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Essential selfadjointness of the Weyl quantized relativistic hamiltonian

by

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Dedicated to Professor Takeyuki Hida on the occasion of his sixtieth birthday

ABSTRACT. — It is shown that the relativistic quantum Hamiltonian H_A^m associated, via the Weyl correspondence, with the relativistic classical Hamiltonian $\sqrt{(p - A(x))^2 + m^2}$ with a general vector potential $A(x)$, is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and bounded from below by m . The core of proof lies in establishing a distributional inequality for H_A^m , an analogue to Kato's inequality for the nonrelativistic quantum Hamiltonian.

RÉSUMÉ. — On démontre que l'hamiltonien relativiste quantique H_A^m associé, via la correspondance de Weyl, à l'hamiltonien relativiste classique $\sqrt{(p - A(x))^2 + m^2}$ avec un potentiel vectoriel général $A(x)$, est essentiellement auto-adjoint sur $C_0^\infty(\mathbb{R}^d)$ et semi-borné inférieurement par m . Le nœud de la preuve consiste à établir une inégalité pour H_A^m au sens de

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distributions, un analogue à l'inégalité de Kato pour l'hamiltonien non relativiste quantique.

1. INTRODUCTION

In the present paper we study the problem of essential selfadjointness of the Weyl quantized relativistic Hamiltonian

$$H^m = H_A^m + \Phi \quad (1.1)$$

corresponding to the classical relativistic Hamiltonian

$$h^m(p, x) = h_A^m(p, x) + \Phi(x) \equiv \sqrt{(p - A(x))^2 + m^2} + \Phi(x), \quad p \in \mathbb{R}^d, \quad x \in \mathbb{R}^d, \quad (1.2)$$

of a spinless particle of mass m interacting with vector and scalar potentials $A(x)$ and $\Phi(x)$. $A(x)$ and $\Phi(x)$ are respectively $\mathbb{R}^d$ -valued and $\mathbb{R}$ -valued measurable functions defined in d -dimensional space $\mathbb{R}^d$.

In the previous paper [12], H_A^m was defined as a Weyl pseudo-differential operator (see Berezin-Subin [1], Hörmander [8])

$$(H_A^m u)(x) = (2\pi)^{-d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y) \cdot p} h_A^m\left(p, \frac{x+y}{2}\right) u(y) dy dp, \quad u \in \mathcal{S}(\mathbb{R}^d), \quad (1.3)$$

the integral on the right being an oscillatory integral. There $A(x)$ was assumed to be bounded and continuous together with its derivatives up to sufficiently higher order, since the usual theory of pseudo-differential operators needs assumption of sufficient regularity of the symbol $h_A^m(p, x)$. Then H_A^m defines a linear operator in $L^2(\mathbb{R}^d)$ with domain $\mathcal{S}(\mathbb{R}^d)$. It was shown ([12], cf. Shubin [22]) that H_A^m is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and [12] that its unique selfadjoint extension is bounded from below by m : $H_A^m \geq m$. Here the proof of the latter result is based on a path integral representation established there for the semigroup $\exp[-t(H^m - m)]$, while in [9], [10] this path integral formula has been further used to discuss the nonrelativistic limit problem. Recently, Nagase-Umeda [20] have proved the essential selfadjointness of H_A^m , assuming $A(x)$ to be continuous with bounded derivatives of the higher order than the first, so as to include the case of constant magnetic fields.

The aim of the present paper is to extend these results to the case of a less regular and unbounded vector potential $A(x)$. Namely we only assume that

$$A(x) \text{ and } \int_{0 < |y| < 1} |A(x-y/2) - A(x)| |y|^{-d} dy \quad \text{are locally bounded.} \quad (1.4)$$

In particular, a *locally Hölder-continuous* function $A(x)$ satisfies (1.4). Then redefining H_A^m with an integral operator which is equivalent to the pseudo-differential operator (1.3) if the latter makes sense, we show that H_A^m is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. Here the mass m is nonnegative; it may be zero. The assumption (1.4) is suggested by the path integral formula for $\exp[-t(H^m - m)]$ obtained in the previous work [12], which is still valid in this case (see the discussion in [12], §5). The problem of essential selfadjointness for (1.1), H^m with both vector and scalar potentials $A(x)$ and $\Phi(x)$, is also discussed. Further it is shown to remain still valid that H_A^m is bounded from below by m .

As the definition of H_A^m with a vector potential $A(x)$ satisfying (1.4), we propose the following:

$$(H_A^m u)(x) = mu(x) - \int_{|y| > 0} [e^{-iy \cdot A(x+y/2)} u(x+y) - u(x) - I_{\{|y| < 1\}} y (\partial_x - iA(x)) u(x)] n^m(dy), \quad u \in \mathcal{S}(\mathbb{R}^d). \quad (1.5)$$

Here $I_{\{|y| < 1\}}$ is the indicator function of the set $\{|y| < 1\}$, and $n^m(dy)$ is a σ -finite measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{|y| > 0} [y^2/(1+y^2)] n^m(dy) < \infty$,

called the Lévy measure. Note that when $A(x)$ identically vanishes, then (1.5) is, via the Fourier transform, equivalent to the Lévy-Khinchin formula for $\sqrt{p^2 + m^2}$ (e. g. [14], [21]). The right-hand side of (1.5) can be shown to coincide with that of (1.3) if $A(x)$ is sufficiently smooth and has bounded derivatives $\partial^\alpha A(x)$. The core of proof of the essential selfadjointness of H_A^m and H^m consists in establishing a distributional inequality for H_A^m : If $v \in L^2(\mathbb{R}^d)$ with $H_A^m v \in L_{loc}^1(\mathbb{R}^d)$, then

$$\operatorname{Re}[(\operatorname{sgn} v) H_A^m v] \geq \sqrt{-\Delta + m^2} |v|, \quad (1.6)$$

in the sense of distributions, where $(\operatorname{sgn} v)(x) = \overline{v(x)}/|v(x)|$, for $v(x) \neq 0$, and $=0$, for $v(x) = 0$. This may be regarded as an analogue to Kato's inequality for the nonrelativistic Schrödinger operator [16].

It should be noted that our H_A^m differs from the square root

$$\sqrt{(-i\partial - A(x))^2 + m^2} \quad (1.7)$$

of the nonnegative selfadjoint operator $(-i\partial - A(x))^2 + m^2$. We are more interested in H_A^m from the path integral point of view, because H_A^m is suited

to path integral but not (1.7) (see [12], cf. [18], IV A b). However, we shall not discuss which is physically more appropriate for a relativistic quantum Hamiltonian of a spinless particle in an magnetic field. We also mention that such a Hamiltonian was also treated, though with no vector potential, by several authors, Weder [25], Herbst [6], Daubechies-Lieb [3], Daubechies [2], Fefferman-de-la Llave [5].

In Section 2 the way of defining H_A^m with $A(x)$ which satisfies (1.4) is presented. Section 3 is concerned with the regularity of solutions u of $H_A^m u = f$, which is needed in Section 4 to prove an analogue to Kato's inequality in the distribution sense. Section 5 is devoted to the main result on the essential selfadjointness and semiboundedness of H^m as well as H_A^m , to the effect that H_A^m and H^m are essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and their unique selfadjoint extensions are bounded from below by m , if $A(x)$ satisfies (1.4) and $\Phi(x)$ is locally square-integrable with $\Phi(x) \geq 0$ a. e.

The result of the present paper has been announced in [11].

2. DEFINITION OF THE WEYL QUANTIZED RELATIVISTIC HAMILTONIAN

We are going to define the Weyl quantized Hamiltonian H_A^m corresponding to the classical relativistic Hamiltonian $h_A^m(p, x)$ in (1.2). m is a nonnegative constant.

Our starting point is the Lévy-Khinchin formula for the conditionally negative definite function $\sqrt{p^2 + m^2} - m$ (e. g. [14], p. 65, or [21] Appendix 2 to XIII. 12, p. 212-222):

$$\sqrt{p^2 + m^2} - m = \int_{|y| > 0} [e^{ipy} - 1 - ipy I_{\{|y| < 1\}}] n^m(dy). \quad (2.1)$$

Here $I_{\{|y| < 1\}}$ is the indicator function of the set $\{|y| < 1\}$, i. e. $I_{\{|y| < 1\}}(z) = 1$, if $|z| < 1$, and $= 0$, if $|z| \geq 1$. $n^m(dy)$ is the Lévy measure which is a σ -finite measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{|y| > 0} y^2 / (1 + y^2) n^m(dy) < \infty.$$

It is given by

$$n^m(dy) = 2(2\pi)^{-(d+1)/2} m^{(d+1)/2} \times |y|^{-(d+1)/2} K_{(d+1)/2}(m|y|) dy, \quad m > 0. \quad (2.2a)$$

$$n^0(dy) = \pi^{-(d+1)} \Gamma\left(\frac{d+1}{2}\right) |y|^{-(d+1)} dy, \quad m = 0, \quad (2.2b)$$

where $K_\nu(z)$ is the modified Bessel function of the third kind of order ν and $\Gamma(z)$ the gamma function. $K_\nu(z)$ satisfies, for $\nu > 0$,

$$0 < K_\nu(z) \leq C [z^{-\nu} \vee z^{-1/2}] e^{-z}, \quad z > 0. \quad (2.3)$$

with a constant $C > 0$ (see [4], chap. VII, 7.2.5, (37), 7.2.6, (41), p. 9-10, and 7.4.1, (1), (4), p. 23-24)).

To see (2.2) recall (see [12], Eq. (4.2), p. 244) that the operator $\exp[-t(\sqrt{-\Delta + m^2} - m)]$ has the kernel $k_0^m(t, x - y)$, where

$$k_0^m(t, z) = 2(2\pi)^{-(d+1)/2} m^{(d+1)/2} e^{mt} t \times (z^2 + t^2)^{-(d+1)/4} K_{(d+1)/2}(m(z^2 + t^2)^{1/2}), \quad m > 0. \quad (2.4a)$$

$$k_0^0(t, z) = \pi^{-(d+1)/2} \Gamma\left(\frac{d+1}{2}\right) t (z^2 + t^2)^{-(d+1)/2}, \quad m = 0, \quad (2.4b)$$

and use the fact (e. g. [14], Example 1) that

$$t^{-1} k_0^m(t, y) dy \rightarrow n^m(dy), \quad \text{as } t \downarrow 0.$$

In this section, unless otherwise specified, $A(x)$ is an $\mathbb{R}^d$ -valued measurable function in $\mathbb{R}^d$ satisfying (1.4) or equivalently

$$a(K) \equiv \sup_{x \in K} |A(x)| < \infty, \quad (2.5a)$$

$$b(K) \equiv \sup_{x \in K} \int_{0 < |y| < 1} |A(x - y/2) - A(x)| |y| n^m(dy) < \infty, \quad (2.5b)$$

for every compact subset K of $\mathbb{R}^d$. It is seen, using the asymptotic behavior (2.3) of $K_\nu(z)$ as $z \rightarrow 0$, that (2.5) is satisfied if $A(x)$ is *locally Hölder-continuous*, i. e. $|A(x - y/2) - A(x)| \leq b'(K) |y/2|^\alpha$, $x \in K$, $0 < |y| < 1$, with constants $0 < \alpha < 1$ and $b'(K)$ depending on K ; in fact, $b(K) \leq 2^{-\alpha} b'(K) n_\alpha^m$. Here and throughout we set

$$n_\infty^m = n^m(|y| \geq 1) \equiv \int_{|y| \geq 1} n^m(dy), \quad (2.6a)$$

$$n_\alpha^m = \int_{0 < |y| < 1} |y|^{1+\alpha} n^m(dy), \quad 0 < \alpha \leq 1. \quad (2.6b)$$

The explicit expression (2.2) of $n^m(dy)$ with (2.3) shows that n_∞^m and n_α^m are uniformly bounded for $m \geq 0$.

We shall denote by $H_0^m \equiv \sqrt{-\Delta + m^2}$ not only the linear map of the Sobolev space $H^s(\mathbb{R}^d)$ into $H^{s+1}(\mathbb{R}^d)$ but also the linear map $\mathcal{F}^{-1} \sqrt{p^2 + m^2} \mathcal{F}$ of $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier and inverse Fourier transforms.

Now let $u \in \mathcal{S}(\mathbb{R}^d)$ and put $\hat{u} = \mathcal{F} u$. Then the inverse Fourier transform of $\hat{u}(p)$ multiplied by (2.1) is

$$\begin{aligned} (H_0^m u)(x) &\equiv (\sqrt{-\Delta + m^2} u)(x) \\ &= mu(x) - \int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}} y \partial_x u(x)] n^m(dy). \end{aligned} \quad (2.7)$$

DEFINITION 2.1. — The Weyl quantized relativistic Hamiltonian H_A^m corresponding to the symbol $h_A^m(p, x)$ in (1.2) is defined to be the integral operator:

$$\begin{aligned} (H_A^m u)(x) &= mu(x) - \int_{|y|>0} [e^{-iy \cdot A(x+y/2)} u(x+y) - u(x) \\ &\quad - I_{\{|y|<1\}} y (\partial_x - iA(x)) u(x)] n^m(dy), \quad u \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \quad (2.8)$$

Obviously, if $A(x) \equiv 0$, (2.8) reduces itself to (2.7). On the other hand, H_A^m may be defined, as in [12], to be the pseudo-differential operator (1.3), i. e.

$$\begin{aligned} (H_A^m u)(x) &= (2\pi)^{-d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y) \cdot p} h_A^m\left(p, \frac{x+y}{2}\right) u(y) dy dp, \\ &\quad u \in \mathcal{S}(\mathbb{R}^d), \end{aligned} \quad (2.9)$$

if the right-hand side exists as an oscillatory integral (e. g. [8]).

Both the definitions yield the same H_A^m , which in the following lemma we content ourselves to check in a case including that of constant magnetic fields.

