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Floquet Operators with Singular Spectrum. I

by

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ABSTRACT. — A positive, discrete Hamiltonian H is perturbed by a time-periodic perturbation $V(t)$. If the gap between successive eigenvalues of H grows sufficiently rapidly, then generically (in a probabilistic sense) $H + \beta V(t)$ has dense pure point Floquet spectrum.

RÉSUMÉ. — Nous perturbons un opérateur Hamiltonien discret par un opérateur $V(t)$ périodique en temps. Si la distance entre les valeurs successives de H croît vite, alors génériquement (en un sens probabiliste) $H + \beta V(t)$ a un spectre de Floquet purement ponctuel dense.

1. INTRODUCTION

Let H be a discrete Hamiltonian operator on \mathcal{H} with eigenvalues λ_n , and $V(t)$ a periodic, time-dependent perturbation of H :

$$V(t+a) = V(t).$$

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The natural object to consider for a periodic Hamiltonian

$$H(t, \beta) = H + \beta V(t)$$

is the *Floquet Hamiltonian*:

$$K(\beta) = i \frac{d}{dt} + H(t, \beta)$$

with periodic boundary condition $u(a) = u(0)$, acting on the space $\mathcal{H} = L_2[0, a] \otimes \mathcal{H}$. (See, e. g., [7], [13] and a vast physics literature.)

If $\beta = 0$, and the period is normalized to $a = 2\pi$, then the operator $K(0)$ has pure point spectrum with eigenvalues

$$\Lambda(n, k) = n + \lambda k$$

($n = 0, \pm 1, \dots$). Except in rare cases, this spectrum will be dense in the line.

The question that we wish to consider is this: *When does the perturbed operator $K(\beta)$ also have dense pure point spectrum?* The question has generated considerable recent interest ([1], [2], [3], [6], [7], [11]), and we refer particularly the reader to article [1] of Bellissard. Most of this work has resulted in operator-theoretic versions of the Kolmogorov-Arnold-Moser theorem, which asserts pure point spectrum for generic values of certain parameters in $H(t)$ and for small coupling β . The essential idea, though, is that K will be pure point if there is no resonance.

The present paper takes a different approach to the problem, based on the author's generalization [8] of the Simon-Wolff-Kotani [12] method from the theory of localization. We shall show under certain conditions that $K(\beta)$ is pure point for "almost every H ". Thus, the method yields generic results on a *probabilistic*, rather than in a metric sense. The essential technical condition, which we believe can be weakened substantially, is that the gap between eigenvalues of H

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n$$

grows like $n^{2+\varepsilon}$, $\varepsilon > 0$.

After recalling some results from [8], we consider in paragraph 3 as an easy consequence, certain *compact* (actually, trace class) perturbations of H , generalizing a result of [7]. We then state the Main Theorem, and show in paragraph 5 how an adiabatic analysis of $H(t, \beta)$ reduces the problem to one like that of paragraph 3. We close with some remarks and conjectures.

2. NOTATION AND PREVIOUS RESULTS

Let H be a *positive definite, discrete* selfadjoint operator of *simple multiplicity* on a separable Hilbert space \mathcal{H} . Let φ_n be a complete orthonormal set of eigenvectors of H :

$$H \varphi_n = \lambda_n \varphi_n$$

with $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$. Let $V(t)$ be a uniformly bounded measurable family of bounded operators, which is 2π -periodic in t :

$$V(t + 2\pi) = V(t)$$

and define the *Floquet operator*

$$K(\beta) = i \frac{d}{dt} + H + \beta V(t)$$

on $\mathcal{H}L^2[0, 2\pi] \otimes \mathcal{H}$, with *periodic boundary condition*:

$$u(0) = u(2\pi).$$

We shall sometimes write $K_0 = K(0)$ and $K = K(1)$.

