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## Semiclassical resolvent estimates

by

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**ABSTRACT.** — We prove estimates in the semiclassical regime of small  $h$  on the boundary values of the resolvent of the Schrödinger operator:  $H(h) = -h^2 \Delta + V$  in a neighborhood of a non-trapping energy  $E$ . The potential  $V$  is bounded, but not necessarily decaying with derivatives decaying at infinity. The method also applies to potentials with local singularities and to a family of Stark Hamiltonians. The proof is based on Mourre theory and decay estimates for wave packets in the classically forbidden region.

**RÉSUMÉ.** — Dans le régime semi-classique (petit  $h$ ), nous estimons les valeurs au bord de la résolvante de l'opérateur de Schrödinger  $H(h) = -h^2 \Delta + V$  dans un voisinage d'une énergie non liante  $E$ . Le potentiel  $V$  est borné mais n'est pas nécessairement décroissant mais ou avec des dérivées décroissantes à l'infini. La méthode s'applique aussi à des potentiels avec des singularités locales et à une famille d'Hamiltoniens de Stark. La preuve repose sur la théorie de Mourre et des estimations de décroissance des paquets d'ondes dans la zone classiquement interdite.

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## 1. INTRODUCTION

Semiclassical estimates on the resolvent of Schrödinger operators are an important technical tool for studying the behavior of observables like the scattering matrix and the total cross-section ([RT-1], [RT-2], [Y], see also [N-1] for an application to the shape resonances). In this note, we give a simple proof of these estimates for a large class of potentials. We give the details for reasonably smooth potentials and discuss the generalization in Section 4. We consider the following conditions:

CONDITION (A). —  $V$  is a real valued function such that  $V = V_1 + V_2$  with  $V_i \in C^i(\mathbb{R}^n)$ ,  $i = 1, 2$  and

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha V_1(x) \right| \leq C \langle x \rangle^{-1-|\alpha|} \quad \text{for } |\alpha| = 0, 1,$$

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha V_2(x) \right| \leq C \langle x \rangle^{-|\alpha|} \quad \text{for } |\alpha| = 0, 1, 2$$

where  $\langle x \rangle = (1 + |x|^2)^{1/2}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = \sum \alpha_i$ .

We will consider fixed energy  $E \in \mathbb{R}$  and let  $G(E) := \{x \in \mathbb{R}^n \mid V(x) - E > 0\}$ .

CONDITION (B). — There are constants  $\delta, \varepsilon_0 > 0$  and a  $C^3$ -vector field  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$(i) \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| \leq C \langle x \rangle^{1-|\alpha|} \quad \text{for } |\alpha| \leq 2$$

and  $|\Delta(\nabla \cdot f)(x)| \leq C$ ;

$$(ii) \quad 2 \left( \inf_{\xi \in \mathbb{R}^n} |\xi|^{-2} \langle \xi, J_f(x) \xi \rangle \right) (E - V(x)) - f(x) \cdot \nabla V(x) \geq \varepsilon_0 \quad (1.1)$$

for any  $x \in G_c(E + \delta) := \mathbb{R}^n \setminus G(E + \delta)$ , where  $J_f$  is the Jacobian of  $f$  and  $\langle \dots \rangle$  denotes the Euclidean inner product.

Condition (A) implies  $H(h)$  is self-adjoint on  $H^2(\mathbb{R}^n)$  (the Sobolev space of order 2). Our main result is:

THEOREM. — Let  $H(h) := -h^2 \Delta + V$  and suppose that  $V$  satisfies Conditions (A) and (B) at energy  $E$ . Then there is an open interval  $I \ni E$  such that for any  $\alpha > 1/2$  and  $\lambda \in I$ ,

$$\lim_{\varepsilon \rightarrow 0} \langle x \rangle^{-\alpha} (H - \lambda \pm i\varepsilon)^{-1} \langle x \rangle^{-\alpha}$$

exists, and

$$\| \langle x \rangle^{-\alpha} (H - \lambda \pm i0)^{-1} \langle x \rangle^{-\alpha} \| \leq C h^{-1} \quad (\lambda \in I) \quad (1.2)$$

if  $h$  is sufficiently small.

