ANNALES DE L'I. H. P., SECTION A

HELLMUT BAUMGÄRTEL

On nets of local algebras on $\mathbb{Z}^4$, covariant with respect to the discrete Poincaré group ; causality and scattering theory

Annales de l'I. H. P., section A, tome 48, nº 4 (1988), p. 311-323
<http://www.numdam.org/item?id=AIHPA_1988__48_4_311_0>

Ann. Inst. Henri Poincaré,
Vol. 48, n° 4, 1988, p. 311-323. *Physique théorique*

On nets of local algebras on $\mathbb{Z}^4$, covariant with respect to the discrete Poincaré group; causality and scattering theory

by

Hellmut BAUMGÄRTEL
Karl-Weierstraß-Institut für Mathematik,
Akademie der Wissenschaften der DDR, Berlin

ABSTRACT. — The paper contains constructions of nets of local algebras on $\mathbb{Z}^4$, which are covariant with respect to the discrete Poincaré group $\mathscr{P}$, using CCR-Weyl-algebras over phase spaces depending on a certain measure. Firstly, there are causal nets in this framework. Secondly, under a certain assumption on the measure, one can apply methods developed in the paper [*1*] and in related papers to establish a $\mathscr{P}$-covariant perturbation theory with convergent LSZ-scattering process.

RÉSUMÉ. — Nous donnons une construction d'un réseau d'algèbres locales sur $\mathbb{Z}^4$ covariantes par rapport au groupe de Poincaré en utilisant des algèbres de commutations de Weyl sur l'espace de phase associées à une mesure. Notre construction permet d'obtenir des réseaux ayant la propriété de consalité. Avec une certaine hypothèse sur la mesure nous pouvons appliquer les méthodes de [*1*] pour établir une théorie de perturbations covariante avec des processus de diffusion L-S-Z convergents.

§ 1. INTRODUCTION

Let $\mathbb{Z}^4$ be the hypercubic lattice considered as a lattice in the Minkowski space $\mathbb{R}^4$. Let $\mathscr{P}$ be a discrete version of the Poincaré group (introduced in § 2) and let $\perp$ denote the usual causal disjointness (in $\mathbb{Z}^4$). We consider nets $\mathscr{A}(\cdot)$ of C*-algebras $\mathscr{A}(\mathscr{O})$, $\mathscr{O} \subset \mathbb{Z}^4$, over $\mathbb{Z}^4$, which are $\mathscr{P}$-covariant and whose quasilocal algebras $\mathscr{A}$ have $\mathscr{P}$-invariant states, which satisfy a certain positivity condition. Obviously, the formulation of a positivity condition causes difficulties, because the dual $\mathbb{T}^4$ of the additive group $\mathbb{Z}^4$ is compact. So, together with the net $\mathscr{A}(\cdot)$ on $\mathbb{Z}^4$ an associated translationally covariant net $\mathscr{B}(\cdot)$ on $\mathbb{R}^4$ is considered such that $\mathscr{A}(\cdot)$ is the restriction of $\mathscr{B}(\cdot)$ to $\mathbb{Z}^4$, $\mathscr{A}(\mathscr{O}) = \mathscr{B}(\mathscr{O})$ for $\mathscr{O} \subset \mathbb{Z}^4$, and such that the translational covariance of the net $\mathscr{A}(\cdot)$ is induced by that of $\mathscr{B}(\cdot)$. Obviously there are translationally invariant states of $\mathscr{B}$. Doplicher [2] formulated a criterion for the existence of a translationally invariant state ω of $\mathscr{B}$ which satisfies a positivity condition (the strongly continuous representation of the translations on the GNS-Hilbert space of ω should be spectral). Here we require additionally that the restriction $\omega \upharpoonright \mathscr{A}$ of ω should be $\mathscr{P}$-invariant. Such nets $\mathscr{A}(\cdot)$ we call admissible. Simple examples of admissible nets $\mathscr{A}(\cdot)$ can be constructed by forming CCR-algebras over suitable phase spaces, using a suitable localization principle. In these examples the Fock state satisfies the mentioned positivity condition. In this note we discuss the existence of causal admissible nets, i. e. admissible nets which are causal with respect to $\perp$ on $\mathbb{Z}^4$. Furthermore, applying methods of [1], a $\mathscr{P}$-covariant perturbation and scattering theory for admissible nets is described briefly.

§ 2. THE DISCRETE POINCARÉ GROUP AND ADMISSIBLE NETS

The elements of $\mathbb{Z}^4$ are denoted by $x = \{ x_0, x_1, x_2, x_3 \}$, sometimes we write $x = \{ x_0, \mathfrak{x} \}$, $x_0 \in \mathbb{Z}$, $\mathfrak{x} \in \mathbb{Z}^3$. As usual, a point $x \in \mathbb{Z}^4$ can be represented by a selfadjoint 2×2-matrix $X = \sum_{j=0}^{3} x_j \sigma_j$, where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_1, \sigma_2, \sigma_3$ denote the Pauli matrices. The elements of X are from $\mathbb{Z} + i\mathbb{Z}$ (Gaussian numbers), one has $\operatorname{tr} X \equiv 0 \pmod 2$. The translation group $\mathbb{Z}^4$ acts on $\mathbb{Z}^4$ by $X \to X + A$. The group $\mathscr{G}$ consisting of all matrices

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det \Lambda = ad - bc = 1,$$
$$a, b, c, d \in \mathbb{Z} + i\mathbb{Z}, |a|^2 + |b|^2 + |c|^2 + |d|^2 \equiv 0 \pmod 2,$$

is a discrete subgroup $\mathscr{G}$ of SL(2, $\mathbb{C}$), which acts on $\mathbb{Z}^4$ by $X \to \Lambda X \Lambda^*$. The semidirect product $\mathscr{P} := \mathbb{Z}^4 \circledS \mathscr{G}$ of $\mathbb{Z}^4$ and $\mathscr{G}$, acting on $\mathbb{Z}^4$ by $X \to \Lambda X \Lambda^* + A$, $\mathscr{P} \ni g = \{ a, \Lambda \}$, is a discrete subgroup of the Poincaré group; we call it the discrete Poincaré group on $\mathbb{Z}^4$. Discrete variants of the Poincaré group, which act on lattices as subsets of the Minkowski space, have been first introduced by A. Schild [3], who discussed some cinematical aspects.

