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Scattering Theory for the Shape Resonance Model II. Resonance Scattering

by

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ABSTRACT. — Continuing the analysis of the part I, we consider the asymptotic behavior of the scattering matrix for the shape resonance model. We give an asymptotic formula for the scattering matrix near the (complex) resonance eigenvalues under the assumptions similar to those in Combes-Duclos-Klein-Seiler [*2*] (exterior scaling analyticity, non-trapping condition, etc.).

RÉSUMÉ. — Nous poursuivons l'analyse de la première partie et nous considérons le comportement asymptotique de la matrice de diffusion pour le modèle de résonance de forme.

Nous donnons une forme asymptotique pour la matrice de diffusion près des valeurs propres résonantes (complexes) sous des hypothèses analogues à celles de Combes-Duclos-Klein-Seiler (2) (analyticité par dilatation, non piégeage, etc.).

§ 1. INTRODUCTION

We consider

$$H = H(h) = H_0(h) + V(x), \qquad H_0 = H_0(h) = -h^2\Delta$$

on $L^2(\mathbb{R}^n)$, $D(H_0) = H^2(\mathbb{R}^n)$ where $h > 0$ is the Planck constant. Instead of $(A)_\alpha$ in (I) (we refer to the part I of this series as (I), we assume for $\alpha > 1$, ASSUMPTION $(A)'_\alpha$. — V is a real-valued continuously differentiable function satisfying

$$|V(x)| \leqq C(1+|x|)^{-\alpha}, \qquad (x \in \mathbb{R}^n) \tag{1.1}$$

and there is $\delta > 0$ such that

$$|x \cdot \nabla V(x)| \leqq C|x|(1+|x|)^{-\delta} \qquad (x \in \mathbb{R}^n) \tag{1.2}$$

We set $\Omega_{\text{int}} = \{ x \in \mathbb{R}^n \mid |x| < R \}$, $K = R \cdot S^{n-1}$ for some $R > 0$, and suppose that $\inf_{x \in K} V(x) = \lambda_0 > 0$. As in (I), $\Omega_{\text{ext}} = \mathbb{R}^n \backslash \overline{\Omega_{\text{int}}}$, $\mathcal{I}(\lambda) = \{ x \in \mathbb{R}^n \mid V(x) > \lambda \}$, Δ^D is the Laplacian with Dirichlet boundary condition on K, and $H^D = H^D(h) = -h^2\Delta^D + V(x) = H_{\text{int}} \oplus H_{\text{ext}}$ on $L^2(\mathbb{R}^n) = L^2(\Omega_{\text{int}}) \oplus L^2(\Omega_{\text{ext}})$.

Next we assume for $\lambda \in (0, \lambda_0)$.

ASSUMPTION $(B)'_\lambda$. — There is $\delta > 0$ such that for any $x \in \Omega_{\text{ext}} \backslash \mathcal{I}(\lambda)$,

$$V(x) + \frac{1}{2} x \cdot \nabla V(x) \leqq \lambda - \delta. \tag{1.3}$$

$(B)'_\lambda$ is a sufficient condition of $(B)_I$ in (I) for some neighborhood I of λ provided $(A)'_\alpha$ $(\alpha > 0)$ (see Proposition A.1 of (I)).

The exterior scaling $U(\theta)$ is defined as follows (cf. Sect. 2 of [2]): for $\theta \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $T(\theta)x = (e^\theta(|x| - R) + R) \cdot \dfrac{x}{|x|}$ if $|x| \geqq R$, $= x$ if $|x| < R$. For $f \in L^2(\mathbb{R}^n)$, $U(\theta)f$ is defined by $(U(\theta)f)(x) = \exp\{ n\theta\chi(|x|)/2 \} \cdot f(T(x))$ where $\chi(r) = 1$ if $r \geqq R$, $= 0$ if $r < R$. $U(\theta)$ is the 1-parameter unitary group generated by $A = 0 \oplus (2i)^{-1} \left\{ (|x| - R)\dfrac{x}{|x|} \cdot \nabla + \nabla \cdot \dfrac{x}{|x|}(|x| - R) \right\}$.

ASSUMPTION (C). — $V(\theta) = U(\theta)VU(\theta)^{-1}$ $(\theta \in \mathbb{R})$ can be extended to a bounded operator valued analytic function on the strip $S_\gamma = \{ \theta \in \mathbb{C} \mid |\text{Im}\,\theta| < \gamma \}$ for some $\gamma > 0$.

It follows from (C) that $H(h, \theta) = U(\theta)H(h)U(\theta)^{-1}$ can be extended to an operator valued function on S_γ and it is analytic in the resolvent sense (cf. Appendix of [2]).

