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# Non crossing walks and interfacial wetting

by

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ABSTRACT. — We compute exactly the generating function of two non crossing random walks on $\mathbb{Z}$, with allowed steps + 1, − 1 or 0. The result is applied to the study of interfacial wetting in 1 + 1 dimensional S. O. S. models.

RÉSUMÉ. — Nous calculons exactement la fonction génératrice de deux marches aléatoires sur $\mathbb{Z}$, sans intersection et de pas + 1, − 1 ou 0. Le résultat est ensuite appliqué à l'étude du mouillage interfacial de modèles S. O. S. bidimensionnels.

## 1. INTRODUCTION

In two dimensions, an interface is a random line, which is almost straight at low temperatures. If it doesn't have overhangs, then it may be considered as a one dimensional random walk $h_i \in \mathbb{Z}$, indexed by « time » $i \in \mathbb{N}$. If three or more phases coexist, then an interface between two of the phases may be wet by bubbles of a third phase. This is described by two random walks $h_i$ and $h'_i$ with $h_i \geq h'_i$: the first phase lies above $h_i$, the second phase lies below $h'_i$, the intruding phase is between $h_i$ and $h'_i$, and is present at $i$ only if $h_i > h'_i$. Suitable weights are given to the steps $h_{i+1} - h_i$ and $h'_{i+1} - h'_i$; the two walks are non crossing ($h_i \geq h'_i$) and also have a contact interaction:

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different weights are given for $h_i > h'_i$ and for $h_i = h'_i$. The physical and mathematical question is then the following: assuming $h_0 = h'_0$ and $h_N = h'_N$, what is the behaviour of the distance $h_i - h'_i$ for $0 < i < N$ when $N \to \infty$? In physical terms, the interface is said to be partially wet by the third phase when $\langle h_i - h'_i \rangle_N$ remains bounded, and totally wet when $\langle h_i - h'_i \rangle_N \to \infty$ as $i, N \to \infty$.

This question and related ones have been answered to a large extent by Fisher, Huse, Szpilka [*1*] [*2*], from whom we borrow the formulation of the problem. The present paper improves upon one aspect of [*1*] [*2*], where the two random walks $h_i$ and $h'_i$ have some unsatisfactory feature: either $h_{i+1} - h_i = \pm 1$ and $h'_{i+1} - h'_i = \pm 1$, which produces a tooth edge interface (lock step), or $| h_{i+1} - h_i | + | h'_{i+1} - h'_i | \leq 1$, which is a long range interaction between the two walks (random turns). Here we allow $h_{i+1} - h_i = 0, \pm 1$ and $h'_{i+1} - h'_i = 0, \pm 1$ (in Fisher's language, one might speak of two *tired* drunken walkers, who randomly walk and have a rest every now and then), and we don't impose « turns ».

The paper is organized as follows: in Section 2, we compute exactly the generating function of two non crossing walks, using a modified reflection principle. We also compute

$$\lim_{N \uparrow \infty} N^{-2} \left\langle \sum_{i=0}^{N} (h_i - h'_i)^2 \right\rangle_N$$

for two walks such that $h_0 = h'_0$, $h_N = h'_N$ and $h_i > h'_i$ for $i = 1, \ldots, N - 1$. In Section 3, we incorporate the necklace representation [*1*] [*3*] [*4*] to give the wetting transition line for a general S. O. S. model of interfacial wetting. In Section 4, we discuss the two dimensional $q$-state Potts model, at the self dual point, in the limit $q \to \infty$. We recover the known fact that the disordered phase wets the interface between two ordered phases. In section 5, we give the full wetting transition line for the S. O. S. chiral Potts model.

Section 2 is rather technical, and may be skipped by the readers interested in the results for the S. O. S. models.

## 2. TWO TIRED VICIOUS WALKERS

The positions of the two walkers are described respectively by $h_i \in \mathbb{Z}$ and $h'_i \in \mathbb{Z}$, at time $i \in \mathbb{N}$. The walkers start at $h_0$, $h'_0$ and die on the first time when they meet:

$$\begin{aligned} &h_0 \geq h'_0 \\ &h_i > h'_i : i = 1, \ldots, N - 1 \\ &h_N = h'_N . \end{aligned}$$

At each tick of the clock, each (tired) walker may move by one step on either side or (more likely) may also remain where he is:

$$h_i - h_{i-1} = 1, 0, -1$$
$$h'_i - h'_{i-1} = 1, 0, -1\,.$$

Our main technical result is the following Theorem:

THEOREM 1. — Let

$$Q_N^b = \Sigma' \exp\left\{ -K \sum_1^N (1 + |h_i - h_{i-1}|) - K' \sum_1^N (1 + |h'_i - h'_{i-1}|) \right\} \quad (2.1)$$

*where the sum is over the configurations* $h_i \in \mathbb{Z}$, $h'_i \in \mathbb{Z}$, *such that*

$$0 = h_0 = h'_0\,; \qquad h_N = h'_N \quad (2.2)$$
$$h_i - h_{i-1} = 1, 0, -1\,; \qquad h'_i - h'_{i-1} = 1, 0, -1 : (i = 1, \ldots, N)$$

*and*

$$h_i > h'_i : i = 1, \ldots, N-1\,. \quad (2.3)$$

*Let*

$$G_b(z) = \sum_{N=2}^{\infty} z^N Q_N^b\,.$$

*Then* $G_b(z)$ *is analytic in*

$$|z| < z_b = (e^{-K} + 2e^{-2K})^{-1}(e^{-K'} + 2e^{-2K'})^{-1}$$

*singular at* $z = z_b$, *and*

$$G_b(z_b) = s\,\frac{\left(\frac{3}{2} + s^{-1}t - \frac{1}{2}(1 + 4s^{-1}t)^{1/2}\right)}{1 + 2s + 4t} \quad (2.4)$$

*with*

$$s = e^{-K} + e^{-K'}\,; \qquad t = e^{-K-K'}\,.$$

*When* $K = K'$, *we obtain*

$$\begin{aligned} G_b(z_b) &= 2e^{-K}(1 + 2e^{-K})^{-2}\left[\frac{3}{2} + \frac{1}{2}e^{-K} - \frac{1}{2}(1 + 2e^{-K})^{1/2}\right] \\ &= 2e^{-K}[1 - 4e^{-K} + 0(e^{-2K})] \qquad \textit{when} \quad K \to \infty\,. \end{aligned} \quad (2.5)$$

*Proof.* — Every couple $(h_i, h'_i)$ may be represented by a point in $\mathbb{Z}^2$; a configuration $(h_i, h'_i)_{i=0\ldots N}$ is an N-step walk on $\mathbb{Z}^2$ starting from $(0, 0)$, then moving below the diagonal, and ending on the diagonal in $(h_N, h_N)$. Each step is either one side of a unit square or a diagonal of a unit square or also may be a stand-still (no move: $(h_{i+1}, h'_{i+1}) = (h_i, h'_i)$). The walk may cross itself but is not allowed to touch the diagonal other than in the end-points (see fig. 1).

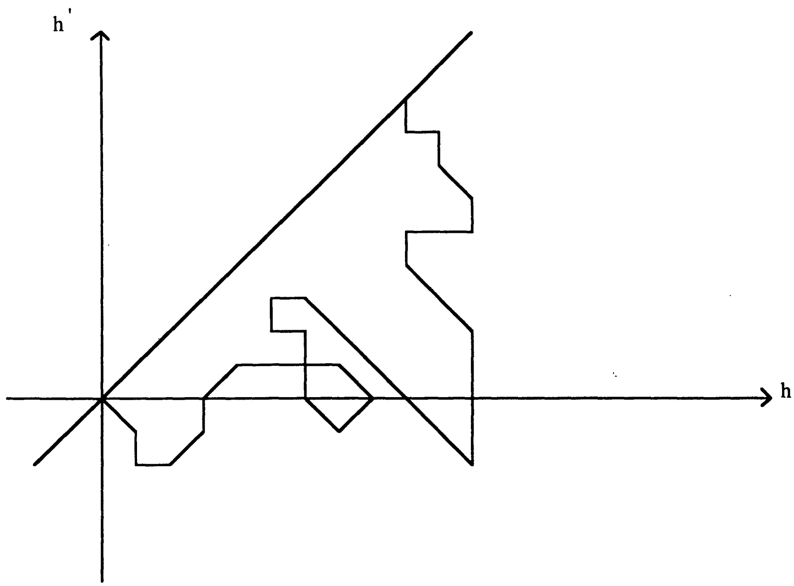


FIG. 1. — Path of a walker on a two-dimensional lattice of points $(h, h')$.

