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by

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ABSTRACT. — We show that on a quantum logic L which has a sufficient set of states S(L) with the property: for every two noncompatible elements a, b of L there is a state s ∈ S(L) such that \( s(a) = s(b) = 1 \), the uncertainty relations cannot be satisfied for any pair of observables on L.

RÉSUMÉ. — Nous montrons que si une logique quantique L a un ensemble d'états S(L) assez grand, c'est-à-dire si pour toute paire a, b d'éléments non compatibles de L il existe un état s ∈ S(L) tel que \( s(a) = s(b) = 1 \), alors la relation d'incertitude ne peut être satisfaite pour aucune paire d'observables de L.

1. INTRODUCTION

A quantum logic (a logic in short) is a partially ordered set L with the first and last elements 0 and 1, respectively, and with the orthocomplementation \( ' : L \to L \) such that

i) \( (a')' = a \),

ii) \( a \leq b \Rightarrow b' \leq a' \),

iii) \( a \lor a' = 1 \),

iv) for any sequence \( \{ a_i \} \subseteq L \) such that \( a_i \leq a_j \) (\( i \neq j, i, j = 1, 2, \ldots \) the supremum \( \bigvee_{i=1}^{\infty} a_i \) exists in L,

v) if \( a \leq b \) then there is \( c \in L \) such that \( c \leq a' \) and \( b = a \lor c \).
Two elements $a, b \in L$ are said to be orthogonal (written $a \perp b$) if $a \leq b'$, and $a, b \in L$ are said to be compatible (written $a \leftrightarrow b$) if there are mutually orthogonal elements $a_1, b_1, c$ in $L$ such that $a = a_1 \lor c$, $b = b_1 \lor c$. We have $a \leq b \Rightarrow a \leftrightarrow b, a \leftrightarrow b \Rightarrow a \leftrightarrow b'$.

A state on $L$ is a map $s : L \to [0, 1]$ such that $s(1) = 1$ and $s\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} s(a_i)$ for any sequence $\{a_i\}$ of mutually orthogonal elements of $L$. Let $S(L)$ denote the set of all states on $L$, i.e. the state space of $L$.

A set $S \subset S(L)$ is said to be sufficient if for every $a \in L, a \neq 0$, there exists $s \in S$ such that $s(a) = 1$, ordering if $a \leq b$ implies that there is $s \in S$ such that $s(a) > s(b)$, strongly ordering if $a \leq b$ implies that there is $s \in S$ such that $s(a) = 1, s(b) \neq 1$.

A strongly ordering set $S$ is ordering and sufficient, but in general, an ordering and sufficient set of states need not be strongly ordering (see e.g. [1] for the proofs of these statements).

A state $s \in S(L)$ such that $s(a) \in \{0, 1\}$ for all $a \in L$ is called dispersion free or a 0-1 state. Let $S_0$ be a set of 0-1 states. The conditions—$S_0$ is ordering—and$—S_0$ is strongly ordering—are equivalent. Indeed, let $S_0$ be ordering and let $a \leq b$. Then there is $s \in S_0$ such that $s(a) > s(b)$. But this means that $s(a) = 1$ and $s(b) = 0$, i.e. $S_0$ is strongly ordering.

Let $B(R)$ denote the family of all Borel subsets of the real line $R$. An observable on a logic $L$ is a map $x : B(R) \to L$ such that

i) $x(R) = 1$,

ii) $x(E^c) = x(E)'$ for any $E \in B(R)$, where $E^c = R - E$,

iii) $x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} x(E_i)$ for any sequence $\{E_i\}$ of mutually disjoint elements of $B(R)$.

If $x$ is an observable and $s \in S(L)$, the map $s_x : E \mapsto s(x(E))$ is a probability measure on $B(R)$. The expectation of $x$ in the state $s$ is defined by

$s(x) = \int ts_x(dt)$,

if the integral on the right exists, and the variance of $x$ in the state $s$ is defined by

$\text{var}_s(x) = \int (t - s(x))^2 s_x(dt)$,

if the integral on the right exists.

Two observables $x, y$ on $L$ are compatible if $x(E) \leftrightarrow y(F)$ for any $E, F \in B(R)$. The spectrum $\sigma(x)$ of an observable $x$ is the smallest closed
subset C of R such that x(C) = 1. An observable x is bounded if σ(x) is compact.

We shall need the following lemma.