This result is a key ingredient in the estimation of the semiclassical behavior of the scattering cross-section  $\sigma_h(E, \omega)$ ,  $E > 0$ ,  $\omega \in S^{n-1}$ . For potentials  $V(x) = O(\langle x \rangle^{-\alpha})$ ,  $\alpha > \frac{n+1}{2}$ , and energies  $E$  such that (1.1)

holds on  $\mathbb{R}^n$  with  $f(x) = x$ , the leading behavior of  $\sigma_h(E, \omega)$  is  $O(h^{-\nu})$ , where  $\nu \equiv (n-1)(\alpha-1)^{-1}$  (cf. [RT-2], [Y]). Using the above theorem, it should be possible to extend [RT-2] to the more general situation where  $V$  satisfies Condition (A) (with possible local singularities, see Section 4) and is non-trapping in the sense of Condition (B). A similar result may hold for Stark Hamiltonians discussed in Section 4. We also remark that our methods apply to generalized N-body Schrödinger operators, although the potential  $V$  does not satisfy Condition (A). The potential

$V = \sum_{j=1}^N V_j \circ \pi_j$ , where  $\{\pi_j\}_{j=1}^N$  is a set of mutually orthogonal projections in  $\mathbb{R}^n$ . We assume that each  $V_j$  satisfies Condition (A) on  $\pi_j(\mathbb{R}^n)$ . Then, if we take  $f(x) = x$  in Condition (B) and consider energies  $E$  for which the resulting nontrapping condition (1.1) holds on  $\mathbb{R}^n$ , the analog of the above theorem holds for  $H = -h^2 \Delta + V$ . To see this, we simply note that all the remainder terms in (3.3)-(3.5) vanish except for  $(x \cdot \nabla)V$  and  $(x \cdot \nabla)^2 V$  because  $\partial^2 f_i / (\partial x_j \partial x_k) = 0$ . (Jensen [J] has recently obtained similar results in this case).

Our proof of this theorem is given in Sections 2-3. In Section 4, we discuss generalizations to potentials with singularities and to Stark Hamiltonians. Our method of proof utilizes the local positive commutator approach of E. Mourre ([M], [CFKS]) to obtain estimates in the nontrapping region  $\mathbb{R}^n \setminus G(E + \delta)$  and semiclassical decay estimates on wave packets localized to  $G(E + \delta)$  (cf. the Appendix).

Some results on semiclassical resolvent estimates are known. These first appeared in a paper by Robert and Tamura [RT-1] who consider nontrapping potential  $V \in C_0^\infty(\mathbb{R}^n)$ . Later, in [RT-2] they obtained semiclassical resolvent estimates at (classically) nontrapping energy  $E$  for smooth potentials decaying at infinity as  $\langle x \rangle^{-\rho}$ ,  $\rho > 0$ , using both Mourre theory and Fourier integral methods. We note that Condition (B) implies the classical condition of [RT-1], [RT-2]. A shorter proof of their result was given by Gérard and Martinez [GM] who constructed an escape function  $a(x, p)$  such that the Poisson bracket  $\{h, a\}$  is globally positive. Yafaev [Y] also used Mourre theory to obtain semiclassical resolvent estimates in the high energy regime for potentials  $C^2$  in the  $|x|$ -variable and satisfying  $|x| \left| \left( \frac{\partial}{\partial |x|} \right)^k V(x) \right| \leq C(k=0, 1, 2)$ . A method of Lavine [L] was also applied to prove estimate (1.2) for decaying potentials under nontrapping condition (1.1) with  $f(x) = x$  [N-1].

We note that the semiclassical resolvent estimate is closely related to the absence of resonances near the real axis in the semiclassical limit. Our nontrapping condition (1.2) first appeared in a proof of the absence of resonance in [N-2] (see also [BCD-1], [DeBH], [HeSj], [K], [S-1]).

## 2. SEMICLASSICAL MOURRE ESTIMATES

We restate the standard assumptions of the Mourre theory for a self-adjoint operator  $H$  and a skew-operator  $A$  in an  $h$ -dependent manner. For  $s \geq 0$ , let  $\mathcal{H}_s := D(|H| + 1)^{s/2}$  with the norm  $\|\psi\|_s := \|(|H| + 1)^{s/2} \psi\|$ , and  $\mathcal{H}_{-s} := \mathcal{H}_s^*$ .  $\|\cdot\|_{s,r}$  denotes the norm of the maps from  $\mathcal{H}_s$  to  $\mathcal{H}_r$ . We let  $C$  denote a  $h$ -independent constant whose value may change from line to line.

(M1)  $D(A) \cap \mathcal{H}_2$  is dense in  $\mathcal{H}_2$ .