We shall now introduce a modified reflection principle applicable to this diagonal boundary. Let $Q_N^b(x, y)$ be defined as in the Theorem except that (2.2) is replaced by

$$h_0 = x, \; h'_0 = 0 \,; \qquad h_N - h'_N = y \,.$$

Let $Q_N^0(x, y)$ be defined similarly without the diagonal boundary (2.3), and $Q_N^d(0, 0)$ similarly with (2.3) replaced by

$$h_i = h'_i : i = 1, \ldots, N - 1 \,.$$

Of course $Q_N^0(x, y)$ and $Q_N^d(0, 0)$ are known explicitly and will be the basic ingredients in the solution for $Q_N^b(x, y)$. We include the possibility that $N = 0$ (or 1) with

$$Q_0^0(x, y) = \delta_{x,y} \,; \qquad Q_0^d(0, 0) = 1$$

$$Q_1^0(x, y) = \delta_{x,y}(e^{-K-K'} + 2e^{-2K-2K'}) + (\delta_{x,y-1} + \delta_{x,y+1})(e^{-2K-K'} + e^{-K-2K'}) + (\delta_{x,y-2} + \delta_{x,y+2})e^{-2K-2K'}$$

$$Q_1^d(0, 0) = e^{-K-K'} + 2e^{-2K-2K'}$$

whereas for $x \geq 0$ and $y \geq 0$

$$Q_0^b(x, y) = (1 - \delta_{x,0})\delta_{x,y}$$
$$Q_1^b(x, y) = (1 - \delta_{x,0}\delta_{x,y})Q_1^0(x, y) \,.$$

We then have, first for $x \geq 1$ and $y \geq 1$

$$Q_N^0(x, y) = Q_N^b(x, y) + \sum_p Q_p^b(x, 0)Q_{N-p}^0(0, y) + e^{-2K-2K'} \sum_p Q_p^b(x, 1)Q_{N-p-1}^0(1, y). \quad (2.6)$$

Indeed the left hand side includes « all » walks. In the right hand side, $Q_N^b(x, y)$ is associated to the walks which remain strictly below the diagonal. The first sum is associated to the walks which stop on the diagonal before possibly going into the other side; the second sum is associated to the walks which jump over the diagonal when they cross it for the first time (such a jump is e. g. $(1, 0) \to (0, -1)$). Similarly, for $x \geq 1$,

$$\begin{aligned} Q_N^0(x, 0) &= Q_N^b(x, 0) + \sum_{p<N} Q_p^b(x, 0)Q_{N-p}^0(0, 0) \\ &+ e^{-2K-2K'} \sum_p Q_p^b(x, 1)Q_{N-p-1}^0(-1, 0) \\ &= \sum_p Q_p^b(x, 0)Q_{N-p}^0(0, 0) + e^{-2K-2K'} \sum_p Q_p^b(x, 1)Q_{N-p-1}^0(-1, 0). \quad (2.7) \end{aligned}$$

Finally,

$$\begin{aligned} Q_N^0(0, 0) = Q_N^d(0, 0) &+ 2 \sum_{p+p'+p''=N} Q_p^d(0, 0)Q_{p'}^b(0, 0)Q_{p''}^0(0, 0) \\ &+ 2e^{-2K-2K'} \sum_{p+p'+p''=N-1} Q_p^d(0, 0)Q_{p'}^b(0, 1)Q_{p''}^0(-1, 0) \quad (2.8) \end{aligned}$$

where the factor 2 indicates that the path may first go either above or below the diagonal.

We now introduce the generating functions

$$\begin{aligned} C_x(z) &= \sum_{N=0}^{\infty} z^N Q_N^0 x, 0 \\ &= \int \frac{d\theta}{2\pi} \frac{e^{-ix\theta}}{1 - ze^{-K-K'}(1 + 2e^{-K}\cos\theta)(1 + 2e^{-K'}\cos\theta)} \quad (2.9) \end{aligned}$$

$$C_0^d(z) = \sum_{N=0}^{\infty} z^N Q_N^d(0, 0) = \frac{1}{1 - z(e^{-K-K'} + 2e^{-2K-2K'})} \quad (2.10)$$

and the unknown

$$G_b(z; x, y) = \sum_{N=2}^{\infty} z^N Q_N^b(x, y).$$

The convolution equations for $Q_N^b(x, y)$ then become linear equations for $G_b(z; x, y)$. For $x \geq 1$ and $y \geq 1$,

$$C_{x-y}(z) = G_b(z; x, y) + G_b(z; x, 0)C_y(z) + ze^{-2K-2K'}G_b(z; x, 1)C_{y+1}(z)$$
$$C_x(z) = G_b(z; x, 0)C_0(z) + ze^{-2K-2K'}G_b(z; x, 1)C_1(z)$$
$$C_0(z) = C_0^d(z) + 2C_0^d(z)G_b(z; 0, 0)C_0(z) + 2ze^{-2K-2K'}C_0^d(z)G_b(z; 0, 1)C_1(z).$$

Fortunately, this infinite set of linear equations indexed by $x, y \in \mathbb{N}$ is partly decoupled. The pairs of equations with $y = 1$ and $y = 0$ may be solved (we simplify the notation by omitting the argument $z$ in $C_x(z)$ and $C_0^d(z)$):

$$C_{x-1} = C_1G_b(z; x, 0) + (1 + ze^{-2K-2K'}C_2)G_b(z; x, 1)$$
$$C_x = C_0G_b(z; x, 0) + ze^{-2K-2K'}C_1G_b(z; x, 1)$$

which yields, still for $x \geq 1$:

$$G_b(z; x, 1) = D^{-1}(C_0C_{x-1} - C_1C_x)$$
$$G_b(z; x, 0) = D^{-1}[C_x - ze^{-2K-2K'}(C_1C_{x-1} - C_2C_x)] \quad (2.11)$$

with

$$D = C_0 + ze^{-2K-2K'}(C_2C_0 - C_1^2). \quad (2.12)$$

All the other $G_b(z; x, y)$ are then readily obtained; we are interested in

$$G_b(z; 0, 0) = (2C_0C_0^d)^{-1}\{C_0 - C_0^d - 2ze^{-2K-2K'}C_0^dC_1G_b(z; 1, 0)\} \quad (2.13)$$

At this point, we note that $C_x(z)$ is analytic in $|z| < z_b$ and diverges at $z = z_b$, with $C_x(z) = C_0(z) - d_x(z)$ and $d_x(z_b)$ bounded; of course $C_0^d(z)$ is analytic up to and above $z_b$. It follows that

$$G_b(z_b; 1, 0) = [1 + z_be^{-2K-2K'}(2d_1 - d_2)]^{-1}(1 - z_be^{-2K-2K'}d_2) \quad (2.14)$$

and

$$G_b(z_b; 0, 0) = (2C_0^d)^{-1}\{1 - 2z_be^{-2K-2K'}C_0^dG_b(z_b; 1, 0)\}. \quad (2.15)$$

Let us now introduce the variables $s = e^{-K} + e^{-K'}$, $t = e^{-K-K'}$. Then

$$z_b^{-1} = t(1 + 2s + 4t)$$
$$d_1(z_b) = \frac{1 + 2s + 4t}{2(s^2 + 4st)^{1/2}}$$
$$d_2(z_b) = \frac{1}{2t}(1 + 2s + 4t)[1 - s(s^2 + 4st)^{-1/2}]$$
$$G_b(z_b; 1, 0) = \frac{2}{1 + (1 + 4s^{-1}t)^{1/2}}$$
$$C_0^d(z_b) = \frac{1 + 2s + 4t}{2s + 2t}$$
$$G_b(z_b; 0, 0) = \frac{s + t - tG_b(z_b; 1, 0)}{1 + 2s + 4t} = \frac{s\left(\frac{3}{2} + s^{-1}t - \frac{1}{2}(1 + 4s^{-1}t)^{1/2}\right)}{1 + 2s + 4t}$$

which is the desired formula (2.4), from which (2.5) follows easily. This completes the proof of Theorem 1.

