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An explicit determination of the non-self-adjoint wave equations on curved space-time that satisfy Huygens' principle. Part I: Petrov type N background space-times

by

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ABSTRACT. — It is shown that the validity of Huygens' principle for the non-self-adjoint wave equation on a general Petrov type N space-time implies that the equation is equivalent to the conformally invariant scalar wave equation on the exact plane wave space-time. This result solves Hadamard's problem for this class of equations since it is known that the latter equation is the only self-adjoint Huygens' equation on type N space-times.

RÉSUMÉ. — On démontre que la validité du principe de Huygens pour l'équation des ondes scalaires non-auto-adjointe sur un espace-temps général de type N de Petrov implique que l'équation est équivalente à l'équation invariante conforme des ondes scalaires sur l'espace-temps des ondes planes. Ce résultat résout le problème de Hadamard pour cette classe des équations puisqu'on sait que la dernière équation est la seule de type Huygens sur les espaces-temps de type N.

1. INTRODUCTION

In a recent series of papers Carminati and McLenaghan [*3*] [*4*] [*5*] (referred to as CM1, CM2 and CM3 respectively in the sequel) have undertaken a programme [2] for the solution of Hadamard's problem for the conformally invariant scalar wave equation, Maxwell's equations, and Weyl's neutrino equation on curved space-time based on the conformally invariant Petrov classification [19] of the Weyl tensor. To date the Petrov types N, D, and III have been considered. The present paper is the first of a series whose purpose is the extension of the above analysis to the general non-self-adjoint scalar wave equation which may be written in coordinate invariant form as

$$\mathrm{F}[u] := \Box u + \mathrm{A}^a \partial_a u + \mathrm{C} u = 0 . \tag{1.1}$$

In the above equation $\Box$ denotes the Laplace-Beltrami operator corresponding to the metric g_{ab} of the background space-time V_4, u the unknown scalar function, A^a the components of a given contravariant vector field, and C a given function. The metric tensor g_{ab}, background space-time V_4, vector field A^a and scalar function C are assumed to be of class C^∞. All considerations in this paper are entirely local.

According to Hadamard [9] *Huygens' principle* (in the strict sense) holds for Eq. (1.1) if and only if for every Cauchy initial value problem and for every $x_0 \in \mathrm{V}_4$, the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of $\mathrm{S} \cap \mathrm{C}^-(x_0)$ where S denotes the initial surface and $\mathrm{C}^-(x_0)$ the past null conoid from x_0. Such an equation is called a *Huygens' differential equation. Hadamard's problem* is that of determining up to equivalence all the Huygens' differential equations. We recall that two equations of the form (1.1) are *equivalent* if and only if one may be transformed into the other by any combination of the following *trivial transformations* which preserve the Huygens' character of the equation.

a) a general coordinate transformation,

b) multiplication of the equation by the function $\exp(-2\phi(x))$, which induces a conformal transformation of the metric:

$$\tilde{g}_{ab} = e^{2\phi} g_{ab} \tag{1.2}$$

c) substitution of λu for the unknown function u, where λ is a non-vanishing function on V_4.

Hadamard's problem for (1.1) has been solved in the case when V_4 is locally conformally flat. In this case it has been shown [12] [10] [1] that a Huygens' equation is necessarily equivalent to the ordinary wave equation

$$\Box u = 0 , \tag{1.3}$$

on flat space-time. For a more detailed description of existing results see the review of the subject by one of us [16]. The problem has also been solved for the self-adjoint equation on Petrov type N, D, and III backgrounds. In CM1 it is shown that every Petrov type N space-time on which the conformally invariant scalar wave equation

$$\Box u + \frac{\mathrm{R}}{6} u = 0 , \tag{1.4}$$

satisfies Huygens' principle is conformally related to an exact plane wave space-time with metric

$$ds^2 = 2dv\{du + [D(v)z^2 + \overline{D}(v)\bar{z}^2 + e(v)z\bar{z}]dv\} - 2dzd\bar{z}, \qquad (1.5)$$

in a special coordinate system. In Eq. (1.4) R denotes the curvature scalar of V_4; in (1.5) D and e denote arbitrary C^∞ functions. Günther [8] has shown that Huygens' principle is satisfied by (1.4) on every exact plane wave space-time. This result when combined with the previous one yields the following theorem which solves Hadamard's problem for the equation (1.4) on a type N space-time [3]:

THEOREM 1. — *The conformally invariant wave equation (1.4) on any Petrov type* N *space-time satisfies Huygens' principle if and only if the space-time is conformally related to a plane wave space-time with metric (1.5) in a special coordinate system.*

In CM2 and CM3 it is shown that there exist *no* Petrov type D or III space-times on which the equation (1.4) satisfies Huygens' principle. The proof of Petrov type III required an additional weak assumption on the covariant derivative of the Weyl tensor C_{abcd}.