Let $\lambda \in (0, \lambda_0)$ and let $h_i \downarrow 0$ $(i \to \infty)$ be a sequence of positive numbers. Suppose that E_i^D is an isolated simple eigenvalue of $H_{\text{int}}(h_i)$ such that $E_i^D \to \lambda$ $(i \to \infty)$. We suppose furthermore that

ASSUMPTION (D). — For some $b > 0$ and $q > 0$,

$$\text{dist}\,(\sigma(H_{\text{int}}(h_i)) \backslash \{ E_i^D \}, E_i^D) \geqq b(h_i)^q \tag{1.4}$$

for any i.

Then it is known that there exists a (resonance) eigenvalue E_i of $H(h_i, \theta)$ $(\text{Im}\,\theta > 0)$ such that E_i is exponentially close to E_i^D in h_i^{-1}:

$$|E_i - E_i^D| \leqq C_\varepsilon \exp\{ -\mathbf{d}_\lambda(K, \Omega_{\text{int}} \backslash \mathcal{I}(\lambda)) - \varepsilon)/h_i \} \tag{1.5}$$

for any $\varepsilon > 0$, where $\mathbf{d}_\lambda$ is the pseudo-distance associated to the Agmon metric $ds^2 = \max(V(x) - \lambda, 0)dx^2$ (cf. Proposition 2.5, or Theorem V-2 of [2]).

Let $\eta_i = \eta_i(\theta; x)$ be an eigenfunction of $H(h_i, \theta)$ corresponding to E_i: $H(h_i, \theta)\eta_i = E_i\eta_i$. $T_{int/ext}$ denotes the trace operator to K:

$$(T_{int/ext}f)(x) = f(x) \quad (x \in K,\ f \in C^\infty(\Omega_{int/ext}));$$
$$T_{int/ext} \in B(H^1(\Omega_{int/ext}), L^2(K)).$$

$\nabla_n = \dfrac{x}{R} \cdot \nabla_x$ is the unit normal derivative on K. Let $\Phi_\pm(h_i; \mu)\varphi \in L^{2,-\alpha/2}(\Omega_{ext})$ be the generalized eigenfunction of $H_{ext}(h_i)$ corresponding to $\varphi \in L^2(S^{n-1})$ defined by (3.3) of (I). We define $\pi_i(\mu) \in B(L^2(S^{n-1}))$ by

$$(\varphi, \pi_i(\mu)\psi) = 2\pi h_i^4(\eta_i(\bar\theta), \eta_i(\theta))^{-1} \times$$
$$\times\ \overline{(T_{int}\eta_i(\theta), T_{ext}\nabla_n\Phi_+(h_i; \mu)\varphi)} \cdot (T_{int}\eta_i(\bar\theta), T_{ext}\nabla_n\Phi_-(h_i; \mu)\psi) \quad (1.6)$$

for $\mu \in (0, \infty)$ and $\operatorname{Im}\theta > 0$. $\eta_i(\bar\theta)$ is an eigenfunction of $H(h_i, \bar\theta) = H(h_i, \theta)^*$ with eigenvalue $\bar{E}_i$. It will be shown that $\pi_i(\mu)$ is independent of θ (Proposition 3.4 and Remark 3.5). We set $\Lambda_i = \{ \mu \in R \mid |\mu - E_i^D| \leqq b(h_i)^q/2 \}$ where b and q are the constants in (D). On the definition of scattering matrices $S(H, H_0; \lambda)$, etc., we refer to Sect. I of (I). Then our main result is:

THEOREM 1. — *Suppose* $(A)'_\alpha$ $(\alpha > 1)$, $(B)'_\lambda$ $(\lambda \in (0, \lambda_0))$ *and* (C). *Let* $h_i \downarrow 0$ $(i \to \infty)$ *and let* $E_i^D \to \lambda$ $(i \to \infty)$ *be a sequence of isolated simple eigenvalues of* $H_{int}(h_i)$ *such that* (D) *holds. Then for any* $\varepsilon > 0$, *there is* $C > 0$ *such that for* $\mu \in \Lambda_i$

$$\| S(H(h_i), H_0(h_i); \mu) - S(H^D(h_i), H_0(h_i); \mu) - (E_i - \mu)^{-1}\pi_i(\mu) \|$$
$$\leqq C \exp\{ -2(\mathbf{d}_\lambda(K, \Omega_{ext} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \}; \quad (1.7)$$

$$\| \pi_i(\mu) \| \leqq C \exp\{ -2(\mathbf{d}_\lambda(K, \Omega_{ext} \backslash \mathscr{I}(\lambda)) + \mathbf{d}_\lambda(K, \Omega_{int} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i, \quad (1.8)$$

where $\mathbf{d}_\lambda$ *is the pseudo-distance associated to the Agmon metric* $ds^2 = \max(V(x) - \lambda, 0) \cdot dx^2$.

