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Time decay of solutions to the Schrödinger equation in exterior domains. II

by

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ABSTRACT. — We continue the study of the time decay of solutions to the Schrödinger equation:

$$(*)\quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, & (t, x) \in (0, \infty) \times \mathrm{D}, \\ u(0, x) = \phi(x), & x \in \mathrm{D}, \\ u(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\mathrm{D}, \end{cases}$$

where D is the complement of a strictly star-shaped, bounded domain in $\mathbb{R}^n$, $n \geqq 3$, and the boundary ∂D is smooth. We improve the result of a previous paper [*1*] with the same title. We prove in particular that for $n \geqq 5$, all solutions of $(*)$ decay in time according to

$$\| u(t) \|_p \leqq \mathrm{C}\mathrm{I}^{\frac{1}{2}} \cdot (1 + t)^{-\frac{n}{2}\left(1 - \frac{2}{p}\right)}$$

for $2 \leqq p \leqq 2n/(n-4)$, where

$$\mathrm{I} = \mathrm{I}(\phi) = \| \, | x |^3 \phi \|^2 + \| \, | x |^2 \phi \|_{1,2}^2 + \| x\Delta\phi \|^2 + \| \phi \|_{2,2}^2 .$$

RÉSUMÉ. — Nous poursuivons l'étude de la décroissance temporelle des solutions de l'équation de Schrödinger:

$$(*)\quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, & (t, x) \in (0, \infty) \times \mathrm{D}, \\ u(0, x) = \phi(x), & x \in \mathrm{D}, \\ u(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\mathrm{D}, \end{cases}$$

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où D est le complément d'un domaine borné strictement étoilé de $\mathbb{R}^n$, $n \geq 3$, et de bord régulier. Nous améliorons le résultat d'un article antérieur [1] de même titre. Nous prouvons en particulier que pour $n \geq 5$ toutes les solutions de (*) décroissent selon

$$\| u(t) \|_p \leqq \mathrm{C I}^{\frac{1}{2}} \cdot (1 + t)^{-\frac{n}{2}\left(1 - \frac{2}{p}\right)}$$

pour $2 \leqq p \leqq 2n/(n-4)$, où

$$\mathrm{I} = \mathrm{I}(\phi) = \| \, |x|^3 \phi \|^2 + \| \, |x|^2 \phi \|_{1,2}^2 + \| x \Delta \phi \|^2 + \| \phi \|_{2,2}^2 .$$

1. INTRODUCTION AND MAIN RESULT

In this paper we study the time decay of solutions to the following Schrödinger equation:

$$i\partial_t u + \frac{1}{2} \Delta u = 0, \qquad (t, x) \in (0, \infty) \times \mathrm{D}, \tag{1.1}$$

$$u(0, x) = \phi(x), \qquad x \in \mathrm{D}, \tag{1.2}$$

$$u(t, x) = 0, \qquad (t, x) \in (0, \infty) \times \partial \mathrm{D}, \tag{1.3}$$

where D is the complement of a strictly star-shaped, bounded domain in $\mathbb{R}^n$, $n \geqq 3$, and the boundary ∂D is smooth. Our main purpose in this paper is to improve the result of a previous paper [1]. In this paper we use the following notations:

NOTATION. — $\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial x_k$, $\nabla = (\partial_1, \ldots, \partial_n)$ $x = (x_1, \ldots, x_n)$, $r = |x|$, $r\partial_r = x \cdot \nabla$, $\Delta = \sum_{k=1}^{n} \partial_k^2$; $\mathrm{S} = \mathrm{S}(t) = \exp(ir^2/2t)$ $(t \in \mathbb{R} \backslash \{0\})$, $\mathrm{J}_k = \mathrm{J}_k(t) = x_k + it\partial_k$, $\mathrm{J} = \mathrm{J}(t) = (\mathrm{J}_1, \ldots, \mathrm{J}_n)$, $\mathrm{K} = r^2 + nit + 2itr\partial_r + 2it^2\partial_t$, $\mathrm{J}^2 = r^2 + nit + 2itr\partial_r - t^2\Delta$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $\mathrm{J}^\alpha = \mathrm{J}_1^{\alpha_1} \ldots \mathrm{J}_n^{\alpha_n}$, $\alpha \in (\mathbb{N} \cup \{0\})^n$, $\partial^0 = x^0 = \mathrm{J}^0 = \mathrm{I}$; $\mathcal{S}$ denotes the space of rapidly decreasing $\mathrm{C}^\infty(\mathrm{D})$-functions from D to $\mathbb{C}$, $\mathcal{S}'$ is the dual space of $\mathcal{S}$; L^p denotes the Lebesque space $\mathrm{L}^p(\mathrm{D})$ or $\mathrm{L}^p(\mathrm{D}) \otimes \mathbb{C}^n$ with the norm $\| \ \|_p$, $1 \leqq p \leqq \infty$; $\| \cdot \| = \| \cdot \|_2$; $(.,.)$ denotes the scalar product ; $\mathrm{H}^{m,p} = \mathrm{H}^{m,p}(\mathrm{D}) = \left\{ \psi \in \mathcal{S}' ; \| \psi \|_{m,p} = \sum_{|\alpha| \leqq m} \| \partial^\alpha \psi \|_p < \infty \right\}$, $\mathrm{H}_0^{m,p} = \mathrm{H}_0^{m,p}(\mathrm{D})$ denotes the completion of $\mathrm{C}_0^\infty(\mathrm{D})$ in $\mathrm{H}^{m,p}$; $\int \cdot \, dx = \int_{\mathrm{D}} \cdot \, dx$;