Theorem 1 will be used in the next sections to study interfacial wetting. Here we first give some further calculations, where for simplicity we now take $K' = K$.

THEOREM 2. — *Suppose that*

$$N^{-2}\left\langle \sum_{i=0}^{N} (h_i - h_i')^2 \right\rangle_N \to \gamma \quad as \quad N \to \infty .$$

*Then*

$$\gamma = \lim_{z \uparrow z_b} \frac{\sum_{x=1}^{\infty} x^2 [G_b(z;0,x)]^2}{\left(z\frac{\partial}{\partial z}\right)^2 G_b(z;0,0)} = e^{-K}(1+2e^{-K})^{-3/2} \frac{[1+(1+2e^{-K})^{1/2}]^3}{3+e^{-K}+(1+2e^{-K})^{1/2}}$$

*Proof.*

$$N^{-2}\left\langle \sum_{i} (h_i - h_i')^2 \right\rangle_N = \frac{\sum_{i}\sum_{x>0} x^2 Q_i^b(0,x) Q_{N-i}^b(x,0)}{N^2 Q_N^b(0,0)} \equiv \frac{a_N}{b_N}.$$

Admitting that $a_N/b_N$ converges as $N \to \infty$, we can obtain the limit from the ratio of the corresponding generating functions. We use the following easy lemma:

LEMMA 1. — *Let* $a(z) = \Sigma a_N z^N$, $b(z) = \Sigma b_N z^N$ *with* $a_N, b_N > 0$ *and* $a_N/b_N \to \gamma$ *as* $N \to \infty$. *Suppose that* $a(z)$ *and* $b(z)$ *are analytic for* $|z| < z_0$, *diverge at* $z = z_0$, *and* $a(z)/b(z) \to \gamma_0$ *when* $z \to z_0$. *Then* $\gamma = \gamma_0$.

We apply lemma 1 with

$$a(z) = \sum_{N} z^N \sum_{i=0}^{N} \sum_{x>0} x^2 Q_i^b(0,x) Q_{N-i}^b(x,0) = \sum_{x>0} x^2 [G_b(z;0,x)]^2$$

$$b(z) = \sum_{N} z^N N^2 Q_N^b(0,0) = \left(z\frac{\partial}{\partial z}\right)^2 G_b(z;0,0).$$

In order to compute $\lim_{z \uparrow z_b} a(z)/b(z)$, we have to estimate $G_b(z;0,x)$ for $x \to \infty$.

Let us begin with the generating functions of two independent walkers:

$$C_x(z) = \sum_{n=0}^{\infty} z^n e^{-2nK} \int \frac{d\theta}{2\pi}(1 + 2e^{-K}\cos\theta)^{2n} e^{-ix\theta}$$

$$= \sum_{n=0}^{\infty} t^n \int \frac{d\theta}{2\pi}\left(\frac{1 + 2e^{-K}\cos\theta}{1 + 2e^{-K}}\right)^{2n} e^{-ix\theta}$$

with $t = z/z_b$. For $t \to 1$, the large $n$'s will dominate the sum and the integrals will be concentrated near $\theta = 0$, where we have

$$\left(\frac{1 + 2e^{-K}\cos\theta}{1 + 2e^{-K}}\right)^{2n} \approx \left(1 - \frac{e^{-K}}{1 + 2e^{-K}}\theta^2\right)^{2n}$$

so that
$$C_x(z) \approx \sum_{n=0}^{\infty} \frac{t^n}{2\pi} \int_{-\infty}^{+\infty} d\theta e^{-2ne^{-K}(1+2e^{-K})^{-1}\theta^2 + ix\theta}$$

$$= \left(\frac{1 + 2e^{-K}}{8\pi e^{-K}}\right)^{1/2} \sum_{n=0}^{\infty} \frac{t^n}{n^{1/2}} e^{-x^2(1+2e^{-K})(8ne^{-K})^{-1}} \qquad (2.16)$$

and one can verify that

$$\sum_{x\in\mathbb{Z}} C_x(z) = (1 - t)^{-1}.$$

For $x = 0$, we have

$$C_0(z) \underset{t\uparrow 1}{\approx} \left(\frac{1 + 2e^{-K}}{8\pi e^{-K}}\right)^{1/2} \int_0^{\infty} \frac{t^n}{n^{1/2}} dn = \frac{1}{2}\left(\frac{1 + 2e^{-K}}{2e^{-K}}\right)^{1/2} |\log t|^{-1/2}. \qquad (2.17)$$

We now compute

$$\sum_{x=1}^{\infty} x^2 [C_x(z)]^2 \approx \frac{1+2e^{-K}}{8\pi e^{-K}} \sum_{n;n'} \frac{t^{n+n'}}{(nn')^{1/2}} \sum_{x=1}^{\infty} x^2 e^{-x^2(1+2e^{-K})(n+n')(8nn'e^{-K})^{-1}}$$

$$\approx \left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{1/2} \frac{1}{2\pi^{1/2}} \sum_{n;n'} t^{n+n'} \frac{nn'}{(n+n')^{3/2}}$$

$$\approx \left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{1/2} \frac{1}{2\pi^{1/2}} \int_0^{\infty}\int_0^{\infty} e^{-(u^2+v^2)|\log t|} \frac{4u^3v^3 du dv}{(u^2+v^2)^{3/2}}$$

$$\approx \frac{1}{16}\left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{1/2} |\log t|^{-5/2}$$

from which we get

$$\sum_{x\geq 1} x^2 \left[\frac{C_x(z)}{C_0(z)}\right]^2 \underset{t\uparrow 1}{\approx} \frac{1}{4}\left(\frac{1 + 2e^{-K}}{2e^{-K}}\right)^{-3/2} |\log t|^{-3/2}. \qquad (2.18)$$

We shall also need

$$\sum_{x=1}^{\infty} x^2 [C_{x-1} - C_x]^2$$

$$\approx \frac{1 + 2e^{-K}}{8\pi e^{-K}} \sum_{n;n'} \frac{t^{n+n'}}{(nn')^{1/2}} \sum_{x=1}^{\infty} \left(\frac{1 + 2e^{-K}}{4e^{-K}}\right)^2 \frac{x^4}{nn'} e^{-x^2(1+2e^{-K})(n+n')(8nn'e^{-K})^{-1}}$$

$$\approx \left(\frac{1 + 2e^{-K}}{2e^{-K}}\right)^{1/2} \frac{3}{4\pi^{1/2}} \sum_{n;n'} t^{n+n'} \frac{nn'}{(n + n')^{5/2}}$$

which gives

$$\sum_{x=1}^{\infty} x^2 [C_{x-1} - C_x]^2 \approx \frac{1}{16}\left(\frac{1 + 2e^{-K}}{2e^{-K}}\right)^{1/2} |\log t|^{-3/2}. \qquad (2.19)$$