If $(A)'_\alpha$ holds for $\alpha > (n + 1)/2$, Theorem 1 can be improved analogously to Theorem 2 in (I). Let $\Psi_\pm(h_i; \lambda, \omega)$ be the solution of the Lippman-Schwinger equation with respect to $H_{ext}(h_i)$ defined by (3.9) of (I). We define $\Xi_i(\mu, \omega, \omega')$ $(\omega, \omega' \in S^{n-1})$ by

$$\Xi_i(\mu, \omega, \omega') = 2\pi i h_i^4(\eta_i(\bar\theta), \eta_i(\theta))^{-1} \times$$
$$\times\ \overline{(T_{int}\eta_i(\theta), T_{ext}\nabla_n\Psi_+(h_i; \mu, \omega))} \cdot (T_{int}\eta_i(\bar\theta), T_{ext}\nabla_n\Psi_-(h_i; \mu, \omega')). \quad (1.9)$$

Then we have

THEOREM 2. — *Let the assumptions of Theorem 1 be satisfied with* $\alpha > (n+1)/2$. *Then for any* $\varepsilon > 0$, *there is* $C > 0$ *such that for* $\mu \in \Lambda_i$

$$| S(H(h_i), H_0(h_i) ; \mu, \omega, \omega') - S(H^D(h_i), H_0(h_i) ; \mu, \omega, \omega') - (E_i - \mu)^{-1} \Xi_i(\mu, \omega, \omega') | \leqq C \exp \{ -2(\mathbf{d}_\lambda(K, \Omega_{ext} \setminus \mathcal{I}(\lambda)) - \varepsilon)/h_i \} ; \quad (1.10)$$

$$| \Xi_i(\mu, \omega, \omega') | \leqq C \exp \{ -2(\mathbf{d}_\lambda(K, \Omega_{ext} \setminus \mathcal{I}(\lambda)) + \mathbf{d}_\lambda(K, \Omega_{int} \setminus \mathcal{I}(\lambda)) - \varepsilon)/h_i \} \quad (1.11)$$

uniformly in $\omega, \omega' \in S^{n-1}$.

REMARK 1.1. — It is expected that $\mathrm{Im}\, E_i$ and $\| \pi_i \|$ are of order

$$h_i^a \exp \{ - \mathbf{d}_\lambda(\Omega_{ext} \setminus \mathcal{I}(\lambda), \Omega_{int} \setminus \mathcal{I}(\lambda))/h_i \}$$

and

$$h_i^c \exp \{ - \mathbf{d}_\lambda(\Omega_{ext} \setminus \mathcal{I}(\lambda), \Omega_{int} \setminus \mathcal{I}(\lambda))/h_i \}$$

respectively for some a and c. If it is true, the order of the influence of resonances to the scattering matrix is of order $h_i^{(c-a)}$.

In Sect. 2 we review the results of Combes-Duclos-Klein-Seiler [2], and prove some related results in Sect. 3. Finally we prove Theorems 1 and 2 using the results of Sect. 3 and (I).

In this paper, we will use freely the symbols and results of (I), and we refer to (I) for historical remarks and further references.

§ 2. PRELIMINARIES: RESONANCE EIGENVALUES

In this section, we review the results of [2] in a slightly extended form, since we are interested not only in low lying eigenvalues but also in highly excited ones. Throughout this paper we suppose $(A)'_\alpha$ $(\alpha > 0)$, $(B)'_\lambda$ $(\lambda \in (0, \lambda_0))$ and (C).

The next proposition asserts the existence of an analytic continuation of the resolvent for H_{ext} :

PROPOSITION 2.1 ([2] Lemma II-3 or [3] Theorem 3.1). — *There are* $\theta_0 \in S_\gamma$ $(\mathrm{Im}\, \theta_0 > 0)$, $h_0 > 0$ *and* $\Theta = \{ z \in \mathbb{C} \mid |\mathrm{Re}\, z - \lambda| < \text{const.}, \mathrm{Im}\, z > -\text{const.} \}$ *: a complex neighborhood of* λ *such that*

$$\| (H_{ext}(h, \theta_0) - z)^{-1} \| \leqq C \quad (2.1)$$

for $0 < h < h_0$ *and* $z \in \Theta$.