$$\| \cdot \|_b^2 = -\int \partial_j (x_j |\cdot|^2) dx = -\sum_{j=1}^{n} \int_{\mathrm{D}} \partial_j (x_j |\cdot|^2) dx = -\int_{\partial \mathrm{D}} |\cdot|^2 (x \cdot n) d\sigma$$

when D is the complement of a star-shaped bounded domain in $\mathbb{R}^n$ with the smooth boundary ∂D, where n denotes the outward normal unit vector at $x \in \partial D$; $\|\cdot\|_{b^*}^2 = \int_{\partial D} |\cdot|^2 d\sigma$. The domain D is said to be the complement of a star-shaped (resp. a strictly star-shaped) domain if $-(x \cdot n) \geqq 0$ (resp. $-(x \cdot n) \geqq \gamma > 0$) holds for all $x \in \partial D$. Re A and Im A denote the real part of A and the imaginary part of A, respectively. Different positive constants might be denoted by the letter C. If necessary, by $C(*, \ldots, *)$ we denote constants depending only on the quantities appearing in parentheses. The following relations will be used in the sequel: $J_k(t) = S(t)(it\partial_k)S(-t)$, $J(t) = S(t)(it\nabla)S(-t)$, $J^2(t) = S(t)(-t^2\Delta)S(-t)$, $L = i\partial_t + \frac{1}{2}\Delta$, $[L, J] = LJ - JL = 0$, $[L, J^2] = LJ^2 - J^2L = 0$, $[L, K] = LK - KL = 4itL$. We let $\delta(p) = n(1-(2/p))/2$, $2n/(n-2) \leqq p \leqq 2n/(n-4)$ if $n \geqq 5$, $2n/(n-2) \leqq p < \infty$ if $n = 4$, $2n/(n-2) \leqq p \leqq \infty$ if $n = 3$. Let $\beta = 2$ if $n \geqq 5$, $1 < \beta < 2$ if $n = 4$, $1 < \beta < 4/3$ if $n = 3$, $a = a(\beta) = 2(2-\beta)/(3-\beta)$, $e = e(\beta) = 2/(3-\beta)$. Finally let $d = d(\beta)$ satisfy $2(2-\beta)/(4-\beta) + (3 - 8/(4-\beta) + d)/2(3-\beta) = d$ if $n = 3$, and $F = F(t) = (1 + \log(1+t))^{e(\beta)}$.

We now state our main result.

THEOREM 1. — Let D be the complement of a strictly star-shaped, bounded domain in $\mathbb{R}^n$ ($n \geqq 3$), with smooth boundary ∂D. Let u be the solution of (1.1)-(1.3) with

$$\phi \in H = \{ \psi \in \mathcal{S}' ; I = I(\psi) = \| \, |x|^3 \psi \|^2 + \| \, |x|^2 \psi \|_{1,2}^2 + \| x\Delta\psi \|^2 + \| \psi \|_{2,2}^2 < \infty \}.$$

Then u satisfies the following decay estimates

$$\| u(t) \|_p \leqq C I^{1/2}(\phi)(1+t)^{-\delta(p) + a(\beta)(\delta(p) - 1)}, \qquad n \geqq 4, \qquad (1.4)$$

$$\| u(t) \|_p \leqq C I^{1/2}(\phi)(1+t)^{-\delta(p) + d(\beta)(\delta(p) - 1)}, \qquad n = 3. \qquad (1.5)$$

REMARK 1. — (1.4) is the same decay rate as that of the solution of the initial value problem for the Schrödinger equation if $n \geqq 5$.

Throughout the paper we assume that the assumptions of theorem 1 are satisfied.

2. PROOF OF THEOREM 1

We first give some lemmas without proof which were proved in a previous paper [*1*].