**Lemma 1.** — Let x be an observable on a logic L. Then t ∈ σ(x) if and only if for any open set U ⊂ R such that t ∈ U we have x(U) ≠ 0.

**Proof.** — Let t ∉ σ(x). As R is a regular topological space, there are disjoint open sets U, V such that t ∈ U and x(U) ⊂ V. This implies that x(U) = 0. Now let there exist an open set U ⊂ R such that t ∈ U and x(U) = 0. Then U^c is closed and x(U^c) = 1. This implies that σ(x) ⊂ U^c, i.e. t ∉ σ(x).

## 2. CLASSES OF LOGICS

Let L denote a quantum logic, S(L) the state space of L and S₀(L) the set of all 0-1 states on L. In [6], the following classes of logics with sufficient state spaces have been studied.

C₁ : a ⊕ b ⇒ there is s ∈ S(L) such that s(a) = 1 and s(b) ≠ 1,
C₂ : a ⊕ b ⇒ to any given ε > 0 there is s ∈ S(L) such that s(a) = 1 and s(b) > 1 - ε,
C₃ : a ⊕ b ⇒ there is s ∈ S(L) such that s(a) = s(b) = 1,
C₄ : S₀(L) is sufficient and a ⊕ b ⇒ there is s ∈ S₀(L) such that s(a) = s(b) = 1.

Clearly, C₁ ⊃ C₂ ⊃ C₃ ⊃ C₄ and by [6], all the inclusions are proper. It is easy to see that C₁ contains exactly logics with strongly ordering state spaces. Indeed, let S(L) be strongly ordering. Since a ⊕ b implies a ≤ b, there is s ∈ S(L) such that s(a) = 1, s(b) ≠ 1, i.e. L ∈ C₁. On the other hand, let L ∈ C₁ and let a ⊆ b. We have only to check the case when a ⊕ b. In this case a = a₁ ∨ c, b = b₁ ∨ c, where a₁, b₁, c₁ are mutually orthogonal. The condition a ⊆ b implies that a₁ ⊂ b. Since S(L) is sufficient, there is s ∈ S(L) such that s(a₁) = 1. This implies that s(a) = 1 and s(b) = 0, hence S(L) is strongly ordering.

Let H be a Hilbert space. Let L(H) denote the quantum logic of all closed subspaces of H (or equivalently, of all projections on H). The logic L(H) is called a Hilbert space logic. For M ∈ L(H), let Pₘ denote the corresponding projection. For any f ∈ H, ∥ f ∥ = 1, the map sₙ : M → ⟨ Pₘ f, f ⟩, where ⟨ ·, · ⟩ is the inner product in H, defines a state on L(H), which is called a vector state. According to Gleason theorem, if dim H ≥ 3 and H is separable, every state on L(H) is a σ-convex combination of vector states. Let M, N ∈ L(H) and let M ≤ N. Then there exists a unit vector f ∈ M, such that f ∉ N, therefore sₙ(M) = 1, sₙ(N) ≠ 1. Hence L(H) belongs to C₁.

Let for any unit vector f ∈ H, [f] denote the one-dimensional subspace
generated by \( f \). As the only state on \( L(H) \) which maps \([ f ]\) to 1 is \( s_f, L(H) \notin C_2. \)

There has been shown in \([7]\), that the logics of the class \( C_2 \) have the following interesting property: for any two bounded observables \( x, y \), the condition \( s(x) = s(y) \) in every state \( s \in S(L) \) implies that \( x = y \). In other words, the logics in \( C_2 \) satisfy the condition \( U(= \text{Uniqueness}, \text{see \([2]\), p. 55}) \). The Hilbert space logics also satisfy the property \( U \). In general, it is not known if the logics in \( C_1 \) satisfy this condition.

A special family of logics is formed by \( \sigma \)-classes. A \( \sigma \)-class is a family of subsets of a nonempty set \( X \) which contains \( X \) and is closed under the formations of set-theoretical complements and countable unions of pairwise disjoint elements. A \( \sigma \)-class ordered by inclusion and orthocomplemented by set-theoretical complementation is a quantum logic. By \([2]\), p. 69, a \( \sigma \)-class can be characterized as a logic possessing an ordering set of \( 0 \)-\( 1 \) states. It is easy to see that the class \( C_4 \) consists exactly of all \( \sigma \)-classes.