We shall also consider families $H(\omega)$ of operators satisfying these conditions, which are measurable on a probability space (P, Ω) . We shall refer to these briefly as "*random Hamiltonians*", and will write

$$H(\omega) \varphi_n(\omega) = \lambda_n(\omega) \varphi_n(\omega).$$

If

$$K(\omega) = i \frac{d}{dt} + H(\omega) + V(t)$$

is the corresponding Floquet operator, we define \mathbf{K} to be the *multiplication operator*

$$\mathbf{K}u(\omega) = K(\omega)u(\omega)$$

on $L^2(P, \Omega) \otimes \mathcal{H}$. If the coupling constant β is included, we obtain $\mathbf{K}(\beta)$, so that $\mathbf{K} = \mathbf{K}(1)$.

We shall next summarize some results of [8]. Let H be pure point and A bounded. We say [8], p. 64, that A is *strongly H-finite on an open interval J* iff

$$\Sigma \{ |A \varphi_n| : \lambda_n \in J \} < \infty.$$

2.1. PROPOSITION [8], pp. 56-58. — *Let H be pure point, A strongly H-finite on J , and W bounded and self-adjoint. Let*

$$H_1 = H + A^*WA.$$

Then there exists a set $N = N(H, A)$ not depending on W such that

- (i) N has Lebesgue measure zero, and
 (ii) N supports the continuous spectrum of H in J .
 Finally, we have the following version of "Kotani's trick".

2.2. PROPOSITION [8], p. 59. — Let $K(\omega)$ be a random self-adjoint operator such that:

- (i) there exists a set N of Lebesgue measure zero, independent ω , which supports the continuous part $K(\omega)$ a. s.,
 (ii) K has absolutely continuous spectral measure. Then $K(\omega)$ is pure point a. s.

3. COMPACT PERTURBATIONS OF FLOQUET OPERATORS

Let H be positive, discrete and of simple multiplicity, and A strongly H -finite. Let K be the Floquet operator

$$K = i \frac{d}{dt} + H.$$

3.1. PROPOSITION. — $1 \otimes A$ is strongly K -finite on any finite interval J .

Proof. — The eigenvalue of K are $\Lambda(n, k) = n + \lambda_k$ with eigenvectors $\Phi(n, k)(t) = e^{int} \varphi_k$. If we assume that J has length less than 1, then $\Lambda(n, k)$ is in J for at most one value of n , which we call n_k . Thus, for each k ,

$$\begin{aligned} \Sigma \{ |(1 \otimes A) \Phi(n, k)| : \Lambda(n, k) \in J \} \\ = \sum_k |(1 \otimes A) \Phi(n_k, k)| \leq \sum_k |A \varphi_k| < \infty. \quad \blacksquare \end{aligned}$$

Let $W(t)$ be a uniformly bounded, 2π -periodic measurable family of self-adjoint operators, and

$$V(t) = A^* W(t) A.$$

Note that the sum of two terms of this form can be written in the same form:

$$\begin{aligned} [A_1^* W_1(t) A_1 + A_2^* W_2(t) A_2] \varphi &= A^* W(t) A \varphi \\ &= (A_1^*, A_2^*) \begin{pmatrix} W_1(t) & 0 \\ 0 & W_2(t) \end{pmatrix} \begin{pmatrix} A_1 \varphi \\ A_2 \varphi \end{pmatrix} \end{aligned}$$

where $A : \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$ (cf. [7]).

3.2. THEOREM. — If $W(t) > 0$, then the Floquet operator for

$$H(t, \beta) = H + \beta V(t)$$

is pure point for a. e. β .

Proof. — It suffices to consider $|\beta| < 1$. From Proposition 2.1, $K(\beta)$ has its continuous part is concentrated on a set $N(H, A)$ of measure zero, independent of β . To be able to apply Proposition 2.2, we write $\beta = \tanh x$, where $-\infty < x < \infty$. The operator K of multiplication by $K(x)$ then has a positive commutator with the bounded operator

$$B = i \arctan(p/2)$$

(where $p = -i \frac{d}{dx}$), and is therefore absolutely continuous by the Putnam-Kato Theorem (cf. [8], p. 60).

We shall next show that for “almost every H ”, $K(\beta)$ is pure point for every β . To be precise, let $X_j(\omega)$ be i. i. d, and uniform on $[-1, 1]$, and $\varepsilon_j > 0$ with

$$\sum_{j=0}^{\infty} \varepsilon_j < \infty.$$

3.3. THEOREM. — *The Floquet operator for*

$$H(t, \omega) = H + \sum_{j=0}^{\infty} \varepsilon_j^2 X_j(\omega) \langle \cdot, \varphi_j \rangle \varphi_j + V(t)$$

is pure point a. s.