LEMMA 2.1. — Let u be the solution of (1.1)-(1.3). Then we have

$$\| Ju(t) \|^2, \int_0^t s \| \nabla u(s) \|_b^2 ds, \quad \| J\partial_t u(t) \|^2, \int_0^t s \| \partial_s \nabla u(s) \|_b^2 ds \leqq CI(\phi), \tag{2.1}$$

$$\| \partial_r \nabla u(t) \|_b^2 \leqq CI(\phi) t^{-1} (1+t)^{-1}, \quad t > 0, \tag{2.2}$$

$$\| Ku(t) \|^2 \leqq CI(\phi)(1+t)^{a(\beta)} F(t), \tag{2.3}$$

$$\| u(t) \|_p^2 \leqq CI(\phi)(1+t)^{-2\delta(p)+2a(\beta)(\delta(p)-1)} F(t)^{(\delta(p)-1)}, \tag{2.4}$$

$$\| \nabla K u(t) \|^2 \leqq CI(\phi) F(t)^{1/e(\beta)}. \tag{2.5}$$

For (2.1), (2.2), (2.3), (2.4) and (2.5) see lemma 2.1, lemma 2.2, lemma 2.6, theorem 1 and lemma 2.3 in a previous paper [*1*], respectively.

LEMMA 2.2. — Let $w \in H_0^{1,2} \cap H^{2,2}$ and $r^2 w \in L^2$. Then we have

$$\| \nabla w \|_b \leqq \begin{cases} Ct^{-4/(4-\beta)} \| J^2 w \|^{2/(4-\beta)} \| w \|^{(2-\beta)/(4-\beta)} + Ct^{-2} \| J^2 w \|, \quad t > 0, & (2.6) \\ C \| w \|_{2,2}, & (2.7) \end{cases}$$

$$\sum_{|\alpha|=2} \| J^\alpha w \| \leqq C(\| J^2 w \| + t^{2(2-\beta)/(4-\beta)} \| J^2 w \|^{2/(4-\beta)} \| w \|^{(2-\beta)/(4-\beta)}). \tag{2.8}$$

For (2.6), (2.7) see [*1*; lemma 2.4] and for (2.8) see [*1*; lemma 2.5]. We next prove some lemmas needed for the proof of theorem 1. Before doing so we give a sketch of the strategy of the proof of theorem 1 which is the same as that of a previous paper [*1*]. The main result follows from Sobolev's inequality

$$\| u(t) \|_p \leqq Ct^{-\delta(p)} \| Ju(t) \|^{2-\delta(p)} \sum_{|\alpha|=2} \| J^\alpha u(t) \|^{\delta(p)-1}, \qquad t > 0.$$

The first norm is estimated by lemma 2.1, the second norm is reduced basically to $\| J^2 u \|$ by lemma 2.2, then $\| J^2 u \| = \| Ku \|$ for the solutions of (1.1)-(1.3), $\| Ku \|$ is estimated in the proof of theorem 1 by using *a priori* estimates of solutions on the boundary which are obtained by making use of lemmas 2.3-2.4. We note that computation stated below is rather formal, but it can be justified by considering regularized problems (see the beginning of section 2 in [*1*]).

LEMMA 2.3. — Let u be the solution of (1.1)-(1.3). Then we have

$$\| \partial_r \nabla u(t) \|_b^2 \leqq CI(\phi) t^{-2\delta(p)+2a(\beta)(\delta(p)-1)} F(t)^{(\delta(p)-1)}, \qquad t \geqq 1, \tag{2.9}$$

$$\| \nabla u(t) \|_b^2 \leqq CI(\phi)(1+t)^{-2(4-a(\beta))/(4-\beta)} F(t)^{2/(4-\beta)}. \tag{2.10}$$

Proof. — We first prove (2.9). In the same way as in the proofs of [*1*; (2.7), (2.9), (2.10)], we have with $\zeta = (1+r)^{-k}$, $k > 2$

$$\| \partial_r \nabla u \|_b^2 \leqq C \| \zeta u \|_{3,2} \| \zeta u \|_{2,2}, \tag{2.11}$$

$$\| \zeta u \|_{2,2} \leqq C(\| \zeta \partial_t u \| + \| \zeta \nabla u \| + \| \zeta u \|), \tag{2.12}$$

$$\| \zeta u \|_{3,2} \leqq C(\| \zeta \nabla \partial_t u \| + \| \zeta u \|_{2,2}). \tag{2.13}$$

By a simple calculation we have

$$\| \zeta u \|_{2,2} \leqq Ct^{-2}(\| \zeta K u \| + \| r^2 \zeta u \| + t \| \zeta u \| + \| x \zeta J u \|) + Ct^{-1}(\| \zeta J u \| + \| x \zeta u \|) + \| \zeta u \|, \tag{2.14}$$

$$\| \zeta \nabla \partial_t u \| \leqq Ct^{-2}(\| \zeta \nabla K u \| + \| r^2 \zeta \nabla u \| + \| r \zeta u \| + t \| \zeta \nabla u \| + t \| r \zeta \nabla \partial_r u \|). \tag{2.15}$$

