a*) *There exist two continuous* $\mathscr{B}(\mathrm{L}^2_\delta, \mathrm{H}^2_{-\delta})$*-valued functions* $\mathrm{R}_0^\pm(k)$ *on* $\tilde{\mathbb{C}}^+$, *analytic in* $\mathbb{C}^+$, *such that for* $k \in \mathbb{R} \setminus \{0\}$,

$$\mathrm{R}_0^\pm(k) = e^{\mp ikr}\mathrm{R}_0(k + i0)e^{\pm ikr}.$$

(*b*) *For any* $z \in \mathrm{S}_\alpha$, *there exist two continuous* $\mathscr{B}(\mathrm{L}^2_\delta, \mathrm{H}^2_{-\delta})$*-valued functions* $R_z^\pm(k)$ *on* $\tilde{\mathbb{C}}^+ \setminus \Sigma(z)$, *meromorphic in* $\mathbb{C}^+$, *such that for* $k \in \mathbb{R} \setminus (\Sigma_r(z) \cup \{0\})$

$$\mathrm{R}_z^\pm(k) = e^{\mp ikr}\mathrm{R}_z(k + i0)e^{\pm ikr}$$

where

$$\mathrm{R}_z(k + i0) = \lim_{\varepsilon \downarrow 0} \mathrm{R}_z(k + i\varepsilon).$$

For $k \in \tilde{\mathbb{C}}^+ \setminus \Sigma(z)$, *and assuming, in addition,* $\delta \leqq \delta_2$,

$$\mathrm{I} + \mathrm{V}_z^\pm(k)\,\mathrm{R}_0^\pm(k) \text{ is invertible in } \mathscr{B}(\mathrm{L}^2_\delta)$$

and

$$\mathrm{R}_z^\pm(k) = \mathrm{R}_0^\pm(k)(\mathrm{I} + \mathrm{V}_z^\pm(k)\mathrm{R}_0^\pm(k))^{-1}.$$

We shall need results proved in [5] on the modified trace operators. The free trace operators $T_0(k)$ is defined in $\mathscr{B}(L^2_\delta, h)$, $\delta > \frac{1}{2}$, $k \in \mathbb{R}\setminus\{0\}$, by

$$(T_0(k) f)(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ik\omega \cdot x} f(x)\, dx.$$

The modified trace operator $T_z(k)$ is defined for

$$z \in S_\alpha, k \in \mathbb{R}\setminus(\Sigma_r(z) \cup \{0\})$$

by

$$T_z(k) = T_0(k)(I + V_z R_0(k + i0))^{-1}.$$

LEMMA 3.3. – *Let* $\delta > \frac{1}{2}$ *be given.*

(*a*) *There exists a continuous* $\mathscr{B}(L^2_\delta, h)$*-valued function* $T_0^+(k)$ on $\tilde{\mathbb{C}}^+$, *analytic in* $\mathbb{C}^+$, *such that for* $k \in \mathbb{R}\setminus\{0\}$

$$T_0^+(k) = T_0(k)\, e^{ikr}.$$

The adjoint of $T_0^+(-\bar{k})$ is a continuous $\mathscr{B}(h, H^2_{-\delta})$*-valued function on* $\tilde{\mathbb{C}}^+$, *analytic in* $\mathbb{C}^+$.

(*b*) *For any* $z \in S_\alpha$, *there exists a continuous* $\mathscr{B}(L^2_\delta, h)$*-valued function* $T_z^+(k)$ *on* $\tilde{\mathbb{C}}^+\Sigma(z)$, *meromorphic in* $\mathbb{C}^+$, *such that for* $k \in \mathbb{R}\setminus(\Sigma_r(z) \cup \{0\})$,

$$T_z^+(k) = T_z(k)\, e^{ikr}.$$

Explicitly, for $k \in \tilde{\mathbb{C}}^+\setminus\Sigma(z)$,

$$T_z^+(k) = T_0^+(k)(I + V_z^+ R_0^+(k))^{-1}.$$

The adjoint of $T_{\bar{z}}^+(-\bar{k})$ *is given by*

$$(T_{\bar{z}}^+(-\bar{k}))^* = (I - R_z^-(k)\, V_z^-(k))(T_0^+(-\bar{k}))^*,$$

and is hence a continuous $\mathscr{B}(h, H^2_{-\delta})$*-valued function on* $\tilde{\mathbb{C}}^+\setminus\Sigma(z)$, *meromorphic in* $\mathbb{C}^+$.