LEMMA 2.2. — Let m be nonnegative. Assume $A(x)$ is a C^∞ function satisfying

$$\begin{aligned} |A(x)| &\leq C_0(1+|x|^2)^{1/2}, \\ |\partial^\alpha A(x)| &\leq C_\alpha, \quad |\alpha| \geq 1, \quad x \in \mathbb{R}^d, \end{aligned}$$

with constants C_0 and C_α . Then: (i) The right-hand side of (2.9) exists as an oscillatory integral. (ii) The pseudo-differential operator H_A^m defined through (2.9) coincides on $\mathcal{S}(\mathbb{R}^d)$ with the integral operator H_A^m defined through (2.8).

Remark. — It is for simplicity that in Lemma 2.2 we have assumed $A(x)$ is C^∞ . It can be seen that both the statements (i) and (ii) are valid for sufficiently smooth $A(x)$ satisfying $|\partial^\alpha A(x)| \leq C_\alpha$, $1 \leq |\alpha| \leq N$, $x \in \mathbb{R}^d$, with N sufficiently large.

Proof of Lemma 2.2. — (i) The assertion is obvious if $m > 0$. We give here a proof which is valid for $m \geq 0$. Let χ be a rotation-invariant C_0^∞ function with $0 \leq \chi(p) \leq 1$ in $\mathbb{R}^d$, $\chi(p) = 1$ on $\{|p| \leq 1/2\}$ and $\chi(p) = 0$ on $\{|p| \geq 1\}$. Put $\chi_R(p) = \chi(p/R)$ for $R > 0$ and $\psi_R(p) = 1 - \chi_R(p)$. Write (2.9)

as a sum of two terms:

$$H_A^m u = H_1 u + H_2 u, \quad (2.10 a)$$

where

$$(H_1 u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p} h_1(x, p, y) u(y) dy dp, \quad (2.10 b)$$

$$(H_2 u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p} h_2(x, p, y) u(y) dy dp, \quad (2.10 c)$$

with

$$h_1(x, p, y) = \chi_R \left(p - A \left(\frac{x+y}{2} \right) \right) h_A^m \left(p, \frac{x+y}{2} \right),$$

$$h_2(x, p, y) = \psi_R \left(p - A \left(\frac{x+y}{2} \right) \right) h_A^m \left(p, \frac{x+y}{2} \right).$$

It is easy to see by change of variables that the integral $H_1 u$ is absolutely convergent. $H_2 u$ is an oscillatory integral, whose existence follows from the basic theory of oscillatory integrals (e. g. [17], Chap. 1), because by assumption on $A(x)$, $h_2(x, p, y)$ is a C^∞ function satisfying: for all multi-indices α, β and β' there exists a constant $C_{\alpha\beta\beta'}$ independent of $m \geq 0$ such that

$$|\partial_p^\alpha \partial_x^\beta \partial_y^{\beta'} h_2(x, p, y)| \leq C_{\alpha\beta\beta'} (1+p^2)^{1/2} (1+x^2+y^2)^{1/2},$$

$$p \in \mathbb{R}^d, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

(ii) In view of (2.7) we can see, for $H_A^m u$ in (2.8), that

$$(H_A^m u)(x) = \left(H_0^m \left(\exp \left[i(x - \cdot) A \left(\frac{x + \cdot}{2} \right) \right] u(\cdot) \right) \right)(x), u \in \mathcal{S}(\mathbb{R}^d), \quad (2.11)$$

which may be said that apply H_A^m to u is nothing but apply H_0^m to the appropriately "gauge transformed" u . Here note by assumption on $A(x)$ that $u \in \mathcal{S}(\mathbb{R}^d)$ implies that, for x fixed, the function

$$y \rightarrow \exp \left[i(x - y) A \left(\frac{x + y}{2} \right) \right] u(y)$$

belongs to $\mathcal{S}(\mathbb{R}^d)$. Now, to show the statement (ii) of the lemma, we have only to show the right-hand side of (2.9) is equal, as oscillatory integrals,

to the right-hand side of (2.11), that is,

$$(2\pi)^{-d} \iint e^{i(x-y)p} \sqrt{p^2+m^2} \exp \left[i(x-y) A \left(\frac{x+y}{2} \right) \right] u(y) dy dp.$$

Changing in (2.10 b, c) the integration variables $p' = p - A \left(\frac{x+y}{2} \right)$ (writing p again instead of p'), we get

$$(H_1 u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p} \chi_R(p) \sqrt{p^2+m^2} \times \exp \left[i(x-y) A \left(\frac{x+y}{2} \right) \right] u(y) dy dp. \quad (2.12 a)$$

and

$$\begin{aligned} (2\pi)^d (H_2 u)(x) &\equiv \lim_{\varepsilon \rightarrow 0} \iint e^{i(x-y)p} \chi(\varepsilon p) h_2(x, p, y) u(y) dy dp, \\ &= \lim_{\varepsilon \rightarrow 0} \iint \exp \left[i(x-y) \left(p + A \left(\frac{x+y}{2} \right) \right) \right] \\ &\quad \times \chi \left(\varepsilon \left(p + A \left(\frac{x+y}{2} \right) \right) \right) \psi_R(p) \sqrt{p^2+m^2} u(y) dy dp, \\ &= \lim_{\varepsilon \rightarrow 0} \iint e^{i(x-y)p} (1+p^2)^{-l} (1-\Delta_y)^l \left\{ \chi \left(\varepsilon \left(p + A \left(\frac{x+y}{2} \right) \right) \right) \right\} \\ &\quad \times \exp \left[i(x-y) A \left(\frac{x+y}{2} \right) \right] \psi_R(p) \sqrt{p^2+m^2} u(y) dy dp. \end{aligned}$$

Here l is an integer $> (d+1)/2$, and the last equality is due to integration by parts based on

$$e^{-iyp} = (1+p^2)^{-l} (1-\Delta_y)^l e^{-iyp}.$$

Since $A(x)$ is continuous and the derivatives of $A(x)$ are all bounded, it is seen for x fixed that as $\varepsilon \rightarrow 0$, $\chi \left(\varepsilon \left(p + A \left(\frac{x+y}{2} \right) \right) \right)$ converges to 1 uniformly on compact sets of both p and y , and $\partial_y^\alpha \chi \left(\varepsilon \left(p + A \left(\frac{x+y}{2} \right) \right) \right)$ with $|\alpha| \geq 1$ converges to zero uniformly in both p and y . Then we have

by the Lebesgue dominated convergence theorem

$$\begin{aligned}
 (2\pi)^d (H_2 u, (x)) &= \iint e^{i(x-y)p} (1+p^2)^{-l} (1-\Delta_y)^l \\
 &\quad \times \left\{ \exp \left[i(x-y) A \left(\frac{x+y}{2} \right) \right] \psi_R(p) \sqrt{p^2+m^2} u(y) \right\} dy dp \\
 &= \lim_{\varepsilon \rightarrow 0} \iint e^{i(x-y)p} \chi(\varepsilon p) (1+p^2)^{-l} (1-\Delta_y)^l \{ \dots \} dy dp \\
 &= \lim_{\varepsilon \rightarrow 0} \iint e^{i(x-y)p} \chi(\varepsilon p) \psi_R(p) \sqrt{p^2+m^2} \\
 &\quad \times \exp \left[i(x-y) A \left(\frac{x+y}{2} \right) \right] u(y) dy dp,
 \end{aligned}$$

where the last equality is due to integration by parts, so that, as an oscillatory integral,

$$\begin{aligned}
 (H_2 u)(x) &= (2\pi)^{-d} \iint e^{i(x-y)p} \psi_R(p) \sqrt{p^2+m^2} \\
 &\quad \times \exp \left[i(x-y) A \left(\frac{x+y}{2} \right) \right] u(y) dy dp. \quad (2.12 b)
 \end{aligned}$$

Thus with (2.12 a, b) we have shown the assertion (ii), completing the proof of Lemma 2.2.

Next we shall see that H_A^m defined by (2.8) maps $C_0^\infty(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$, so that it can define a linear operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$. For $u \in C^\infty \cap L^2(\mathbb{R}^d)$, write (2.8) as

$$H_A^m u = mu + I_1 u + I_2 u, \quad (2.13 a)$$

where

$$(I_1 u)(x) = - \int_{|y| \geq 1} [e^{-iy A(x+y/2)} u(x+y) - u(x)] n^m(dy), \quad (2.13 b)$$

$$\begin{aligned}
 (I_2 u)(x) &= - \int_{0 < |y| < 1} [e^{-iy A(x+y/2)} u(x+y) - u(x) \\
 &\quad - y(\partial_x - i A(x)) u(x)] n^m(dy). \quad (2.13 c)
 \end{aligned}$$

LEMMA 2.3. — Assume $A(x)$ satisfies (2.5) or (1.4). (i) If u is in $C^\infty \cap L^2(\mathbb{R}^d)$, then $H_A^m u$ is in $L_{loc}^2(\mathbb{R}^d)$. More Precisely,

$$\|I_1 u\|_p \leq 2 n_\infty^m \|u\|_p, \quad u \in L^p(\mathbb{R}^d), \quad (2.14 a)$$

for each $1 \leq p \leq \infty$, where $\|\cdot\|_p$ stands for the L^p norm, and for every compact subset K of $\mathbb{R}^d$ there exists a constant C_K such that

$$\|I_2 u\|_{2,K} \leq C_K [\|u\|_{2,K_2} + \|\partial u\|_{2,K_2}], \quad u \in C^\infty \cap L^2(\mathbb{R}^d). \quad (2.14 b)$$

(ii) Let $1 \leq p \leq \infty$. If u is in $C_0^\infty(\mathbb{R}^d)$, then $H_A^m u$ is in $L^p(\mathbb{R}^d)$. For every compact subset K of $\mathbb{R}^d$ there exists a constant C_K such that

$$\|H_A^m u\|_p \leq C_K \left[\|u\|_\infty + \sum_{j=1}^d \|\partial_j u\|_\infty + \sum_{j,k=1}^d \|\partial_j \partial_k u\|_\infty \right], \quad (2.15)$$

for all $u \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp } u \subseteq K$. Here n_∞^m in (2.14 a) is the constant (2.6 a), the C_K in (2.14 b) and (2.15) are constants depending on K and the behavior of $A(x)$ in a neighbourhood of K , and for $r > 0$,

$$K_r = \{x \in \mathbb{R}^d; \text{dist}(x, K) \leq r\}, \quad (2.16)$$

$$\|f\|_{p,K} = \begin{cases} \left(\int_K |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{p \in K} |f(x)|, & p = \infty, \end{cases} \quad (2.17)$$

with K a compact set in $\mathbb{R}^d$.

Proof. — (i) It is easy to verify (2.14 a) by use of the Hölder inequality. To show (2.14 b) let K be a compact set in $\mathbb{R}^d$. Let $\varphi(x)$ be a C_0^∞ function with $0 \leq \varphi(x) \leq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on K_1 and $\text{supp } \varphi \subseteq K_2$. Then (2.13 c) is rewritten for $x \in K$ as

$$\begin{aligned} (I_2 u)(x) &= (I_2(\varphi u))(x) \\ &= \lim_{\varepsilon \downarrow 0} \left\{ - \int_{\varepsilon \leq |y| < 1} [e^{-iy A(x+y/2)} - 1 + iy A(x+y)] \varphi(x+y) u(x+y) n^m(dy) \right. \\ &\quad \left. + \int_{\varepsilon \leq |y| < 1} i[y A(x+y) \varphi(x+y) u(x+y) - y A(x) \varphi(x) u(x)] n^m(dy) \right\} \\ &\quad - \int_{0 < |y| < 1} [\varphi(x+y) u(x+y) - \varphi(x) u(x) - y \partial_x(\varphi(x) u(x))] n^m(dy) \\ &\equiv \lim_{\varepsilon \downarrow 0} \{ i_1(\varepsilon)(x) + i_2(\varepsilon)(x) \} + i_3(x). \end{aligned} \quad (2.18)$$

Then by Fatou's lemma

$$\|I_2 u\|_{2,K} \leq \sum_{j=1}^2 \liminf_{\varepsilon \downarrow 0} \|i_j(\varepsilon)\|_{2,K} + \|i_3\|_{2,K}. \quad (2.19)$$

Now we estimate each term on the right of (2.19). First, for $i_1(\varepsilon)$ we have with $x \in K$,

$$|i_1(\varepsilon)(x)| \leq \int_{\varepsilon \leq |y| < 1} [2^{-1}|y|^2 |A(x+y/2)|^2 + |y| |A(x+y/2) - A(x+y)|] |\varphi(x+y) u(x+y)| n^m(dy),$$

using

$$|e^{-is} - 1 + it| \leq 2^{-1} s^2 + |t - s|, \quad s, t \in \mathbb{R}.$$

We use the Schwarz inequality and then change the integration variables $x+y=x'$ (writing x again instead of x') to get

$$\begin{aligned} \|i_1(\varepsilon)\|_{2,K} &\leq \left\{ \int_{K_1} dx \left(\int_{\varepsilon \leq |y| < 1} [2^{-1}|y|^2 |A(x-y/2)|^2 \right. \right. \\ &\quad \left. \left. + |y| |A(x-y/2) - A(x)|] n^m(dy) \right) \right. \\ &\quad \times \int_{\varepsilon \leq |y| < 1} [2^{-1}|y|^2 |A(x-y/2)|^2 + |y| |A(x-y/2) - A(x)|] \\ &\quad \left. \times |\varphi(x)u(x)|^2 n^m(dy) \right\}^{1/2} \\ &\leq (2^{-1}a(K_2)n_1^m + b(K_1)) \|\varphi u\|_2 \leq C(K_1) \|u\|_{2,K_2}. \quad (2.20) \end{aligned}$$

with a constant $C(K_1)$ independent of $\varepsilon > 0$, where $a(K_2)$ and $b(K_1)$ are the constants in (2.5) with K_2 and K_1 in place of K , respectively, and n_1^m the constant (2.6b) with $\alpha = 1$.