Applying Sobolev's inequality to (2.14) and (2.15) we have

$$\| \zeta u \|_{2,2} \leqq Ct^{-2}\Big(\| K u \| + \| u \| + \sum_{|\alpha| \leqq 2} \| J^\alpha u \| \Big) + \| \zeta u \|,$$

$$\| \zeta u \|_{3,2} \leqq Ct^{-2}\Big(\| \nabla K u \| + \| u \|_{2,2} + \sum_{|\alpha| \leqq 2} \| J^\alpha u \| \Big) + \| \zeta u \|_{2,2}.$$

Thus we have by (2.11)

$$\| \partial_r \nabla u \|_b^2 \leqq Ct^{-4}\Big(\| \nabla K u \|^2 + \| K u \|^2 + \| u \|_{2,2}^2 + \sum_{|\alpha| \leqq 2} \| J^\alpha u \|^2 \Big) + \| u \|_p^2. \tag{2.16}$$

From (2.1), (2.3)-(2.5), (2.7), (2.8) and (2.16) it follows that

$$\| \partial_r \nabla u \|_b^2 \leqq CI \cdot t^{-4}(F^{1/e} + (1+t)^a F + (1+t)^{2a} F^{2/(4-\beta)} + t^4 (1+t)^{-2\delta + 2a\cdot(\delta-1)} F^{\delta-1}). \tag{2.17}$$

It is clear that

$$F^{1/e} \leqq C(1+t)^a F \leqq C(1+t)^{2a} F^{2/(4-\beta)} \leqq Ct^4(1+t)^{-2\delta+2a\cdot(\delta-1)} F^{\delta-1}, \quad t \geqq 1.$$

Therefore we see that (2.9) follows from (2.17). We next prove (2.10). From (2.3), (2.6) and (2.7) it follows that

$$\| \nabla u \|_b^2 \leqq Ct^{-8/(4-\beta)} \| K u \|^{4/(4-\beta)} \| \phi \|^{4(2-\beta)/(4-\beta)} + Ct^{-4} \| K u \|^2 \leqq CI \cdot (t^{-8/(4-\beta)}(1+t)^{2a/(4-\beta)} F^{2/(4-\beta)} + t^{-4}(1+t)^a F), \tag{2.18}$$

and

$$\| \nabla u \|_b^2 \leqq CI. \tag{2.19}$$

(2.18) and (2.19) give

$$\| \nabla u \|_b^2 \leqq CI \cdot ((1+t)^{-2(4-a)/(4-\beta)} F^{2/(4-\beta)} + (1+t)^{-(4-a)} F).$$

Since $2/(4-\beta) \leqq 1$, this gives (2.10). This completes the proof of lemma 2.3.
Q. E. D.

LEMMA 2.4. — Let u be the solution of (1.1)-(1.3). Then we have

$$\frac{d}{dt}\left(\|\mathrm{JK}u(t)\|^2 + 2(n+4)t^4\|\nabla u(t)\|_b^2\right) + \frac{1}{2}t\|\nabla \mathrm{K}u(t)\|_b^2 + 4t^5\|\partial_t \nabla u(t)\|_b^2 \leqq \mathrm{C}t^3\left(\|\nabla u(t)\|_b^2 + \|\partial_r \nabla u(t)\|_b^2\right) + \mathrm{C}t\|\nabla u(t)\|_b^2 .$$

Proof. — We have by (1.1)

$$i\partial_t \mathrm{J}v + \frac{1}{2}\Delta \mathrm{J}v = 0, \tag{2.20}$$

where $v = \mathrm{K}u$. Multiplying (2.20) by $\overline{\mathrm{J}v}$, taking the imaginary part and integrating over D, we have

$$\frac{d}{dt}\|\mathrm{J}v\|^2 + \mathrm{Im}\int \nabla(\nabla \mathrm{J}_j v \cdot \overline{\mathrm{J}_j v})dx = 0. \tag{2.21}$$

By a simple calculation we see that

$$\mathrm{Im}\int \nabla(\nabla \mathrm{J}_j v \cdot \overline{\mathrm{J}_j v})dx = \mathrm{Im}\int \nabla(\mathrm{J}_j \nabla v \cdot \overline{\mathrm{J}_j v}) + \nabla(v \cdot \overline{\mathrm{J}v})dx$$

$$= \mathrm{Im}\int \nabla(r^2\nabla v \cdot \bar{v} - itr\nabla v \cdot \partial_r \bar{v} + itr\partial_r \nabla v \cdot \bar{v} + t^2\partial_j \nabla v \cdot \partial_j \bar{v} - itv\nabla\bar{v})dx. \tag{2.22}$$

We compute the R. H. S. of (2.22). The integration by parts gives

$$\mathrm{Im}\int \nabla(r^2\nabla v \cdot \bar{v})dx = \mathrm{Im}\int \nabla(r^2\nabla v \cdot -2itr\partial_r \bar{u})dx = -2t\,\mathrm{Im}\int \partial_j(x_j r^2 \nabla v \cdot i\nabla\bar{u})dx, \tag{2.23}$$

here we have used the identity

$$\nabla(\nabla a \cdot r\partial_r b) = \nabla(x\Delta a \cdot b - (n-1)\nabla a \cdot b - r\partial_r \nabla a \cdot b) + \sum_{j=1}^{n} \partial_j(x_j \nabla a \cdot \nabla b). \tag{2.24}$$