The causality structure on $\mathbb{Z}^4$ is as usual, $x \perp y$ means

$$\det(X - Y) = (x_0 - y_0)^2 - |\mathfrak{x} - \mathfrak{y}|^2 < 0.$$

As usual, to each subset $\mathcal{O} \subset \mathbb{Z}^4$ there is assigned the causal complement $\mathcal{O} \mapsto \mathcal{O}^\perp = \{ y : y \in \mathbb{Z}^4, \det(Y - X) < 0, x \in \mathcal{O} \}$ with the usual properties $\mathcal{O} \mapsto \mathcal{O}^{\perp\perp}$, $\mathcal{O} \cap \mathcal{O}^\perp = \emptyset$, $(\mathcal{O}_1 \cup \mathcal{O}_2)^\perp = \mathcal{O}_1^\perp \cap \mathcal{O}_2^\perp$, which implies: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{O}_2^\perp \subseteq \mathcal{O}_1^\perp$ and $\mathcal{O}^\perp = \mathcal{O}^{\perp\perp\perp}$. The assignment $\mathcal{O} \mapsto \mathcal{O}^\perp$ is compatible with $\mathcal{P} : (g\mathcal{O})^\perp = g\mathcal{O}^\perp$, $g \in \mathcal{P}$.

We consider isotone and additive nets $\mathcal{A}(\cdot)$ of C*-algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \subset \mathbb{Z}^4$, $\mathcal{O}$ finite, on $\mathbb{Z}^4$, i. e. $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ and

$$\mathcal{A}(\cup_{j=1}^n \mathcal{O}_j) = \bigvee_{j=1}^n \mathcal{A}(\mathcal{O}_j),$$

where $\bigvee_{j=1}^n \mathcal{A}(\mathcal{O}_j)$ denotes the C*-algebra, which is generated by the algebras $\mathcal{A}(\mathcal{O}_j)$. If $\mathcal{O} = \{ x \}$ consists only of one lattice point, we write for brevity $\mathcal{A}(\{ x \}) = \mathcal{A}(x)$. The net $\mathcal{A}(\cdot)$ is uniquely determined by its « atomic » algebras $\mathcal{A}(x)$. The algebra $\mathcal{A} := \operatorname{clo} \bigcup_{\mathcal{O} \subset \mathbb{Z}^4} \mathcal{A}(\mathcal{O})$ is called, as usual, the quasilocal algebra. The net $\mathcal{A}(\cdot)$ is called $\mathcal{P}$-covariant, if there is a representation $\mathcal{P} \ni g \to \alpha_g \in \operatorname{aut} \mathcal{A}$ by automorphisms of $\mathcal{A}$ such that $\mathcal{A}(g\mathcal{O}) = \alpha_g \mathcal{A}(\mathcal{O})$ for all $\mathcal{O} \subset \mathbb{Z}^4$.

Furthermore, we consider isotone and additive nets $\mathcal{B}(\cdot)$ of C*-algebras $\mathcal{B}(\mathcal{M})$, $\mathcal{M} \subset \mathbb{R}^4$, $\mathcal{M}$ compact, on $\mathbb{R}^4$. The net $\mathcal{B}(\cdot)$ is called translationally covariant, if there is a representation $\mathbb{R}^4 \ni x \to \alpha_x \in \operatorname{aut} \mathcal{B}$, $\mathcal{B} = \operatorname{clo} \bigcup_{\mathcal{M} \subset \mathbb{R}^4} \mathcal{B}(\mathcal{M})$, by automorphisms of $\mathcal{B}$ such that $\mathcal{B}(x\mathcal{M}) = \alpha_x \mathcal{B}(\mathcal{M})$ for all $\mathcal{M} \subset \mathbb{R}^4$.

DEFINITION 1. — A net $\mathcal{A}(\cdot)$ of C*-algebras on $\mathbb{Z}^4$ is called *admissible*, if it is isotone, additive, $\mathcal{P}$-covariant, if it is the restriction to $\mathbb{Z}^4$ of an associated net $\mathcal{B}(\cdot)$ on $\mathbb{R}^4$, which is isotone, additive and $\mathbb{R}^4$-covariant and such that there is a translationally invariant state ω of $\mathcal{B}$ with positivity condition and the restricted state $\omega \upharpoonright \mathcal{A}$ is $\mathcal{P}$-invariant.

DEFINITION 2. — An isotone, additive and $\mathcal{P}$-covariant net $\mathcal{A}(\cdot)$ on $\mathbb{Z}^4$ is called *causal*, if $\mathcal{O}_1 \perp \mathcal{O}_2$, $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{Z}^4$, implies that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute by elements.

§ 3. CONSTRUCTION OF ADMISSIBLE NETS

Admissible nets can be easily constructed by construction of CCR-Weyl-algebras over suitable phase spaces and localization. By $\delta_x(\cdot)$ we denote the Dirac measure on $\mathbb{R}^4$ at the point $x \in \mathbb{R}^4$. We denote by $M(\mathbb{R}^4)$ the linear space of all complex-valued Borel measures ν on $\mathbb{R}^4$ with compact support supp ν. The Fourier transform is denoted by

$$\hat{\nu}(p) := \int_{\mathbb{R}^4} \exp(i\langle p, x\rangle)\nu(dx),$$

where $\langle p, x\rangle = p_0 x_0 - (\mathfrak{p}, \mathfrak{x})$ is the Minkowski scalar product. The translations $a \in \mathbb{R}^4$ act on $M(\mathbb{R}^4)$ by $\nu(\cdot) \to \delta_a * \nu = \nu(\cdot - a)$. To produce $\mathscr{P}$-covariance we choose vector-valued phase space elements and put $H := L_{\text{fin}}(\mathscr{G}, M(\mathbb{R}^4))$ to be the pre-phase space. The forward cone in Minkowski space is denoted by V_+. Let μ be a positive finite Borel measure on $\mathbb{R}^4$ with supp $\mu \subseteq \text{clo}\, V_+$, $\mu(\text{clo}\, V_+) = 1$. Then one can define on H a semi-scalar product by

$$(\nu_1, \nu_2) := \sum_{\Lambda\in\mathscr{G}} \int_{\mathbb{R}^4} \overline{\hat{\nu}_1(p, \Lambda)}\, \hat{\nu}_2(p, \Lambda)\mu(\Lambda dp), \quad \|\nu\| := (\nu, \nu)^{1/2},$$

where the shifted measure $\mu(\Lambda^{-1}dp)$ sometimes is denoted by $\mu_\Lambda(dp)$. We choose the *phase space* $H_{\mathbb{R}^4}$ to be $H_{\mathbb{R}^4} := H \bmod \ker \|\cdot\|$. Then

$$\mathfrak{f} := \text{clo}_{\|.\|}\, H_{\mathbb{R}^4} = l^2(\mathscr{G}, L^2(\mathbb{R}^4, d\mu_{\Lambda^{-1}})),$$

i. e. $\mathfrak{f}$ is a discrete direct integral in the l^2-sense, where $\Lambda \to L^2(\mathbb{R}^4, d\mu_{\Lambda^{-1}})$.