$T_{int/ext}$ denotes the trace operator from $H^1(\Omega_{int/ext})$ to $L^2(K)$ and if

$T_{int}\varphi = T_{ext}\varphi$ for $\varphi \in H^1(\Omega_{int}) \oplus H^1(\Omega_{ext})$, we write $T\varphi = T_{int}\varphi$. Then $A(\theta, a)$ and $B(\theta, a)$ $(\theta \in \mathbb{R}, a \in \mathbb{C}: \operatorname{Im} a > 0)$ are defined by

$$A(\theta, a) = T_{int}(H(\theta) - a)^{-1} : L^2(\mathbb{R}^n) \to L^2(K), \tag{2.2}$$

$$B(\theta, a) = B_{int}(a) \oplus B_{ext}(\theta, a) : L^2(\Omega_{int}) \oplus L^2(\Omega_{ext}) \to \Omega^2(K), \tag{2.3}$$

$$B_{int}(a) = - T_{int}\nabla_n(H_{int} - a)^{-1} : L^2(\Omega_{int}) \to L^2(K);$$

$$B_{ext}(\theta, a) = e^{-3\theta/2}T_{ext}\nabla_n(H_{ext}(\theta) - a)^{-1} : L^2(\Omega_{ext}) \to L^2(K),$$

where we have written $H^D(\theta) = U(\theta)H^DU(\theta) = H_{int} \oplus H_{ext}(\theta)$. The above definitions can be extended to $\theta \in S_\gamma$.

PROPOSITION 2.2 (Krein's formula, [2] Appendix II). — *For* $a: \operatorname{Im} a > 0$,

$$\begin{aligned} W(\theta_0, a) = (H(\theta_0) - a)^{-1} - (H^D(\theta_0) - a)^{-1} \\ = h^2 A(\bar{\theta}_0, \bar{a})^* B(\theta_0, a) \\ = h^4 B(\bar{\theta}_0, \bar{a})^* T(H - a)^{-1} T^* B(\theta_0, a). \end{aligned} \tag{2.4}$$

PROPOSITION 2.3 ([2]) Theorem II-3). — *Let* $a(h)$ *be a complex-valued function of* h *such that* $\operatorname{Im} a(h) > 0$; $\operatorname{Re} a(h) \to \lambda$ $(h \downarrow 0)$; $(\operatorname{Im} a(h))^{-1} = O(h^{-p})$ *for some* $p > 0$. *Then* $B(\theta_0, a) = O(h^{-3/2})$; $A(\theta_0, a) = O(h^{-1/2})$ *and* $W(\theta_0, a) = O(1)$ $(h \downarrow 0)$.

The proof of Proposition 2.3 is the same as that of [2] Theorem II-3 except for the choice of the cut off function χ (cf. (I) Proposition 2.2).

Let $h_i \downarrow 0$ $(i \to \infty)$ be a sequence of positive numbers, and let $J_i = \{ E_i^{D,1}, \ldots, E_i^{D,N} \}$ be a set of eigenvalues for $H_{int}(h_i)$ such that $E_i^{D,k} \to \lambda$ $(i \to \infty)$ for each $k \in \{1, \ldots, N\}$. We set $\bar{J}_i = (\inf J_i, \sup J_i)$, $|J_i| = \sup J_i - \inf J_i$ and $\Delta_i = \operatorname{dist}(J_i, \sigma(H_{int}(h_i)) \backslash J_i)$. We suppose

ASSUMPTION (E). — $\bar{J}_i \cap \sigma(H_{int}(h_i)) = J_i$; $b(h_i)^q \leqq \Delta_i$ and $|J_i|^2/\Delta_i = 0(1)$ for some $b > 0$ and $q > 0$.

Let $\underline{J}_i = (\Sigma_{k=1}^N E_i^{D,k})/N$ and let $a_i = \underline{J}_i + i\Delta_i$. $\Gamma_i = \partial G_i$ is defined as the boundary of the region:

$$G_i = \{ z \in \mathbb{C} \mid \operatorname{dist}(\operatorname{Re} z, \bar{J}_i) < 3\Delta_i/4, |\operatorname{Im} z| < \Delta_i/2 \}. \tag{2.5}$$

PROPOSITION 2.4 ([2] Sect. IV). — *Let* a_i, G_i *and* Γ_i *be as above. Then*

1) $$((H^D(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1})^{-1} = 0(1);$$
$$((H(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1})^{-1} = 0(1)$$

uniformly in $z \in \Gamma_i$ *if* $i \to \infty$.

2) *For sufficiently large* i,

$$P_i = - (2\pi i)^{-1} \int_{\Gamma_i} (H(h_i, \theta_0) - z)^{-1} dz \tag{2.6}$$

is well-defined, and has the same dimension as

$$P_i^D = -(2\pi i)^{-1}\int_{\Gamma_i}(H^D(h_i,\theta_0)-z)^{-1}dz.$$

3) *Let* $Q_i^D = 1 - P_i^D$. *Then for sufficiently large i and for any* $z \in G_i$, $Q_i^D((H(h_i,\theta_0)-a_i)^{-1}-(z-a_i)^{-1})Q_i^D$ *is invertible on* Ran Q_i^D. *Furthermore, the inverse is* $0(1)$ *uniformly in* $z \in G_i$ *if* $i \to \infty$.

We next consider the case when $J = \{E_i^D\}$ and E_i^D is simple. Then (E) is equivalent to (D).

