ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 48, nº 4 (1988), p. 325-332
<http://www.numdam.org/item?id=AIHPA_1988__48_4_325_0>

Ann. Inst. Henri Poincaré,
Vol. 48, n° 4, 1988, p. 325-332. *Physique théorique*

Uncertainty relations and state spaces

by

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ABSTRACT. — We show that on a quantum logic L which has a sufficient set of states S(L) with the property: for every two noncompatible elements a, b of L there is a state $s \in S(L)$ such that $s(a) = s(b) = 1$, the uncertainty relations cannot be satisfied for any pair of observables on L.

RÉSUMÉ. — Nous montrons que si une logique quantique L a un ensemble d'états S(L) assez grand, c'est-à-dire si pour toute paire a, b d'éléments non compatibles de L il existe un état $s \in S(L)$ tel que $s(a) = s(b) = 1$, alors la relation d'incertitude ne peut être satisfaite pour aucune paire d'observables de L.

1. INTRODUCTION

A quantum logic (a logic in short) is a partially ordered set L with the first and last elements 0 and 1, respectively, and with the orthocomplementation $' : L \to L$ such that

i) $(a')' = a$,
ii) $a \leq b \Rightarrow b' \leq a'$,
iii) $a \vee a' = 1$,
iv) for any sequence $\{ a_i \} \subset L$ such that $a_i \leq a_j'$ $(i \neq j,\ i, j = 1, 2, \ldots)$ the supremum $\bigvee_{i=1}^{\infty} a_i$ exists in L,
v) if $a \leq b$ then there is $c \in L$ such that $c \leq a'$ and $b = a \vee c$.

Two elements $a, b \in \mathrm{L}$ are said to be orthogonal (written $a \perp b$) if $a \leq b'$, and $a, b \in \mathrm{L}$ are said to be compatible (written $a \leftrightarrow b$) if there are mutually orthogonal elements a_1, b_1, c in L such that $a = a_1 \vee c$, $b = b_1 \vee c$. We have $a \leq b \Rightarrow a \leftrightarrow b$, $a \leftrightarrow b \Rightarrow a \leftrightarrow b'$.

A state on L is a map $s : \mathrm{L} \to [0, 1]$ such that $s(1) = 1$ and $s\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} s(a_i)$ for any sequence $\{ a_i \}$ of mutually orthogonal elements of L. Let S(L) denote the set of all states on L, i. e. the state space of L.

A set $\mathrm{S} \subset \mathrm{S(L)}$ is said to be sufficient if for every $a \in \mathrm{L}$, $a \neq 0$, there exists $s \in \mathrm{S}$ such that $s(a) = 1$, ordering if $a \nleq b$ implies that there is $s \in \mathrm{S}$ such that $s(a) > s(b)$, strongly ordering if $a \nleq b$ implies that there is $s \in \mathrm{S}$ such that $s(a) = 1$, $s(b) \neq 1$.

A strongly ordering set S is ordering and sufficient, but in general, an ordering and sufficient set of states need not be strongly ordering (see e. g. [*1*] for the proofs of these statements).

A state $s \in \mathrm{S(L)}$ such that $s(a) \in \{0, 1\}$ for all $a \in \mathrm{L}$ is called dispersion free or a 0-1 state. Let S_0 be a set of 0-1 states. The conditions—S_0 is ordering—and—S_0 is strongly ordering—are equivalent. Indeed, let S_0 be ordering and let $a \nleq b$. Then there is $s \in \mathrm{S}_0$ such that $s(a) > s(b)$. But this means that $s(a) = 1$ and $s(b) = 0$, i. e. S_0 is strongly ordering.

Let B(R) denote the family of all Borel subsets of the real line R. An observable on a logic L is a map $x : \mathrm{B(R)} \to \mathrm{L}$ such that

i) $x(\mathrm{R}) = 1$,

ii) $x(\mathrm{E}^c) = x(\mathrm{E})'$ for any $\mathrm{E} \in \mathrm{B(R)}$, where $\mathrm{E}^c = \mathrm{R} - \mathrm{E}$,

iii) $x\left(\bigcup_{i=1}^{\infty} \mathrm{E}_i\right) = \bigvee_{i=1}^{\infty} x(\mathrm{E}_i)$ for any sequence $\{ \mathrm{E}_i \}$ of mutually disjoint elements of B(R).

