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Matrix second-order differential equations and hamiltonian systems of quartic type

by

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ABSTRACT. — We show the existence of conserved quantities for matrix differential equations and Hamiltonian systems of quartic type.

RÉSUMÉ. — Nous montrons l'existence de quantités conservées pour des équations différentielles matricielles et des hamiltoniens quartiques.

I. INTRODUCTION

We study second-order differential equations of the form:

$$\ddot{X} = Q(X) = -hh'(X) \quad (1)$$

where $h(X) = \lambda X + \mu X^2$, $(\lambda, \mu) \in \mathbb{R}^2$, and the unknown function X is a $m \times m$ symmetric real matrix.

The method we use is the same which has allowed to prove the existence of a symplectic action of the torus associated to the Olshanetsky-Peregomov systems of the class V ([F1], [F2]).

We associate to (1) a Hamiltonian system for a symplectic form introduced by Kazhdan-Kostant-Sternberg [K-K-S] and used by J. Moser [M]. For a quadratic h , we get a quartic anharmonic flow on the cotangent bundle of the Lie algebra $u(m)$. We show the existence for this system of a $m \times m$ Lax pair whose eigenvalues are in involution for the symplectic form.

The Calogero-Moser Hamiltonian describes an integrable system of m particles on the line interacting pairwise via an inverse-quadratic potential ([C], [M]).

It has been observed by M. Adler [A] that the system stays integrable under the influence of an external quadratic potential.

Following the work of Kazhdan-Kostant-Sternberg, the systems of Calogero-Moser type can be seen as symplectic reductions of harmonic flows on the cotangent bundle of a simple Lie algebra.

The second-order differential equations that we consider here give after a symplectic reduction the Calogero-Moser system with a quartic external potential. We find as a consequence of the main theorem that the reduced system is completely integrable in the Arnol'd-Liouville sense. This last result had been obtained previously by Wojciechowski [W] and independently Inozemtzev [I]. The novelty of our approach is in the use of the Kazhdan-Kostant-Sternberg machinery which makes all the subject more coherent.

Hamiltonian systems of quartic type have been also studied in [F-S-W] in relation with the non-linear Schrödinger equation.

II. SECOND-ORDER MATRIX DIFFERENTIAL EQUATIONS AND HAMILTONIAN SYSTEMS FOR THE KAZHDAN-KOSTANT-STERNBERG SYMPLECTIC FORM

We are interested in second-order differential equations of the form:

$$\ddot{X} = Q(X) = -hh'(X)$$

where h is polynomial with scalar coefficients and the unknown X is a $m \times m$ matrix.

We assume that X varies in a vector space V of matrices so that the trace provides an identification with the dual V^* through the mapping $A \mapsto (B \mapsto \text{Tr}(AB))$. The cotangent bundle $T^*V \simeq V \times V^*$ is equipped with the symplectic form $\omega = \text{Tr}(dX \wedge dY)$ following [K-K-S]. The Hamiltonian

flow of the function $H : T^*V \rightarrow \mathbb{R}$, defined by:
 $H(X, Y) = (1/2) \text{Tr}(h(X)^2 + Y^2)$ is given by

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -hh'(X) \end{aligned} \tag{2}$$

and it coincides with (1) for solutions X in V . Hereafter V will be the space of real symmetric matrices.

III. HAMILTONIAN SYSTEMS OF QUARTIC TYPE

We use the matrices

$$Z = \sqrt{-1} h(X) + L, \quad Z^* = -\sqrt{-1} h(X) + L, \tag{3}$$

In the following, (X, Y) are assumed to be real symmetric matrices so that:

$$Z^* = {}^t\bar{Z} \quad (\text{transposed of the complex conjugated of } Z).$$

We are concerned with the Hamiltonian system

$$H(X, Y) = (1/2) \text{Tr}(h(X)^2 + Y^2), \quad \omega = \text{Tr}(dX \wedge dY) \tag{4}$$

for $h(X) = \lambda X + \mu X^2$.

We use the Hermitian matrix $P = ZZ^*$ that we consider as an element of the Lie algebra $u(m)$.

THEOREM 1. — *The matrix P defines a Lax pair for the Hamiltonian system (4) and its eigenvalues are in involution for the symplectic form ω .*

Proof. — We get from (4):

$$H = (1/2) \text{Tr}(P).$$

Hamilton's equations (2) give:

$$\begin{aligned} \dot{Z} &= \sqrt{-1}(\lambda Y + \mu(XY + YX)) - hh'(X) \\ \dot{Z}^* &= -\sqrt{-1}(\lambda Y + \mu(XY + YX)) - hh'(X) \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{Z} &= \{ \{ (\sqrt{-1/2}) h'(X), Z \} \} \\ \dot{Z}^* &= -\{ \{ (\sqrt{-1/2}) h'(X), Z \} \} \end{aligned} \tag{6}$$

where the symbol $\{ \{ A, B \} \}$ means the anti-commutator of the two matrices A and B .