Thus, if the eigenvalues of H are all wiggled independently by a tiny amount, K will have pure point spectrum.

Proof. — The second term of $H(t, \omega)$ can be written as $EX(\omega)E$ where $E = \sum_j \varepsilon_j \langle \cdot, \varphi_j \rangle \varphi_j$ and $X(\omega) = \sum_j X_j(\omega) \langle \cdot, \varphi_j \rangle \varphi_j$. Since E is strongly H -finite, we find from (3.1) and Proposition 2.1, that $K(\omega)$ has continuous spectrum concentrated on a null set $N = N(H, A, E)$ independent of ω . Absolute continuity of K is obtained as in [8], pp. 67-69. ■

Remarks. — Several improvements in the results are easily made. The distribution of X_j need not be uniform [8], nor is simple multiplicity necessary. The randomness in $H(\omega)$ could be more simply taken as

$$H + \alpha A_1^2$$

with A_1 strongly H -finite. The reader may formulate such results for himself.

4. MAIN THEOREM

As above, let H be positive, discrete and of simple multiplicity, and $V(t)$ be bounded and 2π -periodic satisfying

$$\int_0^{2\pi} V(t) dt = 0. \quad (4.1)$$

Let $X_n(\omega)$ be i. i. d. and uniform on $[-1, 1]$, $\varepsilon_n > 0$ with

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty$$

and define

$$H(t, \omega) = H + \sum_{n=0}^{\infty} \varepsilon_n X_n(\omega) \langle \cdot, \varphi_n \rangle \varphi_n + V(t). \quad (4.2)$$

Let $\Delta\lambda_n$ be the gap between eigenvalues:

$$\Delta\lambda_n = \lambda_n - \lambda_{n-1}.$$

4. 1. THEOREM. — Let $V(t)$ be strongly C^1 , and satisfy for some $c > 0$

$$\Delta\lambda_n \geq cn^\alpha \quad (4.3)$$

and $\alpha > 2$. Then $K(\omega)$ is pure point a. s.

Proof. — Absolute continuity of K follows before. By the adiabatic analysis of $H(t, \omega)$ carried out in the next section, the operator $K(\omega)$ is unitarily equivalent to an operator

$$i \frac{d}{dt} + H + AW(t, \omega)A.$$

The operator A is strongly H -finite since $\gamma > 1$, and $W(t, \omega)$ bounded in norm uniformly in t and ω . Existence of a null set $N(H, A)$, independent of ω and supporting the continuous part of $K(\omega)$, now follows from Proposition 2. 1. ■

5. ADIABATIC ANALYSIS OF $H(t, \beta)$

Let A be a diagonal operator

$$A = \sum_{n=0}^{\infty} a_n \langle \cdot, \varphi_n \rangle \varphi_n$$

with $a_n > 0$. We shall prove the following theorem:

5. 1. THEOREM. — Let H be positive, discrete and of simple multiplicity, and $V(t)$ strongly C^{r+1} , satisfying (4.1). Assume that for some $c > 0$ and

$\alpha > 0$,

$$\Delta\lambda_n \geq cn^\alpha, \quad \alpha > 0 \tag{5.1}$$

and let $a_n = n^{-\gamma}$ for $n \geq 1$ with $0 < 2\gamma < \alpha$.

Then $K(\beta)$ is unitarily equivalent to

$$i \frac{d}{dt} + H + AW(t, \beta)A$$

where $W(t, \beta)$ is strongly C^r in t and uniformly bounded.

Remark. — Actually, $\lambda_n = \lambda_n(\omega)$ is random, but we will suppress ω and assume that (5.1) holds uniformly in ω . In fact, we shall assume that for some η , $|\lambda_n(\omega) - \lambda_n| \leq \eta$ for all ω . Then if (5.1) holds for λ_n , it will hold uniformly for $\lambda_n(\omega)$, with a smaller c .