The proofs of the above results are based on the following set of necessary conditions for (1.1) to be a Huygens' differential equation [11] [7] [13] [22] [15]:

I $$\mathscr{C} := C - \frac{1}{2}A^a{}_{;a} - \frac{1}{4}A_aA^a - \frac{1}{6}R = 0, \qquad (1.6)$$

II $$H^k{}_{a;k} = 0, \qquad (1.7)$$

III $$S_{abk;}{}^k - \frac{1}{2}C^k{}_{ab}{}^l L_{kl} = -5\left(H_{ak}H_b{}^k - \frac{1}{4}g_{ab}H_{kl}H^{kl}\right), \qquad (1.8)$$

IV $$TS(3S_{abk}H^k{}_c + C^k{}_{ab}{}^l H_{ck;l}) = 0, \qquad (1.9)$$

V $$\begin{aligned} TS(3C_{kabl;m}C^k{}_{cd}{}^{l\,m}{}_{;} &+ 8C^k{}_{ab}{}^l{}_{;c}S_{kld} + 40S_{ab}{}^kS_{cdk} \\ &- 8C^k{}_{ab}{}^lS_{klc;d} - 24C^k{}_{ab}{}^lS_{cdk;l} + 4C^k{}_{ab}{}^lC_l{}^m{}_{ck}L_{dm} \\ &+ 12C^k{}_{ab}{}^lC^m{}_{cdl}L_{km} + 12H_{ka;bc}H^k{}_d - 16H_{ka;b}H^k{}_{c;d} \\ &- 84H^k{}_aC_{kbcl}H^l{}_d - 18H_{ka}H^k{}_bL_{cd}) = 0. \end{aligned} \qquad (1.10)$$

In the above conditions

$$H_{ab} := A_{[a,b]}, \qquad (1.11)$$

$$L_{ab} := -R_{ab} + \frac{1}{6}Rg_{ab}, \qquad (1.12)$$

$$S_{abc} := L_{a[b;c]}, \qquad (1.13)$$

$$C_{abcd} := R_{abcd} - 2g_{[a]d}L_{b[c]}, \qquad (1.14)$$

where $A_a = g_{ab}A^b$, R_{abcd} denotes the Riemann curvature tensor of V_4, $R_{ab} := g^{cd}R_{cabd}$ the Ricci tensor, and $R := g^{ab}R_{ab}$. Our sign conventions are the same as those in [*15*]. The symbol TS(...) denotes the trace-free symmetric part of the enclosed tensor [*13*]. It should be noted that the Conditions I-V are necessarily invariant under the trivial transformations.

In the case of Petrov types N and D it was also necessary to invoke a further necessary Condition VII valid for the self-adjoint equation (1.4) derived by Rinke and Wünsch [*21*] in order to complete the proofs.

In this paper we solve Hadamard's problem for (1.1) on a Petrov type N background space-time. We recall that such space-times are characterized by the existence of a null vector field l that satisfies the following conditions [*6*]:

$$C_{abcd}l^d = 0\,, \tag{1.15}$$

$$C_{abcd} \neq 0\,. \tag{1.16}$$

Such a vector field, called a *repeated principal null vector field* of the Weyl tensor, is determined by C_{abcd} up to an arbitrary variable factor.

The main results of this paper are contained in the following theorems:

THEOREM 2. — *Any non-self-adjoint equation (1.1) which satisfies Huygens' principle on any Petrov type* N *background space-time is equivalent to the conformally invariant scalar wave equation (1.4).*

When this theorem is combined with Theorem 1 we obtain the following:

THEOREM 3. — *Any non-self-adjoint equation (1.1) on any Petrov type* N *background space-time satisfies Huygens' principle if and only if it is equivalent to the wave equation*

$$\Box u = 0\,, \tag{1.17}$$

on an exact plane wave space-time with metric (1.5).

The plan of the remainder of the paper is as follows. In Section 2 the formalisms used are briefly described. The proof of Theorem 2 is given in Section 3.