We get:

$$\begin{aligned} \dot{P} &= \dot{Z}Z^* + Z\dot{Z}^*, \\ \dot{P} &= \{ \{ (\sqrt{-1/2}) h'(X), Z \} \} Z^* - Z \{ \{ (\sqrt{-1/2}) h'(X), Z^* \} \}, \\ \dot{P} &= [\sqrt{-1/2} h'(X), P]. \end{aligned} \tag{7}$$

So that P is a Lax matrix for the flow of H . As corollary, we obtain that the eigenvalues of P are constants of the motion.

Let Λ be one the eigenvalues of P and Ψ be the corresponding eigenvector. Let T be the projector on the subspace generated by Ψ . The matrix P being Hermitian, we can assume that Ψ is normalized for the standard Hermitian product $\langle \cdot, \cdot \rangle$ on \mathbb{C}^m preserved by P . We get:

$$d\Lambda = \langle dP\Psi, \Psi \rangle = \text{Tr}(dPT). \quad (8)$$

Let us now consider the Hamiltonian flow generated by Λ and the symplectic form ω . The dot will designate the time-derivative along this flow.

Hamilton's equations give:

$$\text{Tr}(\dot{X} dY - \dot{Y} dX) = d\Lambda = \text{Tr}(dZZ^*T + Z dZ^*T) \quad (9)$$

and so we get first,

$$\begin{aligned} \dot{X} &= Z^*T + TZ \\ \dot{Y} &= -\{(\sqrt{-1/2}) h'(X), Z^*T - TZ\} \end{aligned} \quad (10)$$

then,

$$\begin{aligned} \dot{Z} &= \{(\sqrt{-1}) h'(X), TZ\} \\ \dot{Z}^* &= -\{(\sqrt{-1}) h'(X), Z^*T\}. \end{aligned} \quad (11)$$

This allows to compute \dot{P} ,

$$\begin{aligned} \dot{P} &= \dot{Z}Z^* + Z\dot{Z}^*, \\ \dot{P} &= [T, Z(\sqrt{-1} h'(X)/2) Z^*] + [(\sqrt{-1} h'(X)/2), PT]. \end{aligned} \quad (12)$$

Now we observe that P and T can be codiagonalized so that

$$[P, T] = 0.$$

We deduce then that,

$$\dot{P}P^k = [T, Z(\sqrt{-1} h'(X)/2) Z^* P^k] + [(\sqrt{-1} h'(X)/2) P^k, PT] \quad (13)$$

for all integer k , and then,

$$\text{Tr}(\dot{P}P^k) = 0 \quad (14)$$

which implies that the quantities $\text{tr}(P^k)$ are constants of the motion. Hence, the eigenvalues of P are constants of the motion. But this is true for the flow of any of the eigenvalues of P , so we deduce that the eigenvalues of P are in involution. This ends the proof. \square

Example. — Let us consider the case where X is a 2×2 symmetric real matrix.

$$X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

The equation (1) gives:

$$\begin{aligned} \dot{x}_1 &= -\lambda^2 x_1 - 3 \lambda \mu (x_1^2 + x_3^2) - 2 \mu^2 (x_1 (x_1^2 + x_3^2) + x_3^2 (x_1 + x_2)) \\ \dot{x}_2 &= -\lambda^2 x_2 - 3 \lambda \mu (x_2^2 + x_3^2) - 2 \mu^2 (x_2 (x_2^2 + x_3^2) + x_3^2 (x_1 + x_2)) \\ \dot{x}_3 &= -\lambda^2 x_3 - 3 \lambda \mu_3 (x_1 + x_2) - 2 \mu^2 (x_1 x_3 (x_1 + x_2) + x_3 (x_2^2 + x_3^2)). \end{aligned}$$

From the theorem 1, we know that there are two conserved quantities. It would be interesting to decide in that case if the system is integrable.

Remarks. — 1. Let us consider the special case $\mu=0$ (quadratic case) in order to compare with the general case.

For $\mu=0$, $h'(X)=\lambda \text{Id}$, and the equation (7) becomes $\dot{P}=0$ identically. This means that all the entries of P are constants of the motion.

In general case when $\mu \neq 0$, the equation (7) tells that only the eigenvalues of P are preserved by the flow. Also (12) becomes:

$$\begin{aligned} \dot{P} &= [T, Z(\sqrt{-1 \lambda \text{Id}/2}) Z^*] + [\sqrt{-1 \lambda \text{Id}/2}, PT] \\ \dot{P} &= [T, (\sqrt{-1 \lambda \text{Id}/2}) P] \end{aligned}$$

but $[P,T]=0$, so $\dot{P}=0$ identically also. Thus, all the entries of P are preserved by the flow of any eigenvalue.