It is necessary to choose a finite measure μ in order to get $(\cdot,\cdot)$ defined on H. For example, for the Dirac measure one has $(\delta_x)\hat{}(p) = \exp(i\langle p, x\rangle)$, so, if δ_x is located at $\Lambda_0 \in \mathscr{G}$, and 0 otherwise, then one obtains $\|\delta_{\Lambda\Lambda_0}\delta_x\|^2 = \int_{\mathbb{R}^4} \mu(\Lambda_0 dp)$. Obviously, the translations $a \in \mathbb{R}^4$ are unitarily implemented on $\mathfrak{f}$, the corresponding strongly continuous unitary representation (by multiplication operators: multiplication with $\exp(i\langle a, p\rangle)$) is denoted by $a \to V_a$; spec V_a = supp μ.

The CCR-Weyl-algebra, corresponding to $H_{\mathbb{R}^4}$, is denoted by $\mathscr{W}(H_{\mathbb{R}^4})$. The generating Weyl elements are denoted by $W(\nu)$, $\nu \in H_{\mathbb{R}^4}$, such that the relations $W(0) = 1$, $W(\nu)^* = W(-\nu)$ and

$$W(\nu_1)W(\nu_2) = \exp\left(-\frac{i}{2}\text{Im}(\nu_1, \nu_2)\right)W(\nu_1 + \nu_2)$$

are valid. The localization is as usual, $\mathscr{B}(\mathscr{M}) := \mathscr{W}(W(\nu), \text{supp}\, \nu \subset \mathscr{M})$, where supp $\nu := \bigcup_{\Lambda\in\mathscr{G}} \text{supp}\, \nu(., \Lambda)$.

Now by $M(\mathbb{Z}^4) \subset M(\mathbb{R}^4)$ we denote the set of all measures $\nu \in M(\mathbb{R}^4)$

such that $\operatorname{supp} \nu \subset \mathbb{Z}^4$, i. e. $\nu = \sum_{x \in \mathbb{Z}^4} f(x)\delta_x$, where $f \in \mathrm{L_{fin}}(\mathbb{Z}^4)$. Then the linear space of the Fourier transforms $\hat{\mathrm{M}}(\mathbb{Z}^4)$ is given by

$$\hat{\mathrm{M}}(\mathbb{Z}^4) = \operatorname{spa}\{ \exp(\mathrm{i}\langle x, p \rangle), x \in \mathbb{Z}^4 \},$$

the set of all trigonometric polynomials, periodic modulo $2\pi\mathbb{Z}^4$. We put $\mathrm{H}_0 := \mathrm{L_{fin}}(\mathcal{G}, \mathrm{M}(\mathbb{Z}^4))$, $\mathrm{H}_0 \subset \mathrm{H}$. That is, H_0 consists of all $f \in \mathrm{H}$ such that $\operatorname{supp} f \subset \mathbb{Z}^4$. The *subphase space* $\mathrm{H}_{\mathbb{Z}^4} \subset \mathrm{H}_{\mathbb{R}^4}$ is defined to be

$$\mathrm{H}_{\mathbb{Z}^4} := \mathrm{H}_0 \bmod \ker \|\cdot\| .$$

Note that if $\operatorname{supp} \mu$ is « large enough », so that the system of functions $\{ \exp(\mathrm{i}\langle p, x \rangle) \}_{x \in \mathbb{Z}^4}$ is linearly independent on $\operatorname{supp} \mu_\Lambda$ for all $\Lambda \in \mathcal{G}$, then for H_0 no factorization is needed (example: choose $\operatorname{supp} \mu = \mathrm{H}_m$, $m \geq 0$, where $\mathrm{H}_m = \{ p : p \in \mathbb{R}^4, p_0 = (m^2 + |\mathfrak{p}|^2)^{1/2} \}$ denotes the mass hyperboloid). Note that the Fourier transforms of members of $\mathrm{H}_{\mathbb{Z}^4}$ are (periodic) trigonometric polynomials with coefficients from $\mathrm{L_{fin}}(\mathcal{G})$.

The group $\mathcal{P}$ can be implemented on H_0 by the following action: the element $g = \{ a, \Lambda \} \in \mathcal{P}$ induces the transformation

$$f(x, \Lambda') \rightarrow f(\Lambda^{-1}(x - a), \Lambda'\Lambda)$$

on H_0, which leaves (f_1, f_2) invariant. So one obtains a unitary representation $\mathcal{P} \ni g \rightarrow \mathrm{V}_g$ of $\mathcal{P}$ on $\mathrm{H}_{\mathbb{Z}^4}$ and obviously the induced subrepresentation of $\mathbb{Z}^4$ is the restriction of the unitary representation of all translations (defined above) to $\mathbb{Z}^4$ and to $\mathrm{H}_{\mathbb{Z}^4}$.

Note that in general $\operatorname{clo}_{\|.\|} \mathrm{H}_{\mathbb{Z}^4} \subset \mathfrak{f}$ is a proper subspace $\mathfrak{h}$ of $\mathfrak{f}$ and $\mathfrak{h}$ is invariant only with respect to the discrete translations $a \in \mathbb{Z}^4$. However there are cases $\mathfrak{h} = \mathfrak{f}$.

LEMMA 1. — *If* $\operatorname{supp} \mu = \mathrm{H}_m$, $m \geq 0$, *and if* μ *is absolutely continuous with respect to the Lorentz invariant measure* ν_m *on* H_m *then* $\mathrm{H}_{\mathbb{Z}^4}$ *is dense in* $\mathrm{H}_{\mathbb{R}^4}$, *i. e.* $\mathfrak{h} = \mathfrak{f}$.