Finally we need

$$\sum_{x=1}^{\infty} x^2 C_x [C_{x-1} - C_x]$$

$$\approx \frac{1 + 2e^{-K}}{8\pi e^{-K}} \sum_{n;n'} \frac{t^{n+n'}}{(nn')^{1/2}} \sum_{x=1}^{\infty} \frac{1 + 2e^{-K}}{4e^{-K}} \frac{x^3}{n} e^{-x^2(1+2e^{-K})(n+n')(8nn'e^{-K})^{-1}}$$

$$\approx \frac{1}{2\pi} \sum_{n;n'} t^{n+n'} \frac{(nn')^{1/2}}{n + n'}$$

which gives

$$\sum_{x=1}^{\infty} x^2 \frac{C_x}{C_0} [C_{x-1} - C_x] \approx \frac{1}{8}\left(\frac{2e^{-K}}{1 + 2e^{-K}}\right)^{1/2} |\log t|^{-3/2}. \qquad (2.20)$$

We can now obtain $\sum_{x=1}^{\infty} x^2 [G_b(z;0,x)]^2$ with

$$G_b(z;0,x) = \frac{C_x[1 - ze^{-4K}(d_2 - d_1)] - ze^{-4K}(C_0 - d_1)(C_{x-1} - C_x)}{C_0[1 + ze^{-4K}(2d_2 - d_1)] - ze^{-4K}d_1^2}$$

$$\underset{t\uparrow 1}{\approx} [1 + ze^{-4K}(2d_1 - d_2)]^{-1} \left\{\frac{C_x}{C_0}[1 - ze^{-4K}(d_2 - d_1)] - ze^{-4K}(C_{x-1} - C_x)\right\}.$$

Using (2.17) (2.18) (2.19) (2.20), we find

$$\sum_{x\geq 1} x^2 [G_b(z;0,x)]^2$$
$$\approx [1+ze^{-4K}(2d_1-d_2)]^{-2}\left\{ [1-ze^{-4K}(d_2-d_1)]^2 \frac{1}{4}\left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{3/2}\right.$$
$$-2ze^{-4K}[1-ze^{-4K}(d_2-d_1)]\frac{1}{8}\left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{1/2}$$
$$\left.+z^2e^{-8K}\frac{1}{16}\left(\frac{1+2e^{-K}}{2e^{-K}}\right)^{1/2}\right\}|\log t|^{-3/2}. \quad (2.21)$$

We shall now insert

$$z_b = e^{2K}(1 + 2e^{-K})^{-2}$$
$$d_1 = \frac{1}{4}e^{K}(1 + 2e^{-K})^{3/2}\,; \quad d_2 = \frac{1}{2}e^{2K}(1 + 2e^{-K})^{3/2}\{(1 + 2e^{-K})^{1/2} - 1\}$$

and obtain

$$\sum_{x\geq 1} x^2 [G_b(z;0,x)]^2 \approx 2^{-1/2}e^{-3K/2}(1+2e^{-K})^{-5/2}(1+(1+2e^{-K})^{1/2})^{-4}$$
$$\cdot\left\{ 4(1+2e^{-K})\left(1+\frac{1}{2}e^{-K}+(1+2e^{-K})^{1/2}\right)^2\right.$$
$$\left.-4e^{-K}(1+2e^{-K})^{1/2}\left(1+\frac{1}{2}e^{-K}+(1+2e^{-K})^{1/2}\right)+e^{-2K}\right\}\cdot|\log t|^{-3/2}.$$

The sum in $\{\ \dots\ \}$ happens to be equal to $\frac{1}{4}(1 + (1 + 2e^{-K})^{1/2})^6$ so that we get

$$\sum_{x\geq 1} x^2 [G_b(z;0,x)]^2$$
$$\approx 2^{-5/2}e^{-3K/2}(1+2e^{-K})^{-5/2}(1+(1+2e^{-K})^{1/2})^2\,|\log t|^{-3/2}. \quad (2.22)$$

We now come to the estimates of the derivatives in $\left(z\frac{\partial}{\partial z}\right)^2 G_b(z;0,0)$, with

$$G_b(z;0,0) = \frac{1}{2C_0^d} - \frac{1}{2C_0} - ze^{-4K}\left(1 - \frac{d_1}{C_0}\right)G_b(z;1,0)$$

and

$$G_b(z;1,0) = \frac{\left[1 - \frac{d_1}{C_0}\right][1 - ze^{-4K}d_2]}{1 + ze^{-4K}(2d_2 - d_1) - ze^{-4K}\frac{d_1^2}{C_0}}.$$

The only significant terms in the derivatives when $z \uparrow z_b$ will come from differentiating $\frac{1}{C_0}$. Indeed

$$\frac{1}{C_0} \sim |\log t|^{-1/2} \quad \text{and} \quad \frac{d^2}{dz^2}\frac{1}{C_0} \sim |\log t|^{-3/2}.$$

Let us write

$$G_b(z;0,0) = f_0\left(\frac{1}{C_0}\right) \quad \text{and} \quad G_b(z;1,0) = f_1\left(\frac{1}{C_0}\right)$$

with

$$f_0(u) = cst - \frac{u}{2} - ze^{-4K}(1 - d_1u)f_1(u), \qquad f_1(u) = f_1(0)\frac{1-d_1u}{1-e_1u}$$

$$e_1 = \frac{ze^{-4K}d_1^2}{1 + ze^{-4K}(2d_1 - d_2)}.$$

We then find

$$\left(z\frac{\partial}{\partial z}\right)^2 G_b(z;0,0) \underset{t\uparrow 1}{\approx} f_0'(0)\left(z\frac{\partial}{\partial z}\right)^2 \frac{1}{C_0(z)} \underset{t\uparrow 1}{\approx} -\frac{1}{2}\left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{1/2} |\log t|^{-3/2} f_0'(0)$$

$$\approx \left[\frac{1}{2} - z_b e^{-4K}(2d_1 - e_1)f_1(0)\right]\frac{1}{2}\left(\frac{2e^{-K}}{1+2e^{-K}}\right)^{1/2} |\log t|^{-3/2}. \quad (2.23)$$

Inserting $f_1(0) = G_b(z_b;1,0)$ with the values of $e_1, d_1, d_2, z_b$ as above, we obtain

$$\left(z\frac{\partial}{\partial z}\right)^2 G_b(z;0,0) \underset{t\uparrow 1}{\approx} 2^{-5/2}e^{-K/2}(1 + 2e^{-K})^{-1}(1 + (1 + 2e^{-K})^{1/2})^{-1} \cdot$$
$$\cdot (3 + e^{-K} + (1 + 2e^{-K})^{1/2}) \cdot |\log t|^{-3/2}. \quad (2.24)$$

The ratio of (2.22) by (2.24) gives the desired result:

$$\gamma = \lim_{z\uparrow z_b} \frac{\sum_{x=1}^{\infty} x^2 [G_b(z;0,x)]^2}{\left(z\frac{\partial}{\partial z}\right)^2 G_b(z;0,0)} = e^{-K}(1+2e^{-K})^{-3/2}\frac{[1+(1+2e^{-K})^{1/2}]^3}{3+e^{-K}+(1+2e^{-K})^{1/2}}$$

This result can be compared to the case of two independent random walks with $h_0 = h_0'$, $h_N = h_N'$ and allowed crossing, where

$$\gamma_{\text{indep.}} = \lim N^{-2}\left\langle \sum_{i=0}^{N} (h_i - h_i')^2 \right\rangle_{N,\text{indep.}} = \lim_{z\uparrow z_b} \frac{\sum_{x=-\infty}^{\infty} x^2 C_x(z)^2}{\left(z\frac{\partial}{\partial z}\right)^2 C_0(z)}$$

$$= \frac{2}{3}e^{-K}(1 + 2e^{-K})^{-1}.$$

For $K \to \infty$, we have $\gamma \approx 2e^{-K}$ and $\gamma_{\text{indep.}} \approx \frac{2}{3}e^{-K}$. The ratio $\gamma/\gamma_{\text{indep.}}$ is a measure of the repulsion between the two non crossing walks. The same comparison can be made without the return condition $h_N = h'_N$ and the result is amazingly different, if we then look at the mean square of $h_N - h'_N$:

$$N^{-1} \langle (h_N - h'_N)^2 \rangle_N = \frac{\sum_{x>0} x^2 Q_N^b(0, x)}{N \sum_{x>0} Q_N^b(0, x)}$$

and the same argument as for Theorem 2 leads (after some calculation) to

$$\lim_{N\uparrow\infty} N^{-1} \langle (h_N - h'_N)^2 \rangle_N = \lim_{z\uparrow z_b} \frac{\sum_{x>0} x^2 G_b(z\,; 0, x)}{\sum_{x>0} z \frac{\partial}{\partial z} G_b(z\,; 0, x)} = 4e^{-K}(1+2e^{-K})^{-1}$$

which can be verified to be identical to the analogous result for independent walks...