Indeed, let \( L \) be a \( \sigma \)-class. Since the set of all \( 0 \)-\( 1 \) states on \( L \) is ordering, it is also strongly ordering. Let \( a, b \in L \) be such that \( a \leftrightarrow b \), then surely \( a \perp b' \), and therefore there is \( s \in S_0(L) \) such that \( s(a) = 1 \), \( s(b') = 0 \), i.e. \( s(b) = 1 \). Hence \( L \in C_4 \). On the other hand, if \( L \in C_4 \) then using similar arguments to that used by proving that a logic \( L \in C_1 \) has a strongly ordering state space, we show that \( L \) is a \( \sigma \)-class.

Let \( H \) be a two-dimensional Hilbert space. Then every set of non-zero mutually orthogonal elements in \( L(H) \) is of the form \( \{ a, a' \} \), \( a \in L(H) \). It is easy to see that \( L(H) \) is a \( \sigma \)-class. Indeed, let

\[
S_0 = \{ s : L(H) \rightarrow \{ 0, 1 \} \mid s(a) + s(a') = 1 \}
\]

and \( h(a) = \{ s \in S_0 \mid s(a) = 1 \} \). It is easy to check that the mappings in \( S_0 \) are states on \( L(H) \), \( S_0 \) is ordering and the family \( \Delta = \{ h(a) \mid a \in L(H) \} \) of subsets of \( S_0 \) forms a \( \sigma \)-class. To give a more explicit representation, let \( H = \mathbb{R}^2 \) and let \( X = [0, \pi) \times [0, \pi) \). Put

\[
\tau(\alpha) = \begin{cases} 
[0, \alpha] \times \left[ \alpha + \frac{\pi}{2}, \pi \right) & \text{if } 0 \leq \alpha < \frac{\pi}{2} \\
\left[ \alpha - \frac{\pi}{2}, \pi \right) \times [0, \alpha) & \text{if } \frac{\pi}{2} \leq \alpha < \pi.
\end{cases}
\]

It is easy to check that \( \Delta = \{ \emptyset, X, \tau(\alpha) \mid \alpha \in [0, \pi) \} \) is a \( \sigma \)-class \((\tau(\alpha) \cap \tau(\beta) = 0 \text{ iff } \beta = \alpha + \frac{\pi}{2}, \tau(\alpha') = \tau \left( \alpha + \frac{\pi}{2} \right) \)\). Every one-dimensional subspace in \( \mathbb{R}^2 \) can be characterized by an angle \( \alpha, \alpha \in [0, \pi) \). Denote by \([\alpha] \) the one-dimensional subspace corresponding to \( \alpha \). The map \( h : L(H) \rightarrow \Delta, h(0) = \emptyset, h(H) = X, h([\alpha]) = \tau(\alpha) \), defines an isomorphism between \( L(H) \) and \( \Delta \). Let \( s_{(\beta, \gamma)} \) be the probability measure on \( \Delta \) concentrated in the point \( (\beta, \gamma) \in X \). The set \( S_0 = \{ s_{(\beta, \gamma)} \mid (\beta, \gamma) \in X \} \) represents the set of all \( 0 \)-\( 1 \) states on \( L(H) \).
3. UNCERTAINTY RELATIONS

Let $x$ be an observable on a logic $L$. We put

$$V(x) = \{ s \in S(L) \mid \text{var}_s(x) < \infty \}.$$ 

For any two observables $x, y$ on $L$, one of the following alternative possibilities occurs:

(A) $\forall \varepsilon > 0 (\exists s \in V(x) \cap V(y) (\text{var}_s(x) \cdot \text{var}_s(y) < \varepsilon))$

(B) $\exists \varepsilon > 0 (\forall s \in V(x) \cap V(y) (\text{var}_s(x) \cdot \text{var}_s(y) \geq \varepsilon))$.

If (B) occurs, we say that the uncertainty relation holds for $x$ and $y$ (see [3], [1]).

For $t \in \mathbb{R}, \delta > 0$, put $U(t, \delta) = \{ r \in \mathbb{R} \mid |t - r| < \delta \}$. If $x$ is an observable and $t \in \sigma(x)$, then $x(U(t, \delta) \neq 0$ by Lemma 1.