Proof. — Let $f \in \mathrm{H}_{\mathbb{Z}^4}$, then $\hat{f}(., \Lambda)$ is a periodic trigonometric polynomial. We define $\hat{\mu}(\mathrm{d}p) := \Sigma_{k \in \mathbb{Z}^4} \mu(\mathrm{d}p + 2k\pi)$ as a normalized Borel measure on $\mathbb{T}^4 := \mathbb{R}^4 \bmod 2\pi\mathbb{Z}^4$. Since $\Lambda \in \mathcal{G}$ acts as a diffeomorphism on $\mathbb{T}^4$, also $\hat{\mu}(\Lambda \mathrm{d}p)$ is a measure on $\mathbb{T}^4$. By calculation one obtains

$$\int_{\mathbb{R}^4} \| \hat{f}(p, \Lambda) \|^2 \mu(\Lambda \mathrm{d}p) = \int_{\mathbb{T}^4} \| \hat{f}(p, \Lambda) \|^2 \hat{\mu}(\Lambda \mathrm{d}p) .$$

Now the supports of $\mu(\mathrm{d}p + 2k\pi)$, $k \in \mathbb{Z}^4$, are mutually disjoint (up to a 2-dimensional submanifold, which is a ν_m-zero set). Let $f \in \mathrm{H}_{\mathbb{R}^4}$. Then $\hat{f}(., \Lambda)$ is a bounded C^∞-function. If we consider $\hat{f}(., \Lambda)$ on $\operatorname{supp} \mu_{\Lambda^{-1}}$, it can essentially be « reproduced » in $\mathbb{T}^4$ (on $\operatorname{supp} \hat{\mu}_{\Lambda^{-1}}$) and this function can be approximated in $L^2(\mathbb{T}^4, d\hat{\mu}_{\Lambda^{-1}})$ by a periodic trigonometric polynomial, which is an approximation for f in $L^2(\mathbb{R}^4, d\mu_{\Lambda^{-1}})$, too. ■

REMARK 1. — The CCR-algebra $\mathscr{W}(H_{\mathbb{Z}^4})$, corresponding to $H_{\mathbb{Z}^4}$, satisfies $\mathscr{W}(H_{\mathbb{Z}^4}) \subset \mathscr{W}(H_{\mathbb{R}^4})$. By localization one obtains the net

$$\mathscr{A}(\mathcal{O}) := \mathscr{W}(W(f), f \in H_{\mathbb{Z}^4}, \operatorname{supp} f \subset \mathcal{O}),$$

where $\mathcal{O} \subset \mathbb{Z}^4$, $\mathcal{O}$ finite. Then one can formulate

PROPOSITION 1. — *The net $\mathscr{A}(\cdot)$ over $\mathbb{Z}^4$ is admissible and $\mathscr{B}(\cdot)$ over $\mathbb{R}^4$ is the associated net.*

Proof. — Both nets are isotone and additive. $\mathscr{B}(\cdot)$ is $\mathbb{R}^4$-covariant: the translations $a \in \mathbb{R}^4$ are unitarily implemented on $H_{\mathbb{R}^4}$ by $a \to V_a$. To V_a there corresponds uniquely the so-called Bogoljubov automorphism α_{V_a} of $\mathscr{B}$ with the defining property $W(V_a f) = \alpha_{V_a} W(f)$ (see for example Bratteli/Robinson [*4*]). But $W(V_a f) \in \mathscr{A}(a\mathcal{O})$ iff $\operatorname{supp} V_a f \subset a\mathcal{O}$ iff $\operatorname{supp} f \subset \mathcal{O}$ iff $W(f) \in \mathscr{A}(\mathcal{O})$. Therefore $\mathscr{A}(a\mathcal{O}) = \alpha_{V_a}\mathscr{A}(\mathcal{O})$. Furthermore, $\mathscr{P}$ is unitarily implemented on $H_{\mathbb{Z}^4}$ by $\mathscr{P} \ni g \to V_g$. Thus similar arguments yield: $\mathscr{A}(\cdot)$ is $\mathscr{P}$-covariant. Moreover, if $a \in \mathbb{Z}^4$ then $V_a \upharpoonright H_{\mathbb{Z}^4}$ and $V_{\{a,1\}}$ coincide. Concerning the existence of a state ω with positivity condition choose ω to be the Fock state on $\mathscr{B}$, which is uniquely determined by $\omega(W(f)) = \exp\left(-\frac{1}{4}\| f \|^2\right)$, $f \in H_{\mathbb{R}^4}$. The (strongly continuous) unitary representation $\mathbb{R}^4 \ni a \to U_a$ on the corresponding (GNS-)Fock space, defined by the GNS-representation of α_{V_a}, has its spectrum in $\operatorname{clo} V_+$, since on $\mathfrak{f}$ the representation U_a is given by $(U_a f)\hat{}(p, \Lambda) = \exp(i\langle a, p\rangle)\hat{f}(p, \Lambda)$, $f \in \mathfrak{f}$, and $\operatorname{supp} \mu \subseteq \operatorname{clo} V_+$. If one reduces the state ω to $\mathscr{A}$, one obtains invariance with respect to $\mathscr{P}$, since $\| V_g f \|^2 = \| f \|^2$, $f \in H_{\mathbb{Z}^4}$ (recall that $\mathscr{P}$ acts unitarily on $H_{\mathbb{Z}^4}$), so $\omega \upharpoonright \mathscr{A}$ (Fock state on $\mathscr{A}$) is $\mathscr{P}$-invariant. ■

REMARK 2. — We consider the case that the starting measure μ is quasi-invariant with respect to $\mathscr{G}$, i. e. μ_Λ is equivalent to μ for all $\Lambda \in \mathscr{G}$ (i. e. μ and μ_Λ are mutually absolutely continuous). This is no essential loss of generality, because if a Borel measure μ with $\operatorname{supp} \mu \subseteq \operatorname{clo} V_+$, $\mu(\operatorname{clo} V_+) = 1$ is given, then obviously $\hat{\mu} := \Sigma_{\Lambda \in \mathscr{G}} \alpha_\Lambda \mu_\Lambda$, $\sum_{\Lambda \in \mathscr{G}} \alpha_\Lambda = 1$, $\alpha_\Lambda > 0$, is from the same class and $\hat{\mu}$ is quasi-invariant. So we obtain in this case

$$\mu_\Lambda(dp) = \rho_\Lambda(p)\mu(dp), \qquad \rho_\Lambda(p) \geq 0,$$

$\rho_\Lambda(p) > 0$ a. e. with respect to μ. ρ_Λ is a cocycle, i. e.

$$\rho_{\Lambda_1\Lambda_2}(p) = \rho_{\Lambda_1}(p)\rho_{\Lambda_2}(\Lambda_1^{-1}p).$$