## 3. AN S. O. S. MODEL OF INTERFACIAL WETTING

We consider two interfaces at heights respectively

$$h_0 = 0\,;\ h_i \in \mathbb{Z} : \quad i = 1, \ldots, N-1\,;\ h_N = 0$$

and

$$h'_0 = 0\,;\ h'_i \in \mathbb{Z} : \quad i = 1, \ldots, N-1\,;\ h'_N = 0$$

with

$$h_i \geq h'_i \tag{3.1}$$

and restricted variations:

$$\begin{cases} h_i - h_{i-1} = 1, 0, -1 & i = 1, \ldots, N \\ h'_i - h'_{i-1} = 1, 0, -1 & i = 1, \ldots, N \end{cases} \tag{3.2}$$

The model should describe the possible coexistence of three phases at low temperature: a phase A above $h_i$, a phase B below $h'_i$, and possibly a phase C in between.

The Boltzmann factor is:

$$\exp\left\{ - \mathrm{K} \sum_{1}^{\mathrm{N}} (1 + | h_i - h_{i-1} |) - \mathrm{K}' \sum_{1}^{\mathrm{N}} (1 + | h'_i - h'_{i-1} |) - (\mathrm{K}'' - \mathrm{K} - \mathrm{K}') \sum_{1}^{\mathrm{N}} \delta_{h_i,h'_i}(1 + | h_i - h_{i-1} | \delta_{h_{i-1},h'_{i-1}}) \right\}$$
$$\times \exp\left\{ - \mathrm{K}''' \sum_{1}^{\mathrm{N}} (\delta_{h_i,h'_i} - \delta_{h_{i-1},h'_{i-1}})^2 \right\}. \tag{3.3}$$

The couplings K, K′, K″ are associated respectively to the AC, CB and AB interfaces, and K‴ is associated to the meeting of the three interfaces at a point (see figure 2).

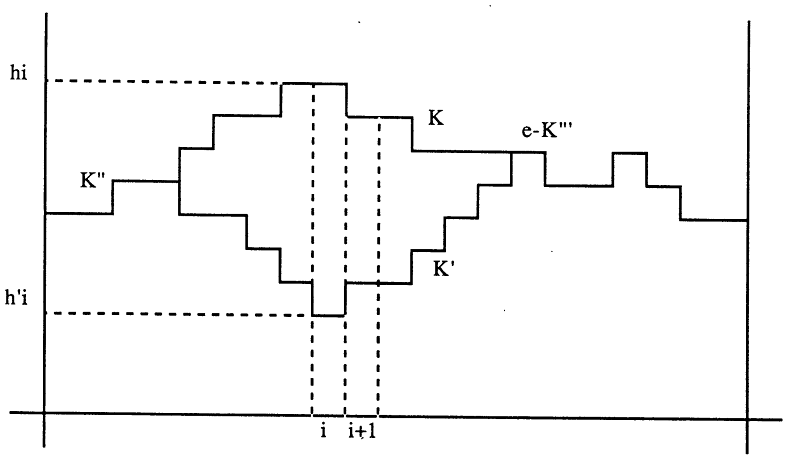


FIG. 2. — A configuration of the interface.

We are interested primarily in the phase diagram of this model, in terms of the existence or non existence of a macroscopic phase C which would wet the AB interface. The order parameter will be

$$\chi_{\mathrm{ACB}} = \lim_{\mathrm{N}\to\infty} \left\langle \frac{1}{\mathrm{N}} \sum_{1}^{\mathrm{N}} \delta_{h_i,h'_i} \right\rangle \tag{3.4}$$

$$\chi_{\mathrm{ACB}} = \begin{cases} = 0 & \text{wetting of AB by C} \\ \neq 0 & \text{dry AB interface} \end{cases}$$

Our general result is the following theorem:

THEOREM 3. — *Let*

$$m(K, K', K'', K''') = \frac{3e^{-K}+3e^{-K'}+2e^{-K-K'}-[e^{-2K}+e^{-2K'}+2e^{-K-K'}+4e^{-2K-K'}+4e^{-K-2K'}]^{1/2}}{2e^{K'''}e^{K''}[(e^{-K}+2e^{-2K})(e^{-K'}+2e^{-2K'})-(e^{-K''}+2e^{-2K''})]}$$

*if* $m < 1$, *then the phase C wets the AB interface and* $\chi_{ACB} = 0$
*if* $m > 1$, *then* $\chi_{ACB} > 0$.

*Proof.* — The model can be considered as a necklace of beads of C in the AB interface [1, 3]. The free energy is given by

$$f(K, K', K'', K''') = -\lim_{N\to\infty} \frac{1}{N} \log Z_N(K, K', K'', K''')$$

where $Z_N$ is the sum of the Boltzmann factor (3.3) over the configuration space

$$h_0 = 0; \quad h_1, h_2, \dots h_N \in \mathbb{Z} \tag{3.5}$$
$$h'_0 = 0; \quad h'_1, h'_2, \dots h'_N \in \mathbb{Z},\ h'_N = h_N$$

with the constraints (3.1) and (3.2). The end point $h'_N = h_N$ is now free, but we shall prove that this does not affect the theorem. We then consider the generating function

$$G(z) = \sum_{N=0}^{\infty} z^N Z_N = 1 + z(e^{-K''}+2e^{-2K''}) + z^2\{(e^{-K''}+2e^{-2K''})^2 + e^{-K'''}(2e^{-2K-2K'}+e^{-K-3K'}+e^{-3K-K'})e^{K''K'''})\} + O(z^3).$$

The necklace representation gives

$$G(z) = \frac{G_a(z)}{1 - v^2 G_a(z) G_b(z)} \tag{3.6}$$

with

$$G_a(z) = \sum_{N=0}^{\infty} z^N Q_N^a$$

where $Q_N^a$ is the sum of the Boltzmann factor (3.3) over the configurations satisfying (3.1), (3.2), (3.5) and

$$h'_i = h_i: \qquad i = 0, \dots N;$$

and

$$v^2 = e^{-2K'''-K''+K+K'} \tag{3.7}$$

$$G_b(z) = \sum_{N=0}^{\infty} z^N Q_N^b$$

where $v^2 Q_N^b$ equals the sum of the Boltzmann factor (3.3) over the configurations satisfying (3.1), (3.2), (3.5) and

$$h_i > h'_i : \qquad i = 1, \dots N - 1 .$$

The function

$$G_a(z) = [1 - z(e^{-K''} + 2e^{-2K''})]^{-1} \tag{3.8}$$

is analytic in $|z| < z_a = (e^{-K''} + 2e^{-2K''})^{-1}$ and diverges at $z = z_a$. The function $G_b(z)$ is analytic in

$$|z| < z_b = (e^{-K} + 2e^{-2K})^{-1}(e^{-K'} + 2e^{-2K'})^{-1} \tag{3.9}$$

and is singular but finite at $z = z_b$ as we proved in Theorem 1. The closest singularity of $G(z)$ will be either $z_b$ or the solution $z = z_{ab}$ of

$$1 - v^2 G_a(z) G_b(z) = 0 .$$

The free energy of the model will therefore be $f = \log \mathrm{Min}\{ z_b, z_{ab} \}$ and the AB interface will be

$$\text{— wet} \quad \text{if} \quad 1 - v^2 G_a(z_b) G_b(z_b) > 0 \tag{3.10}$$

$$\text{— dry} \quad \text{if} \quad 1 - v^2 G_a(z_b) G_b(z_b) < 0 \tag{3.11}$$

The above general discussion is adapted from [*1*]. Using (3.8) for $G_a(z_b)$ and Theorem 1 for $G_b(z_b)$, we find that (3.10) and (3.11) correspond respectively to $m < 1$ and $m > 1$ in Theorem 3. It remains to prove that fixing $h_N = h'_N = 0$ does not affect the transition line, and to check the assertion about $\chi_{ABC}$.