Going back to (11), we get:

$$\begin{aligned} \dot{Z} &= \sqrt{-1 \lambda} TZ \\ \dot{Z}^* &= -\sqrt{-1 \lambda} Z^* T. \end{aligned} \tag{15}$$

One can show [F] that (15) defines a symplectic action of the torus T^m , this is no longer true for $\mu \neq 0$.

2. The proof does not extend as it stands to Hamiltonian systems of the sextic type.

IV. COMPLETE INTEGRABILITY OF THE CALOGERO-MOSER SYSTEM WITH AN EXTERNAL QUARTIC POTENTIAL

Let \mathcal{U} be the vector space of $m \times m$ Hermitian matrices. The cotangent bundle $T^* \mathcal{U}$ may be identified with $\mathcal{U} \times \mathcal{U}^* \simeq \mathcal{U} \times \mathcal{U}$. As a cotangent bundle, it has a symplectic structure which can be written [K-K-S]:

$$\omega = \text{Tr } dX \wedge dL, \quad (X, L) \in \mathcal{U} \times \mathcal{U}.$$

We define on $T^* \mathcal{U}$ the Quartic Anharmonic Flow by the Hamiltonian:

$$H = (1/2) \text{Tr} (h(X)^2 + L^2).$$

The group $G=U(m)$ acts on \mathcal{U} by the adjoint action; this action lifts into an Hamiltonian action on $T^* \mathcal{U}$. The corresponding moment map is given by:

$$T^* \mathcal{U} \cong \mathcal{U} \times \mathcal{U} \ni (X, L) \mapsto \sqrt{-1} [X, L] \in \mathcal{U} \cong \mathcal{U}^*. \tag{16}$$

The Hamiltonian H is invariant under this action.

Following [K-K-S] we proceed to a reduction of $T^*\mathcal{U}$ by the symplectic action of G . By using the identification (16), a fiber of the moment map can be seen as $\{(X, L) : [X, L] = -1gC\}$.

We choose for C the element $(C_{ij} = 1 - \delta_{ij})$. Let G_c be the isotropy subgroup of C . The reduced manifold

$X_c = \{(X, L) : [X, L] = \sqrt{-1}gC\}/G_c$ can be parametrized (see [K-K-S]) by:

$$\begin{aligned} X_{ij} &= x_i \delta_{ij} \\ L_{ij} &= y_i \delta_{ij} + \sqrt{-1}g/(x_i - x_j)(1 - \delta_{ij}) \quad (g > 0), \end{aligned}$$

and is identified with T^*W , $W = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m / x_i - x_j \neq 0, i \neq j\}$ equipped with the standard symplectic form $\omega = \sum_{i=1}^m dx_i \wedge dy_i$. After the reduction, the Hamiltonian $H = (1/2) \text{Tr}(h(X)^2 + L^2)$ becomes the function

$$H = (1/2) \sum_{i=1}^m y_i^2 + g^2 \sum_{i,j; i < j} (x_i - x_j)^{-2} + (1/2) \sum_{i=1}^m (\lambda x_i + \mu x_i^2)^2. \quad (17)$$

This System can be viewed as a perturbation of the Calogero-Moser System by an external quartic potential.

Hamilton's equations define a vector field whose flow is a solution of

$$\begin{aligned} \dot{x}_i &= \partial H / \partial y_i = y_i \\ \dot{y}_i &= -\partial H / \partial x_i = 2g^2 \sum_{j=i}^m (x_i - x_j)^{-3} - \sum_{i=1}^m 2((\lambda + 2\mu x_i)(\lambda x_i + \mu x_i^2)). \end{aligned} \quad (18)$$

The following result is a direct consequence of the theorem proved in the previous paragraph:

COROLLARY 1. — *The Calogero-Moser System with an external quartic potential defined by the Hamiltonian (17) is integrable in the Arnol'd-Liouville sense.*

Proof. — The theorem shows that the m quantities $F_k = \text{tr}(P^k)$ are in involution. They stay in involution after the reduction. Furthermore $dF_1 \wedge \dots \wedge dF_k$ is not identically zero since for $\mu = 0$, we have the usual Calogero-Moser System with an external quadratic potential and we know that the functions F_1, \dots, F_k are generically independent in that case [A]. The symplectic manifold T^*W is $2m$ -dimensional, so we have shown the Arnol'd-Liouville integrability of the System (1). \square

Note that the hypersurfaces of constant energy are compact and so by the Arnol'd-Liouville theorem, there are invariant tori and the solutions of (18) are quasi-periodic functions of the time.

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