Let $|V(t)| \leq M$, $|\dot{V}(t)| \leq M$. Let $\lambda_n(t, \beta)$ be the n -th eigenvalue of

$$H(t, \beta) = H + \beta V(t)$$

and

$$R(z; t, \beta) = (H(t, \beta) - z)^{-1}$$

its resolvent.

The reason for including β will be apparent in the proof of Lemma 5.3 below.

Note that if $n > k$, then

$$\lambda_n - \lambda_k = \Delta\lambda_n + \dots + \Delta\lambda_{k+1} \geq c(n^\alpha + \dots + (k+1)^\alpha) \geq c \int_k^n x^\alpha dx$$

so that for $n > k$,

$$\lambda_n - \lambda_k \geq c(\alpha + 1)^{-1} (n^{\alpha+1} - k^{\alpha+1}). \tag{5.2}$$

In particular, (since $\lambda_0 > 0$)

$$\lambda_n \geq c(\alpha + 1)^{-1} n^{\alpha+1}. \tag{5.3}$$

Let

$$r_n = \frac{c}{2} n^\alpha \leq \frac{1}{2} \min \{ \lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1} \}$$

and let $\Gamma_n (= \Gamma_n(\omega))$ be the positively oriented contour $|z - \lambda_n| = r_n$.

We now choose and fix N such that $r_n \geq 2M$ for $n \geq N$.

5.2. LEMMA. — (a) For $n \geq N$ and $|\beta| \leq 1$,

$$|\lambda_n(\beta, t) - \lambda_n| \leq M \leq r_n/2$$

and hence

$$\text{dist}(\lambda_n(\beta, t), \Gamma_n) \geq r_n - M \geq r_n/2.$$

Moreover, $\lambda_n(\beta, t)$ is the only point of $\sigma(H(t, \beta))$ inside Γ_n .

Proof. — This follows by upper semicontinuity of the spectrum, since the norm of the perturbation does not exceed $|\beta| M$. ■

Note that this gives

$$|R(z, t, \beta)| \leq 2r_n^{-1}$$

for $z \in \Gamma_n$, and

$$|\lambda_n(t, \beta) - \lambda_k(t, \beta)| \geq 1/2 |\lambda_n - \lambda_k|.$$

For $n \geq N$, let the spectral projection for $\lambda_n(t, \beta)$ is

$$P_n(t, \beta) = \frac{1}{2\pi i} \int_{\Gamma_n} R(z; t, \beta) dz = \langle \cdot, \varphi_n(t, \beta) \rangle \varphi_n(t, \beta).$$

The phase of $\varphi_n(t, \beta)$ is fixed by the choice

$$\varphi_n(t, \beta) = |P_n(t, \beta) \varphi_n|^{-1} \cdot P_n(t, \beta) \varphi_n \tag{5.5}$$

which makes $\varphi_n(t, \beta)$ smooth and 2π -periodic in t . Note that the norm of $P_n(t, \beta) \varphi_n$ is never zero; for we have

$$P_n(t, \beta) \varphi_n - \varphi_n = (2\pi i)^{-1} \int_{\Gamma_n} R(z; t, \beta) V R(z, t, 0) dz$$

which yields, by Lemma 3.2, the estimate

$$|P_n(t, \beta) \varphi_n - \varphi_n| \leq M r_n^{-1} \leq 1/2.$$

We now need to separate off the first N eigenvalues in a group. Let

$$Q(t, \beta) = I - \sum_{j=N+1}^{\infty} P_j(t, \beta)$$

be the spectral projection onto the first N eigenvectors of $H(t, \beta)$.

We can write

$$Q(t, \beta) = (2\pi i)^{-1} \int_{\Gamma_0} R(z, t, \beta) dz \tag{5.6}$$

where Γ_0 is a suitable contour encircling $\lambda_j(t, \beta)$ for $0 \leq j \leq N$. From this representation, we obtain immediately the uniform boundedness and continuity of such operators as

$$\frac{\partial Q}{\partial \beta}, \quad \frac{\partial Q}{\partial t} \quad \text{and} \quad \frac{\partial^2 Q}{\partial t \partial \beta}.$$

5.3. LEMMA. — *There exists a bounded operator-valued function $Z(t, \beta)$, defined and 2π -periodic in t for $|\beta| \leq 1$, and satisfying the following:*

- (a) $Z(t, \beta)$ is strongly C^{r+1} in t , and analytic in β , and $|Z(t, \beta)| \leq 1$.
- (b) $Z(t, \beta)$ maps $Q(t, \beta) \mathcal{H}$ isometricly onto $Q(0, 0) \mathcal{H}$ and annihilates the complement of $Q(t, \beta) \mathcal{H}$.
- (c) $\partial Z(t, \beta)/\partial t$ is uniformly bounded.