Let $Z_N^{(k)}$ denote the sum of the Boltzmann factor (3.3) over the configurations (3.5) with the constraints (3.1), (3.2) and $h_N = h'_N = k$. We shall first show that

$$f(K, K', K'', K''') = - \lim_{N \to \infty} \frac{1}{N} \log Z_N^{(0)}(K, K', K'', K''') . \tag{3.12}$$

Clearly

$$\frac{Z_N^{(0)}}{Z_N} \leq 1 . \tag{3.13}$$

On the other hand:

$$Z_{2N+1}^{(0)} \geq \tilde{Z}_{2N+1}$$

where $\tilde{Z}_{2N+1}$ is the sum over the configurations with

$$h_0 = h'_0 = h_{2N+1} = h'_{2N+1} = 0 \qquad \text{and} \qquad h_N = h_{N+1} = h'_N = h'_{N+1} .$$

Therefore:

$$\tilde{Z}_{2N+1} = e^{-K''} \sum_{k=-N}^{N} [Z_N^{(k)}]^2$$

$$\geq e^{-K''} \left[ \sum_{k=-N}^{N} Z_N^{(k)} \right]^2 \times \frac{1}{2N+1} = e^{-K''} \frac{[Z_N]^2}{2N+1} \tag{3.14}$$

where the inequality in (3.14) follows the Schwartz inequality. Then (3.12) follows from (3.13) and (3.14).

Whenever the free energy $f(\mathrm{K}, \mathrm{K}', \mathrm{K}'', \mathrm{K}''')$ is differentiable with respect to K″ we have

$$\frac{\partial f}{\partial \mathrm{K}''} = \lim_{\mathrm{N}\to\infty} \frac{1}{\mathrm{N}} \left\langle \sum_1^{\mathrm{N}} \delta_{h_i,h'_i} + |h_i - h_{i-1}| \, \delta_{h_{i-1},h'_{i-1}} \right\rangle .$$

Thus if $m(\mathrm{K}, \mathrm{K}', \mathrm{K}'', \mathrm{K}''') = v^2 \mathrm{G}_a(z_b)\mathrm{G}_b(z_b) < 1$, then $f$ equals $\log z_b$ which is independant of K″. Therefore $\dfrac{\partial f}{\partial \mathrm{K}''}$ and also $\chi_{\mathrm{ABC}}$ equal zero since $|h_i - h_{i-1}| \leq 1$.

If $m(\mathrm{K}, \mathrm{K}', \mathrm{K}'', \mathrm{K}''') > 1$, since $\mathrm{G}_a(z)$ and $\mathrm{G}_b(z)$ are positive increasing functions of $z$ $(z \geq 0)$, there exists $z_{ab}$, $0 < z_{ab} < z_b$, for which

$$v^2 \mathrm{G}_a(z_{ab})\mathrm{G}_b(z_{ab}) = 1 \tag{3.15}$$

and $f = \log z_{ab}$. Denoting $z'_{ab}$ the derivative of $z_{ab}$ with respect to K″ and $\mathrm{G}'_b(z)$ the derivative of $\mathrm{G}_b(z)$, we get from (3.15)

$$z'_{ab} = \frac{(e^{-\mathrm{K}''} + 4e^{-2\mathrm{K}''})z_{ab} + v^2 \mathrm{G}_b(z_{ab})}{e^{-\mathrm{K}''} + 2e^{-2\mathrm{K}''} + v^2 \mathrm{G}'_b(z_{ab})}$$

provided that $e^{-\mathrm{K}''} + 2e^{-2\mathrm{K}''} + v^2 \mathrm{G}'_b(z_{ab}) \neq 0, \neq +\infty$.

We now use that, $\mathrm{G}_b(z_{ab})$ and $z_{ab}$ are strictly positive; on the other hand since $\mathrm{G}_b(z)$ and $\mathrm{G}'_b(z)$ are analytic in $z < z_b$, then $\mathrm{G}'_b(z_{ab}) < +\infty$.

Therefore $\dfrac{\partial f}{\partial \mathrm{K}''} = \dfrac{z'_{ab}}{z_{ab}}$ and thus also $\chi_{\mathrm{ABC}}$ are strictly positive for $m > 1$.

Analogously we obtain that the averaged number of bubbles

$$\frac{\partial f}{\partial \mathrm{K}'''} = \lim_{\mathrm{N}\to\infty} \frac{1}{\mathrm{N}} \left\langle \sum_1^{\mathrm{N}} [\delta_{h_i,h'_i} - \delta_{h_{i-1},h'_{i-1}}]^2 \right\rangle$$

is zero for $m < 1$, and strictly positive for $m > 1$.

## 4. A WETTING PROBLEM IN THE POTTS MODEL

The 2-dimensional $q$-state Potts model is defined as follows: at each lattice site $i \in \mathbb{Z}^2$ there is a variable $\sigma_i = 0, 1, \ldots q-1$; the hamiltonian in a finite box $\Lambda \subset \mathbb{Z}^2$ is

$$\mathrm{H}_{\Lambda.\mathrm{b.c.}} = \sum_{\langle i;j\rangle \subset \Lambda} [\delta_{\sigma_i \cdot \sigma_j} - 1] - \sum_{\langle i;j\rangle;\, i\in\Lambda;\, j\in\Lambda^c}' [\delta_{\sigma_i \cdot \tilde{\sigma}_j} - 1]. \tag{4.1}$$

The two sums are over nearest neighbour pairs, in the second sum $\tilde{\sigma}_j$ is fixed outside $\Lambda$ and represents boundary conditions (b.c.). In particular we shall consider:

— the ordered b.c. obtained by fixing $\tilde{\sigma}_j = \alpha$ outside $\Lambda$, $\alpha = 0, 1, \ldots q-1$,
— the free b.c. ($f$) where the second sum in (4.1) is omitted,
— the mixed $(\alpha, \alpha')$ b.c. defined for a rectangular box by fixing $\tilde{\sigma}_j = \alpha$ on the top half and $\tilde{\sigma}_j = \alpha'$ on the bottom half.

The Gibbs measures in $\Lambda$ are

$$\mu_{\Lambda,\mathrm{b.c.}} = [Z_\Lambda^{\mathrm{b.c.}}(\beta)]^{-1} e^{-\beta H_{\Lambda,\mathrm{b.c.}}}.$$

Whenever $q > 4$ the model exhibits a first order phase transition in the temperature: the magnetization and the derivative of the free energy with respect to $\beta$ are discontinuous at the self dual point $\beta_t = \log(\sqrt{q}+1)$ ([5] [6] [7]).

The phase diagram for $q$ large enough is as follows [8]:

For $\beta > \beta_t$ there are $q$ phases (translation invariant Gibbs states) obtained as thermodynamic limits of finite volume Gibbs states with ordered b.c.

For $\beta < \beta_t$ there is one state (disordered phase) obtained as thermodynamic limit of finite volume Gibbs states with free b.c.