Similarly we have

$$\mathrm{Im}\int \nabla(-itr\nabla v \cdot \partial_r \bar{v})dx = t\|\nabla v\|_b^2 + 2t^2\,\mathrm{Im}\int \partial_j(x_j \nabla v \cdot r\partial_r \nabla\bar{u})dx - 2t^2\,\mathrm{Im}\int \nabla(r\nabla v \cdot x_j \partial_r \partial_j \bar{u})dx. \tag{2.25}$$

By (2.25) and $i\partial_t v + \frac{1}{2}\Delta v = 0$ we have

$$\begin{aligned}
&\operatorname{Im}\int \nabla(itr\partial_r \nabla v \cdot \bar{v})dx = \\
&= -t \operatorname{Im}\int (n-1)i\nabla(\nabla v \cdot \bar{v}) + 2\partial_j(x_j \partial_t v \cdot \bar{v}) + i\nabla(r\nabla v \cdot \partial_r \bar{v})dx - t \| \nabla v \|_b^2 \\
&= -2t^2(n-1)\operatorname{Im}\int \partial_j(x_j \nabla v \cdot \nabla \bar{u})dx - 8t^3 \operatorname{Im}\int \partial_j(x_j r^2 \partial_r \partial_t u \cdot \partial_r \bar{u})dx \\
&\quad + 2t^2 \operatorname{Im}\int \partial_j(x_j \nabla v \cdot r\partial_r \nabla \bar{u})dx - 2t^2 \operatorname{Im}\int \nabla(r\nabla v \cdot x_j \partial_r \partial_j \bar{u})dx . \qquad (2.26)
\end{aligned}$$

Similarly we see that

$$\begin{aligned}
&\operatorname{Im}\int \nabla(t^2 \partial_j \nabla v \cdot \partial_j \bar{v})dx = t^2 \operatorname{Im}\int \partial_j(-2i\partial_t v \cdot \partial_j \bar{v})dx \\
&= 4t^2 \operatorname{Im}\int \partial_j(x_j(1+t\partial_t)\nabla u \cdot (-nit + r^2 - 2itr\partial_r - 2it(1+t\partial_t))\nabla \bar{u})dx \\
&= 8t^3 \| (1+t\partial_t)\nabla u \|_b^2 + 2n\frac{d}{dt} t^4 \| \nabla u \|_b^2 - 4nt^3 \| \nabla u \|_b^2 \\
&\quad + 4t^3 \operatorname{Im}\int \partial_j(x_j r^2 \partial_t \nabla u \cdot \nabla \bar{u})dx - 8t^3 \operatorname{Im}\int \partial_j(x_j(1+t\partial_t)\nabla u \cdot r\partial_r \nabla \bar{u})dx, \qquad (2.27)
\end{aligned}$$

$$\operatorname{Im}\int \nabla(-itv \cdot \nabla \bar{v})dx = 2t^2 \operatorname{Im}\int \partial_j(x_j \nabla u \cdot \nabla \bar{v})dx . \qquad (2.28)$$

By (2.21)-(2.23), (2.25)-(2.28) we have

$$\begin{aligned}
&\frac{d}{dt} \| Jv \|^2 - 2t \operatorname{Im}\int \partial_j(x_j r^2 \nabla v \cdot i\nabla \bar{u})dx + t \| \nabla v \|_b^2 + \\
&\quad + 4t^2 \operatorname{Im}\int \partial_j(x_j \nabla v \cdot r\partial_r \nabla \bar{u})dx - 4t^2 \operatorname{Im}\int \nabla(r\nabla v \cdot x_j \partial_r \partial_j \bar{u})dx \\
&\quad - 2t^2(n-1)\operatorname{Im}\int \partial_j(x_j \nabla v \cdot \nabla \bar{u})dx - 8t^3 \operatorname{Im}\int \partial_j(x_j r^2 \partial_r \partial_t u \cdot \partial_r \bar{u})dx \\
&\quad + 8t^3 \| (1+t\partial_t)\nabla u \|_b^2 + 2n\frac{d}{dt} t^4 \| \nabla u \|_b^2 - 4nt^3 \| \nabla u \|_b^2 \\
&\quad + 4t^3 \operatorname{Im}\int \partial_j(x_j r^2 \partial_t \nabla u \cdot \nabla \bar{u})dx - 8t^3 \operatorname{Im}\int \partial_j(x_j(1+t\partial_t)\nabla u \cdot r\partial_r \nabla \bar{u})dx \\
&\quad + 2t^2 \operatorname{Im}\int \partial_j(x_j \nabla u \cdot \nabla \bar{v})dx = 0 . \qquad (2.29)
\end{aligned}$$