If x is an observable and $s \in \mathrm{S(L)}$, the map $s_x : \mathrm{E} \mapsto s(x(\mathrm{E}))$ is a probability measure on B(R). The expectation of x in the state s is defined by

$$s(x) = \int t s_x(dt),$$

if the integral on the right exists, and the variance of x in the state s is defined by

$$\mathrm{var}_s(x) = \int (t - s(x))^2 s_x(dt),$$

if the integral on the right exists.

Two observables x, y on L are compatible if $x(\mathrm{E}) \leftrightarrow y(\mathrm{F})$ for any $\mathrm{E}, \mathrm{F} \in \mathrm{B(R)}$. The spectrum $\sigma(x)$ of an observable x is the smallest closed subset C of R such that $x(C) = 1$. An observable x is bounded if $\sigma(x)$ is compact.

We shall need the following lemma.

LEMMA 1. — Let x be an observable on a logic L. Then $t \in \sigma(x)$ if and only if for any open set $U \subset R$ such that $t \in U$ we have $x(U) \neq 0$.

Proof. — Let $t \notin \sigma(x)$. As R is a regular topological space, there are disjoint open sets U, V such that $t \in U$ and $\sigma(x) \subset V$. This implies that $x(U) = 0$. Now let there exist an open set $U \subset R$ such that $t \in U$ and $x(U) = 0$. Then U^c is closed and $x(U^c) = 1$. This implies that $\sigma(x) \subset U^c$, i. e. $t \notin \sigma(x)$.

2. CLASSES OF LOGICS

Let L denote a quantum logic, S(L) the state space of L and $S_0(L)$ the set of all 0-1 states on L. In [6], the following classes of logics with sufficient state spaces have been studied.

$C_1 : a \not\leftrightarrow b \Rightarrow$ there is $s \in S(L)$ such that $s(a) = 1$ and $s(b) \neq 1$,

$C_2 : a \not\leftrightarrow b \Rightarrow$ to any given $\varepsilon > 0$ there is $s \in S(L)$ such that $s(a) = 1$ and $s(b) > 1 - \varepsilon$,

$C_3 : a \not\leftrightarrow b \Rightarrow$ there is $s \in S(L)$ such that $s(a) = s(b) = 1$,

$C_4 : S_0(L)$ is sufficient and $a \not\leftrightarrow b \Rightarrow$ there is $s \in S_0(L)$ such that $s(a) = s(b) = 1$.

Clearly, $C_1 \supset C_2 \supset C_3 \supset C_4$ and by [6], all the inclusions are proper. It is easy to see that C_1 contains exactly logics with strongly ordering state spaces. Indeed, let S(L) be strongly ordering. Since $a \not\leftrightarrow b$ implies $a \not\leq b$, there is $s \in S(L)$ such that $s(a) = 1$, $s(b) \neq 1$, i. e. $L \in C_1$. On the other hand, let $L \in C_1$ and let $a \not\leq b$. We have only to check the case when $a \leftrightarrow b$. In this case $a = a_1 \vee c$, $b = b_1 \vee c$, where a_1, b_1, c_1 are mutually orthogonal. The condition $a \not\leq b$ implies that $a_1 \neq 0$. Since S(L) is sufficient, there is $s \in S(L)$ such that $s(a_1) = 1$. This implies that $s(a) = 1$ and $s(b) = 0$, hence S(L) is strongly ordering.

Let H be a Hilbert space. Let L(H) denote the quantum logic of all closed subspaces of H (or equivalently, of all projections on H). The logic L(H) is called a Hilbert space logic. For $M \in L(H)$, let P^M denote the corresponding projection. For any $f \in H$, $\| f \| = 1$, the map $s_f : M \to \langle P^M f, f \rangle$, where $\langle .,. \rangle$ is the inner product in H, defines a state on L(H), which is called a vector state. According to Gleason theorem, if $\dim H \geq 3$ and H is separable, every state on L(H) is a σ-convex combination of vector states. Let $M, N \in L(H)$ and let $M \not\leq N$. Then there exists a unit vector $f \in M$, such that $f \notin N$, therefore $s_f(M) = 1$, $s_f(N) \neq 1$. Hence L(H) belongs to C_1 Let for any unit vector $f \in H$, $[f]$ denote the one-dimensional subspace generated by f. As the only state on L(H) which maps $[f]$ to 1 is s_f, $L(H) \notin C_2$. There has been shown in [7], that the logics of the class C_2 have the following interesting property: for any two bounded observables x, y, the condition $s(x) = s(y)$ in every state $s \in S(L)$ implies that $x = y$. In other words, the logics in C_2 satisfy the condition U (= Uniqueness, see [2], p. 55). The Hilbert space logics also satisfy the property U. In general, it is not known if the logics in C_1 satisfy this condition.