2. FORMALISMS

We employ the two-component spinor formalism of Penrose [*18*] [*20*] and the spin coefficient formalism of Newman and Penrose (NP) [*17*] whose conventions we follow. In the spinor formalism tensor and spinor indices are related by complex connecting quantities $\sigma_a{}^{A\dot{A}}$ ($a=1, \ldots, 4$; $A=0, 1$) which are Hermitian in the spinor indices $A\dot{A}$. Spinor indices are lowered by the skew symmetric spinors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$ defined by $\varepsilon_{01}=\varepsilon_{\dot{0}\dot{1}}=1$, according to the convention

$$\xi_A = \xi^B \varepsilon_{BA}\,, \tag{2.1}$$

where ξ_A is any 1-spinor. Spinor indices are raised by the respective inverses of these spinors denoted by ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$. The spinor equivalents of the Weyl tensor (1.14), the tensor L_{ab} defined by (1.12), and the tensor H_{ab} of (1.11) are given respectively by

$$C_{abcd}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}}\sigma^c{}_{C\dot{C}}\sigma^d{}_{D\dot{D}} = \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{D}\dot{C}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\varepsilon_{AB}\varepsilon_{DC}, \quad (2.2)$$

$$L_{ab}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}} = 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}), \quad (2.3)$$

$$H_{ab}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}} = \phi_{AB}\varepsilon_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}}\varepsilon_{AB}, \quad (2.4)$$

where $\Psi_{ABCD} = \Psi_{(ABCD)}$ denotes the Weyl spinor, $\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})}$ denotes the trace-free Ricci spinor, $\phi_{AB} = \phi_{(AB)}$ denotes the Maxwell spinor and where

$$\Lambda = (1/24)R. \quad (2.5)$$

The covariant derivative of spinors denoted by « ; » satisfies

$$\sigma_a{}^{A\dot{A}}{}_{;b} = \varepsilon_{AB;b} = 0. \quad (2.6)$$

It will be necessary in the sequel to express spinor equations in terms of a spinor dyad $\{ o_A, \iota_A \}$ satisfying the completeness relation

$$o_A\iota^A = 1. \quad (2.7)$$

Associated to the spinor dyad is the null tetrad $\{ l, n, m, \overline{m} \}$ defined by

$$l^a = \sigma^a{}_{A\dot{A}}o^A\overline{o}^{\dot{A}}, \qquad n^a = \sigma^a{}_{A\dot{A}}\iota^A\overline{\iota}^{\dot{A}}, \qquad m^a = \sigma^a{}_{A\dot{A}}o^A\overline{\iota}^{\dot{A}}. \quad (2.8)$$

The NP spin coefficients associated to the dyad are defined by the equations [*4*]

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}} + \iota_A II_{B\dot{B}}, \quad (2.9)$$

$$\iota_{A;B\dot{B}} = o_A III_{B\dot{B}} - \iota_A I_{B\dot{B}}, \quad (2.10)$$

where

$$I_{B\dot{B}} := \gamma o_B\overline{o}_{\dot{B}} - \alpha o_B\overline{\iota}_{\dot{B}} - \beta\iota_B\overline{o}_{\dot{B}} + \varepsilon\iota_B\overline{\iota}_{\dot{B}}, \quad (2.11)$$

$$II_{B\dot{B}} := -\tau o_B\overline{o}_{\dot{B}} + \rho o_B\overline{\iota}_{\dot{B}} + \sigma\iota_B\overline{o}_{\dot{B}} - \kappa\iota_B\overline{\iota}_{\dot{B}}, \quad (2.12)$$

$$III_{B\dot{B}} := \nu o_B\overline{o}_{\dot{B}} - \lambda o_B\overline{\iota}_{\dot{B}} - \mu\iota_B\overline{o}_{\dot{B}} + \pi\iota_B\overline{\iota}_{\dot{B}}. \quad (2.13)$$

The NP (dyad) components of the Weyl spinor, trace-free Ricci spinor, and Maxwell spinor are defined respectively as follows:

$$\Psi_{ABCD} = \Psi_0\iota_{ABCD} - 4\Psi_1 o_{(A}\iota_{BCD)} + 6\Psi_2 o_{(AB}\iota_{CD)} - 4\Psi_3 o_{(ABC}\iota_{D)} + \Psi_4 o_{ABCD}, \quad (2.14)$$

$$\Phi_{AB\dot{A}\dot{B}} = \frac{1}{2}\Phi_{22}o_{AB}\overline{o}_{\dot{A}\dot{B}} - 2\Phi_{12}o_{(A}\iota_{B)}\overline{\iota}_{\dot{A}\dot{B}} + \Phi_{02}\iota_{AB}\overline{o}_{\dot{A}\dot{B}} + 2\Phi_{11}o_{(A}\iota_{B)}\overline{o}_{(\dot{A}}\overline{\iota}_{\dot{B})} - 2\Phi_{01}\iota_{AB}\overline{o}_{(\dot{A}}\overline{\iota}_{\dot{B})} + \frac{1}{2}\Phi_{00}\iota_{AB}\overline{\iota}_{\dot{A}\dot{B}} + \text{c.c.}, \quad (2.15)$$