(M2) The form  $[H, A]$  defined on  $D(A) \cap \mathcal{H}_2$  extends to a bounded operator from  $\mathcal{H}_2$  to  $\mathcal{H}_{-1}$  and  $\|[H, A]\|_{2,-1} \leq Ch$ .

(M3) There exists a self-adjoint operator  $H_0$  with  $D(H_0) = D(H)$  such that  $[H_0, A]$  extends to a bounded operator from  $\mathcal{H}_2$  to  $\mathcal{H}_0$ ,  $\|[H_0, A](H_0 + i)^{-1}\| \leq C$ ,  $\|H(H_0 + i)^{-1}\| \leq C$  and  $D(A) \cap D(H_0 A)$  is a core for  $H_0$ .

(M4) The form  $[[H, A], A]$  where  $[H, A]$  is as in (M2) extends from  $D(A) \cap D(HA)$  to a bounded operator from  $\mathcal{H}_2$  to  $\mathcal{H}_{-2}$  and  $\|[[H, A], A]\|_{2,-2} \leq Ch$ .

DEFINITION (The semiclassical Mourre estimate). — Let  $g$  be a function such that  $g \in C_0^\infty(\mathbb{R})$ ,  $0 \leq g(x) \leq 1$  and  $g = 1$  on a neighborhood of an interval  $I$ . We say that the semiclassical Mourre estimate holds on  $I$  if there exist such a  $g \in C_0^\infty(\mathbb{R})$ , an operator  $K(h)$  from  $\mathcal{H}_2$  to  $\mathcal{H}_{-2}$  with  $\|K(h)\|_{2,-2} \rightarrow 0$  as  $h \rightarrow 0$  and  $\alpha_0 > 0$  such that

$$M^2 := g(H)[H, A]g(H) \geq \alpha_0 hg(H)^2 + hg(H)K(h)g(H). \quad (2.1)$$

PROPOSITION 2.1. — Let  $H(h)$  be a self-adjoint operator and  $A(h)$  a skew-adjoint operator satisfying (M1)-(M4), and suppose the Mourre estimate (2.1) holds on  $I \subset \mathbb{R}$ . Then there exist  $h_0 > 0$  such that for any  $\alpha > 1/2$ ,  $h \in (0, h_0)$  and  $E \in I$ ,  $\lim_{\varepsilon \rightarrow 0} \langle A \rangle^{-\alpha} (H - E \pm i\varepsilon)^{-1} \langle A \rangle^{-\alpha}$  exists and

$$\|\langle A \rangle^{-\alpha} (H - E \pm i0)^{-1} \langle A \rangle^{-\alpha}\| \leq Ch^{-1}. \quad (2.2)$$

Proof. — (1) We retrace the proof of Mourre as presented in [CFKS] and [PSS] keeping track of the  $h$ -dependence, and we refer Section 4.3 of [CFKS] for details. At first we remark that if  $h$  is sufficiently small, the second term of the RHS of (2.1) is dominated by the first term, and hence it can be omitted.

For  $\varepsilon > 0$  let  $G_\varepsilon(z) := (H - i\varepsilon M^2 - z)^{-1}$  which is analytic in  $z$  for  $\operatorname{Re} z \in I$  and  $\operatorname{Im} z > 0$ . Then we obtain the following estimates (cf. Lemma 4.14 of [CFKS]):

$$\|g(H) G_\varepsilon(z) \varphi\| \leq (2\varepsilon \alpha_0 h)^{-1/2} |(\varphi, G_\varepsilon(z) \varphi)|^{1/2}, \tag{2.3}$$

$$\|(1 - g(H)) G_\varepsilon(z)\| \leq C(1 + \varepsilon h \|G_\varepsilon(z)\|), \tag{2.4}$$

$$\|G_\varepsilon(z)\| \leq C(\varepsilon \alpha_0 h)^{-1}, \tag{2.5}$$

if  $\varepsilon$  is sufficiently small. It follows in the same way as in [CFKS] that the bounds (2.3), (2.4) and (2.5) hold with  $\|\cdot\|_{0,2}$  replacing  $\|\cdot\|$ .