A special family of logics is formed by σ-classes. A σ-class is a family of subsets of a nonempty set X which contains X and is closed under the formations of set-theoretical complements and countable unions of pairwise disjoint elements. A σ-class ordered by inclusion and orthocomplemented by set-theoretical complementation is a quantum logic. By [2], p. 69, a σ-class can be characterized as a logic posessing an ordering set of 0-1 states. It is easy to see that the class C_4 consists exactly of all σ-classes. Indeed, let L be a σ-class. Since the set of all 0-1 states on L is ordering, it is also strongly ordering. Let $a, b \in L$ be such that $a \not\leftrightarrow b$, then surely $a \nleq b'$, and therefore there is $s \in S_0(L)$ such that $s(a) = 1$, $s(b') = 0$, i. e. $s(b) = 1$. Hence $L \in C_4$. On the other hand, if $L \in C_4$ then using similar arguments to that used by proving that a logic $L \in C_1$ has a strongly ordering state space, we show that L is a σ-class.

Let H be a two-dimensional Hilbert space. Then every set of non-zero mutually orthogonal elements in L(H) is of the form $\{a, a'\}$, $a \in L(H)$. It is easy to see that L(H) is a σ-class. Indeed, let

$$S_0 = \{ s : L(H) \to \{0, 1\} \mid s(a) + s(a') = 1 \}$$

and $h(a) = \{ s \in S_0 \mid s(a) = 1 \}$. It is easy to check that the mappings in S_0 are states on L(H), S_0 is ordering and the family $\Delta = \{ h(a) \mid a \in L(H) \}$ of subsets of S_0 forms a σ-class. To give a more explicite representation, let $H = R^2$ and let $X = [0, \pi) \times [0, \pi)$. Put

$$\tau(\alpha) = \begin{cases} [0, \alpha) \times \left[\alpha + \frac{\pi}{2}, \pi\right) & \text{if } 0 \leq \alpha < \frac{\pi}{2} \\ \left[\alpha - \frac{\pi}{2}, \pi\right) \times [0, \alpha) & \text{if } \frac{\pi}{2} \leq \alpha < \pi . \end{cases}$$

It is easy to check that $\Delta = \{ \emptyset, X, \tau(\alpha) \mid \alpha \in [0, \pi) \}$ is a σ-class ($\tau(\alpha) \cap \tau(\beta) = 0$ iff $\beta = \alpha + \frac{\pi}{2}$, $\tau(\alpha)^c = \tau\left(\alpha + \frac{\pi}{2}\right)$). Every one-dimensional subspace in R^2 can be characterized by an angle α, $\alpha \in [0, \pi)$. Denote by $[\alpha]$ the one-dimensional subspace corresponding to α. The map $h : L(H) \to \Delta$, $h(0) = \emptyset$, $h(H) = X$, $h([\alpha]) = \tau(\alpha)$, defines an isomorphism between L(H) and Δ. Let $s_{(\beta,\gamma)}$ be the probability measure on Δ concentrated in the point $(\beta, \gamma) \in X$. The set $S_0 = \{ s_{(\beta,\gamma)} \mid (\beta, \gamma) \in X \}$ represents the set of all 0-1 states on L(H).

3. UNCERTAINTY RELATIONS

Let x be an observable on a logic L. We put

$$V(x) = \{ s \in S(L) \mid \mathrm{var}_s(x) < \infty \} .$$

For any two observables x, y on L, one of the following alternative possibilities occurs:

(A) $(\forall \varepsilon > 0)(\exists s \in V(x) \cap V(y)(\mathrm{var}_s(x) . \mathrm{var}_s(y) < \varepsilon)$
(B) $(\exists \varepsilon > 0)(\forall s \in V(x) \cap V(y)(\mathrm{var}_s(x) . \mathrm{var}_s(y) \geq \varepsilon)$.

If (B) occurs, we say that the uncertainty relation holds for x and y (see [3], [1]).

For $t \in R$, $\delta > 0$, put $U(t, \delta) = \{ r \in R \mid |t - r| < \delta \}$. If x is an observable and $t \in \sigma(x)$, then $x(U(t, \delta) \neq 0$ by Lemma 1.