$$\phi_{AB} = \phi_0\iota_{AB} - 2\phi_1 o_{(A}\iota_{B)} + \phi_2 o_{AB}, \quad (2.16)$$

where the notation $\iota_{A_1 \ldots A} := \iota_{A_1} \ldots \iota_{A_p}$, etc. has been used and « c.c. »

denotes the complex conjugate of the preceeding terms. The NP differential operators are defined by

$$D := l^a \partial_a, \qquad \Delta := n^a \partial_a, \qquad \delta := m^a \partial_a . \tag{2.17}$$

The equations relating the curvature components to the spin coefficients, the commutation relations and the dyad form of Maxwell's equations are given in NP. The Bianchi identities may be found in Pirani [20].

The subgroup of the proper orthochronus Lorentz group $L_+^\uparrow$ preserving the direction of the vector l is given by

$$l' = \exp\left[\frac{1}{2}(w + \overline{w})\right] l, \qquad m' = \exp\left[\frac{i}{2}(\overline{w} - w)\right] m,$$
$$n' = \exp\left[-\frac{1}{2}(w + \overline{w})\right](n + qm + \overline{q}\overline{m} + q\overline{q}l), \tag{2.18}$$

where q and w are complex valued. The transformation formulas for the NP operators, spin coefficients and curvature components induced by (2.18) may be found in CM3. The induced transformation formulas for the Maxwell components are

$$\begin{aligned} \phi_0' &= e^{w}\phi_0, \\ \phi_1' &= \phi_1 + q\phi_0, \\ \phi_2' &= e^{-w}(\phi_2 + 2q\phi_1 + q^2\phi_0). \end{aligned} \tag{2.19}$$

We shall also require the transformation laws for the coefficients of (1.1) induced by the trivial transformations. Excluding consideration of *a*), we consider only the effect of *b*) and the transformation Hadamard calls *bc*) defined as follows:

bc) substitution of λu for u and simultaneous multiplication of the equation by λ^{-1}.

This transformation leaves invariant the space-time metric. The transformations *b*) and *bc*) transform the differential operator F defined in (1.1) into an operator $\overline{F}$ of the same form with different coefficients $\tilde{g}_{ab}$, $\overline{A}^a$ and $\overline{C}$ defined by (see [15] for details).

$$\overline{F}[u] := \lambda^{-1} e^{-2\phi} F[\lambda u]. \tag{2.20}$$

The relation between the coefficients of F and $\overline{F}$ are given by (1.2) and

$$\overline{A}_a = A_a + 2(\log\lambda)_{,a} - 2\phi_{,a}, \tag{2.21}$$
$$\overline{C} = e^{-2\phi}(C + \lambda^{-1}\Box\lambda + A^a(\log\lambda)_{,a}). \tag{2.22}$$

It is well known that $C^a{}_{bcd}$, H_{ab} and $\mathscr{C}$ transform as follows under *b*) and *bc*):

$$\tilde{C}^a{}_{bcd} = C^a{}_{bcd}, \tag{2.23}$$
$$\overline{H}_{ab} = H_{ab}, \tag{2.24}$$
$$\overline{\mathscr{C}} = e^{-2\phi}\mathscr{C}. \tag{2.25}$$

The conformal transformation (1.2) is induced by the following transformation of the null tetrad:

$$\tilde{l}_a = e^{r\phi} l_a, \qquad \tilde{n}_a = e^{(2-r)\phi} n_a, \qquad \tilde{m}_a = e^{\phi} m_a, \tag{2.26}$$

where r is any real constant. Some of the transformation formulas induced by (2.26) are

$$\tilde{\kappa} = e^{(2r-3)\phi}\kappa, \qquad \tilde{\rho} = e^{(r-2)\phi}(\rho - \mathrm{D}\phi),$$
$$\tilde{\sigma} = e^{(r-2)\phi}\sigma, \qquad \tilde{\tau} = e^{-\phi}(\tau - \delta\phi). \tag{2.27}$$

3. PROOF OF THEOREM 2

The idea of the proof is to show

$$\mathrm{H}_{ab} = 0, \tag{3.1}$$

(compare with Lemma 4 of [*16*]). If this equation holds we have by (1.11)

$$\mathrm{A}_{[a,b]} = 0, \tag{3.2}$$

which implies that the 1-form

$$\mathrm{A} = \mathrm{A}_a dx^a \tag{3.3}$$

is closed. Thus by the converse of Poincaré's lemma A is locally exact, that is there exists a function h such that

$$\mathrm{A} := dh. \tag{3.4}$$

It follows from (1.6), (2.21) and (2.25) that for the transformation bc) defined by

$$\lambda = \exp(-h/2) \tag{3.5}$$

one has

$$\overline{\mathrm{A}}^a = 0, \qquad \overline{\mathrm{C}} = \mathrm{R}/6. \tag{3.6}$$

We conclude that if (3.1) holds then the non-self-adjoint equation (1.1) is equivalent to the conformally invariant scalar wave equation (1.4).