Next, for $i_2(\varepsilon)$, we show

$$\|i_2(\varepsilon)\|_{2,K} \leq \|i_2(\varepsilon)\|_2 \leq C_0 \sum_{k=1}^d \|\varphi A_k u\| \leq C_0 a(K_2) \|u\|_{2,K_2}, \quad (2.21)$$

with a constant C_0 independent of $\varepsilon > 0$. To do so, first note that the Lévy measure has a rotation-invariant density: $n^m(dy') = n^m(y') dy'$, as seen from its explicit expression (2.2). Let for $0 < \varepsilon < 1$ and $1 \leq k \leq d$,

$$(N_{k,\varepsilon}f)(x) = \int_{\varepsilon \leq |y| < 1} f(x-y) y_k n^m(y) dy, \quad f \in L^2(\mathbb{R}^d). \quad (2.22)$$

With $N_{k,\varepsilon}$ rewrite $i_2(\varepsilon)$ as

$$i_2(\varepsilon) = -i \sum_{k=1}^d N_{k,\varepsilon}(\varphi A_k u).$$

To establish (2.21) we show that

$$\|N_{k,\varepsilon}f\| \leq C_0 \|f\|, \quad f \in L^2(\mathbb{R}^d), \quad (2.23)$$

with a constant C_0 independent of $0 < \varepsilon < 1$ and $m \geq 0$, and that the L^2 limit of $N_{k,\varepsilon}f$ as $\varepsilon \downarrow 0$ exists. To this end we use the theory of singular integrals; we have only to confirm that each $y_k n^m(y)$ is the Calderón-Zygmund kernel [24], Chap. II, §3, Theorem 2, p. 35, that is,

$$|y_k n^m(y)| \leq B |y|^{-d}, \quad |y| > 0, \quad (2.24a)$$

$$\int_{|y| \geq 2|y'|} |(y_k - y'_k) n^m(y - y') - y_k n^m(y)| dy \leq B, \quad |y'| > 0, \quad (2.24b)$$

$$\int_{R_1 < |y| < R_2} y_k n^m(y) dy = 0, \quad 0 < R_1 < R_2 < \infty, \quad (2.24c)$$

with a constant B independent of $m \geq 0$. (2.24a) is easy to see from (2.2) with the asymptotics (2.3) of the Bessel function $K_\nu(z)$ and (2.24c) is clear from the rotational invariance of the density $n^m(y)$. To see (2.24b), first estimate, when $|y| \geq 2|y'|$, the integrand in the integral to get

$$\begin{aligned} & |(y_k - y'_k) n^m(y - y') - y'_k n^m(y)| \\ &= \left| \int_0^1 (d/d\theta) [(y_k - \theta y'_k) n^m(y - \theta y')] d\theta \right| \\ &= \left| \int_0^1 [y'_k n^m(y - \theta y') + (y_k - \theta y'_k) \sum_{j=1}^d y'_j (\partial/\partial y_j) n^m(y - \theta y')] d\theta \right| \\ &\leq B_0 \int_0^1 \left(|y'_k| |y - \theta y'|^{-(d+1)} + \sum_{j=1}^d |y'_j| |y - \theta y'|^{-(d+1)} \right) d\theta \\ &\leq B_1 |y'| |y|^{-(d+1)}, \end{aligned}$$

with constants B_0 and B_1 independent of $m \geq 0$, because

$$|y - \theta y'| \geq |y| - |y'| \geq 2^{-1}|y|, \quad \text{if } |y| \geq 2|y'|.$$

Then we can bound the left-hand side of (2.24b) by

$$B_1 |y'| \int_{|y| \geq 2|y'|} |y|^{-(d+1)} dy = B_2 |y'| \int_{2|y'|}^\infty |y|^{-2} d|y| = 2^{-1} B_2,$$

using the spherical coordinates, where B_2 is the constant for which B_2/B_1 is the area of the $(d-1)$ -dimensional unit sphere.

Finally, for i_3 , we have by (2.7), for $x \in K$,

$$\begin{aligned} i_3(x) &= (H_0^m(\varphi u))(x) - m \varphi(x) u(x) \\ &\quad + \int_{|y| \geq 1} [\varphi(x+y) u(x+y) - \varphi(x) u(x)] n^m(dy). \end{aligned} \quad (2.25)$$

It follows that

$$\begin{aligned} \|i_3\|_{2, K} &\leq \|i_3\|_2 \leq \|\sqrt{-\Delta + m^2}(\varphi u)\|_2 + (m + 2n_\infty^m) \|\varphi u\|_2 \\ &\leq \|\partial(\varphi u)\|_2 + 2(m + n_\infty^m) \|\varphi u\|_2 \\ &\leq c_K [\|u\|_{2, K_2} + \|\partial u\|_{2, K_2}], \end{aligned} \quad (2.26)$$

with a constant c_K dependent on K . Then (2.14b) follows from (2.20), (2.21) and (2.26) with (2.19).

(ii) Let K be a compact set in $\mathbb{R}^d$, and let u be a C_0^∞ function with $\text{supp } u \subseteq K$. Then we see by (2.14a) that $I_1 u$ is L^2 with

$$\|I_1 u\|_p \leq 2n_\infty^m \|u\|_p \leq 2|K|^{1/p} n_\infty^m \|u\|_\infty,$$

and, since $I_2 u$ has compact support in K_1 , we have with (2.18)

$$\begin{aligned} \|I_2 u\|_p &= \|I_2 u\|_{p, K_1} \leq |K_1|^{1/p} \|I_2 u\|_{\infty, K_1} \\ &\leq |K_1|^{1/p} \left[\sum_{j=1}^2 \liminf_{\varepsilon \downarrow 0} \|i_j(\varepsilon)\|_{\infty, K_1} + \|i_3\|_{\infty, K_1} \right]. \end{aligned}$$

Here $|K|$ and $|K_1|$ are the volumes of K and K_1 . We can estimate, similarly to the proof of (i),

$$\begin{aligned} \|i_1(\varepsilon)\|_{\infty, K_1} &\leq (2^{-1} a(K_2) n_1^m + b(K_1)) \|\varphi u\|_{\infty} \leq C(K_1) \|u\|_{\infty}, \\ \|i_2(\varepsilon)\|_{\infty, K_1} &\leq (b(K) + b(K_1)) \|\varphi u\|_{\infty} + a(K) n_1^m \sum_{j=1}^d \|\partial_j(\varphi u)\|_{\infty}, \\ &\leq c_K \left[\|u\|_{\infty} + \sum_{j=1}^d \|\partial_j u\|_{\infty} \right], \\ \|i_3(\varepsilon)\|_{\infty, K_1} &\leq n_1^m \sum_{j,k=1}^d \|\partial_j \partial_k(\varphi u)\|_{\infty} \leq c_K \sum_{j,k=1}^d \|\partial_j \partial_k u\|_{\infty}. \end{aligned}$$

Thus, recalling (2.13 a), we have shown (2.15). This ends the proof of Lemma 2.3.

3. REGULARITY

Throughout this section, $A(x)$ is assumed to satisfy (2.5) or (1.4). We have seen in Lemma 2.3 among other things that H_A^m defined by (2.8) is a linear operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$. It is easy to see with the rotational invariance of the Lévy measure $n^m(dy)$ that H_A^m is symmetric, i. e.

$$(H_A^m \varphi, \psi) = (\varphi, H_A^m \psi), \quad \varphi, \psi \in C_0^\infty(\mathbb{R}^d). \quad (3.1)$$

For $u \in L^2(\mathbb{R}^d)$ we can define, in view of Lemma 2.3, a distribution $H_A^m u$ in $\mathcal{D}'(\mathbb{R}^d)$ through

$$(H_A^m u, \varphi) = (u, H_A^m \varphi), \quad \varphi \in C_0^\infty(\mathbb{R}^d). \quad (3.2)$$

In this section we shall show regularity of the function $v \in L^2(\mathbb{R}^d)$ with $H_A^m v \in L_{\text{loc}}^1(\mathbb{R}^d)$. To this end we give a kind of integral representation of such v , the proof of which needs some task. The main result of this section is Theorem 3.6, which is needed in the next section.

Let $G_0^m(x-y)$ be the fundamental solution for the operator $H_0^m + 1 \equiv \sqrt{-\Delta + m^2} + 1$. A direct calculation with (2.4) yields

$$\begin{aligned} G_0^m(z) &= (2\pi)^{-d} \int_0^\infty dt \int e^{izp} e^{-t[(p^2 + m^2)^{1/2} + 1]} dp \\ &= \int_0^\infty e^{-(m+1)t} k_0^m(t, z) dt \\ &= 2(2\pi)^{-(d+1)/2} m^{(d+1)/2} \int_0^\infty t e^{-t} (z^2 + t^2)^{-(d+1)/4} \\ &\quad \times K_{(d+1)/2}(m(z^2 + t^2)^{1/2}) dt. \end{aligned} \quad (3.3)$$

Hence $G_0^m(z)$ is a positive C^∞ function where $|z| > 0$, and satisfies

$$G_0^m(z) \leq C |z|^{1-d}, \quad d > 1, \quad (3.4a)$$

$$G_0^m(z) \leq C(1 + |\log |z||), \quad d = 1, \quad (3.4b)$$

$$|\partial_z G_0^m(z)| \leq C |z|^{-d}, \quad d > 0, \quad (3.4c)$$

near $z=0$, with a constant C independent of $m \geq 0$, and $\int G_0^m(z) dz = 1/(m+1)$, which can be seen with the aid of (2.3) (cf. [24], Chap. V).

Now put for $\varepsilon \geq 0$,

$$G_{R,\varepsilon}^m(x) = (2\pi)^{-d/2} \chi_R(x) \left(\mathcal{F}^{-1} \left(\frac{\exp[-\varepsilon((p^2+m^2)^{1/2}+1)]}{(p^2+m^2)^{1/2}+1} \right) \right)(x) \\ = e^{-(m+1)\varepsilon} \chi_R(x) \int G_0^m(x-y) k_0^m(\varepsilon, y) dy, \quad \varepsilon > 0, \quad (3.5a)$$

$$G_{R,0}^m(x) = \chi_R(x) G_0^m(x), \quad \varepsilon = 0, \quad (3.5b)$$

where χ_R is the same cutoff function as in the proof of Lemma 2.2, $k_0^m(t, z)$ the same as (2.4) and $\mathcal{F}^{-1}$ means the inverse Fourier transform. From (3.5) it is evident that, for $\varepsilon \geq 0$, $G_{R,\varepsilon}^m(x)$ is a nonnegative C^∞ function where $|x| > 0$ which identically vanishes outside $|x| < R$, while, for $\varepsilon > 0$, it is a C_0^∞ function in $\mathbb{R}^d$. In view of (2.7), put for $\varepsilon > 0$.

$$S_{R,\varepsilon}^m(x-y) = ([\sqrt{-\Delta+m^2}+1] G_{R,\varepsilon}^m(x-\cdot))(y) \\ = ([\sqrt{-\Delta+m^2}+1] G_{R,\varepsilon}^m)(x-y) \\ = (m+1) G_{R,\varepsilon}^m(x-y) - \int_{|y'|>0} [G_{R,\varepsilon}^m(x-y-y') - G_{R,\varepsilon}^m(x-y) \\ - I_{\{|y'|<1\}} y' \partial_y G_{R,\varepsilon}^m(x-y)] n^m(dy'), \quad (3.6)$$

which is a real-valued, bounded C^∞ function.

With $u \in \mathcal{S}(\mathbb{R}^d)$ let for $\varepsilon \geq 0$

$$(G_{R,\varepsilon}^m u)(x) = \int G_{R,\varepsilon}^m(x-y) u(y) dy, \quad (3.7)$$

and for $\varepsilon > 0$

$$(S_{R,\varepsilon}^m u)(x) = \int S_{R,\varepsilon}^m(x-y) u(y) dy. \quad (3.8)$$

Remark. — Here, to define $G_{R,\varepsilon}^m(x)$ we have made a large-momentum cutoff of $G_0^m(x)$. We might as well use another cutoff of $G_0^m(x)$ near the origin $x=0$ (e. g. [13], Appendix):

$$G_{R,\varepsilon}^{m'}(x) = \chi_R(x) (1 - \chi_\varepsilon(x)) G_0^m(x), \quad 0 < \varepsilon < R.$$

However, the merit of such a choice of $G_{R,\varepsilon}^m(x)$ as in (3.5) lies in allowing one to appeal more to Fourier analysis so as to simplify the proofs which follow.

In the following two lemmas we shall observe some properties of $G_{R,\varepsilon}^m$, (3.7), and $S_{R,\varepsilon}^m$, (3.8), as operators in the L^2 space. So $\|\cdot\|$ stands for the L^2 norm there.

LEMMA 3.1. — *Let $\varepsilon \geq 0$. The constants C below are, though all different in general, independent of $m \geq 0$ and ε .*

(i)

$$0 \leq G_{R,\varepsilon}^m(x) \leq C|x|^{1-d}, \quad d > 1, \quad (3.9a)$$

$$0 \leq G_{R,\varepsilon}^m(x) \leq C(1 + |\log|x||), \quad d = 1, \quad (3.9b)$$

near $x=0$.

(ii)

$$\|\sqrt{-\Delta + m^2} G_{R,\varepsilon}^m u\| \leq C\|u\|, \quad u \in \mathcal{S}(\mathbb{R}^d). \quad (3.10)$$

In particular, $G_{R,\varepsilon}^m$ defines a bounded linear operator on $L^2(\mathbb{R}^d)$.