(2) Let  $D_\varepsilon := (1 + |A|)^{-\alpha} (\varepsilon |A| + 1)^{\alpha-1}$  for  $\alpha \in (1/2, 1]$ ,  $\varepsilon > 0$  and let  $F_\varepsilon(z) := D_\varepsilon G_\varepsilon(z) D_\varepsilon$  for  $z : \operatorname{Re} z \in I, \operatorname{Im} z > 0$ . By (2.5) and the definition of  $F_\varepsilon(z)$ ,

$$\|F_\varepsilon(z)\| \leq \|D_\varepsilon\|^2 \cdot \|G_\varepsilon(z)\| \leq C(\varepsilon \alpha_0 h)^{-1}. \tag{2.6}$$

From (2.3) and (2.4) with  $\varphi = D_\varepsilon \psi$ , we have

$$\|G_\varepsilon D_\varepsilon\| \leq C((\alpha_0 \varepsilon h)^{-1/2} \|F_\varepsilon\|^{1/2} + 1).$$

The derivative of  $F_\varepsilon(z)$  in  $\varepsilon$  is estimated using (2.3)-(2.6) (cf. [CFKS], Lemma 4.15), and we obtain

$$\left\| \frac{dF_\varepsilon}{d\varepsilon} \right\| \leq C\varepsilon^{\alpha-1} (1 + (\alpha_0 \varepsilon h)^{-1/2} \|F_\varepsilon\|^{1/2} + \|F_\varepsilon\|). \tag{2.8}$$

It follows from (2.6) and (2.8) that there exists  $C > 0$  such that

$$\overline{\lim}_{\varepsilon \downarrow 0} \sup_{\lambda \in I} \|\langle A \rangle^{-\alpha} (H - \lambda \pm i\varepsilon) \langle A \rangle^{-\alpha}\| \leq C h^{-1} \tag{2.9}$$

after integrating a finite number of times ([CFKS], Proposition 4.11).

(3) By differentiating  $F_\varepsilon(z)$  in  $z$ , we have

$$\|F_\varepsilon(z) - F_\varepsilon(z')\| \leq |z - z'| \sup_z \|D_\varepsilon G_\varepsilon(z)^2 D_\varepsilon\| \leq C\varepsilon^{-1} |z - z'| \tag{2.10}$$

for sufficiently small fixed  $h$ . Here we used estimates (2.7) and  $\|F_\varepsilon\| \leq C$ . (2.8) and (2.9) imply

$$\begin{aligned} \|F_0(z) - F_0(z')\| &\leq \|F_0(z) - F_\varepsilon(z)\| \\ &\quad + \|F_\varepsilon(z) - F_\varepsilon(z')\| + \|F_\varepsilon(z') - F_0(z')\| \\ &\leq C\varepsilon^{\alpha-1/2} + \varepsilon^{-1} |z - z'|. \end{aligned} \tag{2.11}$$

If we set  $\varepsilon = |z - z'|^\beta$  with  $\beta = (\alpha - 1/2)^{-1}$ , then we obtain the Hölder continuity of order  $(\alpha - 1/2)/(\alpha + 1/2)$  for  $F_0(z)$ . The existence of the limit of  $F_0(z)$  as  $\operatorname{Im} z \rightarrow 0$ ,  $\operatorname{Re} z \in I$  follows from this. Consequently, (2.2) follows from (2.9). ■

*Remark 2.2.* — It follows from (M2), (M4) and Lemma 4.12 of [CFKS], i.e. that  $\|[A, g(H)]\|_{-1,1} \leq C$  in our situation, that for any  $g \in C_0^\infty(\mathbb{R})$ ,  $[g(H)[H, A]g(H), A]$  extends to a bounded operator and is

$O(h)$ . As an alternative to (M4) we can take

$$(M4') \quad \text{for any } g \in C_0^\infty(\mathbb{R}), \quad \|[g(H)[H, A]g(H), A]\| \leq Ch.$$

### 3. PROOF OF THEOREM

In this section, we prove that Conditions (A) and (B) imply that  $H(h)$  satisfies (M1)-(M4) and the semiclassical Mourre estimate for  $\text{supp } g$  sufficiently small and containing the nontrapping energy  $E$ . The conjugate operator is

$$A := \frac{h}{2} [\nabla \cdot f(x) + f(x) \cdot \nabla] \tag{3.1}$$

where  $f$  is the vector field of Condition B.