Since $\| (1+t\partial_t)\nabla u \|_b^2 = \| \nabla u \|_b^2 + t^2 \| \partial_t \nabla u \|_b^2 + t \dfrac{d}{dt} \| \nabla u \|_b^2$, we have by (2.29)

$$\frac{d}{dt} \| \mathrm{J}v \|^2 + t \| \nabla v \|_b^2 + 8t^5 \| \partial_t \nabla u \|_b^2 + 2(n+4) \frac{d}{dt} t^4 | \nabla u \|_b^2 \leqq 4(n-2)t^3 \| \nabla u \|_b^2$$
$$+ \mathrm{C}(t \| \nabla v \|_b \| \nabla u \|_b + t^3 \| \partial_t \nabla u \|_b \| \nabla u \|_b + t^2 \| \nabla v \|_b \| \partial_r \nabla u \|_b$$
$$+ t^2 \| \nabla v \|_{b*} \| \partial_r \nabla u \|_{b*} + t^2 \| \nabla v \|_b \| \nabla u \|_b + t^3 \| \nabla u \|_b \| \partial_r \nabla u \|_b$$
$$+ t^4 \| \partial_t \nabla u \|_b \| \partial_r \nabla u \|_b).$$

We have $\gamma \| \cdot \|_{b*} \leqq \| \cdot \|_b$ with some $\gamma > 0$ since D is the complement of a strictly star-shaped, bounded domain in $\mathbb{R}^n$. Therefore we have the desired result by the above inequality and the Schwarz inequality. Q. E. D.

Proof of Theorem 1. — We first prove the case $n \geqq 5$. We put $\delta(p) = 2$ in (2.9). Then we have by (2.2) and (2.9)

$$\| \partial_r \nabla u \|_b^2 \leqq \mathrm{CI} t^{-1}(1+t)^{-3}\mathrm{F}, \tag{2.30}$$
$$\| \nabla u \|_b^2 \leqq \mathrm{CI} \cdot (1+t)^{-4}\mathrm{F}, \tag{2.31}$$

where $\mathrm{F} = (1 + \log(1+t))^2$. From lemma 2.4 we get for $0 < \varepsilon < 1$

$$\int_0^t s^{5-\varepsilon} \| \partial_s \nabla u \|_b^2 \, ds \leqq \mathrm{CI} \cdot \left(1 + \int_0^t (1+s)^{-1} s^{-\varepsilon} \mathrm{F} ds\right) \leqq \mathrm{CI}. \tag{2.32}$$

In the same way as in the proof of [*1*; (2.36)] we have

$$\frac{d}{dt} \| \mathrm{K}u \|^2 \leqq \mathrm{C}t^2 (\| \partial_r \nabla u \|_b + t \| \partial_t \nabla u \|_b) \| \nabla u \|_b. \tag{2.33}$$

By (2.3), (2.30), (2.31) and (2.33)

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI} \cdot \left(1 + \int_1^t (1+s)^{-2} \mathrm{F} \, ds\right) + \mathrm{CI}^{1/2} \int_1^t s \mathrm{F}^{1/2} \| \partial_s \nabla u \|_b ds$$
$$\leqq \mathrm{CI} + \mathrm{CI}^{1/2} \cdot \left(\int_1^t s^{5-\varepsilon} \| \partial_s \nabla u \|_b^2 ds\right)^{1/2} \left(\int_1^t s^{-3-\varepsilon} \mathrm{F} ds\right)^{1/2}, \tag{2.34}$$

where $0 < \varepsilon < 1$. Thus we have by (2.32) and (2.34)

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI}. \tag{2.35}$$

Theorem 1 for the case $n \geqq 5$ follows from (2.35), Sobolev's inequality and (2.8). We next prove the case $n = 4$. It is sufficient to prove (2.35).

We can take $\beta = 2 - \varepsilon_1$, $\delta(p) = 2 - \varepsilon_1$ in (2.9), where ε_1 is a sufficiently small positive constant. Then we have instead of (2.30) and (2.31)

$$\| \partial_r \nabla u \|_b^2 \leqq \mathrm{C}\mathrm{I}t^{-1}(1 + t)^{-3+\varepsilon_2}, \tag{2.36}$$

$$\| \nabla u \|_b^2 \leqq \mathrm{C}\mathrm{I} \cdot (1 + t)^{-4+\varepsilon_3}, \tag{2.37}$$

where ε_2 and ε_3 are sufficiently small positive constants depending only on ε_1. In the same way as in the proof of (2.35), we see that (2.35) holds valid for the case $n = 4$. This completes the proof of theorem 1 for the case $n = 4$. Finally we prove the case $n = 3$. We put $\delta(\infty) = 3/2$ in (2.9), then we have