Proposition 2.5 ([2] Theorem V-2). — *Let* E_i *be the eigenvalue of* $H(\theta_0, h_i)$ *corresponding to* P_i, *then for any* $\varepsilon > 0$ *there is* C *such that*

$$|E_i - E_i^D| \leqq C\exp\{-(\mathbf{d}_\lambda(K,\Omega_{\text{int}}\backslash\mathscr{I}(\lambda))-\varepsilon)/h_i\}. \quad (2.8)$$

§ 3. RESONANCE EIGENFUNCTION AND EIGENPROJECTION

In this section we suppose $(A)'_\alpha$ ($\alpha > 0$), $(B)'_\lambda$ ($\lambda \in (0,\lambda_0)$), (C) and (D).

Let $\eta_i = \eta_i(\theta_0)$ be an (E_i)-eigenfunction of $H(h_i,\theta_0)$ such that $\eta_i = \eta_i^D + \zeta_i$ where η_i^D is a normalized E_i^D-eigenfunction of $H_{\text{int}}(h_i)$ and $(\zeta_i) \perp (\eta_i^D)$. Then

Proposition 3.1. — *For any* $\varepsilon > 0$, *there is* C *such that*

$$\|\zeta_i\| \leqq C\exp\{-(\mathbf{d}_\lambda(K,\Omega_{\text{int}}\backslash\mathscr{I}(\lambda))-\varepsilon)/h_i\}. \quad (3.1)$$

Proof. — We write $R = (H(h_i,\theta_0)-a_i)^{-1}$, $R^D = (H^D(h_i,\theta_0)-a_i)^{-1}$, $F = (E_i - a_i)^{-1}$, $F^D = (E_i^D - a_i)^{-1}$ and $W = R - R^D$. Then since $R\eta_i = F\eta_i$, we have

$$(F-R)\zeta_i = W\eta_i + (F^D - F)\eta_i^D \quad (3.2)$$

by easy computations. Applying $Q^D = 1 - P_i^D$ to the both sides of (3.2), and noting that $(R - F)$ is invertible on Ran Q^D by Proposition 2.4-(3), we see

$$\zeta_i = (Q^D(F-R)Q^D)^{-1}Q^DW\eta_i^D. \quad (3.3)$$

By Proposition 2.2 and a trace estimate (see e. g. Lemma III-4 of [2]),

$$\begin{aligned}\|W\eta_i^D\| &\leqq \|A(\theta_0,a_i)\|\cdot\|B_{\text{int}}(a_i)\eta_i^D\| \\ &\leqq \|A(\theta_0,a_i)\|\cdot(\operatorname{Im}a_i)^{-1}\cdot C\cdot(\|\chi\nabla_n\eta_i^D\|\cdot\|\nabla_n\chi\nabla_n\eta_i^D\|)^{1/2} \quad (3.4)\end{aligned}$$

for a smooth cut off function such that $\chi = 1$ on K. For any $\varepsilon > 0$, if χ is supported in a sufficiently small neighborhood of K, the Agmon estimate gives ([5], [1], see also Proposition 3.3 of (I))

$$\|\chi\nabla_n\eta_i^D\| + \|\nabla_n\chi\nabla_n\eta_i^D\| \leqq C\exp\{-(\mathbf{d}_\lambda(K,\Omega_{\text{int}}\backslash\mathscr{I}(\lambda))-\varepsilon)/h_i\}. \quad (3.5)$$

Combining (3.4) with Proposition 2.3, Assumption (D) and (3.5), we have

$$\| W\eta_i^{D} \| \leqq C_\varepsilon h_i^{-1/2} \cdot h_i^{-q} \exp \{ -(\mathbf{d}_\lambda(K, \Omega_{int} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \} \quad (3.6)$$

for any $\varepsilon > 0$. (3.1) follows from (3.3), (3.6) and Proposition 2.4-(3). □

COROLLARY 3.2. — *For any* $\varepsilon > 0$, *there is* C *such that*

$$\| T_{int}\eta_i \| = \| T_{int}\zeta_i \| \leqq C \exp \{ -(\mathbf{d}_\lambda(K, \Omega_{int} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \}. \quad (3.7)$$

Proof. — Since $\eta_i = \eta_i^D + \zeta_i$ is an E_i-eigenfunction of $-h_i^2\Delta + V$ in the distribution sense, we have

$$\begin{aligned}(-h_i^2\Delta + V)\zeta_i &= -(-h_i^2\Delta + V)\eta_i^D + E_i\eta_i \\ &= (E_i - E_i^D)\eta_i^D + E_i\zeta_i \end{aligned} \quad (3.8)$$

and hence

$$\| \zeta_i \|_{H^2(\Omega_{int})} \leqq C_\varepsilon \exp \{ -(\mathbf{d}_\lambda(K, \Omega_{int} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \} \quad (3.9)$$

for any $\varepsilon > 0$ by (2.8) and (3.1). (3.7) follows immediately from (3.9) and the trace estimate. □