(iii) For $u \in L^2(\mathbb{R}^d)$,

$$\|\sqrt{-\Delta + m^2} (G_{R,\varepsilon}^m u - G_{R,0}^m u)\| \rightarrow 0, \quad \varepsilon \downarrow 0, \quad (3.11)$$

uniformly on bounded subsets of m in $[0, \infty)$.

Proof. — To simplify notation we suppress the superscript “ m ” to write $G_{R,\varepsilon}$, $G_0(x)$ and $k_0(t, x)$ for $G_{R,\varepsilon}^m$, $G_0^m(x)$ and $k_0^m(t, x)$, respectively.

(i) (3.9) follows from (3.4), since $G_{R,\varepsilon}(x) \leq G_0(x)$.

(ii) Let $\hat{\chi}_R$ be the Fourier transform of χ_R . By the Plancherel theorem we have with $\|u\|=1$,

$$\begin{aligned} \|(-\Delta + m^2)^{1/2} G_{R,\varepsilon} u\| &= \|(p^2 + m^2)^{1/2} \mathcal{F}(G_{R,\varepsilon} u)\| \\ &\leq (2\pi)^{d/2} \sup_p [(p^2 + m^2)^{1/2} |\hat{G}_{R,\varepsilon}(p)|] \\ &= \sup_p \left| \int \frac{(p^2 + m^2)^{1/2} \exp[-\varepsilon((p-\xi)^2 + m^2)^{1/2} - \varepsilon]}{((p-\xi)^2 + m^2)^{1/2} + 1} \hat{\chi}_R(\xi) d\xi \right| \\ &\leq \int (1 + |\xi|) |\hat{\chi}_R(\xi)| d\xi, \end{aligned}$$

because $|((p-\xi)^2 + m^2)^{1/2} - (p^2 + m^2)^{1/2}| \leq |\xi|$.

(iii) Similarly to the proof of (ii) we have

$$\begin{aligned} \|(-\Delta + m^2)^{1/2} (G_{R,\varepsilon} u - G_{R,0} u)\|^2 &= \|(p^2 + m^2)^{1/2} [\mathcal{F}(G_{R,\varepsilon} u) - \mathcal{F}(G_{R,0} u)]\|^2 \\ &\leq (2\pi)^{-d} \int \left(\int (1 + |\xi|) |\hat{\chi}_R(\xi)| \right. \\ &\quad \left. \times \left(1 - \exp[-\varepsilon((p-\xi)^2 + m^2)^{1/2} - \varepsilon] \right) d\xi \right)^2 |\hat{u}(p)|^2 dp, \end{aligned}$$

which tends to zero as $\varepsilon \downarrow 0$, uniformly for bounded $m \geq 0$, by the Lebesgue dominated convergence theorem. This proves Lemma 3.1.

LEMMA 3.2. — (i) If $\varepsilon > 0$, then

$$\|S_{R,\varepsilon}^m u\| \leq C \|u\|, \quad u \in \mathcal{S}(\mathbb{R}^d), \quad (3.12)$$

with a constant C independent of $m \geq 0$ and ε . In particular, $S_{R,\varepsilon}^m$ defines a bounded linear operator on $L^2(\mathbb{R}^d)$.

(ii) For $u \in L^2(\mathbb{R}^d)$,

$$\|S_{R,\varepsilon}^m u - S_{R,\varepsilon'}^m u\| \rightarrow 0, \quad \varepsilon, \varepsilon' \downarrow 0, \quad (3.13 a)$$

uniformly on bounded subsets of m in $[0, \infty)$, so that

$$S_R^m \equiv \text{S-lim}_{\varepsilon \rightarrow 0} S_{R,\varepsilon}^m \quad (3.13 b)$$

is a bounded linear operator on $L^2(\mathbb{R}^d)$. The bounded linear operator Q_R^m on $L^2(\mathbb{R}^d)$ defined by

$$Q_R^m u \equiv \int Q_R^m(x-y) u(y) dy = u - S_R^m u \quad (3.14)$$

satisfies that

$$\|\sqrt{-\Delta + m^2} Q_R^m u\| \leq C \|u\|, \quad u \in L^2(\mathbb{R}^d), \quad (3.15)$$

with a constant C independent of $m \geq 0$.

Proof. — We suppress “ m ” of $S_{R,\varepsilon}^m$, $G_{R,\varepsilon}^m$ and Q_R^m . Since by (3.6)

$$S_{R,\varepsilon} = [\sqrt{-\Delta + m^2} + 1] G_{R,\varepsilon}, \quad (3.16)$$

(3.12) and (3.13 a) follow from (3.10) and (3.11) in Lemma 3.1. Obvious is that S_R and so Q_R is a bounded operator on $L^2(\mathbb{R}^d)$. We have (3.15), since Q_R is a pseudo-differential operator with symbol

$$q_R(p) = (2\pi)^{-d} \int \left(1 - \frac{(p^2 + m^2)^{1/2} + 1}{((p - \xi)^2 + m^2)^{1/2} + 1}\right) \hat{\chi}_R(\xi) d\xi$$

and so $(p^2 + m^2)^{1/2} q_R(p)$ is uniformly bounded. This proves Lemma 3.2.

The following lemma studies some further properties of $G_{R,\varepsilon}^m$ as an operator in the L^p space. Recall K_r , (2.16), and $\|f\|_{p,K}$, (2.17).

LEMMA 3.3. — Let $1 \leq p < \infty$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ and $K = \text{supp } \psi$. (i) If $\varepsilon \geq 0$, then

$$\|\psi G_{R,\varepsilon}^m f\|_p \leq C \|f\|_{p,K_{R+1}}, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (3.17 a)$$

and

$$\|\sqrt{-\Delta + m^2} \psi G_{R,\varepsilon}^m f\|_p \leq C \|f\|_{p,K_{R+1}}, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (3.17 b)$$

with a constant C dependent on ψ but independent of $m \geq 0$ and ε . In particular, $\psi G_{R,\varepsilon}^m$ and $\sqrt{-\Delta + m^2} \psi G_{R,\varepsilon}^m$ define bounded linear operators of $L_{\text{loc}}^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$.

(ii) For $f \in L^p_{\text{loc}}(\mathbb{R}^d)$,

$$\|\psi G_{R,\varepsilon}^m f - \psi G_{R,0}^m f\|_p \rightarrow 0, \quad \varepsilon \downarrow 0, \quad (3.18)$$

uniformly on bounded subsets of m in $[0, \infty)$.

Proof. — We write $G_{R,\varepsilon}$, $G_0(x)$ and $k_0(t, x)$ for $G_{R,\varepsilon}^m$, $G_0^m(x)$ and $k_0^m(t, x)$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Put $K = \text{supp } \psi$ and let φ be a C_0^∞ function with $0 \leq \varphi(x) \leq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on K_R and $\text{supp } \varphi \subseteq K_{R+1}$.

(i) *Proof of (3.17 a).* — Let $f \in \mathcal{S}(\mathbb{R}^d)$. Since $G_{R,\varepsilon}(x-y)$ identically vanishes outside $|x-y| < R$, we have $G_{R,\varepsilon} f = G_{R,\varepsilon}(\varphi f)$ on K , so that

$$\|\psi G_{R,\varepsilon} f\|_p = \|\psi G_{R,\varepsilon}(\varphi f)\|_p \leq \|\psi\|_\infty \|G_{R,\varepsilon}(\varphi f)\|_p.$$

Therefore, the proof of (3.17 a) is reduced to that of

$$\|G_{R,\varepsilon} u\|_p \leq \|u\|_p, \quad u \in L^p(\mathbb{R}^d). \quad (3.19)$$

since $\|\varphi f\|_p \leq \|f\|_{p, K_{R+1}}$. But this follows by Young's inequality with

$$\int G_{R,\varepsilon}(x) dx \leq (m+1)^{-1} \leq 1.$$

Proof of (3.17 b). — By the same argument as above, the proof is reduced to that of

$$\|(-\Delta + m^2)^{1/2} \psi G_{R,\varepsilon} u\|_p \leq C \|u\|_p, \quad u \in L^p(\mathbb{R}^d), \quad (3.20)$$

with a constant C independent of $m \geq 0$ and $\varepsilon \geq 0$. In view of (3.5), let with $u \in L^p(\mathbb{R}^d)$,

$$(K_{R,\varepsilon} u)(x) = e^{-(m+1)\varepsilon} \int \chi_R(x-y) k_0(\varepsilon, x-y) u(y) dy \quad (3.21 a)$$

or

$$(\mathcal{F}(K_{R,\varepsilon} u))(p) = (2\pi)^{-d/2} \int \hat{\chi}_R(\eta) \exp[-\varepsilon((p-\eta)^2 + m^2)^{1/2} - \varepsilon] d\eta \hat{u}(p). \quad (3.21 b)$$

It is easy to see by Young's inequality that $K_{R,\varepsilon}$ is a bounded linear operator on $L^p(\mathbb{R}^d)$:

$$\|K_{R,\varepsilon} u\|_p \leq e^{-(m+1)\varepsilon} \|u\|_p \leq \|u\|_p. \quad (3.22)$$

Therefore showing (3.20) is equivalent to showing

$$\| [(-\Delta + m^2)^{1/2} \psi G_{R,\varepsilon} - \psi K_{R,\varepsilon}] u \|_p \leq C \|u\|_p, \quad u \in L^p(\mathbb{R}^d), \quad (3.23)$$

with a constant C independent of $m \geq 0$ and $\varepsilon \geq 0$. The operator in the bracket $[\dots]$ on the left of (3.23) is represented as the pseudo-differential operator

$$(b(X, D) u)(x) = (2\pi)^{-d/2} \int e^{ixp} b(x, p) \hat{u}(p) dp, \quad u \in \mathcal{S}(\mathbb{R}^d), \quad (3.24 a)$$

with symbol

$$b(x, p) = (2\pi)^{-d} \iint e^{ix\xi} \left(\frac{((p+\xi)^2 + m^2)^{1/2}}{((p-\eta)^2 + m^2)^{1/2} + 1} - 1 \right) \\ \times \exp[-\varepsilon((p-\eta)^2 + m^2)^{1/2} - \varepsilon] \hat{\psi}(\xi) \hat{\chi}_R(\eta) d\xi d\eta \\ \equiv b_1(x, p) + b_2(x, p). \quad (3.24 b)$$

Here

$$b_1(x, p) = (2\pi)^{-d} \iint e^{ix\xi} (\chi_R(p+\xi) + \psi_R(p+\xi) \chi_R(p-\eta)) (\dots) \\ \times \exp[-\varepsilon((p-\eta)^2 + m^2)^{1/2} - \varepsilon] \hat{\psi}(\xi) \hat{\chi}_R(\eta) d\xi d\eta, \quad (3.25 a)$$

$$b_2(x, p) = (2\pi)^{-d} \iint e^{ix\xi} \psi_R(p+\xi) \psi_R(p-\eta) (\dots) \\ \times \exp[-\varepsilon((p-\eta)^2 + m^2)^{1/2} - \varepsilon] \hat{\psi}(\xi) \hat{\chi}_R(\eta) d\xi d\eta, \quad (3.25 b)$$

with the same cutoff functions $\chi_R(p)$ and $\psi_R(p) = 1 - \chi_R(p)$ as in the proof of Lemma 2.2. Note here that

$$(\chi_R(p+\xi) + \psi_R(p+\xi) \chi_R(p-\eta)) + \psi_R(p+\xi) \psi_R(p-\eta) = 1.$$

To show (3.23) we need to show for $1 \leq p < \infty$ that

$$\|b_1(X, D)u\|_p \leq C \|u\|_p, \quad (3.26 a)$$

$$\|b_2(X, D)u\|_p \leq C \|u\|_p, \quad u \in \mathcal{S}(\mathbb{R}^d), \quad (3.26 b)$$

with a constant C independent of $m \geq 0$ and $\varepsilon \geq 0$. We can show (3.26 b), using a general result [19], Theorem 3, since the symbol $b_2(x, p)$ satisfies that for every multi-index α there exists a constant C_α independent of $m \geq 0$ and $\varepsilon \geq 0$ such that for $|\alpha| \leq d+1$,

$$|\partial_p^\alpha b_2(x, p)| \leq C_\alpha (1+p^2)^{-(|\alpha|+1)/2}, \quad x \in \mathbb{R}^d, \quad p \in \mathbb{R}^d.$$

To get (3.26 a) write $b_1(X, D)$ as

$$(b_1(X, D)u)(x) = \int K(x, x-y) u(y) dy, \quad (3.27)$$

where

$$K(x, z) = (2\pi)^{-d} \int e^{izp} b_1(x, p) dp \\ = (2\pi)^{-2d} \iiint e^{izp} a(x, p, \xi, \eta) d\xi d\eta dp. \quad (3.28)$$

with

$$a(x, p, \xi, \eta) = e^{ix\xi} (\chi_R(p + \xi) + \psi_R(p + \xi) \chi_R(\eta)) \left(\frac{((p + \xi)^2 + m^2)^{1/2}}{(\eta^2 + m^2)^{1/2} + 1} - 1 \right) \\ \times \exp[-\varepsilon((\eta^2 + m^2)^{1/2} + 1)] \hat{\psi}(\xi) \hat{\chi}_R(p - \eta).$$

Then it is easy to verify that $K(x, z)$ is a bounded function. Therefore to show (3.26a) we have only to show that for $0 < \delta < 1$ there exists a constant C_δ depending on δ such that

$$|K(x, z)| \leq C_\delta |z|^{-d-1+\delta}, \quad (x, z) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (3.29)$$

To do so let α be a multi-index with $|\alpha| = d$. Then we have by integration by parts

$$(2\pi)^{2d} z^\alpha K(x, z) = i^d \iiint e^{izp} \partial_p^\alpha a(x, p, \xi, \eta) d\xi d\eta dp \\ = -i^{d+1} \iiint \sum_{j=1}^d z_j |z|^{-2} \partial_{p_j} (e^{izp} - e^{-iz\xi}) \partial_p^\alpha a(x, p, \xi, \eta) d\xi d\eta dp \\ = i^{d+1} \iiint \sum_{j=1}^d z_j |z|^{-2} (e^{izp} - e^{-iz\xi}) \partial_{p_j} \partial_p^\alpha a(x, p, \xi, \eta) d\xi d\eta dp.$$

Hence, noting that $|e^{izp} - e^{-iz\xi}| \leq 2|z|^\delta |p + \xi|^\delta$ for $0 < \delta < 1$, we obtain

$$|z^\alpha K(x, z)| \leq (2\pi)^{-2d} |z|^{-1+\delta} \\ \times \iiint_{\{|p + \xi| \leq R \text{ or } |\eta| \leq R\}} |p + \xi|^\delta |\partial_{p_j} \partial_p^\alpha a(x, p, \xi, \eta)| d\xi d\eta dp.$$

The integral on the right is convergent, because

$$|\partial_{p_j} \partial_p^\alpha a(x, p, \xi, \eta)| \leq C |\hat{\psi}(\xi)| \left[\sum_{l=0}^d |p + \xi|^{-l} \right] \left[\sum_{l=0}^{d+1} \sum_{|\beta|=l} |\partial_p^\beta \hat{\chi}_R(p - \eta)| \right],$$

with a constant C independent of $m \geq 0$ and $\varepsilon \geq 0$. This yields (3.29).