Let x and y be observables. The following two possibilities can occur:

(a) $(\forall(u, v, \delta) : u \in \sigma(x),\ v \in \sigma(y),\ \delta > 0)\ (\exists \eta_0 > 0)\ (\forall \eta, 0 < \eta < \eta_0)$ $(x(U(u, \delta)) \leftrightarrow y(U(v, \eta)))$.

(b) $(\exists(u, v, \delta) : u \in \sigma(x),\ v \in \sigma(y), \delta > 0)\ (\forall \eta_0 > 0)\ (\exists \eta, 0 < \eta < \eta_0)$ $(x(U(u, \delta)) \not\leftrightarrow y(U(v, \eta)))$.

THEOREM 1. — Let L be a logic with a sufficient state space. If for the observables x and y on L the condition (a) holds, then the uncertainty relation does not hold. In other words, (a) $\Rightarrow$ (A).

Proof. — Let (a) hold for the observables x and y. We show that the following holds:

$$(\forall(u, \delta) : u \in \sigma(x), \delta > 0)\ (\exists v \in \sigma(y)\ (\forall \eta < \eta_0)\ (x(U(u, \delta)) \wedge y(U(v, \eta)) \neq 0)$$

(η_0 exists by (a)). Suppose that the opposite holds, i. e.

$$(\exists(u, \delta) : u \in \sigma(x), \delta > 0)\ (\forall v \in \sigma(y))\ (\exists \eta < \eta_0)\ (x(U(u, \delta)) \wedge y(U(v, \eta)) = 0) .$$

Since by (a) $x(U(u, \delta) \leftrightarrow y(U(v, \eta)$, it is $x(U(u, \delta)) \perp y(U(v, \eta))$. We have $\sigma(y) \subset \cup \{ U(v, \eta(v)) \mid v \in \sigma(y) \}$. By the second countability of the topology of R, there is a countable set $\{ v_i \}$ such that

$$\sigma(y) \subset \bigcup_{i=1}^{\infty} U(v_i, \eta_i) \qquad \text{and}$$

$$y\left(\bigcup_{i=1}^{\infty} U(v_i, \eta_i)\right) = \bigvee_{i=1}^{\infty} y(U(v_i, \eta_i)) \geq y(\sigma(y)) = 1 .$$

Then $x(U(u, \delta)) \leq y(U(v_i, \eta_i))'$ for all $i = 1, 2, \ldots$ implies that

$$x(U(u, \delta)) \leq \bigwedge_{i=1}^{\infty} y(U(v_i, \eta_i))' = \left(\bigvee_{i=1}^{\infty} y(U(v_i, \eta_i))\right)' = 0,$$

which contradicts the supposition that $u \in \sigma(x)$. Let us choose $u \in \sigma(x)$ and $\delta > 0$. Then there is $v \in \sigma(y)$ such that for any $\eta < \eta_0$ ($\eta_0 = \eta_0(u, v, \delta)$) we have $x(U(u, \delta)) \wedge y(U(v, \eta)) \neq 0$. By the sufficiency of S(L) there is $s \in S(L)$ such that

$$s(x(U(u, \delta)) \wedge y(U(v, \eta))) = 1.$$

Hence $\mathrm{var}_s(x) = \int_{U(u,\delta)} (t - s(x))^2 s_x(dt) < 4\delta^2$, and similarly $\mathrm{var}_s(y) < 4\eta^2$.

By choosing $\eta < \min\left(\eta_0, \frac{\sqrt{\varepsilon}}{2\delta}\right)$, we obtain that for any given $\varepsilon > 0$ there exists a state $s \in S(L)$ such that $\mathrm{var}_s(x) . \mathrm{var}_s(y) < \varepsilon$.

Remark. — Condition (a) can be weakened to (a'), where

(a') $(\exists(u, \delta) : u \in \sigma(x), \delta > 0)$ $(\forall v \in \sigma(y))$ $(\exists \eta_0 > 0)$ $(\forall \eta, 0 < \eta < \eta_0)$ $(x(U(u, \delta)) \leftrightarrow y(U(v, \eta)))$

and (a') $\Rightarrow$ (A).

THEOREM 2. — Let L be a logic which is a lattice. If for the observables x and y the condition (a) holds, then x and y are compatible.