PROPOSITION 3.3. — *Let* $\eta_i(\theta_0)$ *be an* (E_i)*-eigenfunction of* $H(h_i, \theta_0)$ *and let* $\eta_i(\bar\theta_0)$ *be an* $(\bar E_i)$*-eigenfunction of* $H(h_i, \bar\theta_0) = H(h_i, \theta_0)^*$. *Then*

$$P_i\varphi = (\eta_i(\bar\theta_0), \eta_i(\theta_0))^{-1}\eta_i(\theta_0) \cdot (\eta_i(\bar\theta_0), \varphi). \quad (3.10)$$

Proof. — Since P_i is a rank one operator, P_i can be written as $P_i\varphi = f \cdot (g, \varphi)$ for some f, $g \in L^2(\mathbb{R}^n)$ by the Riesz representation theorem. But since P_i is an eigenprojection of $H(h_i, \theta_0)$, $f = \text{const.}\ \eta_i(\theta_0)$. On the other hand, P_i^* is an eigenprojection of $H(h_i, \bar\theta_0) = H(h_i, \theta_0)^*$, and hence $g = \text{const.}\ \eta(\bar\theta_0)$. Because P_i is projective, the constant is easily determined to conclude (3.10). □

Now, $\pi_i(\mu) = \pi_i(\theta_0 ; \mu)$ is defined by

$$(\varphi, \pi_i(\theta_0 ; \mu)\psi) = 2\pi i h_i^4(\eta_i(\bar\theta_0), \eta_i(\theta_0))^{-1} \times$$
$$\times \overline{(T_{int}\eta_i(\theta_0), T_{ext}\nabla_n\Phi_+(h_i, \mu)\varphi)} \cdot (T_{int}\eta_i(\bar\theta_0), T_{ext}\nabla_n\Phi - (h_i, \mu)\psi) \quad (3.11)$$

for $\varphi, \psi \in L^2(\mathbb{R}^n)$ and $\mu \in (0, \infty)$. In fact, E_i and π_i is independent of θ. We remark that Proposition 2.1, and hence all the results of Sect. 2 holds for θ in a neighborhood of θ_0, and one can define $E_i(\theta)$ and $\pi_i(\theta, \mu)$ similarly (cf. [*3*], Theorem 3.1).

PROPOSITION 3.4. — $E_i(\theta)$ *and* $\pi_i(\theta ; \mu)$ *is independent of* θ *in a neighborhood of* θ_0.

Proof. — Let θ be in the neighborhood of θ_0 and let $\eta_i(\theta_0)$ be an (E_i)-eigenvector of $H(h_i, \theta_0)$. If $\text{Im}\,\theta = \text{Im}\,\theta_0$, we see

$$(H(h_i, \theta) - E_i)U(\theta - \theta_0)\eta_i(\theta_0) = U(\theta - \theta_0)(H(h_i, \theta_0) - E_i)\eta_i(\theta_0) = 0$$

and hence $U(\theta - \theta_0)\eta_i(\theta_0)$ is an (E_i)-eigenvector of $H(h_i, \theta)$. By the analytic perturbation theory, $H(h_i, \theta)$ has only one eigenvalue $E_i(\theta)$ near E_i if $|\theta - \theta_0|$ is sufficiently small, and $E_i(\theta)$ is analytic in θ. But if $\operatorname{Im}\theta = \operatorname{Im}\theta_0$ then $E_i(\theta) = E_i(\theta_0)$, and hence $E_i(\theta) = E_i(\theta_0)$ in the neighborhood of θ_0 (cf. [*4*], Theorem XIII-36). Similarly, if $\operatorname{Im}\theta = \operatorname{Im}\theta_0$,

$$\begin{aligned}(\varphi, \pi_i(\theta, \mu)\psi) = 2\pi i h_i^4 (U(\bar\theta - \bar\theta_0)\cdot\eta_i(\bar\theta_0), U(\theta - \theta_0)\cdot\eta_i(\theta_0))^{-1} \times \\ \times \overline{(T_{int}U(\theta - \theta_0)\eta_i(\theta_0), T_{ext}\nabla_n\Phi_+(h_i, \mu)\varphi)} \times \\ \times (T_{int}U(\bar\theta - \bar\theta_0)\eta_i(\bar\theta_0), T_{ext}\nabla_n\Phi_-(h_i, \mu)\psi) = (\varphi, \pi_i(\theta_0, \mu)\psi)\end{aligned}$$

since $U(\bar\theta - \bar\theta_0) = (U(\theta - \theta_0)^{-1})^*$ and $T_{int}U(\theta - \theta_0) = T_{int}$. The eigenfunction $\eta_i(\theta)$ can be taken to be analytic in θ, and hence $\pi_i(\theta\,;\mu)$ is also analytic in θ. Thus $\pi_i(\theta, \mu) = \pi_i(\theta_0, \mu)$ in the neighborhood of θ_0. □