We recall the following formulas: The S-matrix is given for $z \in S_\alpha$, $k \in \mathbb{R}\setminus(\Sigma_r(z) \cup \{0\})$ by

$$S_z(k) = I - \pi i k^{n-2} T_0(k)\{V_z - V_z R_z(k + i0) V_z\}(T_0(k))^*. \quad (3.1)$$

Let R denote reflection in h, *i.e.*, $(R\sigma)(\omega) = \sigma(-\omega)$. Then ([5], (2.6)]) for $z \in S_\alpha$, $k \in \mathbb{R}^+\setminus\Sigma_r(z)$, $-k \notin \Sigma_r(z)$

$$e^{-ikr} R_z(k + i0)\, e^{-ikr} = e^{-ikr} R_z(-k + i0)\, e^{-ikr} + \pi i k^{n-2} (T_{\bar{z}}^+ z(\bar{z}k))^* S_z(k)\, RT_z^+(-zk). \quad (3.2)$$

We are now ready to prove

LEMMA 3.4. — *Let* $\delta > \frac{1}{2}$ *and* $k \in S_\alpha \setminus \mathscr{R}$ *be given. Then the* $\mathscr{B}(L^2_\delta, H^2_{-\delta})$*-valued function* $e^{-izkr} R_z(zk + i0)\, e^{-izkr}$ *has a continuous extension from* $z \in k^{-1}\mathbb{R}^+$ *to* $\{z \in S_\alpha \mid \operatorname{Im} zk \leq 0\}$, *analytic in the interior.*

Explicitly,

$$e^{-izkr} \tilde{R}_z(zk)\, e^{-izkr} = e^{-izkr} R_z(-zk)\, e^{-izkr} + \pi i (zk)^{n-2} (T^+_{\bar{z}}(\overline{zk}))^* \tilde{S}(k) R T^+_z(-zk). \quad (3.3)$$

Proof. — We use formula (3.2), together with the identity $\tilde{S}(k) = S_k(zk)$. The latter follows from (3.1) as proved in [3]. The explicit formulas in Lemmas 3.2 and 3.3, together with (A.2) assure joint analyticity in z and k. The singular points are controlled by Lemma 3.1. ■

In order to utilize Lemma 3.4, the following decomposition of V is convenient:

Define for $\varepsilon > 0$ the function $q(r) = \exp(-\varepsilon r^\beta)$, $\beta = \frac{\pi}{2\alpha}$. Let $W = Vq$, $V_2 = qW$ and $V_1 = V - V_2$.

We remark that V_1, V_2 satisfy (A.2), and that $q(rz)$ for fixed $z \in S_\alpha$ decays faster than any exponential. Moreover, for any given open set K with the closure of K contained in S_α, $H_1 = H_0 + V_1$ does not have resonances in K for small enough ε, *cf.* [5], Lemma 2.8.

Let $k_0 \in \mathscr{R}$ be given and choose any K as above with $k_0 \in K$, and $\varepsilon > 0$ accordingly. We can apply Lemma 3.4 with V replaced by V_1 and $k \in K$. We use it to conclude that the dilation-analytic $\mathscr{C}(L^2)$-valued function $W_z R_{1\,z}(zk)\, q(rz)$ defined for $k \in K$ and $z \in \{z' \in S_\alpha \mid \operatorname{Arg} z'k > 0\}$ actually continues analytically to $z \in S_\alpha$. Here, and in what follows, the notation $R_{1\,z}(zk)$, $\tilde{R}_{1\,z}(zk)$ is used with obvious meaning. By a standard dilation-analytic argument, the spectrum of the analytically continued operator, $W_z \tilde{R}_{1\,z}(zk)\, q(rz)$, is independent of $z \in S_\alpha$.

Specifying to $k = k_0$, $\mathcal{N}(I + W_z \tilde{R}_{1\,z}(zk_0)\, q(rz)) \neq 0$, since for z with $\operatorname{Arg} zk_0 > 0$, this space is isomorphic to $\mathcal{N}(H(z) - k_0^2)$ by the map $\varphi(z) \to z^2 R_{1\,z}(zk_0)\, q(rz)\, \varphi(z)$. The analogue for $z = 1$ is given by the following:

LEMMA 3.5. — $\mathscr{F}(k_0)$ *is the isomorphic image of* $\mathcal{N}(I + W\tilde{R}_1(k_0)\, q)$ *via the map*

$$\mathcal{N}(I + W\tilde{R}_1(k_0)\, q) \ni \varphi \to \tilde{R}_1(k_0)\, q\varphi = f \in \mathscr{F}.$$