Proof. — Given the projection valued function $Q(\beta, t)$ defined for $|\beta| \leq 1$ and $t \in \mathbb{R}$, we proceed as in Kato's proof of the adiabatic theorem [5], p. 99 (see also [14]), to define an operator $Z_1(\beta, t)$ as the solution of the linear initial value problem:

$$\begin{aligned} \frac{\partial Z_1}{\partial \beta} &= \left[\frac{\partial Q}{\partial \beta}, Q \right] Z_1 \\ Z_1(0, t) &= I. \end{aligned} \tag{5.7}$$

Since Z_1 is the sum of a uniformly convergent Volterra series, Z_1 will be analytic in β and C^1 and 2π -periodic in t .

We will have, as in [5],

$$Q(\beta, t) Z_1(\beta, t) = Z_1(\beta, t) Q(0, t) = Z_1(\beta, t) Q(0, 0).$$

For part (c), we have the equation

$$\frac{\partial \dot{Z}_1}{\partial \beta} = \left[\frac{\partial \dot{Q}}{\partial \beta}, Q \right] Z_1 + \left[\frac{\partial Q}{\partial \beta}, \dot{Q} \right] Z_1 + \left[\frac{\partial Q}{\partial \beta}, Q \right] \dot{Z}_1. \tag{5.8}$$

By using Gronwall's inequality, we can obtain a bound on \dot{Z}_1 depending only on bounds on the coefficients. In particular, we can get bounds independent of a parameter ω in $H(\omega)$. ■

For $n, k \geq N$, define

$$a_{n,k}(t, \beta) = \langle \dot{\varphi}_k(t, \beta), \varphi_n(t, \beta) \rangle$$

where the dot denotes differentiation with respect to t .

5.4. LEMMA. — *For $n, k \geq N$ and $|\beta| \leq 1$, we have*

$$|a_{n,k}(t, \beta)| \leq 8 \beta \dot{M} |\lambda_n - \lambda_k|^{-1} \tag{5.9}$$

for $n \neq k$

$$|a_{nn}(t, \beta)| \leq 8 \beta \dot{M} r_n^{-1} \tag{5.10}$$

and

$$|Q(t, \beta) \dot{\varphi}_n(t, \beta)| \leq \beta \dot{M} C(N) \lambda_n^{-1}. \tag{5.11}$$

Remark. — If $H(t, \beta)$ commutes with an antiunitary C , such as complex conjugation, one can choose $\varphi_n(t)$ with $C \varphi_n(t) = \varphi_n(t)$, which implies that

$\langle \varphi_n(t), \varphi_n(t) \rangle$ is real. In this case,

$$a_{nn}(t) = 1/2 \frac{d}{dt} |\varphi_n(t)|^2 = 0.$$

Proof. — For simplicity, we suppress β throughout most of the proof. Differentiate

$$P_n(t) = \langle \cdot, \varphi_n(t) \rangle \varphi_n(t)$$

to obtain

$$P_n(t) = \langle \cdot, \dot{\varphi}_n(t) \rangle \varphi_n(t) + \langle \cdot, \varphi_n(t) \rangle \dot{\varphi}_n(t)$$

and hence

$$\dot{P}_n(t) P_k(t) = a_{n,k}(t) \langle \cdot, \varphi_k(t) \rangle \varphi_n(t).$$