At $\beta_t$ there are exactly $q + 1$ phases, the $q$ ordered ones and the disordered one. In higher dimensions the phase diagram is the same, with $\beta_t \approx \frac{1}{d} \log q$.

The mixed $(\alpha, \alpha')$ b.c. produce an interface and the question is whether this interface is wetted or not by the disordered (free) phase at $\beta_t$.

The 3-dimensional case where the interface is rigid (for $q$ large enough) has been discussed by Bricmont and Lebowitz [9], for $q$ large; their argument does not apply in dimension 2 because the interface fluctuates. However wetting is also expected as well as by Monte-Carlo simulations [10]. We shall consider here the 2-dimensional case.

The surface tension between two ordered phases is microscopically defined by:

$$\tau^{\alpha,\alpha'}(\beta) = \lim_{N\to\infty} \lim_{M\to\infty} -\frac{1}{N} \log \frac{Z_\Lambda^{\alpha,\alpha'}(\beta)}{[Z_\Lambda^{\alpha}(\beta) Z_\Lambda^{\alpha'}(\beta)]^{1/2}} \tag{4.2}$$

where $\Lambda$ is a rectangular box: $\Lambda \equiv \{(i^1, i^2) \in \mathbb{Z}^2 / 0 \le i^1 \le N, |i^2| \le M\}$.

By contour estimates based on the duality transformation [7] one can show that the surface tension between two ordered phases is strictly positive for $\beta \ge \beta_t$: $Z_\Lambda^{\alpha,\alpha'}(\beta)$ can be expanded in terms of contours as follows (see ref. [7]):

$$Z_\Lambda^{\alpha,\alpha'}(\beta) = \sum_{\gamma,\gamma'} g(\gamma, \gamma') Z_U^0(\beta) Z_D^0(\beta) Z_I^f(\beta) e^{-\beta(|\gamma\cap\gamma'| + |\gamma\cup\gamma'\setminus\gamma\cap\gamma'|)}. \tag{4.3}$$

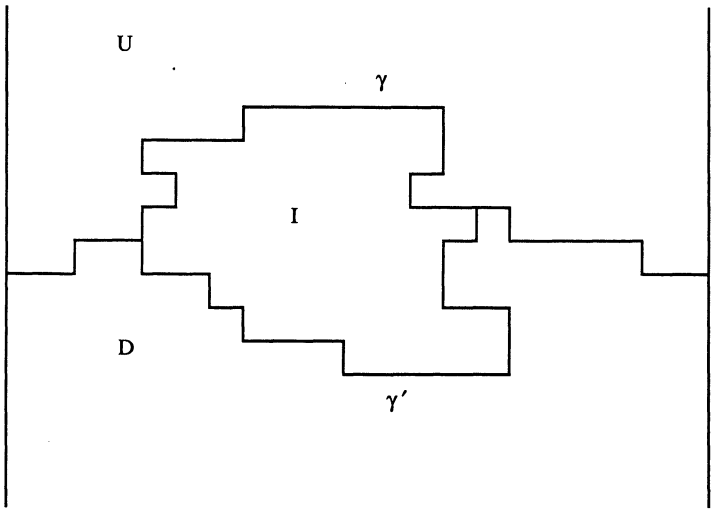


FIG. 3. — Interface in the Potts model separating ordered-ordered (U-D) and ordered-disordered (U-I, D-I) configurations.

The contours are lines in the dual lattice

$$(\mathbb{Z}^2)^* = \left\{ \left( i^1 + \frac{1}{2}, i^2 + \frac{1}{2} \right) \Big/ (i^1, i^2) \in \mathbb{Z}^2 \right\}$$

with endpoints $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(N+\frac{1}{2}, -\frac{1}{2}\right)$; $|\gamma|$ is the number of bonds of $\gamma$. Each bond of $\gamma$ is dual of a bond $ij$ with $\sigma_i = \alpha$, $\sigma_j \neq \alpha$ and each bond of $\gamma'$ is dual of a bond $ij$ with $\sigma_i = \alpha'$, $\sigma_j \neq \alpha'$. The configurations in the region U above $\gamma$ have ordered $\alpha$ b.c., the configuration in the region D below $\gamma'$ have ordered $\alpha'$ b.c. while in the region I between $\gamma$ and $\gamma'$ the spins on the boundary take all values exept $\alpha$ or $\alpha'$; for $q$ large this corresponds to free b.c. up to the corrective tem $g(\gamma, \gamma')$ which is expected to behave like $e^{-O(1/q)}$ and for which one can easily proof a lower bound $e^{-O(1/\sqrt{q})}$ and an upper bound 1.

By duality transformation on $Z_\Lambda^f(\beta)$, (4.3) gives ([7]):

$$\frac{Z_\Lambda^{\alpha.\alpha'}(\beta_t)}{[Z_\Lambda^{\alpha}(\beta_t)Z_\Lambda^{\alpha'}(\beta_t)]^{1/2}} = \frac{Z_\Lambda^{\alpha.\alpha'}(\beta_t)}{Z_\Lambda^{\alpha}(\beta_t)} = \sum_{\gamma;\gamma'} g(\gamma, \gamma')h(\gamma, \gamma')e^{-\beta_t|\gamma_a|}e^{-\left[\frac{\beta_t}{2} + O(1/\sqrt{q})\right].|\gamma_b|} \quad (4.4)$$

where

$$\gamma_a = |\gamma \cap \gamma'|; \qquad \gamma_b = |\gamma \cup \gamma' \backslash \gamma \cap \gamma'|$$

ant the term

$$h(\gamma, \gamma') = [Z^0_\Lambda(\beta_t)]^{-1} Z^0_U(\beta_t) Z^0_D(\beta_t) Z^0_{I^*}(\beta_t)$$

has a lower bound $e^{-.O(1/\sqrt{q})|\gamma \cup \gamma'|}$ and an upper bound 1 (the lattice sites of I* are the centers of the squares whose corners are sites of I).

Whenever $q$ is large enough the overhangs are unprobable and thus a necklace S. O. S. of interfacial wetting should be a good approximation, and thus also the model considered in the preceding section: if we restrict in (3.4) the summation over R. S. O. S. contours we obtain the model of section 2 with:

$$K = K' = \frac{\beta_t}{2} + O\left(\frac{1}{\sqrt{q}}\right)$$

$$K'' = \beta_t + O\left(\frac{1}{\sqrt{q}}\right)$$

$$K''' = 0 .$$

Since we are interested in $q$ very large one sees from theorem 3 that we are in the wetting case since

$$m = \frac{1}{2}[1 + O(q^{-1/4})] < 1 .$$

The above discussion strongly suggests that this is also the case for the original model.

## 5. THE WETTING TRANSITION IN THE S. O. S. CHIRAL POTTS MODEL

Consider the 3-state chiral Potts model on $\mathbb{Z}^2$, with the Boltzmann factor

$$\exp\left\{\beta J \sum_{\langle i;j\rangle}\left(\cos\left[\frac{2\pi}{3}(n_i - n_j + \vec{\Delta}\cdot\overrightarrow{i-j})\right] - \cos\left[\frac{2\pi}{3}\vec{\Delta}\cdot\overrightarrow{i-j}\right]\right)\right\}$$

where the spin variables $n_i$ take value 0, 1 or 2, and the chiral field $\vec{\Delta}$ is chosen parallel to one axis of the lattice. The mirror symmetry with respect to an axis perpendicular to $\vec{\Delta}$ is broken, whence the name « chiral » (especially in the 3-dimensional analog of the model). We are interested in the case $0 \leq |\vec{\Delta}| < \frac{1}{2}$ and $\beta$ large, where three translation invariant states are associated respectively to

$$\langle n_i \rangle \approx 0, 1 \text{ or } 2$$

To investigate interfaces and non translation invariant states, let us first look at the weight of a piece of contour separating nearest neighbour spins $n_i \neq n_j$. For definiteness, we choose $\vec{\Delta}$ in the positive vertical direction. Then the elementary Boltzmann factor for a piece of contour of lenght one will be

$$'' \cdot | \cdot '' = \exp\left(-\frac{3}{2}\beta J\right) \equiv e^{-K}$$

$$''\frac{1}{0}'' = ''\frac{2}{1}'' = ''\frac{0}{2}'' = \exp\left\{-\sqrt{3}\beta J \sin\frac{2\pi}{3}\left(\frac{1}{2}-\Delta\right)\right\} \equiv w \quad (5.1)$$

$$''\frac{0}{1}'' = ''\frac{1}{2}'' = ''\frac{2}{0}'' = \exp\left\{-\sqrt{3}\beta J \sin\frac{2\pi}{3}\left(\frac{1}{2}+\Delta\right)\right\} \equiv u$$