Proof. — Let U be any open subset of R. We have $y(U) = y(U \cap \sigma(y))$ and $U \cap \sigma(y) \subset \cup \{ U(v, \eta(v)) \mid v \in \sigma(y) \cap U \} \subset U$, where $\eta(v) > 0$. By the second countability of R, there is a countable subfamily $\{ U(v_i, \eta(v_i)) \}$ such that

$$U \cap \sigma(y) \subset \bigcup_{i=1}^{\infty} U(v_i, \eta(v_i))$$

and

$$y(U) = \bigvee_{i=1}^{\infty} y(U(v_i, \eta(v_i))).$$

By the property (a), to any $u \in \sigma(x)$ and $\delta > 0$, and to any open set U there are $v_i \in \sigma(y)$, $\eta(v_i) > 0$ such that $y(U) = \bigvee_{i=1}^{\infty} y(U(v_i, \eta(v_i)))$ and

$$x(U(u, \delta)) \leftrightarrow y(U(v_i, \eta(v_i)))$$

for $i = 1, 2, \ldots$, which implies that $x(U(u, \delta)) \leftrightarrow y(U)$. Now let V be an open subset of R. Then there are $u_i \in \sigma(x)$ and $\delta_i > 0$, $i = 1, 2, \ldots$ such that $x(V) = \bigvee_{i=1}^{\infty} x(U(u_i, \delta_i))$. Since $x(U(u_i, \delta_i)) \leftrightarrow y(U)$, we get $x(V) \leftrightarrow y(U)$ for any open subsets U, V of R, and this implies that $x \leftrightarrow y$.

THEOREM 3. — Let $L \in C_3$. Then the uncertainty relation (B) does not hold for any pair of observables on L.

Proof. — Let x, y be observables on L. By Theorem 1, (*a*) $\Rightarrow$ (A). Suppose that (*b*) holds for x and y. Then there are $u \in \sigma(x)$, $\delta > 0$, $v \in \sigma(y)$ such that to any $\eta_0 > 0$ there is $\eta < \eta_0$ such that $x(U(u, \delta)) \not\leftrightarrow y(U(v, \eta))$. As L belongs to C_3, there is a state $s \in S(L)$ such that $s(x(U(u, \delta))) = s(y(U(v, \eta))) = 1$. Choosing η sufficiently small we obtain $\mathrm{var}_s(x) \,.\, \mathrm{var}_s(y) < \varepsilon$ for any given $\varepsilon > 0$.

Let $H^2 = L(R)$ be the set of all square-integrable complex valued functions defined on R with respect to the Lebesgue measure. Let q and p be the « position » and « momentum » observables corresponding to the self-adjoint operators P, Q, where $(Qf)(r) = rf(r)$, $(Pf)(r) = -ih\frac{d}{dr}f(r)$ for $r \in R$. It can be shown that $\mathrm{var}_{s_f}(q) \cdot \mathrm{var}_{s_f}(p) \geq \frac{h^2}{4}$ for all $f \in D(Q) \cap D(P)$, where D(A) denotes the domain of the operator A (see e. g. [*8*], p. 77, 393-394 for the proof). For any self-adjoint operator A its domain

$$D(A) = \left\{ f \in H \,\middle|\, \int t^2 \langle E^A(dt)f, f \rangle < \infty \right\} = \{ f \in H \mid s_f \in V(E^A) \},$$

where E^A is the spectral measure (which can be identified with the observable corresponding to A by the spectral theorem). Owing to Gleason theorem, every state on L(H) is a σ-convex combination of vector states. From this we may conclude that the observables p and q satisfy the uncertainty relation in the sense of our definition.

The above example shows that there are couples of observables on the logics of the class C_1 which satisfy the uncertainty relations. It remains an open question if there exist couples of observables on the logics of the class C_2 satisfying the uncertainty relations.

In [*3*], the notion of complementarity has been introduced as follows. Let x, y be observables on a logic L. We say that x, y are complementary if $x(E) \wedge y(F) = 0$ for every bounded subsets E, F of R such that $x(E) \neq 1$ and $y(F) \neq 1$. It is a well-known fact that the observables q, p in the above example are complementary (see e. g. [*4*], [*5*]). Now let us consider the logic L(H) of the two-dimensional Hilbert space H. It is easy to see that any two noncompatible observables on L(H) are complementary. This example shows that complementarity is not excluded on the logics of the class C_3 or even C_4. However, it would be interesting to find less trivial examples of unbounded complementary observables on the logics of the class C_3.

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(Manuscrit reçu le 3 décembre 1987)

(Version révisée reçue le 9 mars 1988)