$$\| \partial_r \nabla u \|_b^2, \qquad \| \nabla u \|_b^2 \leqq \mathrm{C}\mathrm{I}t^{-1}(1 + t)^{-2+a}\mathrm{F}^{1/2}, \tag{2.38}$$

since $-2(4 - a)/(4 - \beta) \leqq -3 + a$ if $\beta > 1$. From lemma 2.4 and (2.38) we have

$$\frac{d}{dt}\left(\| \mathrm{J}\mathrm{K}u \|^2 + 2(n + 4)t^4 \| \nabla u \|_b^2 \right) + \frac{1}{2} t \| \nabla \mathrm{K}u \|_b^2 + 4t^5 \| \partial_t \nabla u \|_b^2$$
$$\leqq \mathrm{C}\mathrm{I} \cdot (1 + t)^a \mathrm{F}^{1/2} + \mathrm{C}t \| \nabla u \|_b^2. \tag{2.39}$$

We multiply (2.39) by $(1 + t)^{-1-a}\mathrm{F}^{-1/2}$ to obtain

$$\int_0^t s^5(1 + s)^{-1-a}\mathrm{F}^{-1/2} \| \partial_s \nabla u \|_b^2 ds \leqq \mathrm{C}\mathrm{I} \cdot (1 + \log(1 + t)), \tag{2.40}$$

here we have used (2.1). From (2.6), (2.33) and (2.38) we obtain

$$\begin{aligned} \frac{d}{dt} \| \mathrm{K}u \|^2 &\leqq \mathrm{C}t^2(\mathrm{I}^{1/2}t^{-1/2}(1+t)^{-1+(a/2)}\mathrm{F}^{1/4} + t \| \partial_t \nabla u \|_b) \\ &\quad \times (t^{-4/(4-\beta)} \| \mathrm{K}u \|^{2/(4-\beta)} \| \phi \|^{(2-\beta)/(4-\beta)} + t^{-2} \| \mathrm{K}u \|) \\ &\leqq \mathrm{C}t^{2b}(\mathrm{I}^{1/2}t^{-1/2}(1+t)^{-1+(a/2)}\mathrm{F}^{1/4} + t \| \partial_t \nabla u \|_b) \| \mathrm{K}u \|^{2/(4-\beta)} \| \phi \|^{b_2} \\ &\quad + \mathrm{C} \cdot (\mathrm{I}^{1/2}t^{-1/2}(1+t)^{-1+(a/2)}\mathrm{F}^{1/4} + t \| \partial_t \nabla u \|_b) \| \mathrm{K}u \|, \end{aligned}$$

where $b_2 = (2 - \beta)/(4 - \beta)$. From this we see that

$$\begin{aligned} \frac{d}{dt} \| \mathrm{K}u \|^{b_1} &\leqq \mathrm{C}t^{2b_2} \| \phi \|^{b_2}(\mathrm{I}^{1/2}t^{-1/2}(1+t)^{-1+(a/2)}\mathrm{F}^{1/4} + t \| \partial_t \nabla u \|_b) \\ &\quad + \mathrm{C} \cdot (\mathrm{I}^{1/2}t^{-1/2}(1+t)^{-1+(a/2)}\mathrm{F}^{1/4} + t \| \partial_t \nabla u \|_b) \| \mathrm{K}u \|^{b_2}, \end{aligned} \tag{2.41}$$

where $b_1 = 2(3 - \beta)/(4 - \beta)$. We have $\| \mathrm{K}u \|^2 \leqq \mathrm{C}\mathrm{I} \cdot (1 + t)^4$ by (2.3). Therefore from (2.41) it follows that

$$\frac{d}{dt} \| \mathrm{K}u \|^{b_1} \leqq \mathrm{C}\mathrm{I}^{b_2/2}(1+t)^{2b_2}(\mathrm{I}^{1/2}t^{-1/2}(1+t)^{-1+(a/2)}\mathrm{F}^{1/4} + t \| \partial_t \nabla u \|_b). \tag{2.42}$$

On the other hand, by the Schwarz inequality and (2.40) we have

$$\int_1^t (1+s)^{2b_2} s \, \| \partial_s \nabla u \|_b ds \leqq \left(\int_1^t s^5 (1+s)^{-1-a} \mathrm{F}^{-1/2} \| \partial_s \nabla u \|_b^2 ds \right)^{1/2}$$

$$\times \left(\int_1^t s^{-3} (1+s)^{1+a+4b_2} \mathrm{F}^{1/2} ds \right)^{1/2}$$

$$\leqq \mathrm{CI}^{1/2} \cdot (1+t)^{-(1-a)+2b_2} \mathrm{F}^{(1/2+1/e)/2}, \qquad t \geqq 1. \quad (2.43)$$

(2.42) and (2.43) show that

$$\| \mathrm{K}u \|^{b_1} \leqq \| \mathrm{K}u(1) \|^{b_1} + \mathrm{CI}^{b_1/2} \cdot (1+t)^{-(1-a)/2+2b_2} \times \mathrm{F}^{(1/2+1/e)/2}.$$