(ii) *Proof of (3.18).* — By the same argument as used with ϕ in the proof of (i) it suffices to show that for $u \in L^p(\mathbb{R}^d)$,

$$\|G_{R, \varepsilon} u - G_{R, 0} u\|_p \rightarrow 0, \quad \varepsilon \downarrow 0. \quad (3.30)$$

To get (3.30) we see by applying Young's inequality

$$\|G_{R, \varepsilon} u - G_{R, 0} u\|_p \leq \int |G_{R, \varepsilon}(x) - G_{R, 0}(x)| dx \|u\|_p.$$

Therefore we have only to show that

$$g(\varepsilon) \equiv \int |G_{R, \varepsilon}(x) - G_{R, 0}(x)| dx \rightarrow 0, \quad \varepsilon \downarrow 0. \quad (3.31)$$

uniformly on bounded $m \geq 0$. We have from (3.5)

$$\begin{aligned} g(\varepsilon) &\leq \int k_0(\varepsilon, y) dy \int \chi_R(x) | [e^{-(m+1)\varepsilon} G_0(x-y) - G_0(x)] | dx \\ &\leq (1 - e^{-(m+1)\varepsilon}) + \int k_0(\varepsilon, y) dy \int \chi_R(x) | G_0(x-y) - G_0(x) | dx, \end{aligned}$$

because $\int k_0(\varepsilon, y) dy = (m+1) \int G_0(x) dx = 1$. The first term in the last member above tends to zero as $\varepsilon \downarrow 0$, uniformly on bounded $m \geq 0$. Therefore we have to prove that so does the second term, which is equal, by changing the integration variables $y = \varepsilon^{1/2} y'$ (writing y instead of y'), to

$$\begin{aligned} \int dy k_0(\varepsilon, \varepsilon^{1/2} y) \varepsilon^{d/2} \int \chi_R(x) | [G_0(x - \varepsilon^{1/2} y) - G_0(x)] | dx \\ = \int_{|y| > r} dy \dots + \int_{|y| \leq r} dy \dots \equiv g_1(\varepsilon) + g_2(\varepsilon). \end{aligned}$$

Let $\delta > 0$ be arbitrary. Then there exists $r_0 > 0$ such that for $r \geq r_0$,

$\int_{|y| > r} k_0(\varepsilon, \varepsilon^{1/2} y) \varepsilon^{d/2} dy < \delta/2$, uniformly in $0 < \varepsilon < 1$ and $m \geq 0$. For the proof see [9], Lemma 3.3, where the uniformity in m is not mentioned, but the proof there is still valid. It follows that $g_1(\varepsilon) < \delta$. On the other hand, we can see $g_2(\varepsilon)$ satisfies that for $|y| \leq r_0$,

$$g_2(\varepsilon) \leq \int_{-\mathbf{R}}^{\mathbf{R}} | [G_0(x - \varepsilon^{1/2} y) - G_0(x)] | dx,$$

which tends to zero as $\varepsilon \downarrow 0$, since $G_0(x)$ is integrable. Thus we have shown (3.31), ending the proof of Lemma 3.3.

Now we are in a position to derive an integral representation for v in L^2 with $H_A^m v$ in L_{loc}^1 . Let $\varepsilon > 0$. By Lemma 3.1, $G_{R, \varepsilon}^m(x-y)$ is, for x fixed, a real-valued C_0^∞ function in y , so that by (3.2),

$$([H_A^m + 1] v, G_{R, \varepsilon}^m(x - \cdot)) = (v, [H_A^m + 1] G_{R, \varepsilon}^m(x - \cdot)) \quad (3.32a)$$

or

$$\begin{aligned} \int G_{R, \varepsilon}^m(x-y) ([H_A^m + 1] v)(y) dy \\ = \int \overline{([H_A^m + 1] G_{R, \varepsilon}^m(x - \cdot)) (y)} v(y) dy. \end{aligned} \quad (3.32b)$$

We use the expression (2.8) for H_A^m to rewrite the kernel of the integral on the right-hand side of (3.32b):

$$([H_A^m + 1] G_{R, \varepsilon}^m(x - \cdot)) (y) = S_{R, \varepsilon}^m(x-y) - \overline{E_{R, \varepsilon}^m(x, y)} - \overline{F_{R, \varepsilon}^m(x, y)}, \quad (3.33)$$

where $S_{R,\varepsilon}^m(x-y)$ is defined by (3.6), $\overline{E_{R,\varepsilon}^m(x,y)}$ and $\overline{F_{R,\varepsilon}^m(x,y)}$ are respectively the complex conjugates of

$$E_{R,\varepsilon}^m(x,y) = \int_{|y'| > 0} [e^{iy' A(y+y'/2)} - 1 - I_{\{|y'| < 1\}} iy' A(y)] \times G_{R,\varepsilon}^m(x-y-y') n^m(dy'), \quad (3.34)$$

and

$$F_{R,\varepsilon}^m(x,y) = \int_{0 < |y'| < 1} iy' A(y) [G_{R,\varepsilon}^m(x-y-y') - G_{R,\varepsilon}^m(x-y)] n^m(dy'). \quad (3.35)$$

The following two lemmas are concerned with the integral operators with kernels $E_{R,\varepsilon}^m(x,y)$ and $F_{R,\varepsilon}^m(x,y)$:

$$(E_{R,\varepsilon}^m u)(x) = \int E_{R,\varepsilon}^m(x,y) u(y) dy, \quad (3.36)$$

$$(F_{R,\varepsilon}^m u)(x) = \int F_{R,\varepsilon}^m(x,y) u(y) dy. \quad (3.37)$$

There $\|\cdot\|$ stands for the L^2 norm.

LEMMA 3.4. — Let $\psi \in C_0^\infty(\mathbb{R}^d)$. (i) If $\varepsilon \geq 0$, then $E_{R,\varepsilon}^m u$ exists with $u \in L^2(\mathbb{R}^d)$, and

$$\|\psi E_{R,\varepsilon}^m u\| \leq C_1 \|u\|, \quad u \in L^2(\mathbb{R}^d), \quad (3.38a)$$

while if $\varepsilon > 0$, then

$$\|\partial(\psi E_{R,\varepsilon}^m u)\| \leq C_2 \|u\|, \quad u \in L^2(\mathbb{R}^d), \quad (3.38b)$$

with constants C_1 and C_2 dependent on ψ but independent of $m \geq 0$ and ε .

(ii) For every $u \in L^2(\mathbb{R}^d)$,

$$\|\psi E_{R,\varepsilon}^m u - \psi E_{R,0}^m u\| \rightarrow 0, \quad \varepsilon \downarrow 0, \quad (3.39a)$$

$$\|\partial(\psi E_{R,\varepsilon}^m u) - \partial(\psi E_{R,\varepsilon'}^m u)\| \rightarrow 0, \quad \varepsilon, \varepsilon' \downarrow 0, \quad (3.39b)$$

uniformly on bounded subsets of m in $[0, \infty)$.

LEMMA 3.5. — Let $\psi \in C_0^\infty(\mathbb{R}^d)$. (i) If $\varepsilon \geq 0$, then $F_{R,\varepsilon}^m u$ exists with $u \in L^2(\mathbb{R}^d)$, and

$$\|\psi F_{R,\varepsilon}^m u\| \leq C_1 \|u\|, \quad u \in L^2(\mathbb{R}^d), \quad (3.40a)$$

while if $\varepsilon > 0$, then

$$\|\partial(\psi F_{R,\varepsilon}^m u)\| \leq C_2 \|u\|, \quad u \in L^2(\mathbb{R}^d), \quad (3.40b)$$

with constants C_1 and C_2 dependent on ψ but independent of $m \geq 0$ and ε .

(ii) For every $u \in L^2(\mathbb{R}^d)$,

$$\|\psi F_{R,\varepsilon}^m u - \psi F_{R,0}^m u\| \rightarrow 0, \quad \varepsilon \downarrow 0, \quad (3.41a)$$

$$\|\partial(\psi F_{R,\varepsilon}^m u) - \partial(\psi F_{R,\varepsilon'}^m u)\| \rightarrow 0, \quad \varepsilon, \varepsilon' \downarrow 0, \quad (3.41 b)$$

uniformly on bounded subsets of m in $[0, \infty)$.

Based on Lemmas 3.1-5, we now give the main theorem of this section, Theorem 3.6. The proofs of Lemmas 3.4 and 3.5 will be postponed to the end of the proof of Theorem 3.6. We shall write for $x \neq y$,

$$G_R^m(x-y) = G_{R,0}^m(x-y), \quad E_R^m(x, y) = E_{R,0}^m(x, y), \\ F_R^m(x, y) = F_{R,0}^m(x, y).$$

THEOREM 3.6. — *Let $v \in L^2(\mathbb{R}^d)$ and let $H_A^m v \in L_{\text{loc}}^1(\mathbb{R}^d)$. Then v admits an integral representation*

$$v(x) = \int G_R^m(x-y) ([H_A^m + 1]v)(y) dy + \int Q_R^m(x-y) v(y) dy \\ + \int E_R^m(x, y) v(y) dy + \int F_R^m(x, y) v(y) dy \\ \equiv G_R^m[H_A^m + 1]v + Q_R^m v + E_R^m v + F_R^m v. \quad (3.42)$$

Here $G_R^m \equiv G_{R,0}^m$, Q_R^m , $E_R^m \equiv E_{R,0}^m$ and $F_R^m \equiv F_{R,0}^m$ are defined by (3.7), (3.14), (3.36) and (3.37), respectively. Consequently, v has a decomposition $v = v_1 + v_2$ such that, for every $\psi \in C_0^\infty(\mathbb{R}^d)$, both ψv_1 and $\sqrt{-\Delta + m^2} \psi v_1$ are in $L^1(\mathbb{R}^d)$, and both ψv_2 and $\sqrt{-\Delta + m^2} \psi v_2$ are in $L^2(\mathbb{R}^d)$.

Proof. — We simplify notation to suppress the superscript “ m ” of H_A^m , $G_{R,\varepsilon}^m$, $Q_{R,\varepsilon}^m$, $S_{R,\varepsilon}^m$, $E_{R,\varepsilon}^m$ and $F_{R,\varepsilon}^m$. Let $v \in L^2$ and $H_A v \in L_{\text{loc}}^1$. Then we get for $\varepsilon > 0$ from (3.32) and (3.33)

$$S_{R,\varepsilon} v = G_{R,\varepsilon} [H_A + 1]v + E_{R,\varepsilon} v + F_{R,\varepsilon} v. \quad (3.43)$$

By Lemma 3.2 (ii), (3.13) and (3.14), the left-hand side of (3.43), $S_{R,\varepsilon} v$, converges to $v - Q_R v$ in L^2 , as $\varepsilon \downarrow 0$. As to the right-hand side of (3.43), $G_{R,\varepsilon} [H_A + 1]v$ converges to $G_R [H_A + 1]v$ in L_{loc}^1 , by Lemma 3.3 (ii), (3.18), $E_{R,\varepsilon} v$ to $E_R v$ in L_{loc}^2 , by Lemma 3.4 (ii), (3.39a) and $F_{R,\varepsilon} v$ to $F_R v$ in L_{loc}^2 , by Lemma 3.5 (ii), (3.41a). This proves (3.42). Once (3.42) is established, the remaining assertion is immediately seen by taking $v_1 = G_R [H_A + 1]v$ and $v_2 = Q_R v + E_R v + F_R v$, since $(-\Delta + m^2)^{1/2} \psi G_R [H_A + 1]v$ is in $L^1(\mathbb{R}^d)$ [by Lemma 3.3 (i), (3.17b)], and the three $Q_R v$, $\psi E_R v$ and $\psi F_R v$ are all in the Sobolev space $H^1(\mathbb{R}^d)$ [by Lemma 3.2 (ii), (3.15), Lemma 3.4 (ii), (3.39a, b) and Lemma 3.5 (ii), (3.41a, b)]. Thus Theorem 3.6 is proved.

Finally we prove Lemmas 3.4 and 3.5.

Proof of Lemma 3.4. — We write $G_{R,\varepsilon}$, $E_{R,\varepsilon}$ and $n(dy')$ for $G_{R,\varepsilon}^m$, $E_{R,\varepsilon}^m$ and $n^m(dy')$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Put $K = \text{supp } \psi$ and let φ be a C_0^∞ function with $0 \leq \varphi(x) \leq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on K_R and $\text{supp } \varphi \subseteq K_{R+1}$.