REMARK 3.5. — For each $\theta: 0 < \operatorname{Im}\theta < \operatorname{Im}\theta_0$, $\pi_i(\theta\,;\mu)$ is well-defined if i is sufficiently large since Proposition 2.1 can be considerably improved ([*3*]). Then the above argument shows that E_i and π_i is independent of such θ. On the other hand, the similar result holds for $\tilde\pi_i$ defined by

$$(\varphi, \tilde\pi_i(\mu)\psi) = (T_{ext}\nabla_n\Phi_+(h_i, \mu)\varphi, (T_{int}P_iT_{int}^*)T_{ext}\nabla_n\Phi_-(h_i, \mu)\psi)$$

$(\varphi, \psi \in L^2(\mathbb{R}^n))$, instead of π_i under Assumption (E).

§ 4. PROOF OF THEOREMS 1 AND 2

In this section we assume $(A)'_\alpha$ $(\alpha > 1)$, $(B)'_\lambda$ $(\lambda \in (0, \lambda_0))$ and (C).

PROPOSITION 4.1. — *(A representation of the scattering matrix)*

$$\begin{aligned}&(\varphi, \{ S(H, H_0\,;\mu) - S(H^D, H_0\,;\mu) \}\psi) \\ &= 2\pi i h_i^4 (T_{ext}\nabla_n\Phi_+(h_i, \mu)\varphi, \{ T_{int}(H(h_i, \theta_0) - \mu)^{-1}T_{int}^* \} T_{ext}\nabla_n\Phi_-(h_i, \mu)\psi)\end{aligned} \tag{4.1}$$

for μ in a neighborhood of λ and $\varphi, \psi \in L^2(S^{n-1})$.

Proof. — $T_{int}(H(h_i, \theta) - \mu)^{-1}T_{int}^*$ is analytic in $\theta \in S_\gamma \cap \mathbb{C}_+$ $(\mathbb{C}_+ = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \})$ and if $\operatorname{Im}\theta = \operatorname{Im}\theta'$,

$$\begin{aligned}T_{int}&(H(h_i, \theta) - (\mu + i\varepsilon))^{-1}T_{int}^* \\ &= T_{int}U(\theta - \theta')(H(h_i, \theta') - (\mu + i\varepsilon))^{-1}U(\theta - \theta')T_{int}^* \\ &= T_{int}(H(h_i, \theta') - (\mu + i\varepsilon))^{-1}T_{int}^*.\end{aligned} \tag{4.2}$$

Hence $T_{int}(H(h_i, \theta) - (\mu + i\varepsilon))^{-1}T_{int}^*$ is independent of θ. Letting $\varepsilon \to 0$, $\theta = 0$ and $\theta' = \theta_0$, we have

$$T(H(h_i) - (\mu + i0))^{-1}T^* = T_{int}(H(h_i, \theta_0) - \mu)^{-1}T_{int}^*. \tag{4.3}$$

(4.3) and Proposition 5.1 in (I) conclude (4.1). □

PROPOSITION 4.2. — *Suppose* (E), *then for any* $\varepsilon > 0$ *there is* C *such that for* $\mu \in \mathrm{M}_i = \{ \mu \in \mathbb{R} \mid \operatorname{dist}(\bar{\mathrm{J}}_i, \mu) \leqq b \cdot g_i^q/2 \}$ *(b and q are the constants in* (E)*) and* $\varphi, \psi \in \mathrm{L}^2(\mathrm{S}^{n-1})$,

$$
\begin{aligned}
&| (\varphi, \{ \mathrm{S}(\mathrm{H}(h_i), \mathrm{H}_0(h_i); \mu) - \mathrm{S}(\mathrm{H}^{\mathrm{D}}(h_i), \mathrm{H}_0(h_i); \mu) \} \psi) \\
&\quad - 2\pi i h_i^4 (\mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_+(h_i, \mu)\varphi, \{ \mathrm{T}_{\mathrm{int}}(\mathrm{H}(h_i, \theta_0) - \mu)^{-1} \mathrm{P}_i \mathrm{T}_{\mathrm{int}}^* \} \mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_-(h_i, \mu)\psi) | \\
&\quad \leqq \mathrm{C} \exp \{ -2(\mathbf{d}_\lambda(\mathrm{K}, \Omega_{\mathrm{ext}} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \} \| \varphi \| \| \psi \|. \qquad (4.4)
\end{aligned}
$$