LEMMA 3.1. — *Let  $H(h) := -h^2 \Delta + V$  where  $V$  satisfies Conditions (A) and (B). Let  $g \in C_0^\infty(I)$ ,  $I \subset \mathbb{R}$  compact and  $E \in I$ . Then*

(i) *A and H satisfy (M1)-(M4) with  $H_0 := H$  in (M3).*

(ii) *There exist  $\alpha_0 > 0$  and a bounded operator  $K(h)$  with  $\|K(h)\| \rightarrow 0$  as  $h \rightarrow 0$  such that for  $|I|$  sufficiently small,*

$$g(H)[H, A]g(H) \geq \alpha_0 hg(H)^2 + hg(H)K(h)g(H). \tag{3.2}$$

The operator  $K(h)$  is given explicitly in (3.8) below.

In the proof of this lemma, we use a decay result for wave packets in the classically forbidden region  $G(E)$ . This result, in its optimal form due to [BCD-2], is discussed in the Appendix.

Let  $\delta$  be as in Condition (B). The function:  $K(x) := \inf_{\xi \in \mathbb{R}^n} \{ |\xi|^{-2} \langle \xi, J_f(x)\xi \rangle \}$  is easily seen to be uniformly Lipschitz continuous, and let  $c_0$  be the Lipschitz constant.

LEMMA 3.2. — *Let  $K(x)$  be as above and  $\varepsilon_0$  be as in (1.1). Then there exists  $\tilde{K}(x) \in C^\infty(\mathbb{R}^n)$  such that*

- (i)  $\tilde{K}(x) \leq K(x), x \in \mathbb{R}^n;$
- (ii)  $2\tilde{K}(x)(V(x) - E) - f(x) \cdot \nabla V(x) \geq \varepsilon_0/2, x \in G_c(E + \delta).$

*Proof.* — Let  $c_x$  be a mollifier:  $c_x \in C_0^\infty(\{|x| \leq \kappa\})$ ,  $\int c_x(x) dx = 1$ . Let  $K_x := c_x * K$ , so  $K_x \in C^\infty$ . Since  $K$  is uniformly Lipschitz, it follows that

$$K(x) - c_0 \kappa \leq K_x(x) < K(x) + c_0 \kappa$$

for  $x \in G_c(E + \delta)$ . Set  $\tilde{K}(x) := K_x(x) - c_0 \kappa$ , then this proves (i). For (ii),

$$2\tilde{K}(x)(V(x) - E) - f(x) \cdot \nabla V(x) \geq \varepsilon_0 - 2c_0 \kappa (V(x) - E)$$

for  $x \in G_c(E + \delta)$ . If  $\kappa < \varepsilon_0(4c_0 \sup |V(x) - E|)^{-1}$ , (ii) holds. ■

*Proof of lemma 3.1.* — (1) Since  $C_0^\infty(\mathbb{R}^n)$  is a common core for  $D(H)$ ,  $D(A)$ , etc., it is sufficient to prove the estimates. By a simple calculation, as a quadratic form on  $C_0^\infty(\mathbb{R}^n)$ :

$$[H, A] = h \left\{ 2h^2 p J_f p - f \cdot \nabla V - \frac{h^2}{2} \Delta(\nabla \cdot f) \right\} \tag{3.3}$$

where  $J_f = (\partial f_i / \partial x_j)$  is the Jacobian matrix of  $f$  and  $p = -i \nabla$ . By Conditions (A) and (B),  $\| [H, A] \|_{2,0} \leq ch$ , hence (M1)-(M3) are satisfied. As for (M4),  $[[H, A], A]$  as a quadratic form on  $C_0^\infty(\mathbb{R}^n)$  has the form:

$$[[H, A], A] = h^2 \left\{ -2p_i J_{ij,k} f_k p_j + 2p_k J_{ki} J_{ij} p_j + 2p_i J_{ij} J_{kj} p_k + i(f_{k,ik} J_{ij} p_j - p_i J_{ij} f_{k,jk}) - [(f \cdot \nabla V), f \cdot \nabla] + \frac{h^2}{2} [(f \cdot \nabla), \Delta(\nabla \cdot f)] \right\} \tag{3.4}$$

where  $J_{ij,k} := \frac{\partial}{\partial x_k} (J_f)_{ij}$ ,  $f_{i,k} := (\partial f_i) / (\partial x_k)$ , etc. The term  $h^2 \{ \dots \}$  is clearly uniformly bounded by  $H$ , and the last is also uniformly bounded by  $H$ . The second term is

$$-h^2 [(f \cdot \nabla V), f \cdot \nabla] = h^2 \left\{ f \cdot \nabla (f \cdot \nabla V_2) + [\nabla_j, f_j (f \cdot \nabla V_1)] - (\nabla \cdot f) \cdot (f \cdot \nabla V_1) \right\} = h^2 \left\{ f \cdot \nabla (f \cdot \nabla V_2) - (\nabla \cdot f) (f \cdot \nabla V_1) - h [(h \nabla_j), f_j (f \cdot \nabla V_1)] \right\} = I_1 + I_2. \tag{3.5}$$

Clearly,  $I_1$  is  $O(h^2)$ , and  $(H+i)^{-1} I_2 (H+i)^{-1}$  is  $O(h)$  since  $h \cdot \nabla_j$  is uniformly  $H$ -bounded. Thus  $\| [[H, A], A] \|_{2,-2} = O(h)$ .