(i) *Proof of (3.38a).* Changing, in (3.34), the integration variables $y + y' = z$ (writing y instead of z) and using the rotational invariance of

the Lévy measure $n(dy')$, we have

$$(E_{R,\varepsilon} u)(x) = \int G_{R,\varepsilon}(x-y) f_1(y) dy + \int G_{R,\varepsilon}(x-y) f_2(y) dy \\ \equiv (E_{R,\varepsilon}^1 u)(x) + (E_{R,\varepsilon}^2 u)(x), \quad (3.44)$$

where

$$f_1(y) = \int_{|y'| \geq 1} [e^{-iy' A(y+y'/2)} - 1] u(y+y') n(dy'), \quad (3.45a)$$

$$f_2(y) = \int_{0 < |y'| < 1} [e^{-iy' A(y+y'/2)} - 1 + iy' A(y+y')] u(y+y') n(dy'). \quad (3.45b)$$

Then we see by Lemma 3.1 (ii), (3.10), that $E_{R,\varepsilon}^1 u$ on the right of (3.44) satisfies

$$\|E_{R,\varepsilon}^1 u\| = \|G_{R,\varepsilon} f_1\| \leq C \|f_1\| \leq 2 C n_\infty \|u\|,$$

with the constants $n_\infty \equiv n_\infty^m$ in (2.6a) and C in (3.10).

Similarly we have for $E_{R,\varepsilon}^2 u$,

$$\|\Psi E_{R,\varepsilon}^2 u\| = \|\Psi G_{R,\varepsilon}(\varphi f_2)\| \leq C \|\Psi\|_\infty \|\varphi f_2\| \leq C \|\Psi\|_\infty \|f_2\|_{2, K_{R+1}},$$

where $\|\cdot\|_{2, K_{R+1}}$ stands for the local L^2 norm in (2.17). Therefore the proof of (3.38a) will be accomplished if we can show

$$\|f_2\|_{2, K_{R+1}} \leq C(K_{R+1}) \|u\|_{2, K_{R+2}} \leq C(K_{R+1}) \|u\|, \quad (3.46)$$

with a constant $C(K_{R+1})$. The proof of (3.46) will be done by an analogous argument used to prove (2.20).

Proof of (3.38b). — Let $\varepsilon > 0$ and put $\partial_j = \partial_{x_j}$, $1 \leq j \leq d$. We can see similarly to the proof of (3.38a) above

$$\|\partial_j E_{R,\varepsilon}^1 u\| = \|\partial_j G_{R,\varepsilon} f_1\| \leq C \|f_1\| \leq 2 C n_\infty \|u\|,$$

and by Lemma 3.1 (ii) and by (3.46),

$$\|\partial_j(\Psi E_{R,\varepsilon}^2 u)\| = \|\partial_j(\Psi G_{R,\varepsilon}(\varphi f_2))\| \\ \leq \|\partial_j \Psi\|_\infty \|G_{R,\varepsilon}(\varphi f_2)\| + \|\Psi\|_\infty \|\partial_j(G_{R,\varepsilon}(\varphi f_2))\| \\ \leq C \|\partial_j \Psi\|_\infty \|\varphi f_2\| + C \|\Psi\|_\infty \|\varphi f_2\| \\ \leq C_\Psi C \|f_2\|_{2, K_{R+1}} \leq C_\Psi C C(K_{R+1}) \|u\|,$$

with $C_\Psi = \|\Psi\|_\infty + \|\partial_j \Psi\|_\infty$.

(ii) *Proof of (3.39a, b).* — By the same arguments as in the proof of (i), we can see that for $\varepsilon > 0$,

$$\|E_{R,\varepsilon}^1 u - E_{R,\varepsilon'}^1 u\| = \|G_{R,\varepsilon} f_1 - G_{R,\varepsilon'} f_1\|, \\ \|\Psi E_{R,\varepsilon}^2 u - \Psi E_{R,\varepsilon'}^2 u\| = \|\Psi E_{R,\varepsilon}^2(\varphi f_2) - \Psi E_{R,\varepsilon'}^2(\varphi f_2)\|,$$

and for $\varepsilon, \varepsilon' > 0$,

$$\|\partial_j E_{R,\varepsilon}^1 u - \partial_j E_{R,\varepsilon'}^1 u\| = \|\partial_j G_{R,\varepsilon} f_1 - \partial_j G_{R,\varepsilon'} f_1\|, \\ \|\partial_j(\Psi E_{R,\varepsilon}^2 u) - \partial_j(\Psi E_{R,\varepsilon'}^2 u)\| = \|\partial_j(\Psi E_{R,\varepsilon}^2(\varphi f_2)) - \partial_j(\Psi E_{R,\varepsilon'}^2(\varphi f_2))\|,$$

all the four of which tend to zero as $\varepsilon \downarrow 0$ and $\varepsilon, \varepsilon' \downarrow 0$, uniformly on bounded $m \geq 0$, by Lemma 3.1 (iii). This shows (3.39 a, b), ending the proof of Lemma 3.4.

Proof of Lemma 3.5. — We write again $G_{R, \varepsilon}$, $F_{R, \varepsilon}$ and $n(dy) = n(y) dy$ for $G_{R, \varepsilon}^m$, $F_{R, \varepsilon}^m$ and $n^m(dy) = n^m(y) dy$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Put $K = \text{supp } \psi$ and let φ be a C_0^∞ function with $0 \leq \varphi(x) \leq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on K_{R+1} and $\text{supp } \varphi \subseteq K_{R+2}$.

(i) First we show that for $\varepsilon \geq 0$, $F_{R, \varepsilon} u$ exists with $u \in L^2(\mathbb{R}^d)$. Put

$$F_{R, \varepsilon}(x, y) = i \sum_{k=1}^d A_k(y) L_{k, R, \varepsilon}(x - y), \quad (3.47 a)$$

with

$$L_{k, R, \varepsilon}(x) = \int_{0 < |y| < 1} y_k [G_{R, \varepsilon}(x - y) - G_{R, \varepsilon}(x)] n(dy), 1 \leq k \leq d. \quad (3.47 b)$$

Since $L_{k, R, \varepsilon}$ has compact support, we have only to show that for some constant $0 < \delta < 1$,

$$|L_{k, R, \varepsilon}(x)| \leq c_0 |x|^{\delta-d}, \quad (3.48)$$

near $x=0$, with a constant c_0 independent of $m \geq 0$ and $\varepsilon \geq 0$. Write (3.47 b) as

$$L_{k, R, \varepsilon}(x) = \int_{Y_1} + \int_{Y_2} + \int_{Y_3} \equiv L_1(x) + L_2(x) + L_3(x), \quad (3.49)$$

dividing the integration region $0 < |y| < 1$ into the three parts:

$$Y_1 = \{y; 0 < |y| < 1 \wedge (|x|/2)\}, Y_2 = \{y; 0 < |y| < 1, |x - y| < |x|/2\}$$

and the rest Y_3 .

In the following argument, the constants such as c_1, c_2, c_0 , are independent of $m \geq 0$ and $\varepsilon \geq 0$. Since

$$L_1(x) = - \int_{Y_1} n(dy) \int_0^1 y_k \sum_{j=1}^d y_j \partial_{x_j} G_{R, \varepsilon}(x - \theta y) d\theta,$$

we have by (3.4 c) and (2.2) together with (2.3),

$$\begin{aligned} |L_1(x)| &\leq c_1 \int_{Y_1} |y|^{1-d} dy \int_0^1 |x - \theta y|^{-d} d\theta \\ &\leq 2^d c_1 |x|^{-d} \int_{0 < |y| < |x|/2} |y|^{1-d} dy \leq c_0 |x|^{1-d}, \end{aligned} \quad (3.50)$$

because $|y| < |x|/2$ implies $|x - \theta y| \geq |x| - |y| \geq |x|/2$. For $L_2(x)$, we have by (3.4a, b) and (2.2), in case $d > 1$,

$$\begin{aligned} |L_2(x)| &\leq \int_{Y_2} |y| (|G_{R,\varepsilon}(x-y)| + |G_{R,\varepsilon}(x)|) n(dy) \\ &\leq c_1 \int_{Y_2} |y|^{-d} (|x-y|^{1-d} + |x|^{1-d}) dy \\ &\leq 2^d c_1 |x|^{-d} \int_{|x-y| < |x|/2} (|x-y|^{1-d} + |x|^{1-d}) dy \leq c_0 |x|^{1-d}, \quad (3.51a) \end{aligned}$$

because $|x-y| < |x|/2$ implies $|y| > |x|/2$, and similarly, in case $d = 1$,

$$|L_2(x)| \leq c_1 (1 + |\log |x||) \leq c_0 |x|^{\delta-1}, \quad (3.51b)$$

with $0 < \delta < 1$. Since $|y| \geq |x|/2$ and $|x-y| \geq |x|/2$ in Y_3 , we have by (3.4a, b) and (2.2), in case $d > 1$,

$$\begin{aligned} |L_3(x)| &\leq c_1 \int_{Y_3} |y|^{-d} (|x-y|^{1-d} + |x|^{1-d}) dy \\ &\leq (2^d + 1) c_1 |x|^{1-d} \int_{\substack{1 > |y| \geq |x|/2 \\ |x-y| \geq |x|/2}} |y|^{-d} dy \\ &\leq c_2 |x|^{1-d} (1 + |\log |x||) \leq c_0 |x|^{\delta-d}, \quad (3.52a) \end{aligned}$$

with $0 < \delta < 1$, and similarly, in case $d = 1$,

$$|L_3(x)| \leq c_1 (1 + |\log |x||)^2 \leq c_0 |x|^{\delta-1}, \quad (3.52b)$$

with $0 < \delta < 1$. Thus (3.50), (3.51a, b) and (3.52a, b) with (3.49) yield (3.48).

Next we show (3.40a, b). We give a unified proof using the theory of singular integrals, though it is seen that (3.47) and (3.48) imply (3.40a).

Proof of (3.40a). — For $\varepsilon \geq 0$, we have

$$\begin{aligned} \psi(x)(F_{R,\varepsilon}u)(x) &= \psi(x)(F_{R,\varepsilon}(\varphi u))(x) \\ &= i \psi(x) \sum_{k=1}^d \int_{0 < |y'| < 1} (f_{k,\varepsilon}(x-y') - f_{k,\varepsilon}(x)) y'_k n(dy') \\ &= i \psi(x) \sum_{k=1}^d \lim_{\delta \downarrow 0} \int_{\delta \leq |y'| < 1} (f_{k,\varepsilon}(x-y') - f_{k,\varepsilon}(x)) y'_k n(dy') \\ &= i \psi(x) \sum_{k=1}^d \lim_{\delta \downarrow 0} N_{k,\delta}(f_{k,\varepsilon}), \quad (3.53) \end{aligned}$$

with $N_{k,\delta}$ in (2.22) with δ in place of ε , because $n(y')$ is rotation-invariant. Here

$$f_{k,\varepsilon}(x) = (G_{R,\varepsilon}(\varphi A_k u))(x) = \int G_{R,\varepsilon}(x-y) \varphi(y) A_k(y) u(y) dy, \quad (3.54)$$

which is a C_0^∞ function. Then it follows from (3.53) with (2.23) that for $\varepsilon \geq 0$,

$$\begin{aligned} \|\psi F_{R,\varepsilon} u\| &= \|\psi F_{R,\varepsilon}(\varphi u)\| \leq \|\psi\|_\infty \|F_{R,\varepsilon}(\varphi u)\| = C_\psi \lim_{\delta \downarrow 0} \left\| \sum_{k=1}^d N_{k,\delta}(f_{k,\varepsilon}) \right\| \\ &\leq C_\psi C_0 \sum_{k=1}^d \|f_{k,\varepsilon}\| = C_\psi C_0 \sum_{k=1}^d \|G_{R,\varepsilon}(\varphi A_k u)\| \\ &\leq C_\psi C_0 C \sum_{k=1}^d \|\varphi A_k u\| \leq C_\psi C_0 C a(K_{R+2}) \|u\|, \end{aligned}$$

with $C_\psi = \|\psi\|_\infty$, C_0 a constant independent of $0 < \delta < 1$ and $m \geq 0$, and $a(K_{R+2})$ in (2.5a) with K_{R+2} in place of K . Here we have used, in the second last inequality, Lemma 3.1 (ii). This shows (3.40a).

Proof of (3.40b). — The proof proceeds in the same way as above. We have, this time for $\varepsilon > 0$,

$$\begin{aligned} \|\partial_j(\psi F_{R,\varepsilon} u)\| &= \|\partial_j(\psi F_{R,\varepsilon}(\varphi u))\| \\ &\leq \|\partial_j \psi\|_\infty \|F_{R,\varepsilon}(\varphi u)\| + \|\psi\|_\infty \|\partial_j(F_{R,\varepsilon}(\varphi u))\|. \end{aligned}$$

Here $F_{R,\varepsilon}(\varphi u)$ in the first term of the third member above has already been dealt with in the proof of (3.40a). For the second term, since $\partial_j N_{k,\delta}(f_{k,\varepsilon}) = N_{k,\delta}(\partial_j f_{k,\varepsilon})$ and the L^2 -limits of $N_{k,\delta}(f_{k,\varepsilon})$ and $N_{k,\delta}(\partial_j f_{k,\varepsilon})$ for $\delta \downarrow 0$ exist, we have

$$\begin{aligned} \|\partial_j(F_{R,\varepsilon}(\varphi u))\| &= \lim_{\delta \downarrow 0} \left\| \sum_{k=1}^d N_{k,\delta}(\partial_j f_{k,\varepsilon}) \right\| \\ &\leq C_0 \sum_{k=1}^d \|\partial_j f_{k,\varepsilon}\| = C_0 \sum_{k=1}^d \|\partial_j(G_{R,\varepsilon}(\varphi A_k u))\| \\ &\leq C_0 C \sum_{k=1}^d \|\varphi A_k u\| \leq C_0 C a(K_{R+2}) \|u\|. \end{aligned}$$

This shows (3.40b).