Let $x$ and $y$ be observables. The following two possibilities can occur:

(a) $\forall(u, v, \delta) : u \in \sigma(x), \ v \in \sigma(y), \ \delta > 0 \ (\exists \eta_0 > 0) \ (\forall \eta, 0 < \eta < \eta_0) \ (x(U(u, \delta)) \leftrightarrow y(U(v, \eta)))$

(b) $\exists(u, v, \delta) : u \in \sigma(x), \ v \in \sigma(y), \ \delta > 0 \ (\forall \eta_0 > 0) \ (\exists \eta, 0 < \eta < \eta_0) \ (x(U(u, \delta)) \leftrightarrow y(U(v, \eta)))$.

**Theorem 1.** — Let $L$ be a logic with a sufficient state space. If for the observables $x$ and $y$ on $L$ the condition (a) holds, then the uncertainty relation does not hold. In other words, (a) $\Rightarrow$ (A).

**Proof.** — Let (a) hold for the observables $x$ and $y$. We show that the following holds:

$$\forall(u, \delta) : u \in \sigma(x), \ \delta > 0 \ (\exists \eta \in \sigma(y)) \ (\forall \eta < \eta_0) \ (x(U(u, \delta)) \land y(U(v, \eta)) \neq 0)$$

($\eta_0$ exists by (a)). Suppose that the opposite holds, i.e.

$$\forall(u, \delta) : u \in \sigma(x), \ \delta > 0 \ (\exists \eta \in \sigma(y)) \ (\forall \eta < \eta_0) \ (x(U(u, \delta)) \land y(U(v, \eta)) = 0).$$

Since by (a) $x(U(u, \delta) \leftrightarrow y(U(v, \eta))$, it is $x(U(u, \delta)) \perp y(U(v, \eta))$. We have $\sigma(y) \subseteq \bigcup_{i=1}^{\infty} U(v_i, \eta_i)$ and

$$y\left(\bigcup_{i=1}^{\infty} U(v_i, \eta_i)\right) = \bigvee_{i=1}^{\infty} y(U(v_i, \eta_i)) \geq y(\sigma(y)) = 1.$$
Then $x(U(u, \delta)) \leq y(U(v, \eta_i))'$ for all $i = 1, 2, \ldots$ implies that

$$x(U(u, \delta)) \leq \bigcap_{i=1}^{\infty} y(U(v, \eta_i))' = \left( \bigcup_{i=1}^{\infty} y(U(v, \eta_i)) \right)' = 0,$$

which contradicts the supposition that $u \in \sigma(x)$. Let us choose $u \in \sigma(x)$ and $\delta > 0$. Then there is $v \in \sigma(y)$ such that for any $\eta < \eta_0 = \eta_0(u, v, \delta)$ we have $x(U(u, \delta)) \land y(U(v, \eta)) \neq 0$. By the sufficiency of $S(L)$ there is $s \in S(L)$ such that

$$s(x(U(u, \delta)) \land y(U(v, \eta))) = 1.$$

Hence $\text{var}_s(x) = \int_{U(u, \delta)}(t - s(x))^2s_x(dt) < 4\delta^2$, and similarly $\text{var}_s(y) < 4\eta^2$.

By choosing $\eta < \min\left(\eta_0, \frac{\sqrt{\varepsilon}}{2\delta}\right)$, we obtain that for any given $\varepsilon > 0$ there exists a state $s \in S(L)$ such that $\text{var}_s(x), \text{var}_s(y) < \varepsilon$.

**Remark.** — Condition (a) can be weakened to (a'), where

(a') $(\exists u, \delta : u \in \sigma(x), \delta > 0) \ (\forall v \in \sigma(y)) \ (\exists \eta_0 > 0) \ (\forall \eta, 0 < \eta < \eta_0) \ (x(U(u, \delta)) \leftrightarrow y(U(v, \eta)))$

and (a') $\Rightarrow$ (A).

**Theorem 2.** — Let $L$ be a logic which is a lattice. If for the observables $x$ and $y$ the condition (a) holds, then $x$ and $y$ are compatible.