This is proved in [5], Theorem 2.4 (*a*). The following analytically continued formula is intimately related to the proof:

Let b_2 be given such that $b_2 > -\operatorname{Im} k$ for all $k \in K$. Then the meromorphic $\mathscr{B}(L^2_{0,\,b_2}, H^2_{0,\,-b_2})$-valued continuation $\tilde{R}(k)$ of $R(k)$ from $\mathbb{C}^+$ across $\mathbb{R}^+$ to K is given for $k \in K$ by

$$\tilde{R}(k) = \tilde{R}_1(k) - \tilde{R}_1(k)\, q(I + W\tilde{R}_1(k)\, q)^{-1} W\tilde{R}_1(k), \quad (3.4)$$

where $\tilde{R}_1(k)$ is given by (3.3) with $z=1$.

Specifying again to $k=k_0$, we let γ be a circle separating -1 from the rest of the spectrum of $W\tilde{R}_1(k_0)\,q$ and set

$$Q(z) = -\frac{1}{2\pi i}\int_\gamma (-\lambda + W_z \tilde{R}_{1\,z}(zk_0)\,q(rz))^{-1}\,d\lambda, \qquad z\in S_\alpha.$$

This is a projection onto the finite-dimensional algebraic null space corresponding to the eigenvalue -1.

Proof of Lemma 2.3. – Let $f\in\mathcal{F}(k_0)$ be given and choose φ in accordance with Lemma 3.5. Since φ is in the range of Q $(z=1)$, there exists a dilation-analytic vector $\eta\in L^2$ such that $\varphi=Q(1)\,\eta$. Because $Q(z)$ is dilation-analytic, we conclude the same for φ. Next we define

$$\chi(z) = \begin{cases} z^2 R_{1\,z}^{+}(zk_0)\,e^{-izk_0 r}\,q(rz)\,\varphi(z), & \operatorname{Im} zk_0>0;\\ z^2 (e^{-izk_0 r}\tilde{R}_{1\,z}(zk_0)\,e^{-izk_0 r})\,e^{izk_0 r}\,q(rz)\,\varphi(z), & \\ \qquad \operatorname{Im} zk_0\leqq 0. & \end{cases}$$

It follows from Lemmas 3.1-4 with V replaced by V_1, that $\chi(z)$ is a dilation-analytic $H^2_{-\delta}$-valued function, $\delta>\frac{1}{2}$.

By definition,

$$f(r,.)=e^{ik_0 r}\chi(1)(r,.).$$

Since $\varphi(z)\in\mathcal{N}(I+W_z R_{1\,z}(zk_0)\,q(rz))$ for any $z\in S_\alpha$ with $\operatorname{Arg} zk_0>0$, we get that for such z,

$$\psi(z)(r,.)=e^{ik_0 rz}\chi(z)(r,.)\in\mathcal{N}(H(z)-k_0^2).$$

The argument can easily be reversed, *i.e.*, we can prove that any $\psi(z)\in\mathcal{N}(H(z)-k_0^2)$ can be constructed as above from some $f\in\mathcal{F}$. ■

4. GENERALIZED EIGENFUNCTIONS

In this section, we assume to simplify the presentation that (A 1) and (A 2) hold with $b_1=\infty$.

The generalized eigenfunctions $\psi_z(k,\sigma)$ of the operators H_z are defined for $\sigma\in h$, $z\in S_\alpha$ and $k\in\tilde{\mathbb{C}}^+\setminus\Sigma(z)$ by

$$\psi_z(k,\sigma)=T_z^*(-\bar{k})\,R\,\sigma \qquad \text{where} \qquad T_z^*(-\bar{k})=e^{-ikr}(T_z^+(-\bar{k}))^*.$$

Notice that for $k\in\mathbb{R}\setminus(\{0\}\cup\Sigma_r)$, $\psi_z(k,\sigma)$ is given by $(I-R_z(k+i0)\,V_z)\,T_0^*(k)\,\sigma$. Hence [by Lemma 3.3 (*b*)], the functions $e^{ikr}\psi_z(k,\sigma)$ are analytic extensions to $k\in\tilde{\mathbb{C}}^+\setminus\Sigma(z)$ of the usual generalized eigenfunctions (multiplied by e^{ikr}). The functions $\psi_z(k,\sigma)$ belong to

$H^2_{-\delta, -|\operatorname{Im} k|}$ and can therefore be regarded as h-valued, continuous functions $\psi(k, \sigma; r, .)$ of $r \in \mathbb{R}^+$. This is useful for studying their analytic and asymptotic properties. The following result also establishes the connection between $\psi(k, \sigma, .)$ and $\psi_z(k, \sigma, .)$ by scaling of k and r.