We prove (3.1) by showing that the contrary

$$\mathrm{H}_{ab} \neq 0, \tag{3.7}$$

is incompatible with the Conditions II-V.

We begin the proof of this assertion by giving the spinor form of Conditions II-V:

IIs
$$\phi_{\mathrm{AK};}{}^{\mathrm{K}}{}_{\dot{\mathrm{A}}} = 0, \tag{3.8}$$

IIIs
$$\Psi_{\mathrm{ABKL};}{}^{\mathrm{K}}{}_{\dot{\mathrm{A}}}{}^{\mathrm{L}}{}_{\dot{\mathrm{B}}} + \overline{\Psi}_{\dot{\mathrm{A}}\dot{\mathrm{B}}\dot{\mathrm{K}}\dot{\mathrm{L}};}{}^{\dot{\mathrm{K}}}{}_{\mathrm{A}}{}^{\dot{\mathrm{L}}}{}_{\mathrm{B}} + \Psi_{\mathrm{AB}}{}^{\mathrm{KL}}\Phi_{\mathrm{KL}\dot{\mathrm{A}}\dot{\mathrm{B}}} + \overline{\Psi}_{\dot{\mathrm{A}}\dot{\mathrm{B}}}{}^{\dot{\mathrm{K}}\dot{\mathrm{L}}}\Phi_{\dot{\mathrm{K}}\dot{\mathrm{L}}\mathrm{AB}}$$
$$= -10\phi_{\mathrm{AB}}\overline{\phi}_{\dot{\mathrm{A}}\dot{\mathrm{B}}}, \tag{3.9}$$

$$\text{IVs}\quad 3\Psi_{ABCK;}{}^{K}{}_{(\dot{A}}\bar{\phi}_{\dot{B}\dot{C})} + 3\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{K};}{}^{\dot{K}}{}_{(A}\phi_{BC)} - \Psi_{ABC}{}^{K}\bar{\phi}_{(\dot{A}\dot{B};\dot{C})K} - \bar{\Psi}_{\dot{A}\dot{B}\dot{C}}{}^{\dot{K}}\phi_{(AB;C)\dot{K}} = 0\,, \tag{3.10}$$

$$\begin{aligned}\text{Vs}\quad & 3\Psi_{ABCD;K\dot{K}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D};}{}^{\dot{K}K} + 4\Psi^{K}{}_{(ABC;D)(\dot{A}}\bar{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{K};}{}^{\dot{K}}{}_{K} \\ & + 4\bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D})(A}\Psi_{BCD)K;}{}^{K}{}_{\dot{K}} - 40\Psi_{(ABC|K|;}{}^{K}{}_{(\dot{A}}\bar{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{K};}{}^{\dot{K}}{}_{D)} \\ & - 4\Psi^{K}{}_{(ABC}\bar{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{|K|\dot{D})D)} - 4\bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{ABC|K|;}{}^{K}{}_{|\dot{K}|D)\dot{D})} \\ & + 12\Psi^{K}{}_{(ABC}\bar{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;}{}^{\dot{K}}{}_{D)\dot{D})K} + 12\bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|K|;}{}^{K}{}_{\dot{D})D)\dot{K}} \\ & - 16\Psi_{K(ABC}\Phi_{D)}{}^{K}{}_{(\dot{A}}{}^{\dot{K}}\bar{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{K}} - 32\Lambda\Psi_{ABCD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \\ & - 6\bar{\phi}_{(\dot{A}\dot{B}}\phi_{(AB;C\dot{C}D)\dot{D})} - 6\phi_{(AB}\bar{\phi}_{(\dot{A}\dot{B};\dot{C}C\dot{D})D)} + 16\phi_{(AB;C(\dot{A}}\bar{\phi}_{\dot{B}\dot{C};\dot{D})D)} \\ & - 42\phi_{(AB}\phi_{CD)}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} - 42\bar{\phi}_{(\dot{A}\dot{B}}\bar{\phi}_{\dot{C}\dot{D})}\Psi_{ABCD} + 36\phi_{(AB}\Phi_{CD)(\dot{A}\dot{B}}\bar{\phi}_{\dot{C}\dot{D})} = 0\,.\end{aligned} \tag{3.11}$$