(2) In the sense of quadratic forms, it follows from Lemma 3.2 that  $p J_f p \geq p K p \geq p \tilde{K} p$  and  $2 \cdot p \tilde{K} p = \tilde{K} p^2 + p^2 \tilde{K} + \Delta \tilde{K}$ . We obtain from (3.3):

$$[H, A] \geq h \left[ \tilde{K} (H - E) + (H - E) \tilde{K} + 2 \tilde{K} (E - V) - f \cdot \nabla V + h^2 \left\{ \Delta \tilde{K} - \frac{1}{2} \Delta(\nabla \cdot f) \right\} \right]. \tag{3.6}$$

Let  $g \in C_0^\infty(I)$ ,  $E \in I$  and let  $\chi$  be the characteristic function of  $G_c(E + \delta)$ . By Lemma 3.2,

$$(2 \tilde{K} (E - V) - f \cdot \nabla V) \chi \geq (\varepsilon_0/2) \chi.$$



Let  $\beta := \sup_{x \in G(E+\delta)} |2\tilde{K}(E-V) - f \cdot \nabla V|$  and  $\gamma = \sup_{x \in \mathbb{R}^n} |\tilde{K}(x)|$ . Then for  $|I|$  sufficiently small,

$$\begin{aligned}
 g(H)[H, A]g(H) &\geq h \left( \frac{\varepsilon_0}{2} - 2\gamma|I| \right) g(H)^2 \\
 &\quad - hg(H) \left[ (1-\chi) \left( \beta + \frac{\varepsilon_0}{2} \right) + h^2 \Delta(\nabla \cdot f) - h^2 \Delta \tilde{K} \right] g(H) \\
 &\geq \frac{h\varepsilon_0}{4} g(H)^2 + hg(H) K(h)g(H) \quad (3.7)
 \end{aligned}$$

where

$$K(h) := \left( \beta + \frac{\varepsilon_0}{2} \right) E_1(H)(\chi-1)E_1(H) - \frac{h^2}{2} \Delta(\nabla \cdot f) + h^2 \Delta \tilde{K} \quad (3.8)$$

and  $E_1(H)$  is the spectral projection for  $H$  and  $I$ . By Lemma A.2,  $\|E_1(H)(\chi-1)\| = O(h^N)$  for any  $N$ , so we have  $\|K(h)\| = O(h^2)$ . This completes the proof. ■

*Proof of Theorem.* — By Lemma 3.1, the hypothesis of Proposition 2.1 are satisfied, so the resolvent of  $H(h)$  satisfies (2.2). To pass to (1.3), we use the fact there exists a constant  $C$  independent of  $h$  such that

$$\|\langle x \rangle^{-\alpha} (H+i)^{-1} \langle A \rangle^\alpha\| \leq C \quad (3.9)$$

for  $\alpha \in [0,1]$  (cf. Lemma 8.2 of [PSS]). Estimate (3.9) is proved directly for  $\alpha=1$  using the fact  $|\langle x \rangle^{-1} f(x)| \leq C$  which follows from Condition (B), and extended by complex interpolation. ■

*Remark 3.3.* — In certain cases, a more precise propagation estimate results from (2.2) if we replace  $\langle A \rangle^{-\alpha}$  by  $\langle f \rangle^{-\alpha}$ . This is the case when  $f$  vanishes on some unbounded set.

*Remark 3.4.* — Instead of Lemma A.2, we can also apply the cut and paste technique (or so-called geometric method) to isolate the classically forbidden region. In fact, if the semiclassical resolvent estimate is proved for  $H$  on  $L^2(G_c(E+\delta))$ , the estimate on  $L^2(\mathbb{R}^n)$  follows (cf. (A.5) or [BCD-2]). Since nontrapping inequality (1.1) holds globally on  $G_c(E+\delta)$ , the semiclassical resolvent estimate on  $L^2(G_c(E+\delta))$  can be proved by the above argument.