(ii) *Proof of (3.41a).* We have for $\varepsilon > 0$,

$$\begin{aligned} \|\psi F_{R,\varepsilon} u - \psi F_R u\| &= \|\psi F_{R,\varepsilon}(\varphi u) - \psi F_R(\varphi u)\| \\ &\leq \|\psi\|_\infty \|F_{R,\varepsilon}(\varphi u) - F_R(\varphi u)\| \\ &= C_\psi \lim_{\delta \downarrow 0} \left\| \sum_{k=1}^d N_{k,\delta}(f_{k,\varepsilon} - f_{k,0}) \right\| \\ &\leq C_\psi C_0 \sum_{k=1}^d \|f_{k,\varepsilon} - f_{k,0}\| \\ &= C_\psi C_0 \sum_{k=1}^d \|G_{R,\varepsilon}(\varphi A_k u) - G_R(\varphi A_k u)\|, \end{aligned}$$

with $C_\psi = \|\psi\|_\infty$, which tends to zero as $\varepsilon \downarrow 0$, uniformly for bounded $m \geq 0$, by Lemma 3.1 (iii). This shows (3.41 a).

Proof of (3.41 b). — The proof is similar to the above. In fact, we have for $\varepsilon, \varepsilon' > 0$,

$$\begin{aligned} \|\partial_j(\psi F_{R, \varepsilon} u) - \partial_j(\psi F_{R, \varepsilon'} u)\| &= \|\partial_j(\psi F_{R, \varepsilon}(\varphi u)) - \partial_j(\psi F_{R, \varepsilon'}(\varphi u))\| \\ &\leq \|\partial_j \psi\|_\infty \|F_{R, \varepsilon}(\varphi u) - F_{R, \varepsilon'}(\varphi u)\| \\ &\quad + \|\psi\|_\infty \|\partial_j F_{R, \varepsilon}(\varphi u) - \partial_j F_{R, \varepsilon'}(\varphi u)\|. \end{aligned}$$

As ε and ε' go to zero, the first term in the last member tends to zero, uniformly for bounded $m \geq 0$, because $F_{R, \varepsilon}(\varphi u) \rightarrow F_R(\varphi u)$, $\varepsilon \downarrow 0$, as seen above. The second term tends to zero, too. In fact, we have

$$\begin{aligned} \|\partial_j F_{R, \varepsilon} u - \partial_j F_{R, \varepsilon'} u\| &= \lim_{\delta \downarrow 0} \left\| \sum_{k=1}^d N_{k, \delta} (\partial_j f_{k, \varepsilon} - \partial_j f_{k, \varepsilon'}) \right\| \\ &\leq C_0 \sum_{k=1}^d \|\partial_j f_{k, \varepsilon} - \partial_j f_{k, \varepsilon'}\| \\ &= C_0 \sum_{k=1}^d \|\partial_j (G_{R, \varepsilon}(\varphi A_k u)) - \partial_j (G_{R, \varepsilon'}(\varphi A_k u))\|, \end{aligned}$$

which tends to zero, as $\varepsilon, \varepsilon' \downarrow 0$, uniformly for bounded $m \geq 0$, by Lemma 3.1 (iii), again. This proves (3.41 b), ending the proof of Lemma 3.5.

4. KATO'S INEQUALITY

For the nonrelativistic quantum Hamiltonian, *i.e.* the nonrelativistic Schrödinger operator, with magnetic fields, Kato [16] established a distributional inequality, which is now called Kato's inequality. In this section we are going to show an analogue for our relativistic quantum Hamiltonian H_A^m with magnetic fields. *The vector potential $A(x)$ is assumed to satisfy (2.5) or (1.4) as in the previous sections.*

In [12], §4.4, we established, though assuming a sufficient regularity and boundedness of $A(x)$, a path integral representation for the semigroup $\exp[-t(H_A^m - m)]$ to derive a domination relation between two positive selfadjoint semigroups $\exp[-t(H_A^m - m)]$ and $\exp[-t(H_0^m - m)]$:

$$(f, \exp[-t(H_A^m - m)]f) \leq (|f|, \exp[-t(H_0^m - m)]|f|), \quad t \geq 0, \quad f \in L^2(\mathbb{R}^d),$$

which is equivalent to an abstract operator version of Kato's inequality (see [23], [7]). The distributional inequality is, however, a stronger statement in the sense that we are before knowing H_A^m to be essentially selfadjoint or selfadjoint. The following theorem may be considered as Kato's inequality for the relativistic quantum Hamiltonian H_A^m in (1.5).

THEOREM 4.1. — Assume $A(x)$ satisfies (2.5) or (1.4). Let $m \geq 0$. If v is in $L^2(\mathbb{R}^d)$ with $H_A^m v$ in $L_{loc}^1(\mathbb{R}^d)$, then the following distributional inequality holds

$$\operatorname{Re}[(\operatorname{sgn} v) H_A^m v] \geq H_0^m |v| = \sqrt{-\Delta + m^2} |v|. \quad (4.1)$$

Here $\operatorname{sgn} v$ is a bounded function in $\mathbb{R}^d$ defined by

$$(\operatorname{sgn} v)(x) = \begin{cases} \overline{v(x)}/|v(x)|, & \text{if } v(x) \neq 0, \\ 0, & \text{if } v(x) = 0. \end{cases}$$

Thanks to the expression (1.5) or (2.8) for H_A^m , the proof of Theorem 4.1 proceeds in a similar way to that of the original Kato's inequality [16]. We need the following lemma, whose proof needs Theorem 3.6. We write $f^\delta = \rho_\delta * f$ for a function $f \in L_{loc}^1$, where $\rho_\delta(x) = \delta^{-d} \rho(x/\delta)$, and $\rho(x)$

is a nonnegative C_0^∞ function with $\operatorname{supp} \rho \subseteq \{|x| \leq 1\}$ and $\int \rho(x) dx = 1$.

LEMMA 4.2. — Let $v \in L^2(\mathbb{R}^d)$ and let $H_A^m v \in L_{loc}^1$. Then $v^\delta \rightarrow v$ in L^2 and $H_A^m v^\delta \rightarrow H_A^m v$ in L_{loc}^1 as $\delta \downarrow 0$.

Proof. — We simply write H_A , H_0 and $n(dy)$ for H_A^m , H_0^m and $n^m(dy)$. Let $v \in L^2(\mathbb{R}^d)$ and let $H_A v \in L_{loc}^1(\mathbb{R}^d)$. Since v^δ is C^∞ and L^2 , we know by Lemma 2.3 that $H_A v^\delta$ is L_{loc}^2 and so L_{loc}^1 . It is well-known that $v^\delta \rightarrow v$ in L^2 as $\delta \downarrow 0$. It suffices to show that $H_A v^\delta \rightarrow H_A v$ in L_{loc}^1 . Then, with the decomposition (2.13 a) of H_A , $H_A = m + I_1 + I_2$, the assumption implies that $v \in L^2$ and $I_2 v \in L_{loc}^1$, because $I_1 v \in L^2$ by (2.14 a). Since I_1 is, by (2.14 a) again, a bounded linear operator on $L^2(\mathbb{R}^d)$, so that $I_1 v^\delta \rightarrow I_1 v$ in L^2 , we have only to show that

$$I_2 v^\delta \rightarrow I_2 v \quad \text{in } L_{loc}^1, \quad \delta \downarrow 0, \quad (4.2)$$

or equivalently,

$$I_2 v^\delta \rightarrow I_2 v \quad \text{in } \mathscr{D}', \quad \delta \downarrow 0, \quad (4.3 a)$$

$$I_2 v^\delta - I_2 v^{\delta'} \rightarrow 0 \quad \text{in } L_{loc}^1, \quad \delta, \delta' \downarrow 0. \quad (4.3 b)$$

It is easy to verify (4.3 a), since $v^\delta \rightarrow v$ in L^2 and $(I_2 v^\delta, \varphi) = (v^\delta, I_2 \varphi)$, $(I_2 v, \varphi) = (v, I_2 \varphi)$, for $\varphi \in C_0^\infty(\mathbb{R}^d)$, by the rotational invariance of the Lévy measure $n(dy)$. To see (4.3 b) let K be an arbitrary compact subset of $\mathbb{R}^d$. We show first that there exists a constant C_K [dependent on K and the behavior of $A(x)$ in a neighbourhood of K] such that

$$\|I_2 u\|_{1,K} \leq C_K [\|H_0 u\|_i + \|u\|_2], \quad i=1, 2, \quad (4.4)$$

for all $u \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u \subseteq K_4$. Recall that $u \in C_0^\infty(\mathbb{R}^d)$ implies $H_0 u \in L^i(\mathbb{R}^d)$, by Lemma 2.3 with $A(x) \equiv 0$. Here $\|\cdot\|_i$ stands for the L^i norm, $i=1, 2$. The following argument proceeds similarly to the proof of Lemma 2.3 (i). Let φ be a C_0^∞ function with $0 \leq \varphi(x) \leq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on K_4 and $\operatorname{supp} \varphi \subseteq K_5$. Then from (2.18) we have by Fatou's lemma,

this time, in the local L^1 norm,

$$\|I_2 u\|_{1, K} \leq \sum_{j=1}^2 \liminf_{\varepsilon \downarrow 0} \|i_1(\varepsilon)\|_{1, K} + \|i_3\|_{1, K}. \quad (4.5)$$

For $i_1(\varepsilon)$ and $i_2(\varepsilon)$ we obtain from (2.20) and (2.21) with the Schwarz inequality

$$\|i_1(\varepsilon)\|_{1, K} \leq |K|^{1/2} \|i_1(\varepsilon)\|_{2, K} \leq c_1 \|u\|_2, \quad (4.6a)$$

$$\|i_2(\varepsilon)\|_{1, K} \leq |K|^{1/2} \|i_2(\varepsilon)\|_{2, K} \leq c_2 \|u\|_2, \quad (4.6b)$$

where c_1 and c_2 are constants independent of $\varepsilon > 0$. For i_3 we see from (2.25) that if $H_0 u$ is L^1 , then

$$\begin{aligned} \|i_3\|_{1, K} &\leq \|i_3\|_1 \leq \|H_0(\varphi u)\|_1 + (m + 2n_\infty) \|\varphi u\|_1, \\ &\leq \|H_0 u\|_1 + |K_4|^{1/2} (m + 2n_\infty) \|u\|_2, \end{aligned} \quad (4.6c)$$

while if $H_0 u$ is L^2 , then by (2.26)

$$\begin{aligned} \|i_3\|_{1, K} &\leq |K|^{1/2} \|i_3\|_{2, K} \leq |K|^{1/2} \|i_3\|_2 \\ &\leq |K|^{1/2} [\|H_0 u\|_2 + (m + 2n_\infty) \|u\|_2], \end{aligned} \quad (4.6d)$$

with the constant $n_\infty \equiv n_\infty^m$ in (2.6a) and $|K|$ the volume of K . Then (4.4) follows from (4.6a, b, c, d) with (4.5).

Now we shall finish the proof of (4.3b). Let ψ be a C_0^∞ function with $0 \leq \psi(x) \leq 1$ in $\mathbb{R}^d$, $\psi(x) = 1$ on K_2 and $\text{supp } \psi \subseteq K_3$. It is here that we need Theorem 3.6, by which $v = v_1 + v_2$ where $\psi v_1, H_0(\psi v_1) \in L^1$ and $\psi v_2, H_0(\psi v_2) \in L^2$. As ψv and ψv_2 are L^2 , so is ψv_1 . Let $0 < \delta, \delta' < 1$. Then $(\psi v_1)^\delta$ and $(\psi v_2)^\delta$ are C_0^∞ with support in K_4 , $H_0(\psi v_1)^\delta = (H_0(\psi v_1))^\delta$ is L^1 , and $H_0(\psi v_2)^\delta = (H_0(\psi v_2))^\delta$ is L^2 . Noting $v_i^\delta = (\psi v_i)^\delta$ on K_1 and $I_2 v_i^\delta = I_2(\psi v_i)^\delta$ on K , we use (4.4) with $u = (\psi v_i)^\delta - (\psi v_i)^{\delta'}$, $i = 1, 2$, to get

$$\begin{aligned} \|I_2 v^\delta - I_2 v^{\delta'}\|_{1, K} &= \left\| \sum_{i=1}^2 (I_2(\psi v_i)^\delta - I_2(\psi v_i)^{\delta'}) \right\|_{1, K} \\ &\leq \sum_{i=1}^2 \|I_2(\psi v_i)^\delta - I_2(\psi v_i)^{\delta'}\|_{1, K} \\ &\leq C_K \sum_{i=1}^2 [\|H_0(\psi v_i)^\delta - H_0(\psi v_i)^{\delta'}\|_i + \|(\psi v_i)^\delta - (\psi v_i)^{\delta'}\|_2] \\ &= C_K \sum_{i=1}^2 [\|(H_0(\psi v_i))^\delta - (H_0(\psi v_i))^{\delta'}\|_i + \|(\psi v_i)^\delta - (\psi v_i)^{\delta'}\|_2], \end{aligned}$$

which tends to zero as $\delta, \delta' \downarrow 0$. This proves (4.3b), completing the proof of Lemma 4.2.

Proof of Theorem 4.1. — We write H_A, H_0 and $n(dy)$ for H_A^m, H_0^m and $n^m(dy)$.