This and (2.3) imply

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI} \cdot ((1+t)^{3-8(4-\beta)+a} \mathrm{F}^{1/2+1/e})^{1/b_1}. \qquad (2.44)$$

By Sobolev's inequality (2.1), (2.3), (2.8) and (2.44) we have

$$\| u \|_\infty^2 \leqq \mathrm{C} \cdot (1+t)^{-3} (\| \mathrm{J}u \| + \| \nabla u \|) \left(\sum_{|\alpha| \leqq 2} \| \mathrm{J}^\alpha u \| + \| u \|_{2,2} \right)$$

$$\leqq \mathrm{CI}^{1/2} \cdot (1+t)^{-3} (\mathrm{I}^{1/2} + (1+t)^{2b_2} \| \mathrm{K}u \|^{1-b_2} \mathrm{I}^{b_2/2})$$

$$\leqq \mathrm{CI} \cdot (1+t)^{-3+2b_2+(3-8/(4-\beta)+a)(1-b_2)/2b_1} \mathrm{F}^{(1/2+1/e)(1-b_2)/2b_1}$$

$$\leqq \mathrm{CI} \cdot (1+t)^{-3+a_1(\beta)} \mathrm{F}_1^{1/2}, \qquad (2.45)$$

where $\mathrm{F}_1 = \mathrm{F}^{(1/2+1/e)(1-b_2)/b_1} = \mathrm{F}^{1/b_1} = \mathrm{F}^{(4-\beta)/2(3-\beta)}$ and

$$a_1 = a_1(\beta) = 2b_2 + (3 - 8/(4-\beta) + a)(1-b_2)/2b_1$$

$$= 2(2-\beta)/(4-\beta) + (3 - 8/(4-\beta) + a)/2(3-\beta) \quad (\leqq a).$$

In the same way as in the proof of lemma 2.3 we have by (2.45)

$$\| \partial_r \nabla u \|_b^2, \qquad \| \nabla u \|_b^2 \leqq \mathrm{CI}t^{-1}(1+t)^{-2+a_1} \mathrm{F}_1^{1/2}, \qquad (2.46)$$

since $-2(4-a_1)/(4-\beta) \leqq -3 + a_1$ for $\beta > 1$. In the same way as in the proof of (2.44) we obtain by (2.46)

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI}((1+t)^{3-8/(4-\beta)+a_1} \mathrm{F}_1^{1/2+1/e})^{1/b_1}. \qquad (2.47)$$

We iterate this procedure, then we have

$$\| \partial_r \nabla u \|^2, \qquad \| \nabla u \|_b^2 \leqq \mathrm{CI}t^{-1}(1+t)^{-2+a_n} \mathrm{F}_n^{1/2}, \qquad (2.48)$$

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI} \cdot ((1+t)^{3-8/(4-\beta)+a_n} \mathrm{F}_n^{1/2+1/e})^{1/b_1}, \qquad (2.49)$$

where $a_n = 2(2-\beta)/(4-\beta) + (3 - 8/(4-\beta) + a_{n-1})/2(3-\beta)$, $a_0 = a$ and $\mathrm{F}_n = \mathrm{F}^{((4-\beta)/2(3-\beta))^n}$, here we have used the fact that $-2(4-a_n)/(4-\beta) \leqq -3 + a_n$ for $\beta > 1$. (2.48) and (2.49) give

$$\| \partial_r \nabla u \|_b^2, \qquad \| \nabla u \|_b^2 \leqq \mathrm{CI}t^{-1}(1+t)^{-2+d}, \qquad (2.50)$$

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI} \cdot (1+t)^{(3-8/(4-\beta)+d)/b_1}, \qquad (2.51)$$

since $(4-\beta)/2(3-\beta) < 1$.

By Sobolev's inequality, (2.1), (2.8) and (2.51) we have

$$\| u(t) \|_p \leqq C \cdot (1 + t)^{-\delta(p)} (\| Ju \| + \| \nabla u \|)^{2-\delta(p)} \times \Big(\sum_{|\alpha|=2} \| J^\alpha u \| + \| u \|_{2,2} \Big)^{\delta(p)-1}$$

$$\leqq C \cdot (1+t)^{-\delta(p)} I^{1-\delta(p)/2} (I^{1/2} + (1+t)^{2b_2} \| Ku \|^{1-b_2} I^{b_2/2})^{\delta(p)-1}$$

$$\leqq CI^{1/2} \cdot (1+t)^{-\delta(p)+(2b_2+(3-8/(4-\beta)+d)(1-b_2)/2b_1)(\delta(p)-1)}$$

$$\leqq CI^{1/2} \cdot (1 + t)^{-\delta(p)+d(\beta)(\delta(p)-1)}.$$

This completes the proof of theorem 1. Q. E. D.

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