Thus,

$$|a_{n,k}(t)| = |\dot{P}_n(t) P_k(t)|. \quad (5.12)$$

Differentiate

$$P_n(t) = (2\pi i)^{-1} \int_{\Gamma_n} R(z, t) dz$$

to obtain

$$\dot{P}_n(t) = -(2\pi i)^{-1} \int_{\Gamma_n} R(z, t) \dot{V}(t) R(z, t) dz.$$

Hence

$$\begin{aligned} \dot{P}_n(t) P_k(t) &= -(2\pi i)^{-2} \int_{\Gamma_k} \int_{\Gamma_n} R(z, t) \dot{V}(t) R(z, t) R(z', t) dz dz' \\ &= -(2\pi i)^{-2} \int_{\Gamma_n} dz R(z, t) \\ &\quad \times \int_{\Gamma_n} \dot{V}(t) (z-z')^{-1} [R(z, t) - R(z', t)] dz'. \end{aligned} \quad (5.13)$$

Now (for $k \neq n$), $z \in \Gamma_n$ is fixed, so $(z-z')^{-1} R(z, t)$ is analytic inside Γ_k as a function of z' . So the first term drops out and we obtain

$$\dot{P}_n(t) P_k(t) = (2\pi i)^{-2} \int_{\Gamma_n} \int_{\Gamma_k} (z'-z)^{-1} R(z, t) \dot{V}(t) R(z', t) dz' dz. \quad (5.14)$$

Estimating gives

$$|\dot{P}_n(t) P_k(t)| \leq 8\beta M |\lambda_n - \lambda_k|^{-1}$$

which is (5.9).

For (5.10), estimate (5.13) directly with $k = n$. For (5.11), write

$$Q(t) \dot{\varphi}_n(t) = \sum_{k=0}^N P_k(t) \dot{P}_n(t) \varphi_n = - \sum_{k=0}^N \dot{P}_k(t) P_n(t) \varphi_n \quad (5.15)$$

where the second step results from the identity

$$0 = \frac{d}{dt} [P_n, P_k] = \dot{P}_n P_k + P_n \dot{P}_k \quad (n \neq k).$$

But (5.15) is equal to the right side of (5.14) with Γ_n replaced by the contour Γ_0 , encircling the first $(N + 1)$ eigenvalues. Since Γ_0 can be chosen with

$$\text{dist}(\Gamma_0, \sigma(H(t, \beta))) \geq r_N$$

we obtain

$$|Q(t) \dot{\varphi}_n(t)| \leq \left(\frac{8 \beta \dot{M} L}{2 \pi r_N} \right) (\lambda_n - \lambda_N)^{-1} \quad (5.16)$$

where L is the length of Γ_N .

But

$$\frac{\lambda_n - \lambda_N}{\lambda_n} = 1 - \frac{\lambda_N}{\lambda_n} \geq 1 - \frac{\lambda_N}{\lambda_{N+1}} = \frac{\lambda_{N+1} - \lambda_N}{\lambda_{N+1}} \geq \frac{1}{N+1}$$

by (5.1). Thus (5.16) does not exceed

$$8 \beta \dot{M} L (N+1) (2 \pi r_N)^{-1} \lambda_n^{-1} = \beta \dot{M} C(N) \lambda_n^{-1}.$$

If we now define

$$U_1(t, \beta) = \sum_{k=N+1}^{\infty} \langle \cdot, \varphi_k(t, \beta) \rangle \varphi_n \quad (5.17)$$

then the operator

$$U(t, \beta) = Z(t, \beta) + U_1(t, \beta)$$

is unitary, and maps $Q(t, \beta) \mathcal{H}$ onto $Q(0, 0) \mathcal{H}$ and $\varphi_n(t, \beta)$ to φ_n for $n > N$. Let $U(\beta)$ be the operator-valued multiplication operator on \mathcal{H} defined by

$$(U(\beta) u)(t) = U(t, \beta) u(t)$$

and compute that

$$U(\beta) K(\beta) U^*(\beta) = i \frac{d}{dt} + \sum_{k=N+1}^{\infty} \lambda_k(t, \beta) P_k + \Delta(t, \beta) \quad (5.18)$$

where

$$\Delta = ZHQZ^* + i \{U_1 \dot{U}_1^* + Z\dot{Z}^* + U_1 \dot{Z}^* + Z\dot{U}_1^*\}. \quad (5.19)$$