In this case $\mathfrak{f}$ is isometrically isomorphic to $l^2(\mathscr{G}) \otimes \mathscr{K}$, where $\mathscr{K} = \mathrm{L}^2(\mathbb{R}^4, d\mu)$ via the isomorphism Φ

$$\mathfrak{f} \ni f \to \Phi f : (\Phi f)(p, \Lambda) = \rho_{\Lambda^{-1}}(p)^{1/2} f(p, \Lambda).$$

Φ transforms the representation V_g for $g = \{0, \Lambda\}$ as follows:

$$\mathrm{V}_{\{0,\Lambda\}} = \Phi^{-1}(\mathrm{R}_\Lambda \otimes \tilde{\mathrm{V}}_\Lambda)\Phi,$$

where R_Λ denotes the pure right shift on $l^2(\mathscr{G})$, i. e. $(\mathrm{R}_\Lambda g)(\Lambda') = g(\Lambda'\Lambda)$, $g \in l^2(\mathscr{G})$ and $\tilde{\mathrm{V}}_\Lambda$ is the following unitary representation on $\mathscr{K}$:

$$\mathscr{K} \ni f \to \tilde{\mathrm{V}}_\Lambda f : (\tilde{\mathrm{V}}_\Lambda f)(p) = \rho_\Lambda(p)^{1/2} f(\Lambda^{-1}p), \tag{1}$$

i. e. up to the factor $\rho_\Lambda(p)^{1/2}$, $\tilde{\mathrm{V}}_\Lambda$ acts pure « geometrically » on the functions.

In particular, if one chooses a measure μ which is equivalent to a Lorentz invariant measure ν (supp $\nu \subseteq$ clo V_+), i. e. $\mu(dp) = \sigma(p)\nu(dp)$ with $\sigma(p) > 0$ a. e. with respect to ν and $\int_{\mathbb{R}^4} \sigma(p)\nu(dp) = 1$, then μ is quasi-invariant and one calculates

$$\rho_\Lambda(p) = \sigma(\Lambda^{-1}p)/\sigma(p). \tag{2}$$

Then $\mathfrak{f}$ is isometrically isomorphic to $l^2(\mathscr{G}) \otimes \mathrm{L}^2(\mathbb{R}^4, d\nu)$ and the corresponding isomorphism Ψ is defined by

$$(\Psi f)(p, \Lambda) = \rho_{\Lambda^{-1}}(p)^{1/2}\sigma(p)^{1/2} f(p, \Lambda) = \sigma(\Lambda p)^{1/2} f(p, \Lambda),$$

where
$$\Psi = \Phi_0 \Phi \quad \text{and}$$
$$\mathscr{K} \ni g \to \Phi_0 g \in \mathrm{L}^2(\mathbb{R}^4, d\nu), (\Phi_0 g)(p) = \sigma(p)^{1/2} g(p).$$

Shifting $\tilde{\mathrm{V}}_\Lambda$, defined on $\mathscr{K}$, to $\mathrm{L}^2(\mathbb{R}^4, d\nu)$, one obtains $\tilde{\tilde{\mathrm{V}}}_\Lambda = \Phi_0 \tilde{\mathrm{V}}_\Lambda \Phi_0^{-1}$ and from (1)

$$(\tilde{\tilde{\mathrm{V}}}_\Lambda f)(p) = \sigma(p)^{1/2} \rho_\Lambda(p)^{1/2} \sigma(\Lambda^{-1}p)^{-1/2} f(\Lambda^{-1}p)$$

follows. Using (2), one obtains

$$(\tilde{\tilde{\mathrm{V}}}_\Lambda f)(p) = f(\Lambda^{-1}p),$$

so that in this case the corresponding representation on $\mathrm{L}^2(\mathbb{R}^4, d\nu)$ is simply given by the geometric action of Λ.

§ 4. CAUSALITY

Now we take into account the causality of admissible nets (see Definition 2). As in § 3, by $\mathscr{A}(\cdot)$ we denote the admissible net of Proposition 1, defined in Remark 1 by localization of $\mathscr{W}(\mathrm{H}_{\mathbb{Z}^4})$.

LEMMA 2. — *If*

$$\int_{\mathbb{R}^4} \sin \langle x, p \rangle \mu(\mathrm{d}p) = 0, \; x \in \mathbb{Z}^4, \; x_0^2 - |\mathfrak{x}|^2 < 0, \tag{3}$$

then the admissible net $\mathscr{A}(\cdot)$ *is causal.*

Proof. — The algebra $\mathscr{A}$ is a CCR-algebra. Therefore it is sufficient to prove that $\operatorname{supp} f \perp \operatorname{supp} g$, $f, g \in \mathrm{H}_{\mathbb{Z}^4}$, implies $\operatorname{Im}(f, g) = 0$. Note that $\operatorname{supp} f \perp \operatorname{supp} g$ implies $\operatorname{supp} f(., \Lambda) \perp \operatorname{supp} g(., \Lambda)$ for all $\Lambda \in \mathscr{G}$. It is no loss of generality to choose only members f, g of $\mathrm{H}_{\mathbb{Z}^4}$ with real coefficients $f(x, \Lambda)$, $g(x, \Lambda)$, $f = \sum_x f(x, \cdot)\delta_x$, $g = \sum_x g(x, \cdot)\delta_x$. One calculates

$$\begin{aligned}
\operatorname{Im}(f, g) &= \frac{1}{2\mathrm{i}}((f, g) - (g, f)) \\
&= \sum_\Lambda \sum_{x,y} f(x, \Lambda) g(y, \Lambda) \int_{\mathbb{R}^4} \sin \langle p, \Lambda(y - x) \rangle \mu(\mathrm{d}p) \\
&= \sum_\Lambda \sum_{\langle y-x, y-x \rangle \geq 0} f(x, \Lambda) g(y, \Lambda) \int_{\mathbb{R}^4} \sin \langle p, \Lambda(y - x) \rangle \mu(\mathrm{d}p) = 0,
\end{aligned}$$

since $f(x, \Lambda) \cdot g(y, \Lambda) = 0$ for $\langle y - x, y - x \rangle \geq 0$. ■