We next make the hypothesis that the space-time V_4 is of Petrov type N. The conditions for this are given by (1.15) and (1.16) which are equivalent to the existence of a principal null spinor o_A of the Weyl spinor such that

$$\Psi_{ABCD} = \Psi o_{ABCD}\,, \tag{3.12}$$

where $\Psi := \Psi_4 \neq 0$. We select o_A to be the first spinor in a spinor dyad $\{ o_A, \iota_A \}$ which implies by (2.14) that

$$\Psi_i = 0 \tag{3.13}$$

for $i = 0, \ldots, 3$. We use the transformation (2.18) to obtain a spinor dyad in which the Weyl spinor has the form

$$\Psi_{ABCD} = o_{ABCD}\,. \tag{3.14}$$

We proceed by substituting for ϕ_{AB} from (2.16) and for Ψ_{ABCD} from the above in (3.8)-(3.11). The covariant derivatives of o_A and ι_A that appear are eliminated using Eqs. (2.9)-(2.13); derivatives of the form $S_{;A\dot{A}}$ are expressed as

$$S_{;A\dot{A}} = \Delta S o_A \bar{o}_{\dot{A}} - \bar{\delta} S o_A \bar{\iota}_{\dot{A}} - \delta S \iota_A \bar{o}_{\dot{A}} + D S \iota_A \bar{\iota}_{\dot{A}}\,. \tag{3.15}$$

The dyad form of Eqs. (3.1)-(3.4) is obtained by contracting the resulting equations with all possible products of o^A, ι^A and their complex conjugates. In view of the invariance of Conditions II-V [*15*] [*23*] under the trivial transformations *b*) and *bc*) it follows that each dyad equation must be invariant.

The first contraction to consider is $o^A \iota^{BC} \bar{o}^{\dot{A}\dot{B}\dot{C}}$ with Condition IVs which yields the condition

$$\kappa\bar{\phi}_0 = 0\,. \tag{3.16}$$

We first assume $\kappa \neq 0$. It then follows that

$$\phi_0 = 0\,. \tag{3.17}$$

The contraction $o^A \iota^{BC} \bar{o}^{\dot{A}\dot{B}} \bar{\iota}^{\dot{C}}$ with IVs now gives

$$\kappa\bar{\phi}_1 = 0\,, \tag{3.18}$$

which implies

$$\phi_1 = 0. \tag{3.19}$$

Finally the contraction $o^{AB}\iota^{C}\bar{\iota}^{\dot{A}\dot{B}\dot{C}}$ yields

$$\kappa\phi_2 = 0. \tag{3.20}$$

This equation implies

$$\phi_2 = 0. \tag{3.21}$$

However, this is impossible since the inequality (3.7) implies that not all of ϕ_0, ϕ_1, and ϕ_2 are zero. We thus conclude that (3.10) implies

$$\kappa = 0. \tag{3.22}$$

We note that this condition is invariant under the tetrad transformation (2.18) and the conformal transformation (2.26).
We next observe that (3.22) and the $o^{AB}\bar{o}^{\dot{A}\dot{B}}$ contraction with IIIs implies immediately that

$$\phi_0 = 0. \tag{3.21}$$

The dyad form of Condition II (Maxwell's equations), frequently required in the sequel, now take the form [*17*]

$$D\phi_1 = 2\rho\phi_1, \tag{3.22}$$

$$D\phi_2 - \bar{\delta}\phi_1 = 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2, \tag{3.23}$$

$$\delta\phi_1 = 2\tau\phi_1 - \sigma\phi_2, \tag{3.24}$$

$$\delta\phi_2 - \Delta\phi_1 = 2\mu\phi_1 + (\tau - 2\beta)\phi_2. \tag{3.25}$$

We proceed with the proof writing the $o^{AB}\iota^{C}\bar{\iota}^{\dot{A}\dot{B}\dot{C}}$ contraction with IVs:

$$D\phi_1 + (\rho - 3\bar{\rho} + 12\bar{\varepsilon})\phi_1 = 0. \tag{3.26}$$

Combining this with (3.22) we obtain

$$(\rho - \bar{\rho} + 4\bar{\varepsilon})\phi_1 = 0, \tag{3.27}$$

which implies

$$\phi_1 = 0. \tag{3.28}$$

This follows in the case

$$\rho - \bar{\rho} + 4\bar{\varepsilon} = 0, \tag{3.29}$$

from the $o^{A}\iota^{B}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}$ contraction with IIIs which reads

$$\sigma(\bar{\rho} - \rho + 4\varepsilon) + \bar{\sigma}(\rho - \bar{\rho} + 4\bar{\varepsilon}) = -10\phi_1\bar{\phi}_1. \tag{3.30}$$