I. First suppose v is C^∞ and L^2 . Then by Lemma 2.2, $H_A v$ is L^2_{loc} , and so L^1_{loc} . For $\varepsilon > 0$ let

$$v_\varepsilon(x) = \sqrt{|v(x)|^2 + \varepsilon^2}. \quad (4.7)$$

Then v_ε is C^∞ and $v_\varepsilon(x) \geq \varepsilon$. A direct calculation shows that

$$(v_\varepsilon(x)^2 - \varepsilon^2)(v_\varepsilon(x+y)^2 - \varepsilon^2) \leq (v_\varepsilon(x)v_\varepsilon(x+y) - \varepsilon^2)^2$$

or

$$|v(x)| |v(x+y)| \leq v_\varepsilon(x)v_\varepsilon(x+y) - \varepsilon^2.$$

Subtracting $|v(x)|^2$ yields

$$-|v(x)| |v(x+y)| + |v(x)|^2 \geq -v_\varepsilon(x)v_\varepsilon(x+y) + v_\varepsilon(x)^2. \quad (4.8)$$

Now we use (2.8) to get

$$\begin{aligned} 2 \operatorname{Re} \overline{v(x)} ([H_A - m] v)(x) &= \overline{v(x)} ([H_A - m] v)(x) + v(x) \overline{([H_A - m] v)(x)} \\ &= - \int_{|y| > 0} \overline{v(x)} [e^{-iy \cdot A(x+y/2)} v(x+y) - v(x) - I_{\{|y| < 1\}} y (\partial_x - iA(x)) v(x)] \\ &\quad + v(x) [e^{iy \cdot A(x+y/2)} \overline{v(x+y)} - \overline{v(x)} - I_{\{|y| < 1\}} y (\partial_x + iA(x)) \overline{v(x)}] n(dy) \\ &\geq 2 \int_{|y| > 0} [-|v(x)| |v(x+y)| + |v(x)|^2 + 2^{-1} I_{\{|y| < 1\}} y \partial |v(x)|^2] n(dy). \end{aligned}$$

Since $\partial v_\varepsilon(x)^2 = \partial(|v(x)|^2 + \varepsilon^2) = \partial|v(x)|^2$, it is seen with (4.8) that

$$\begin{aligned} \operatorname{Re} \overline{v(x)} ([H_A - m] v)(x) &\geq \int_{|y| > 0} [-v_\varepsilon(x)v_\varepsilon(x+y) + v_\varepsilon(x)^2 + 2^{-1} I_{\{|y| < 1\}} y \partial v_\varepsilon(x)^2] n(dy) \\ &= v_\varepsilon(x) ([H_0 - m] v_\varepsilon)(x), \quad (4.9) \end{aligned}$$

pointwise and so in the distribution sense. It follows that

$$\operatorname{Re} [\overline{v(x)/v_\varepsilon(x)}] [H_A - m] v \geq [H_0 - m] v_\varepsilon. \quad (4.10)$$

II. In the general case where $v \in L^2$ and $H_A v \in L^1_{\text{loc}}$, let $v^\delta = \rho_\delta * v$. Then v^δ is C^∞ and L^2 , so that by Lemma 2.3, $H_A v^\delta$ is L^2_{loc} and hence L^1_{loc} . Then by (4.10) we have

$$\operatorname{Re} [\overline{v^\delta/(v^\delta)_\varepsilon}] [H_A - m] v^\delta \geq [H_0 - m] (v^\delta)_\varepsilon, \quad (4.11)$$

for each $\varepsilon > 0$ and $\delta > 0$. For $\varepsilon > 0$ fixed let $\delta \downarrow 0$. Then $v^\delta \rightarrow v$ in L^2 . Taking a subsequence if necessary, we may suppose that $v^\delta \rightarrow v$ in L^2 as well as a. e. It is easy to see that $|(v^\delta)_\varepsilon - v_\varepsilon| \leq \|v^\delta - v\| \leq |v^\delta - v|$, so that $(v^\delta)_\varepsilon \rightarrow v_\varepsilon$ in L^2 as well as a. e., as $\delta \downarrow 0$. Hence $v^\delta/(v^\delta)_\varepsilon \rightarrow \bar{v}/v_\varepsilon$ a. e. and $H_0(v^\delta)_\varepsilon \rightarrow H_0 v_\varepsilon$ in $\mathcal{D}'$. On the other hand, we know by Lemma 4.2 that $H_A v^\delta \rightarrow H_A v$ in L^1_{loc} . Since $|v^\delta/(v^\delta)_\varepsilon| \leq 1$, it follows by the Lebesgue dominated convergence theorem that $(\bar{v}/(v^\delta)_\varepsilon) [H_A - m] v^\delta \rightarrow (\bar{v}/v_\varepsilon) [H_A - m] v$ in L^1_{loc} , as $\delta \downarrow 0$, so that (4.10) holds for v . Now let $\varepsilon \downarrow 0$. Then $\bar{v}/v_\varepsilon \rightarrow \operatorname{sgn} v$ a. e. with $|\bar{v}/v_\varepsilon| \leq 1$, so that the left-hand side of (4.10) converges to $\operatorname{Re}[(\operatorname{sgn} v) [H_A - m] v]$ a. e.,

while the right-hand side of (4.10) converges to $[H_0 - m]|v|$ in $\mathcal{D}'$. Hence we get

$$\operatorname{Re}[(\operatorname{sgn} v)[H_A - m]v] \geq [H_0 - m]|v|$$

and hence (4.1), having proved Theorem 4.1.

Remark. — When both $A(x)$ and $v(x)$ are C^∞ , Theorem 4.1 follows from that in the case $A(x) \equiv 0$ together with the fact (2.11) with v in place of u . The proof of Theorem 4.1 with $A(x) \equiv 0$ is comparatively easy, because then Theorem 3.6 and so Lemma 4.2 is evident.

5. ESSENTIAL SELFADJOINTNESS

We now show the essential selfadjointness of the Weyl quantized relativistic Hamiltonian $H^m = H_A^m + \Phi$ in (1.1), with H_A^m in (1.5) or (2.8).

THEOREM 5.1. — *Let A be an $\mathbb{R}^d$ -valued measurable function in $\mathbb{R}^d$ satisfying (2.5) or (1.4) and let Φ be in $L^2_{\text{loc}}(\mathbb{R}^d)$ with $\Phi(x) \geq 0$ a.e. Then:*

(i) $H^m = H_A^m + \Phi$, and, in particular, H_A^m is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.

(ii) *The unique selfadjoint extension of H_A^m , denoted again by the same H_A^m , is bounded from below by m : $H_A^m \geq m$.*

Proof. — We write H_A and H_0 for H_A^m and H_0^m . (i) We already know H_A is symmetric on $C_0^\infty(\mathbb{R}^d)$ as in (3.1). From the proof of Theorem 4.1, $[H_A - m]u, u \geq 0$ for $u \in C_0^\infty(\mathbb{R}^d)$. In fact, integrating (4.9) with u in place of v , we have with $u_\varepsilon(x) = \sqrt{|u(x)|^2 + \varepsilon^2}$,

$$\operatorname{Re}([H_A - m]u, u) \geq ([H_0 - m]u_\varepsilon, u_\varepsilon) \geq 0,$$

because $w \Phi u_\varepsilon - \varepsilon$ is C_0^∞ and $(H_0 - m)u_\varepsilon = (H_0 - m)w$.

Now we show that $H_A + \Phi$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. Since $H_A + \Phi + 1$ is a strictly positive symmetric operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$, it suffices to show that $v \in L^2(\mathbb{R}^d)$ with

$$(H_A + \Phi + 1)^* v = 0 \tag{5.1}$$

implies that $v = 0$. (5.1) means

$$(H_A + \Phi + 1)v = 0 \tag{5.2}$$

in the sense of distributions. It follows that

$$H_A v = -\Phi v - v$$

is L^1_{loc} , because $\Phi + 1$ is L^2_{loc} and v is L^2 . By Theorem 4.1,

$$H_0|v| \leq \operatorname{Re}[(\operatorname{sgn} v)H_A v] = -\operatorname{Re}((\operatorname{sgn} v)(\Phi v + v)) = -(\Phi + 1)|v|.$$

Hence $(H_0 + 1)|v| \leq 0$ in the distribution sense, or $((H_0 + 1)|v|, \varphi) \leq 0$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi \geq 0$. Then $(H_0 + 1)^{-1}\psi$ is in the Sobolev space $H^1(\mathbb{R}^d)$ and nonnegative. Choose a sequence $\{\varphi_n\}$ in $C_0^\infty(\mathbb{R}^d)$ with $\varphi_n \geq 0$ for all n which converges to $(H_0 + 1)^{-1}\psi$ as $n \rightarrow \infty$. Since $(H_0 + 1)|v|$ is in the Sobolev space $H^{-1}(\mathbb{R}^d)$, we have

$$0 \leq (|v|, \psi) = ((H_0 + 1)|v|, (H_0 + 1)^{-1}\psi) = \lim_{n \rightarrow \infty} ((H_0 + 1)|v|, \varphi_n) \leq 0,$$

whence $(|v|, \psi) = 0$. It follows that $|v| = 0$ or $v = 0$.

(ii) As seen in the above proof of (i), H_A is, on $C_0^\infty(\mathbb{R}^d)$, bounded from below by m and essentially selfadjoint. So the assertion is obvious. This proves Theorem 5.1.

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REFERENCES

- [1] F. A. BEREZIN and M. A. ŠUBIN, *Symbols of Operators and Quantization*, Coll. Math. Soc. Janos Bolyai 5, Hilbert Space Operators, 21-52, Tihany, 1970.
- [2] I. DAUBECHIES, One-Electron Molecules with Relativistic Kinetic Energy: Properties of the Discrete Spectrum, *Commun. Math. Phys.*, Vol. **94**, 1984, pp. 523-535.
- [3] I. DAUBECHIES and E. H. LIEB, One Electron Relativistic Molecules with Coulomb Interaction, *Commun. Math. Phys.*, Vol. **90**, 1983, pp. 497-510.
- [4] A. ERDÉLYI, *Higher Transcendental Functions*, Vol. **2**, McGraw-Hill, New York, 1953.
- [5] C. FEFFERMAN and R. DE LA LLAVE, Relativistic Stability of Matter-I, *Rev. Mat. Iberoamericana*, Vol. **2**, 1986, pp. 119-213.
- [6] I. W. HERBST, Spectral Theory of the Operator $(p^2 + m^2)^{1/2} - Ze^2/r$, *Commun. Math. Phys.*, Vol. **53**, 1977, pp. 285-294; Errata, *Ibid.*, Vol. **55**, 1977, p. 316.
- [7] H. HESS, R. SCHRADER and D. A. UHLENBROCK, Domination of Semigroups and Generalization of Kato's Inequality, *Duke Math. J.*, Vol. **44**, 1977, pp. 893-904.
- [8] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators III*, Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
- [9] T. ICHINOSE, The Nonrelativistic Limit Problem for a Relativistic Spinless Particle in an Electromagnetic Field, *J. Functional Analysis*, **73**, 1987, pp. 233-257.
- [10] T. ICHINOSE, Path Integral for a Weyl Quantized Relativistic Hamiltonian and the Nonrelativistic Limit Problem, in *Differential Equations and Mathematical Physics; Lecture Notes in Mathematics*, Springer, No. **1285**, 1987, pp. 205-210.
- [11] T. ICHINOSE, Kato's Inequality and Essential Selfadjointness for the Weyl Quantized Relativistic Hamiltonian, *Proc. Japan Acad.*, **64A**, 1988, pp. 367-369.
- [12] T. ICHINOSE and H. TAMURA, Imaginary-Time Path Integral for a Relativistic Spinless Particle in an Electromagnetic Field, *Commun. Math. Phys.*, Vol. **105**, 1986, pp. 239-257; Path Integral for the Weyl Quantized Relativistic Hamiltonian, *Proc. Japan Acad.*, Vol. **62A**, 1986, pp. 91-93.

- [13] T. IKEBE and T. KATO, Uniqueness of the Self-Adjoint Extension of Singular Elliptic Differential Operators, *Arch. Rat. Mech. Anal.*, Vol. **9**, 1962, pp. 77-92.
- [14] N. IKEDA and S. WATANABE, *Stochastic Differential Equations and Diffusion Processes*, North-Holland/Kodansha, Amsterdam, Tokyo, 1981.
- [15] N. IKEDA and S. WATANABE, On Some Relations Between the Harmonic Measure and the Lévy Measure for a Certain Class of Markov Processes, *J. Math. Kyoto Univ.*, Vol. **2**, 1962, pp. 79-95.
- [16] T. KATO, Schrödinger Operators with Singular Potentials, *Israel J. Math.*, Vol. **13**, 1972, pp. 135-148.
- [17] H. KUMANO-GO, *Pseudo-Differential Operators*, The M.I.T. Press, Cambridge, Massachusetts, 1981.
- [18] M. M. MIZRAHI, The Weyl Correspondence and Path Integrals, *J. Math. Phys.*, Vol. **16**, 1975, pp. 2201-2206.
- [19] M. NAGASE, The L^p -Boundedness of Pseudo-Differential Operators with Non-Regular Symbols, *Comm. in P.D.E.*, Vol. **2**, 1977, pp. 1045-1061.
- [20] M. NAGASE and T. UMEDA, On the Essential Self-Adjointness of Quantum Hamiltonians of Relativistic Particles in Magnetic Fields, *Sci. Rep. Col. Gen. Educ. Osaka Univ.*, Vol. **36**, 1987, pp. 1-6.
- [21] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1975.
- [22] M. A. SHUBIN, *Pseudodifferential Operators and Spectral Theory*. Springer, Berlin-Heidelberg, 1987.
- [23] B. SIMON, An Abstract Kato's Inequality for Generators of Positivity Preserving Semigroups, *Indiana Univ. Math. J.*, Vol. **26**, 1977, pp. 1067-1073.
- [24] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.
- [25] R. A. WEDER, Spectral Analysis of Pseudodifferential Operators, *J. Functional Analysis*, **20**, 1975, pp. 319-337.

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