REMARK 3. — The existence of positive Borel measures μ with $\operatorname{supp} \mu \subseteq \operatorname{clo} \mathrm{V}_+$, $\mu(\operatorname{clo} \mathrm{V}_+) = 1$, equipped with property (3), can be easily shown. For example, start with a closed subset $\mathrm{B} \subset \operatorname{clo} \mathrm{V}_+$, invariant with respect to all Lorentz transformations (union of hyperboloids H_m, $m \in [m_1, m_2]$, $0 \leq m_1 \leq m_2 < \infty$) and consider the linear manifold $\mathscr{T} := \operatorname{spa} \{ \sin \langle x, p \rangle, p \in \mathrm{B}, x_0^2 - |\mathfrak{x}|^2 < 0 \}$. Let $\mathrm{C}_b(\mathrm{B})$ be the C*-algebra of all bounded continuous functions g on B with $\| g \| := \sup_{p \in \mathrm{B}} | g(p) |$. Let e be the identity in $\mathrm{C}_b(\mathrm{B})$, $e(p) \equiv 1$. Take a bounded closed subset $\mathrm{B}_0 \subset \mathrm{B}$, for example $\mathrm{B}_0 := \{ p \in \mathrm{B}, \; p = \{ p_0, \mathfrak{p} \}, \; |\mathfrak{p}| \leqslant \mathrm{R}, \; p_0 = (m^2 + |\mathfrak{p}|^2)^{1/2} \}$ where $m \in [m_1, m_2]$ is fixed. Then it is sufficient to construct a positive Borel measure μ_0 on B_0 with $\mu_0(\mathrm{B}_0) = 1$, $\int_{\mathrm{B}_0} t(p) \mu_0(\mathrm{d}p) = 0$ for all $t \in \mathscr{T}$ and to extend μ_0 to $\operatorname{clo} \mathrm{V}_+$ by zero. Now choose $k \in \mathbb{N}$ such that $k^2\pi^2 > m^2$ and take $\mathrm{R} := 2(k^2\pi^2 - m^2)^{1/2}$. Then the points $\{ (m^2 + |\mathfrak{p}|^2)^{1/2}, \mathfrak{p} \}$ with $|\mathfrak{p}| \leqslant (k^2\pi^2 - m^2)^{1/2}$ are members of B_0. Take an arbitrary $t \in \mathscr{T}$, $t(p) = \sum_x a_x \sin \langle x, p \rangle$ (finite sum), it is sufficient to consider real coefficients a_x. Then $t(\{ p_0, \mathfrak{p} \}) + t(\{ p_0, -\mathfrak{p} \}) = 2 \sum_x a_x \sin(x_0 p_0) \cos(\mathfrak{x}, \mathfrak{p})$, where $x = \{ x_0, \mathfrak{x} \}$, $p_0 = (m^2 + |\mathfrak{p}|^2)^{1/2}$. For points $\mathfrak{p}$ from the sphere $|\mathfrak{p}| = (k^2\pi^2 - m^2)^{1/2}$ one obtains immediately

$$t(\{ p_0, \mathfrak{p} \}) + t(\{ p_0, -\mathfrak{p} \}) = 2 \sum_x a_x \sin(x_0 k\pi) \cos(\mathfrak{x}, \mathfrak{p}) = 0$$

because of $\sin(x_0 k\pi) = 0$ for all x_0 (a finite subset of $\mathbb{Z}$). So one obtains $t(\{ p_0, \mathfrak{p} \}) = - t(\{ p_0, -\mathfrak{p} \})$. That is, one has either $t(\{ p_0, \mathfrak{p} \}) = 0$ or on the linear segment between $\mathfrak{p}$ and $-\mathfrak{p}$ (which belongs to $|\mathfrak{p}| \leqslant R$), resp. on the corresponding trajectory $\{ (m^2 + |\mathfrak{p}'|^2)^{1/2}, \mathfrak{p}' \}$, given by this segment, there is at least one point $q = \{ (m^2 + |\mathfrak{q}|^2)^{1/2}, \mathfrak{q} \}$ such that $t(q) = 0$. But one has $q \in B_0$. Thus every $t \in \mathcal{T}$ has at least one zero within B_0. So, with with respect to $C(B_0)$, one obtains

$$\operatorname{dis}(e, \operatorname{clo}_{\|.\|} \mathcal{T}) = \operatorname{dis}(e, \mathcal{T}) = \inf_{t \in \mathcal{T}} \sup_{p \in B_0} |1 - t(p) = 1 .$$

Hence the (positive) linear form l_0 on $\mathbb{R}e + \operatorname{clo}_{\|.\|} \mathcal{T}$ defined by $l_0(\alpha e + t) := \alpha$ has the property $\| l_0 \| = \sup_{\|\alpha e + t\| \leqslant 1} | l_0(\alpha e + t) | = \sup_{\|\alpha e + t\| \leqslant 1} |\alpha| = 1$. Then the Hahn-Banach extension theorem yields an extension l of l_0 to the whole space $C_b(B_0)$ such that $\| l \| = \| l_0 \|$, i. e. $\| l \| = 1$. But $l(e) = l_0(e) = 1$, i. e. $\| l \| = l(e)$, i. e. l is positive and the corresponding positive Borel measure μ_0 satisfies $\emptyset \subset \operatorname{supp} \mu_0 \subseteq B_0$, $\mu_0(B_0) = 1$. Extending μ_0 to B by zero one gets a positive Borel measure μ satisfying $\operatorname{supp} \mu \subseteq \operatorname{clo} V_+$, $\emptyset \subset \operatorname{supp} \mu$, $\mu(\operatorname{clo} V_+) = 1$ and condition (3).

Then one obtains that μ_Λ satisfies the same conditions, since

$$\int_{\mathbb{R}^4} \sin \langle x, p \rangle \mu_\Lambda(dp) = \int_{\mathbb{R}^4} \sin \langle x, p \rangle \mu(\Lambda^{-1} dp)$$
$$= \int_{\mathbb{R}^4} \sin \langle x, \Lambda q \rangle \mu(dq) = \int_{\mathbb{R}^4} \sin \langle \Lambda^{-1} x, p \rangle \mu(dp) .$$

Hence the measure $\hat{\mu} = \sum_{\Lambda \in \mathcal{G}} \alpha_\Lambda \mu_\Lambda$, $\Sigma \alpha_\Lambda = 1$, $\alpha_\Lambda > 0$, which is quasi-invariant with respect to $\mathcal{G}$, also satisfies the mentioned properties.

A more detailed inspection of $\mathcal{T}$ leads to stronger results. For example, in the case $B = H_m$, $m \geq 0$, one can prove that there is a corresponding measure with unbounded support.