### 4. GENERALIZATIONS

#### A. Stark Hamiltonians

The methods developed here can be extended to a class of Stark Hamiltonians as we now indicate. In place of Condition (A) we assume  $V \in C^2(\mathbb{R}^n)$ ,  $|V(x)| \leq C \langle x \rangle$  and  $\left| \left( \frac{\partial}{\partial x} \right)^\alpha V(x) \right| \leq C$ ,  $|\alpha| = 1, 2$ . The vector field in Condition (B) must satisfy  $f \in C^4(\mathbb{R}^n)$ ,  $|f(x)| \leq C$  and

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| \leq C \langle x \rangle^{-1} \quad \text{for } |\alpha| = 1, 2, 3, 4.$$

The nontrapping condition is as in (1.1). Note that the proof of Lemma 3.2 must be improved to show that  $|K(x) - \tilde{K}(x)| \leq \kappa \langle x \rangle^{-1}$  with small  $\kappa > 0$  using the fact that  $K(x) = 0 \langle x \rangle^{-1}$ . We also need the following lemma:

LEMMA 4.1. — *Let  $V \in C(\mathbb{R}^n)$  and suppose that  $|V(x)| < C \langle x \rangle^\gamma$  for some  $\gamma : 0 \leq \gamma \leq 2$ . Then  $-h^2 \langle x \rangle^{-\gamma} \Delta$  is relatively  $H(h)$ -bounded uniformly in  $h$ .*

It follows from the assumptions and this lemma that  $\| [H, A](H+i)^{-1} \| = O(h)$  and  $\| [[H, A], A](H+i)^{-1} \| = O(h^2)$ . With these modifications, one proves (M1)-(M4) and the semiclassical Mourre estimate (2.1). As a consequence, we obtain the semiclassical resolvent estimate

$$\| (H - E \pm i0)^{-1} \|_{B(H^1, H^{-1})} \leq Ch^{-1}$$

where  $H^1(\mathbb{R}^n)$  is the usual Sobolev space with norm  $\| \varphi \|_{H^1}^2 = \| \varphi \|^2 + h^2 \| \nabla \varphi \|^2$ . Here we used the fact that  $D(A) \rightarrow H^1(\mathbb{R}^n)$ , and the inclusion map is bounded uniformly in  $h$ .

#### B. Local Singularities

The results of Section 3 apply if  $V$  is singular in the classically forbidden region for an interval of nontrapping energies around  $E$ . In this case, we require  $V \in L^p(G(E+\delta))$  for  $\delta$  as in Condition (B), with  $p=2$  for  $n \leq 3$  and  $p > n/2$  for  $n \geq 4$ . As is easily seen from the proof, we only need  $V$  to be bounded away from  $G(E+\delta)$  so the decay estimate  $\| (1-\chi) E_1(H) \| = O(h^N)$  holds for this class of potentials.

**C. Exploding potentials**

We can also treat potentials of the type

$$V \in C^2(\mathbb{R}^n), \left| \left( \frac{\partial}{\partial x} \right)^\alpha V(x) \right| \leq C \langle x \rangle^{2-|\alpha|}, |\alpha| \leq 2, \quad \text{and} \quad V(x) \rightarrow -\infty$$

as  $|x| \rightarrow \infty$ . Again, we must take vector fields  $f$  such that  $f \in C^4(\mathbb{R}^n)$  and  $\left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| \leq C \langle x \rangle^{-1-|\alpha|}, |\alpha| \leq 4$ . Following modifications similar to those described in Part A above, we obtain a semiclassical Mourre estimate and the result that  $\| (H - E \pm iO)^{-1} \|_{B(H^1, H^{-1})} \leq Ch^{-1}$ .

**APPENDIX**

**Decay of wave packets**

The purpose of this section is to prove Lemma A.2 the result of which is used in equation (3.8). We use a perturbation idea of [BCD-2] and a simple iteration argument on the localized resolvent. Although Lemma A.2 is sufficient for our purposes, we mention a result of [BCD-2] which states that there exists  $\sigma > 0$ , where  $\sigma$  is described in terms of a distance in the Agmon metric, such that  $\| (1 - \chi) E_1(H) \| = O(e^{-\sigma/h})$ .