**Proof.** — Let $U$ be any open subset of $\mathbb{R}$. We have $y(U) = y(U \cap \sigma(y))$ and $U \cap \sigma(y) \subset \bigcup \{U(v, \eta(v)) \mid v \in \sigma(y) \cap U\} \subset U$, where $\eta(v) > 0$. By the second countability of $\mathbb{R}$, there is a countable subfamily $\{U(v_i, \eta(v_i))\}$ such that

$$U \cap \sigma(y) \subset \bigcup_{i=1}^{\infty} U(v_i, \eta(v_i))$$

and

$$y(U) = \bigcup_{i=1}^{\infty} y(U(v_i, \eta(v_i))).$$

By the property (a), to any $u \in \sigma(x)$ and $\delta > 0$, and to any open set $U$ there are $v_i \in \sigma(y), \eta(v_i) > 0$ such that $y(U) = \bigcup_{i=1}^{\infty} y(U(v_i, \eta(v_i)))$ and

$$x(U(u, \delta)) \leftrightarrow y(U(v_i, \eta(v_i)))$$

for $i = 1, 2, \ldots$, which implies that $x(U(u, \delta)) \leftrightarrow y(U)$. Now let $V$ be an open subset of $\mathbb{R}$. Then there are $u_i \in \sigma(x)$ and $\delta_i > 0$, $i = 1, 2, \ldots$ such
that \( x(V) = \bigvee_{i=1}^{\infty} x(U(u_i, \delta_i)) \). Since \( x(U(u_i, \delta_i)) \leftrightarrow y(U) \), we get \( x(V) \leftrightarrow y(U) \) for any open subsets \( U, V \) of \( \mathbb{R} \), and this implies that \( x \leftrightarrow y \).

**Theorem 3.** — Let \( L \in C_3 \). Then the uncertainty relation (B) does not hold for any pair of observables on \( L \).

*Proof.* — Let \( x, y \) be observables on \( L \). By Theorem 1, \((a) \Rightarrow (A)\). Suppose that \((b)\) holds for \( x \) and \( y \). Then there are \( u \in \sigma(x), \delta > 0, v \in \sigma(y) \) such that to any \( \eta_0 > 0 \) there is \( \eta < \eta_0 \) such that \( x(U(u, \delta)) \leftrightarrow y(U(v, \eta)) \). As \( L \) belongs to \( C_3 \), there is a state \( s \in S(L) \) such that \( s(x(U(u, \delta))) = s(y(U(v, \eta))) = 1 \). Choosing \( \eta \) sufficiently small we obtain \( \text{var}_s(x) \cdot \text{var}_s(y) \leq \varepsilon \) for any given \( \varepsilon > 0 \).

Let \( H^2 = L(\mathbb{R}) \) be the set of all square-integrable complex valued functions defined on \( \mathbb{R} \) with respect to the Lebesgue measure. Let \( q \) and \( p \) be the « position » and « momentum » observables corresponding to the self-adjoint operators \( P, Q \), where \( (Qf)(r) = rf(r), (Pf)(r) = -i\hbar \frac{d}{dr} f(r) \) for \( r \in \mathbb{R} \). It can be shown that \( \text{var}_f(q) \cdot \text{var}_f(p) \geq \frac{\hbar^2}{4} \) for all \( f \in D(Q) \cap D(P) \), where \( D(A) \) denotes the domain of the operator \( A \) (see e.g. [8], p. 77, 393-394 for the proof). For any self-adjoint operator \( A \) its domain

\[
D(A) = \left\{ f \in H \left| \int t^2 \langle E^A(dt)f, f \rangle < \infty \right\} = \{ f \in H \mid s_f \in V(E^A) \},
\]

where \( E^A \) is the spectral measure (which can be identified with the observable corresponding to \( A \) by the spectral theorem). Owing to Gleason theorem, every state on \( L(H) \) is a \( \sigma \)-convex combination of vector states. From this we may conclude that the observables \( p \) and \( q \) satisfy the uncertainty relation in the sense of our definition.

The above example shows that there are couple of observables on the logics of the class \( C_1 \) which satisfy the uncertainty relations. It remains an open question if there exist couple of observables on the logics of the class \( C_2 \) satisfying the uncertainty relations.

In [3], the notion of complementarity has been introduced as follows. Let \( x, y \) be observables on a logic \( L \). We say that \( x, y \) are complementary if \( x(E) \land y(F) = 0 \) for every bounded subsets \( E, F \) of \( \mathbb{R} \) such that \( x(E) \neq 1 \) and \( y(F) \neq 1 \). It is a well-known fact that the observables \( q, p \) in the above example are complementary (see e.g. [4], [5]). Now let us consider the logic \( L(H) \) of the two-dimensional Hilbert space \( H \). It is easy to see that any two noncompatible observables on \( L(H) \) are complementary.
example shows that complementarity is not excluded on the logics of the class $C_3$ or even $C_4$. However, it would be interesting to find less trivial examples of unbounded complementary observables on the logics of the class $C_3$.

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