We wish to choose $a_n > 0$ so that if

$$A = \sum_{n=0}^{\infty} a_n P_n$$

then $A^{-1} \Delta(t, \beta) A^{-1}$ is uniformly bounded. Observe first that

$$A^{-1} Q(p) = \sum_{k=0}^N a_k^{-1} P_k$$

is bounded. Second, note that

$$Z(t) = Q(0) Z(t) = Z(t) Q(t) = Q(0) Z(t) Q(t) \quad (5.20)$$

and hence that

$$\dot{Z}(t) = Q(0) \dot{Z}(t). \quad (5.21)$$

Third, differentiate

$$U_1(t) Z^*(t) = 0 \quad (5.22)$$

to obtain

$$\dot{Z}(t) U_1^*(t) = -Z(t) \dot{U}_1^*(t). \quad (5.23)$$

Consider now the five terms of $A^{-1} \Delta(t, \beta) A^{-1}$. The two terms

$$A^{-1} Z(t) H(t) Q(t) Z^*(t) A^{-1} \\ = (A^{-1} Q(0)) (Z(t) H(t) Q(t) Z^*(t)) (A^{-1} Q(0))^* \quad (5.24)$$

and

$$A^{-1} Z(t) \dot{Z}(t)^* A^{-1} = (A^{-1} Q(0)) (Z(t) \dot{Z}^*(t)) (A^{-1} Q(0))^* \quad (5.25)$$

are uniformly bounded [Lemma 5.2(c) and $H(t) Q(t) \leq \lambda_N(t)$]. The two terms

$$A^{-1} Z(t) \dot{U}_1^*(t) A^{-1} = (A^{-1} Q(0)) Z(t) (Q(t) \dot{U}_1^*(t) A^{-1}) \quad (5.26)$$

and

$$A^{-1} U_1(t) \dot{Z}^*(t) A^{-1} = (\dot{Z}(t) U_1^*(t) A^{-1})^* (A^{-1} Q(0))^* \\ = -(Z(t) \dot{U}_1^*(t) A^{-1} Q(0))^* \\ = (Q(t) \dot{U}_1^*(t) A^{-1})^* Z^*(t) (A^{-1} Q(0))^* \quad (5.27)$$

will both be uniformly bounded if

$$Q(t) \dot{U}_1^*(t) A^{-1} \quad (5.28)$$

is bounded. Thus we need only estimate this operator and

$$A^{-1} U_1(t) \dot{U}_1^*(t) A^{-1}. \quad (5.29)$$

5.5. LEMMA. — Let $a_n = n^{-\gamma}$ where $0 < 2\gamma < \alpha$. Then (5.28) and (5.29) are norm bounded uniformly in t and β , $|\beta| \leq 1$.

Proof. — We compute that

$$Q(t) \dot{U}_1^*(t) A^{-1} = \sum_{j=N+1}^{\infty} a_j^{-1} \langle \cdot, \varphi_j \rangle Q(t) \varphi_j(t).$$

Hence by (5.11), its norm does not exceed,

$$\beta MC(N) \sum_{j=N+1}^{\infty} j^\gamma \lambda_j^{-1}$$

which is finite by (5.1).

Similarly, (5.29) is equal to

$$\sum_{j,l=N+1}^{\infty} a_j^{-1} \bar{a}_{jl}(t) a_l^{-1} \langle \cdot, \varphi_j \rangle \varphi_l.$$

By (5.2) and (5.9), it therefore suffices to show boundedness of the infinite matrix (b_{jl}) with

$$b_{jl} = \frac{j^\gamma l^\gamma}{j^{\alpha+1} - l^{\alpha+1}}, \quad j \neq l \tag{5.30}$$

$$b_{jj} = j^{2\gamma-\alpha}.$$

Since b_{kl} is symmetric, the Schur-Holmgren condition for boundedness is simply

$$\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty.$$

The diagonal is bounded, and so causes no problem; thus we require finiteness of

$$\sup_n n^\gamma \sum_{k=n+1}^{\infty} \frac{k^\gamma}{(k^{\alpha+1} - n^{\alpha+1})} + \sup_n n^\gamma \sum_{k=1}^{n-1} \frac{k^\gamma}{(n^{\alpha+1} - k^{\alpha+1})}. \tag{5.31}$$