Proof. — By virtue of Proposition 4.1, it is sufficient to prove

$$
\begin{aligned}
&| (\mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_+(h_i, \mu)\varphi, \{ \mathrm{T}_{\mathrm{int}}(\mathrm{H}(h_i, \theta_0) - \mu)^{-1}(1 - \mathrm{P}_i)\mathrm{T}_{\mathrm{int}}^* \} \mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_-(h_i, \mu)\psi) | \\
&\qquad \leqq \mathrm{C} \exp \{ -2(\mathbf{d}_\lambda(\mathrm{K}, \Omega_{\mathrm{ext}} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \} \| \varphi \| \| \psi \|. \qquad (4.5)
\end{aligned}
$$

By easy computations, we have

$$
\begin{aligned}
&(\mathrm{H}(h_i, \theta_0) - z)^{-1} \\
&\quad = -(\mathrm{H}(h_i, \theta_0) - a_i)^{-1}(z - a_i)^{-1} \{ (\mathrm{H}(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1} \}^{-1} \qquad (4.6)
\end{aligned}
$$

for $z \in \Gamma_i$ and hence

$$
\begin{aligned}
&\| (\mathrm{H}(h_i, \theta_0) - z)^{-1} \| \\
&\quad \leqq \| (\mathrm{H}(h_i, \theta_0) - a_i)^{-1} \| \cdot | z - a_i |^{-1} \cdot \| \{ (\mathrm{H}(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1} \}^{-1} \| \\
&\quad \leqq \mathrm{C} \cdot h_i^{-q} \qquad (4.7)
\end{aligned}
$$

uniformly in $z \in \Gamma_i$ by Propositions 2.1, 2.3 and 2.4-(1). Since $(\mathrm{H}(h_i, \theta_0) - z)^{-1}(1 - \mathrm{P}_i)$ is analytic in G_i (= the interior of Γ_i),

$$
(\mathrm{H}(h_i, \theta_0) - \mu)^{-1}(1 - \mathrm{P}_i) = -(2\pi i)^{-1} \int_{\Gamma_i} \frac{(\mathrm{H}(h_i, \theta_0) - z)^{-1}(1 - \mathrm{P}_i)}{\mu - z} dz \qquad (4.8)
$$

by Cauchy's integral formula for $\mu \in \mathrm{G}_i$. (4.7) and (4.8) implies

$$
\| (\mathrm{H}(h_i, \theta_0) - \mu^{-1}(1 - \mathrm{P}_i) \| \leqq \mathrm{C} \cdot h_i^{-q} \cdot h_i^{-q} = \mathrm{C} \cdot h_i^{-2q} \qquad (4.9)
$$

uniformly in $\mu \in \mathrm{M}_i$. The trace estimate and the standard argument for elliptic operators give

$$
\| \mathrm{T}_{\mathrm{int}}(\mathrm{H}(h_i, \theta_0) - \mu)^{-1}(1 - \mathrm{P}_i)\mathrm{T}_{\mathrm{int}}^* \| \leqq \mathrm{C} \cdot h_i^{-2-2q}. \qquad (4.10)
$$

(4.10) and Corollary 3.4 of (I) conclude (4.5). □

Proof of Theorem 1. — (1.7) follows immediately from Propositions 3.3 and 4.2. By Corollary 3.2 and Corollary 3.4 of (I), we see

$$
\begin{aligned}
&| (\mathrm{T}_{\mathrm{int}} \eta_i(\theta_0), \mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_+(h_i, \mu)\varphi) | \leqq \| \mathrm{T}_{\mathrm{int}} \eta_i(\theta_0) \| \, \| \mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_+(h_i, \mu)\varphi \| \\
&\quad \leqq \mathrm{C} \exp \{ -(\mathbf{d}_\lambda(\mathrm{K}, \Omega_{\mathrm{int}} \backslash \mathscr{I}(\lambda)) + \mathbf{d}_\lambda(\mathrm{K}, \Omega_{\mathrm{ext}} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \} \| \varphi \|; \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
&| (\mathrm{T}_{\mathrm{int}} \eta_i(\bar{\theta}_0), \mathrm{T}_{\mathrm{ext}} \nabla_n \Phi_-(h_i, \mu)\psi) | \\
&\quad \leqq \mathrm{C} \exp \{ -(\mathbf{d}_\lambda(\mathrm{K}, \Omega_{\mathrm{int}} \backslash \mathscr{I}(\lambda)) + \mathbf{d}_\lambda(\mathrm{K}, \Omega_{\mathrm{ext}} \backslash \mathscr{I}(\lambda)) - \varepsilon)/h_i \} \| \psi \|. \qquad (4.12)
\end{aligned}
$$

On the other hand, $(\eta_i(\bar{\theta}_0), \eta(\theta_0)) = 1 + o(1)$ by Proposition 3.1, (1.8) follows from (4.11) and (4.12). □

Theorem 2 can be proved quite analogously by using Corollary 3.8 (instead of Corollary 3.4) of (I) and Corollary 5.2 (instead of Proposition 5.1) of (I), so we omit it.

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