Since we have shown that

$$\phi_0 = \phi_1 = 0, \tag{3.31}$$

it follows from (3.7) that

$$\phi_2 \neq 0. \tag{3.32}$$

We are now able to conclude from (3.24) that

$$\sigma = 0. \tag{3.33}$$

It should be noted that the conditions (3.31) and (3.33) are invariant under the tetrad transformation (2.18) and the conformal transformation (2.26).

The next step consists in combining (3.23) with the $o^{A}\iota^{BC}\bar{\iota}^{\dot{A}\dot{B}\dot{C}}$ contraction with IVs to obtain (3.29). Thus

$$\varepsilon = \frac{1}{4}(\rho - \bar{\rho}). \tag{3.34}$$

We now invoke Condition Vs for the first time. The $o^{AB}\iota^{CD}\bar{o}^{\dot{A}\dot{B}}\bar{\iota}^{\dot{C}\dot{D}}$ contraction yields after a lengthy calculation

$$D\rho + \rho(3\rho - \varepsilon - \bar{\varepsilon}) - 4\rho\bar{\rho} - \Phi_{00} + \text{c.c.} = 0, \tag{3.35}$$

where (3.23) has been used. The above equation and NP Eq. (4.2 *a*) imply

$$\rho = \bar{\rho}, \tag{3.36}$$

from which we obtain

$$\varepsilon = 0. \tag{3.37}$$

We observe that (3.36) is invariant under the dyad transformation (2.18) and the conformal transformation (2.26) while (3.37) is not. The dyad and conformally invariant form of this equation is (see CM1)

$$D\Psi + 4\varepsilon\Psi = 0. \tag{3.38}$$

This result is based in part on the following transformation formula induced by (2.26):

$$\tilde{D}\tilde{\Psi} + 4\tilde{\varepsilon}\tilde{\Psi} = e^{-(r+2)\phi}[D\Psi + 4\varepsilon\Psi]. \tag{3.39}$$

We may now use the conformal transformations (2.27) to set

$$\rho = \tau = 0. \tag{3.40}$$

This is done by choosing the function ϕ in (2.26) to be a solution of the differential equations

$$D\phi = \rho, \qquad \delta\phi = \tau, \qquad \bar{\delta}\phi = \bar{\tau}. \tag{3.41}$$

This system of equations has a solution since it may be shown, in a manner similar to that in CM1, that the required integrability conditions are satisfied. We note that the conditions (3.40) are invariant under a general dyad transformation (2.18).

The results obtained to this point may be summarized as follows: *Conditions IIs-Vs imply that with respect to any null tetrad* $\{l, n, m, \bar{m}\}$ *for which l* *is a principal null vector of the type* N *Weyl tensor there exists a conformally related tetrad in which*

$$\kappa = \sigma = \rho = \tau = 0, \tag{3.42}$$

$$D\Psi + 4\varepsilon\Psi = 0, \tag{3.43}$$

$$\phi_0 = \phi_1 = 0, \tag{3.44}$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \tag{3.45}$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0. \tag{3.46}$$

The Eqs. (3.46) above follow from (3.42) and NP Eqs. (4.2). Following CM1 we may write the conditions (3.42), (3.43) and (3.45) in spinorial form as

$$\Psi_{ABCD;E\dot{E}} = \Psi_{ABCD}K_{E\dot{E}}, \tag{3.47}$$

where

$$K_{E\dot{E}} := \Psi^{-1}(\Theta_1 o_E \bar{o}_{\dot{E}} + \Theta_2 o_E \bar{\iota}_{\dot{E}} + \Theta_3 \iota_E \bar{o}_{\dot{E}}), \tag{3.48}$$

where

$$\Theta_1 := \Delta\Psi + 4\gamma\Psi, \tag{3.49}$$

$$\Theta_2 := -(\bar{\delta}\Psi + 4\alpha\Psi), \tag{3.50}$$

$$\Theta_3 := -(\delta\Psi + 4\beta\Psi). \tag{3.51}$$

We recall that (3.47) is the defining equation for a complex recurrent space-time [*14*] [*16*].