LEMMA A.1. — *Suppose  $F > E$  and  $\sup |\nabla V| = C < \infty$ . Then*

$$\text{dist}(G(F), G_c(E)) \geq C^{-1}(F - E) \tag{A.1}$$

where  $G_c(E) := \mathbb{R}^n \setminus G(E)$  and  $\text{dist}(\cdot, \cdot)$  is the Euclidean distance.

*Proof.* — Let  $x \in G(F), y \in G_c(E)$ , then

$$F - E \leq V(x) - V(y) = \int_0^1 \frac{d}{dt} V(\gamma(t)) dt \tag{A.2}$$

for the path  $\gamma : \gamma(t) = tx + (1 - t)y$ . By the assumption,

$$\begin{aligned} [\text{the RHS of (A.2)}] &= \int_0^1 \frac{d\gamma}{dt} \cdot (\nabla V)(\gamma(t)) dt \\ &\leq C \int_0^1 \left| \frac{d\gamma}{dt} \right| dt = C \text{dist}(x, y). \end{aligned} \tag{A.3}$$

This proves the lemma. ■

We note that the assumption  $\sup |\nabla V| < \infty$  is necessary only on the convex hull of  $G(E)$  in order to apply the method to exploding potentials [Section 4 (C)].

LEMMA A.2. — Suppose  $\sup |\nabla V| < \infty$ . Let  $\chi$  be the characteristic function of  $G_c(F)$  and  $I = [D, E]$  with  $D < E < F$ . Then

$$\| (1 - \chi) E_1(H) \| \leq C_N \cdot h^N \tag{A.4}$$

for any  $N$ , where  $E_1(H)$  is the spectral projection of  $H$ .

*Proof.* — Let  $\varepsilon := (F - E)/(2N + 4)$ . By virtue of Lemma A.1, there exist  $C^\infty$ -functions  $\{J_j\}_{j=1, \dots, N}$  such that (i)  $0 \leq J_j(x) \leq 1$ ; (ii)  $\sup |\nabla J_j(x)| < \infty$ ; (iii)  $J_j(x) = 1$  if  $x \in G(F - 2j\varepsilon)$  and  $= 0$  if  $x \in G_c(F - (2j + 1)\varepsilon)$ . Let  $V_0(x) := \max\{V(x), E + 2\varepsilon\}$ , and let  $H_0 = -h^2 \Delta + V_0(x)$ . Then  $\sigma(H_0) \subset [E + 2\varepsilon, \infty)$ . We have the geometric resolvent equation:

$$J_N R(z) = R_0(z) J_N + R_0(z) M_N R(z) \tag{A.5}$$

where

$$R(z) = (H - z)^{-1},$$

$$R_0(z) = (H_0 - z)^{-1} \quad \text{and} \quad M_j = -h^2 \{ \nabla (\nabla J_j) + (\nabla J_j) \nabla \}.$$

It is easy to see  $\text{supp } M_j \subset G(F - (2j + 1)\varepsilon) \cap G_c(F - 2j\varepsilon)$ , and hence  $M_{j+1} J_j = 0$ . Using this identity, we obtain

$$\begin{aligned} (1 - \chi) R_0(z) M_N R(z) &= (1 - \chi) J_{N-1} R_0(z) M_N R(z) \\ &= (1 - \chi) [J_{N-1}, R_0(z)] M_N R(z) \\ &= (1 - \chi) R_0(z) M_{N-1} R_0(z) M_N R(z) \\ &= (1 - \chi) R_0(z) M_1 R_0(z) M_2 \dots R_0(z) M_N R(z). \end{aligned} \tag{A.6}$$

Let  $\Gamma$  be a positively oriented, simple closed around  $I$ , and away from  $[E + 2\varepsilon, \infty)$ . Then, as the first term of the RHS of (A.5) is analytic in  $\Gamma$ , we conclude

$$\begin{aligned} (1 - \chi) E_1(H) &= -\frac{1}{2\pi i} \int_{\Gamma} (1 - \chi) J_N R(z) E_1(H) dz \\ &= -\frac{1}{2\pi i} \int_{\Gamma} (1 - \chi) R_0(z) M_1 \dots R_0(z) M_N R(z) E_1(H) dz. \end{aligned} \tag{A.7}$$

Now, since  $\|M_j R_0(z)\| = O(h)$  and  $\|R(z) E_1(H)\| \leq C$  on  $\Gamma$ , it follows immediately from (A.7) that  $\|(1 - \chi) E_1(H)\| = O(h^N)$ . ■

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