THEOREM 4.1. – *For every $\sigma \in h$, there exists a meromorphic, h-valued function $\tilde{\psi}(k, \sigma, z)$ of k and z defined for $k \in \mathbb{C}^+ \cup S_\alpha$, $z \in S_\alpha$, with poles at most at $\{(k, z) \mid k \in \mathcal{R} \cup \Sigma\}$., such that for $r \in \mathbb{R}^+$:*

(*a*) $$\tilde{\psi}(k, \sigma, r) = \psi(k, \sigma; r, .), \qquad k \in \tilde{\mathbb{C}}^+ \setminus \Sigma\,(1);$$

(*b*) $$\tilde{\psi}(k, \sigma, zr) = \begin{cases} \psi_z(zk, \sigma; r, .) & \text{for } \operatorname{Arg} zk \geqq 0 \\ \psi_z(-zk, R\tilde{S}(k)\,\sigma; r, .) & \text{for } \operatorname{Arg} zk \leqq 0, \end{cases}$$

and $z \in S_\alpha$, $k \in (\mathbb{C}^+ \cup S_\alpha) \setminus (\mathcal{R} \cup \Sigma\,(1))$;

(*c*) *for $z \to \infty$ in S_α, uniformly for $|\operatorname{Im} kz| \geqq \delta > 0$, $|\operatorname{Arg} z| \leqq \alpha - \delta$*

$$\tilde{\psi}(k, \sigma, z) = (2\pi)^{-1/2} (ikz)^{(1-n)/2} \{(-1)^{(n-1)/2} e^{-ikz} (R\sigma + o(1)) + e^{ikz} (\tilde{S}(k)\,\sigma + o(1))\}$$

$$\frac{d}{dz} \tilde{\psi}(k, \sigma, z) = (2\pi)^{-1/2} (ikz)^{(1-n)/2} \times ik \{(-1)^{(n+1)/2} e^{-ikz} (R\sigma + o(1)) + e^{ikz} (\tilde{S}(k)\,\sigma + o(1))\}.$$

Proof. – For $k \in \mathbb{R}^+ \setminus \Sigma_r$, r, $z \in \mathbb{R}^+$, $\sigma \in h$,

$$(T^*(-k)\,\sigma)(zr, .) = (T_z^*(-zk)\,\sigma)(r, .). \tag{4.1}$$

From the identity

$$T_z(zk) = S_z(zk)\, RT_z(-zk) = S(k)\, RT_z(-zk),$$

we obtain

$$T_z^*(-zk) = T_z^*(zk)\, S(k)\, R. \tag{4.2}$$

For fixed r, $z \in \mathbb{R}^+$, both sides of (4.1) extend meromorphically to $(\mathbb{C}^+ \cup S_\alpha)$. The extensions are given by

$$(\tilde{T}^*(-\bar{k})\,\sigma)(zr, .) = (\tilde{T}_{\bar{z}}^*(-\overline{zk})\,\sigma)(r, .), \tag{4.3}$$

where

$$\tilde{T}^*(-\bar{k}) = \begin{cases} T^*(-\bar{k}) & \text{for } \operatorname{Arg} k \geqq 0, \\ T^*(\bar{k})\, \tilde{S}(k)\, R & \text{for } \operatorname{Arg} k \leqq 0; \end{cases}$$

$$\tilde{T}_{\bar{z}}^*(-\overline{zk}) = \begin{cases} T_{\bar{z}}^*(-\overline{zk}) & \text{for } \operatorname{Arg} k \geqq 0, \\ T_{\bar{z}}^*(\overline{zk})\, \tilde{S}(k)\, R & \text{for } \operatorname{Arg} k \leqq 0. \end{cases}$$

Here we have used (4.2) and Lemma 3.3 (*b*).

Now the r.h.s. of (4.3) extends for fixed $r \in \mathbb{R}^+$ and fixed $k \in (\mathbb{C}^+ \cup S_\alpha) \setminus (\mathcal{R} \cup \Sigma)$, analytically to $z \in S_\alpha$. We define $\tilde{\psi}(k, \sigma, z)$ by

$$\tilde{\psi}(k, \sigma, z) = (\tilde{T}_{\bar{z}}^*(-\overline{zk})\, R\sigma)(1, .). \tag{4.4}$$