At this point of the proof it is advantageous to employ a different choice of spinor dyad than the one in which (3.14) holds. The appropriate choice is one for which

$$\phi_2 = 1, \tag{3.52}$$

which is always possible in view of (2.19). It follows immediately from (3.23) and (3.25) that

$$\varepsilon = \beta = 0, \tag{3.53}$$

which implies by (3.43) and NP Eq. (4.2 *d*) that

$$D\Psi = D\alpha = 0. \tag{3.54}$$

The $\iota^{ABCD}\bar{o}^{\dot{A}\dot{B}}\bar{\iota}^{\dot{C}\dot{D}}$ contraction with Vs now yields

$$\bar{\delta}\alpha = -3\alpha^2, \tag{3.55}$$

while the $o^{A}\iota^{BCD}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ contraction gives

$$3(D\gamma + \delta\alpha - \alpha\bar{\pi}) - 11\alpha\bar{\alpha} - 18\Phi_{11} + \text{c.c.} = 0. \tag{3.56}$$

When the latter equation is combined with NP Eqs. (4.2*f*) and (4.2 *l*) we obtain

$$\Phi_{11} = -\frac{2}{3}\alpha\bar{\alpha}. \tag{3.57}$$

The NP Eq. (4.2 l) now reads

$$\delta\alpha = \frac{1}{3}\alpha\bar{\alpha}. \tag{3.58}$$

In view of (3.55) and (3.58) we are now able to compute $[\bar{\delta}, \delta]\alpha$; it follows from this expression, Eq. (3.54) and the NP Eqs. (4.4) that

$$\alpha = 0, \tag{3.59}$$

which implies

$$\Phi_{11} = D\gamma = 0. \tag{3.60}$$

We next compute the $\iota^{ABCD}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ contraction with Vs obtaining

$$\bar{\delta}\gamma + \bar{\delta}\bar{\gamma} - 6\Phi_{21} = 0. \tag{3.61}$$

When this equation is combined with NP Eqs. (4.2 o) and (4.2 r) we find

$$\delta\gamma = \Phi_{12} = 0. \tag{3.62}$$

The remaining equations in Conditions IIIs, IVs and Vs obtained with the help of (3.47)-(3.51) are respectively

$$\delta^2\Psi + \bar{\delta}^2\bar{\Psi} = -10, \tag{3.63}$$

$$\delta\Psi + \bar{\delta}\bar{\Psi} = 0, \tag{3.64}$$

$$24\bar{\Psi}\bar{\delta}\delta\bar{\Psi} + 3\bar{\delta}\Psi\delta\bar{\Psi} + 43\delta\Psi\bar{\delta}\bar{\Psi} + 84\Psi + 24(\Delta\gamma + 3\gamma^2) - 40\gamma\bar{\gamma} - 36\Phi_{22} + \text{c.c.} = 0, \tag{3.65}$$

while the Bianchi identities reduce to

$$D\Phi_{22} = 0, \tag{3.66}$$

$$\delta\Psi = \bar{\delta}\Phi_{22}. \tag{3.67}$$

From (3.66), (3.67) and the NP Eqs. (4.4) it follows that

$$\delta^2\Psi = \delta\bar{\delta}\Phi_{22} = \bar{\delta}\delta\Phi_{22}. \tag{3.68}$$

The above and (3.63) imply that

$$\bar{\delta}^2\bar{\Psi} = \delta^2\Psi = -5. \tag{3.69}$$

It is a consequence of (3.64), (3.69) and NP Eqs. (4.4) that

$$\bar{\delta}\delta\bar{\Psi} = \delta\bar{\delta}\bar{\Psi} = 5. \tag{3.70}$$

The Eq. (3.65) may now be written as

$$3\bar{\delta}\Psi\delta\bar{\Psi} + 43\delta\Psi\bar{\delta}\bar{\Psi} + 102(\Psi + \bar{\Psi}) - 36\Phi_{22} + 12(\Delta\gamma + \Delta\bar{\gamma} + 3\gamma^2 + 3\bar{\gamma}^2) - 40\gamma\bar{\gamma} = 0. \tag{3.71}$$

Applying the δ operator to this equation we obtain

$$3\bar{\delta}\Psi\delta^2\bar{\Psi} + 568\delta\Psi + 117\delta\bar{\Psi} = 0, \tag{3.72}$$

where the Eqs. (3.66), (3.68), (3.69) and NP Eqs. (4.2) and (4.4) have been used. Noting that $\bar{\delta}\delta^2\bar{\Psi} = 0$, by (3.54), (3.70) and NP Eqs. (4.4), the application of $\bar{\delta}$ to (3.72) finally yields the contradiction

$$3\bar{\delta}^2\Psi\delta^2\bar{\Psi} + 3425 = 0. \qquad (3.73)$$

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