PROPOSITION 2. — *Let* $B = H_m$, $m \geq 0$. *There is a positive Borel measure* μ *with* $\mu(H_m) = 1$ *and with unbounded support* supp μ, *which satisfies condition* (3).

For a proof of this result see [5].

REMARK 4. — An important question refers to the existence of positive measures in the case $B = H_m$, which are absolutely continuous with respect to ν (for example, cf. Lemma 1). If there is such a measure then the proof of Proposition 2 can be modified and the existence of a corresponding absolutely continuous measure with unbounded support supp μ can be proved.

At first glance it seems that one can construct such measures explicitly with the help of the Pauli-Jordan function

$$\int_{\mathbb{R}^4} \exp\{-\mathrm{i}\langle x, p\rangle\}\, \nu_m(\mathrm{d}p), \quad m \geq 0, \tag{4}$$

of the free scalar field as follows: Let $\alpha \in C_0^\infty(\mathbb{R}^4)$ and $\operatorname{supp}\alpha \subset K$, where $K = \{x : |x| < \varepsilon\}$ is a small ball at the origin and choose the measure $\mu(\mathrm{d}p) := |\hat{\alpha}(p)|^2 \nu_m(\mathrm{d}p)$, which is finite and absolutely continuous with respect to ν_m. Moreover one has supp $\mu = H_m$. From the support properties of (4) one obtains immediately

$$\int_{\mathbb{R}^4} \sin\langle a, p\rangle\, |\hat{\alpha}(p)|^2 \nu_m(\mathrm{d}p) = 0, \text{ a spacelike, } a \in \mathbb{Z}^4,$$

provided that K and $K + a$ are causally disjoint. However, there are spacelike lattice points $a \in \mathbb{Z}^4$, which have arbitrarily small distance to the light cone $x_0^2 = |\mathfrak{x}|^2$ in $\mathbb{R}^4$. Therefore, there are always many spacelike lattice points $a \in \mathbb{Z}^4$ such that $(y + a) - x = (y - x) + a$ fails to be spacelike for all $x, y \in K$.

§ 5. PERTURBATION AND SCATTERING THEORY

In this § we restrict to the case that supp $\mu = H_m$, $m \geq 0$, and that μ is absolutely continuous with respect to ν_m, i. e. $\mu(\mathrm{d}p) = \sigma(p)\nu_m(\mathrm{d}p)$, $\sigma(p) > 0$ a. e. on H_m. Note that the following considerations work without the causality property of the admissible net, i. e. property (3) is not necessarily required.

Then, according to Lemma 1, $H_{\mathbb{Z}^4}$ is dense in $H_{\mathbb{R}^4}$, i. e.

$$\mathfrak{f} = \mathrm{clo}_{\|.\|}\, H_{\mathbb{R}^4} = \mathrm{clo}_{\|.\|}\, H_{\mathbb{Z}^4}.$$

Moreover, according to Remark 2, $\mathfrak{f}$ is isometrically isomorphic to $l^2(\mathscr{G}) \otimes \mathscr{K}$, where $\mathscr{K} := L^2(\mathbb{R}^4, \mathrm{d}\nu_m)$ and the isomorphism Φ considered in Remark 2 has the following properties:

$$V_a = \Phi^{-1}(1_{l^2(\mathscr{G})} \otimes \tilde{V}_a)\Phi,\ (\tilde{V}_a\varphi)(p) := \exp(\mathrm{i}\langle a, p\rangle)\varphi(p),\ \varphi \in \mathscr{K},\ a \in \mathbb{R}^4,$$

$$V_{\{0,\Lambda\}} = \Phi^{-1}(R_\Lambda \otimes \tilde{V}_{\{0,\Lambda\}})\Phi,\ (R_\Lambda g)(\Lambda') := g(\Lambda'\Lambda),\ g \in l^2(\mathscr{G}),$$

$$(\tilde{V}_{\{0,\Lambda\}}\varphi)(p) := \varphi(\Lambda^{-1}p),\ \varphi \in \mathscr{K},\ \Lambda \in \mathscr{G},$$

such that $\tilde{V}_{\{a,\Lambda\}} := \tilde{V}_a \tilde{V}_{\{0,\Lambda\}}$ turns out to be the restriction of the irreducible representation of SL(2, $\mathbb{C}$) of type $s = 0$, $m \geq 0$, $+$, to $\mathscr{P}$ if $a \in \mathbb{Z}^4$.

Now we choose the Fock state ω defined by $\omega(W(v)) := \exp\left(-\frac{1}{4}\|v\|^2\right)$, $v \in H_{\mathbb{R}^4}$. Then one has the Fock space $\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}_n$, $\mathscr{F}_0 = \mathbb{C}\Omega$ (Ω Fock vacuum), $\mathscr{F}_n = S_n(\underbrace{\mathfrak{f} \otimes \mathfrak{f} \otimes \ldots \otimes \mathfrak{f}}_{n \text{ times}})$ and the Fock representation π of $\mathscr{A} = \mathscr{W}(H_{\mathbb{Z}^4})$ resp. $\mathscr{B} = \mathscr{W}(H_{\mathbb{R}^4})$ on $\mathscr{F}$, where each $\pi(W(v))$, $v \in H_{\mathbb{R}^4}$, can be strongly approximated by the $\pi(W(f))$, $f \in H_{\mathbb{Z}^4}$. One has the unitary implementations $\Gamma(V_a)$ and $\Gamma(V_g)$ of V_a, V_g on $\mathscr{F}$. Note that the n-particle spaces $\mathscr{F}_n$ are invariant subspaces, i. e. one can form $\Gamma(V_a) \upharpoonright \mathscr{F}_n$, $\Gamma(V_g) \upharpoonright \mathscr{F}_n$ and one obtains

$$\Gamma(V_a) \upharpoonright \mathscr{F}_n = \Phi_n^{-1}(1_{S_n l^2(\mathscr{G})^n} \otimes \bigotimes_{n \text{ times}} \tilde{V}_a)\Phi_n, \tag{5}$$

$$\Gamma(V_g) \upharpoonright \mathscr{F}_n = \Phi_n^{-1}(\bigotimes_{n \text{ times}} R_\Lambda \otimes \bigotimes_{n \text{ times}} \tilde{V}_g)\Phi_n, \tag{6}$$

where $S_n l^2(\mathscr{G})^n := S_n(\underbrace{l^2(\mathscr{G}) \otimes l^2(\mathscr{G}) \otimes \ldots \otimes}_{n \text{ times}} l^2(\mathscr{G}))$ and where Φ_n denotes the isomorphism induced by Φ. Moreover, one has the usual covariance formulas