Of course

$$e^{-K} = u = w \qquad \text{if} \quad \Delta = 0$$

and more important for our purposes

$$u < w \qquad \text{if} \quad 0 < \Delta < \frac{1}{2}$$

and

$$u = w^2 \qquad \text{if} \quad \Delta = \frac{1}{4}.$$

We now choose boundary conditions $n_i = 0$ on the lower boundary and $n_i = 2$ on the upper boundary. At low enough temperature, a reasonable approximation is the S. O. S. limit where overhangs of interfaces and bubbles in the bulk phases are excluded. More precisely, the allowed configurations here will be such that $n_i$ is non decreasing in the direction of the chiral field. The question then is whether the 0-2 interface is wet, or not, by an intruding layer of the 1-phase. This question has been addressed by Huse Szpilka and Fisher [*2*], who solve the model for $\vec{\Delta}$ along the diagonal, and also give a low temperature expansion for $\vec{\Delta}$ vertical (or in fact horizontal).

Here we extend the result of [*2*] by considering an arbitrary temperature, still in the S. O. S. model. We do not have the unphysical condition in [*2*] whereby the 0-1 and 1-2 interfaces were not allowed to have simultaneous excitations. For this reason, our results, taken at low temperature, differ slightly from [*2*].

The model is essentially the same as in section 2, but the parameters are different and the Boltzmann factor (3.3) should be replaced by

$$\exp\left\{(\log u)\sum_1^N \delta_{h_i,h'_i} + 2(\log w)\sum_1^N (1-\delta_{h_i,h'_i})\right\}$$

$$\times \exp\left\{-K\sum_1^N [\,|h_i - h_{i-1}| + (1-\delta_{h_i,h'_i}\delta_{h_{i-1},h'_{i-1}})\,|h'_i - h'_{i-1}|\,]\right\}. \quad (5.2)$$

This leads to $G_b^\Delta(z)$ and $z_b^\Delta$ different from $G_b$ and $z_b$ of section 3, with

$$G_b^\Delta(z_b^\Delta) = G_b(z_b) = 2e^{-K}(1+2e^{-K})^{-2}\left[\frac{3}{2}+\frac{1}{2}e^{-K}-\frac{1}{2}(1+2e^{-K})^{1/2}\right]$$

$$z_b^\Delta = w^{-2}(1+2e^{-K})^{-2}$$

and

$$G_a^\Delta(z) = [1 - zu(1+2e^{-K})]^{-1}$$

$$G_a^\Delta(z_b^\Delta) = [1 - w^{-2}u(1+2e^{-K})]^{-1}$$

and

$$v^2 = w^{-2}u\,.$$

We then have the following result:

THEOREM 4. — *Let $Z_N(u,w,K)$ be the sum of the Boltzmann factor (5.2) over the configuration space*

$$h_0 = h'_0 = 0;\quad h_N = h'_N$$

$$h_i \geq h'_i :\quad i = 1, \dots N-1$$

$$h_i - h_{i-1} = 1, 0, -1;\quad h'_i - h'_{i-1} = 1, 0, -1 :\quad i = 1, \dots N$$

*and*

$$\beta\sigma_{0,2}(u, w, K) = -\lim_{N\to\infty}\frac{1}{N}\log Z_N(u, w, K)\,.$$

*Then $\beta\sigma_{0,2}(u, w, K)$ is singular on the « wetting transition line » (see fig. 4) :*

$$u^{-1}w^2 = (1+2e^{-K})^{-2}[1 + 5e^{-K} + e^{-2K} - e^{-K}(1+2e^{-K})^{1/2}]\,. \quad (5.3)$$

*If $u$, $w$, K are given by (5.1), then*

$$u^{-1}w^2 = \exp\left\{-3\beta J\sin\frac{2\pi}{3}\left(\frac{1}{4}-\Delta\right)\right\} \quad \text{and} \quad e^{-K} = \exp\left(-\frac{3}{2}\beta J\right)$$

*and, for $\beta \to \infty$, the transition line obeys*

$$\Delta \approx \frac{1}{4} - \frac{2}{\pi\beta J}e^{-3\beta J}\,. \quad (5.4)$$

*Proof.* — According to section 3, the wetting transition is given by

$$v^2 G_a^\Delta(z_b^\Delta)G_b^\Delta(z_b^\Delta) = 1$$

which gives (5.3), from which (5.4) follows easily.

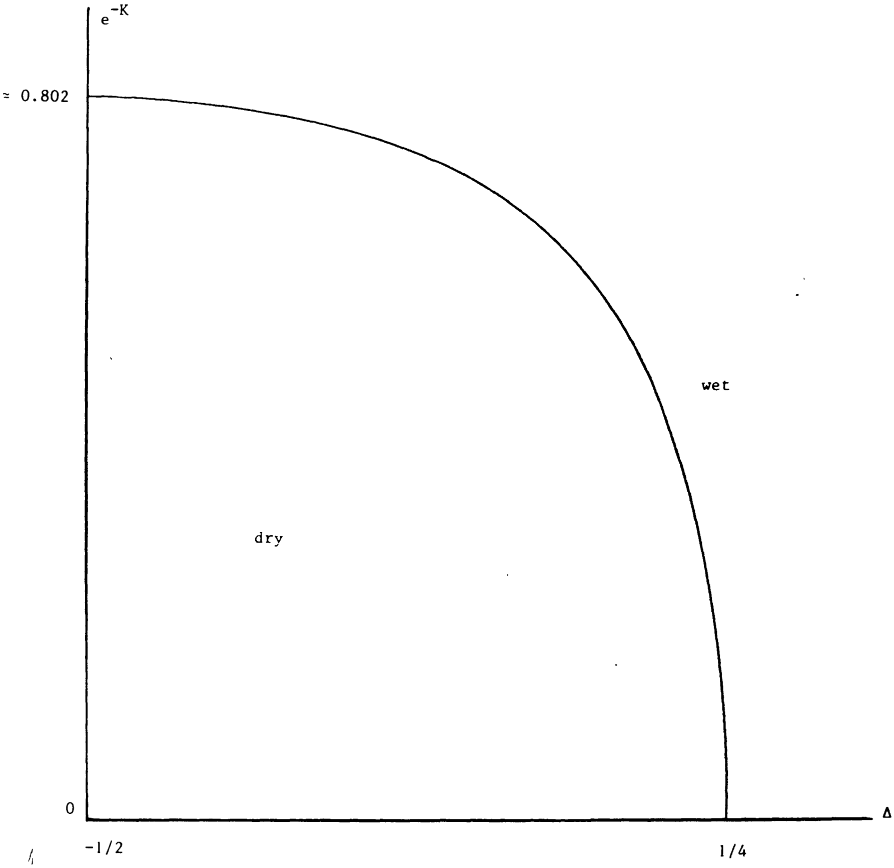


FIG. 4. — Phase diagram of the S. O. S. chiral clock model, where the 0-2 interface may be wet by a layer of the 1 phase. The symmetries in this case are $\Delta \to \Delta + 3$ and $\Delta + \frac{1}{2} \to -\left(\Delta + \frac{1}{2}\right)$.


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