For the first term, we have

$$n^\gamma \sum_{k=n+1}^{\infty} \frac{k^\gamma}{(n^{\alpha+1} - k^{\alpha+1})} \leq n^\gamma \int_{n+1}^{\infty} \frac{x^\gamma}{x^{\alpha+1} - n^{\alpha+1}} dx$$

$$= n^{2\gamma-\alpha} \int_{1+1/n}^{\infty} \frac{s^\gamma}{s^{\alpha+1} - 1} ds \leq c(\alpha, \gamma) n^{2\gamma-\alpha} \log n$$

which goes to zero if $0 < 2\gamma < \alpha$. Similarly,

$$n^\gamma \sum_{k=1}^{n-1} \frac{k^\gamma}{n^{\alpha+1} - k^{\alpha+1}} = n^{2\gamma-\alpha} \sum_{k=1}^{n-1} \frac{1}{1 - (k/n)^{\alpha+1}} \left(\frac{k}{n}\right)^\gamma \frac{1}{n}$$

$$\leq n^{2\gamma-\alpha} \left\{ \int_0^{1-1/n} \frac{s^\gamma}{s^{\alpha+1} - 1} ds + c(\alpha) \right\} \leq c(\sigma, \gamma) n^{2\gamma-\alpha} \log n$$

[we have estimated the $k = n - 1$ term by $c(\alpha) = 2^\gamma (\alpha + 1)^{-1}$]. ■

We have now shown that $K(\beta)$ is uniformly equivalent to an operator of the form (4.3), with H replaced by the diagonal operator

$$\sum_{n=0}^{\infty} \lambda_n(t, \beta) P_n.$$

From perturbation theory [5], p. 88, we have

$$\lambda_n(t, \beta) = \lambda_n + \beta (V(t) \varphi_n, \varphi_n) + E_n(t, \beta)$$

where the error term satisfies

$$|E_n(t, \beta)| \leq 2\beta^2 M^2 r_n^{-1}.$$

Since $n^{2\gamma} r_n^{-1}$ is bounded, the term

$$\sum_{n=0}^{\infty} E_n(t, \beta) P_n$$

can be absorbed into the $AW(t, \beta)A$ term in (4.3). To eliminate the remaining term, note that by (4.1),

$$(V(t) \varphi_n, \varphi_n) = \dot{g}_n(t)$$

where $g_n(t)$ is 2π -periodic. Let $G(t)$ be the unitary transformation

$$G(t) = \sum_{n=0}^{\infty} e^{i\beta g_n(t)} P_n$$

and $Gu(t) \equiv G(t)u(t)$. If we now transform by the "gauge transformation" G , the term

$$\sum_{n=0}^{\infty} \beta \langle V(t) \varphi_n, \varphi_n \rangle P_n$$

disappears, while the form of $AW(t, \beta)A$ is preserved, since $G(t)$ commutes with A . This completes the proof. ■

6. CONCLUDING REMARKS

(1) Our theorem is unsatisfactory in several ways. In the first place, one would like to reduce the value of α . Hamiltonians like the one-dimensional rotor considered by Bellisard ([1], [2]) corresponds to $\Delta\lambda_n \simeq n$, or $\alpha = 1$. (It also has multiplicity two.) The harmonic oscillator ([3], [6]) has $\alpha = 0$, and is doubtless more delicate.

(2) One would also like to be able to randomize $H(\omega)$ within a natural class. For example, if H is a Schroedinger operator, we would like to have

$H(\omega)$ a Schroedinger operator as well. The chief problem here is the *difficulty of proving that operator multiplications like \mathbf{K} are absolutely continuous*. Theorems of this type would be very useful, both here and in localization theory [10].

(3) The theorem here seems essentially one-dimensional in its assumption of increasing gap $\Delta\lambda_n$. For example, if H represents the particle in a box in d dimensions, then for $d \geq 3$, the density of eigenvalues becomes *larger* as energy increase, rather than smaller. *Does this result in a different spectral type for \mathbf{K} ?* An answer to this question would be very interesting.

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