$$\pi(W(V_a v)) = \Gamma(V_a)\pi(W(v))\Gamma(V_a)^{-1}, \quad v \in H_{\mathbb{R}^4}, a \in \mathbb{R}^4,$$
$$\pi(W(V_g f)) = \Gamma(V_g)\pi(W(f))\Gamma(V_g)^{-1}, f \in H_{\mathbb{Z}^4}, g \in \mathscr{P},$$

and corresponding transformation formulas for $\pi(\mathscr{A}(g\mathscr{O}))$ resp. $\pi(\mathscr{B}(a\mathscr{O}))$ such that, for example,

$$\Gamma(V_{\{0,\Lambda\}})\pi(\mathscr{A}(\{0\}))\Gamma(V_{\{0,\Lambda\}})^{-1} = \pi(\mathscr{A}(\{0\})),$$

i. e. the algebra $\pi(\mathscr{A}(\{0\}))$ at the origin is invariant with respect to the group $\mathscr{G}$. We put $\mathscr{N}(\mathscr{O}) := \text{clo}_w \pi(\mathscr{A}(\mathscr{O}))$ (clo_w means weak closure) and consider perturbed algebras $\hat{\mathscr{N}}(\{0\})$, defined by an operator $C \in \mathscr{L}(\mathscr{F})$ and by

$$\hat{\mathscr{N}}(\{0\}) := \text{clo}_w \{1, C^*NC, N \in \mathscr{N}(\{0\})\}. \tag{7}$$

Obviously, if $C\Gamma(V_{\{0,\Lambda\}}) = \Gamma(V_{\{0,\Lambda\}})C$ for all $\Lambda \in \mathscr{G}$, then $\hat{\mathscr{N}}(\{0\})$ generates (via translational covariance) a $\mathscr{P}$-covariant net again. Now the formulas (5), (6) suggest the application of methods and results concerning the existence of time-asymptotic constants, commuting with Lorentz transformations, pointed out in [*1*]. We do not repeat these considerations. The application of these methods yields the following results.

LEMMA 3. — *Let* $\Gamma(V_{\{0,\Lambda\}})$, $\Gamma(V_a)$, $\Lambda \in \mathscr{G}$, $a \in \mathbb{R}^4$, *be defined on* $\mathscr{F}$ *as before. Then there exists an operator* A *and a dense linear manifold* $\mathscr{D} \subset \mathscr{F}_{\text{fin}}$

(the finite particle vectors in $\mathscr{F}$), invariant with respect to $\Gamma(V_{\{0,\Lambda\}})$, $\Gamma(V_a)$ *such that* $A \in \mathscr{L}(\mathscr{F})$, A *selfadjoint*, $AE = EA$ *(E the projection onto* $\bigoplus_{n=2}^{\infty} \mathscr{F}_n$*)*, $A\Gamma(V_{\{0,\Lambda\}}) = \Gamma(V_{\{0,\Lambda\}})A$, $\underset{t \to \pm\infty}{s\text{-lim}}\, \Gamma(V_{(-t,0)})A\Gamma(V_{(t,0)}) := A_\pm$, $A_+ = E$, $A_- = 0$, $\| (A - A_\pm)\Gamma(V_a)u \| \leqslant c^{\pm}_{m,u} \langle a, a \rangle^{-m}$, $m = 1, 2, \ldots, \langle a, a \rangle > 1$, $\pm\, t > 1$, $u \in \mathscr{D}$.

Proof. — Use the ansätze $A := 0 \oplus 0 \oplus A_2 \oplus A_3 \oplus \ldots$, $A_n := \Phi_n^{-1} B_n \Phi_n$, $B_n := 1_{S_n l^2(\mathscr{G})^n} \otimes C_n$, $C_n \in \mathscr{L}(S_n L^2(\mathbb{R}^{4n}, \underbrace{\otimes}_{n \text{ times}} dv_m)$ and choose C_n according to Proposition 4 of [*1*]. ■

Now one considers operators S on $\mathscr{F}$, as candidates for scattering operators, with the properties

I) S unitary,

II) $S \restriction \mathscr{F}_0 \oplus \mathscr{F}_1 = 1$,

III) $S\Gamma(V_a) = \Gamma(V_a)S$, $a \in \mathbb{R}^4$, $S\Gamma(V_{\{0,\Lambda\}}) = \Gamma(V_{\{0,\Lambda\}})S$, $\Lambda \in \mathscr{G}$,

and one obtains, similarly as in [*1*],

PROPOSITION 3. — *If the operator* S *satisfies (I)-(III) then there is an operator* $C \in \mathscr{L}(\mathscr{F})$ *and a dense linear manifold* $\mathscr{D}_c \subset \mathscr{F}$ *such that for each generating element* C^*NC *of* $\hat{\mathscr{N}}(\{0\})$ *(defined by* (7)*) the LSZ-scattering process is convergent with respect to* $\mathscr{D}_C$ *and such that the scattering operator coincides with* S.

Proof. — Choose $C := S^{1/2}AS^{1/2} + S^{-1/2}(1 - A)S^{1/2}$, $\mathscr{D}_C = S^{-1/2}\mathscr{D}$ $\left(S^{1/2} := \exp\left(\frac{i}{2} B\right), \text{ if } S = \exp(iB), \text{ B bounded selfadjoint} \right)$, where A, $\mathscr{D}$ are chosen according to Lemma 3 and proceed similarly as in [*1*]. ■

REMARK 5. — See [*1*] for details. It is also possible to include CPT-invariance in this framework. Furthermore, for special operators S one can obtain stronger results, using the existence of *unitary* operators C with the properties of Proposition 3 in these cases (for example, see Wollenberg [*6*]). Then $\hat{\mathscr{N}}(\{0\}) = C^*\mathscr{N}(\{0\})C$ and the LSZ-scattering process is convergent not only for the generating elements of the perturbed algebra $\hat{\mathscr{N}}(\{0\})$ but for all its elements.

REFERENCES

[*1*] H. BAUMGÄRTEL, M. WOLLENBERG, A Class of Nontrivial Weakly Local Massive Wightman Fields with Interpolating Properties. *Comm. Math. Phys.*, t. **94**, 1984, p. 331-352.

[*2*] S. DOPLICHER, An algebraic spectrum condition. *Comm. Math. Phys.*, t. **1**, 1965, p. 1-5.