

ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 48, n° 2 (1988), p. 97-104

http://www.numdam.org/item?id=AIHPA_1988__48_2_97_0

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Quantized fields and operators on a partial inner product space

by

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ABSTRACT. — We investigate the connection between the space $\text{Op}V$ of all operators on a partial inner product space V and the weak sequential completion of the $*$ -algebra $L^+(V^\#)$ of all operators X such that $V^\# \subset D(X) \cap D(X^*)$ and both X and its adjoint X^* leave $V^\#$ invariant. This connection allows us to describe quantized fields at a point as mappings from the Minkowski space-time into $\text{Op}V$.

RÉSUMÉ. — Nous analysons la relation entre l'espace $\text{Op}V$ de tous les opérateurs sur un espace à produit interne partiel V et la complétion séquentiellement faible de l' $*$ -algèbre $L^+(V^\#)$ de tous les opérateurs X tels que $V^\# \subset D(X) \cap D(X^*)$ et tels que X et son adjoint X^* laissent $V^\#$ invariant. Cette relation nous permet de décrire des champs quantiques en un point comme des applications de l'espace-temps de Minkowski dans $\text{Op}V$.

1. INTRODUCTION

The fundamental concept of Wightman axiomatics is the concept of quantized field $A(x)$ at a point x , which is usually defined [1] as an operator-valued distribution on some space of test functions (x is the four-dimensional coordinate of space-time).

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Let \mathcal{D} be a dense linear manifold of a Hilbert space \mathcal{H} and denote by $L^+(\mathcal{D})$ the $*$ -algebra of all operators X such that $\mathcal{D} \subset D(X) \cap D(X^*)$ and both X and its adjoint X^* leave \mathcal{D} invariant. It has been first proposed by Haag [2] that a quantized field $A(x)$ at any point x should be described in terms of sesquilinear forms on $\mathcal{D} \times \mathcal{D}$, corresponding to the heuristically defined mapping $(f, g) \mapsto (A(x)f, g)$. This idea has been particularized by Ascoli, Epifanio and Restivo [3] in such a way that these sesquilinear forms may be considered as elements of the weak sequential completion $\overline{L^+(V^\#)}^w$ of $L^+(V^\#)$.

On the other hand, it is well known that if V is an arbitrary partial inner product (PIP) space [4], which is quasi complete in its canonical Mackey topology $\tau(V, V^\#)$, then the space $L^+(V^\#)$ is isomorphic to the $*$ -algebra $\text{Reg } V$ of all regular operators on V [5].

In this note, after a brief recall in Section 2 of basic facts on PIP spaces and operators on them [4-7] we investigate in Section 3 the connection between the space $\text{Op}V$ of all operators on a PIP space V , and the weak sequential completion of $L^+(V^\#)$. In particular, we show that if V is an arbitrary PIP space, and $\langle V^\#, V \rangle$ is a reflexive dual pair, then $\text{Op}V$ is isomorphic to $\overline{L^+(V^\#)}^w$, which means that a quantized field at a point may be considered as a mapping from the Minkowski space-time M into $\text{Op}V$. This corresponds to the idea that a field at a point is a limit of observables localized in a shrinking sequence of space-time regions [8] i.e. $A(x) = w - \lim_{n \rightarrow \infty} A(f_n)$ where $f_n \rightarrow \delta_x$ (Dirac delta at the point $x \in M$) in the topology of the dual $\mathcal{S}'(M)$ of the Schwartz space $\mathcal{S}(M)$ of fast decreasing C^∞ -functions on M .

At this stage we should mention some related works on the mathematical formulation of point like fields as operators on some PIP space. In [9], extending the machinery of Fock space (a symmetric tensor algebra over a Hilbert space), Grossmann defines the unsmeared free field at a point as an operator on some nested Hilbert space [10]. Grossmann's approach is summarized in [4]. In [11], Nelson defines a Euclidian free field as an operator on the PIP space corresponding to the scale built from the Hamiltonian. This fact was already noticed by Antoine and Karwowski [12] and extensively used by Fredenhagen and Hertel [8].

Consider on \mathcal{D} (a dense linear subspace of \mathcal{H}) a topology t finer than the norm-topology and let $\mathcal{D}'[t]$ be the topological dual of \mathcal{D} , equipped with the strong dual topology t' . Let $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ be the set of all continuous operators from $\mathcal{D}[t]$ into $\mathcal{D}'[t']$. It has been shown [13] that if

$$\mathcal{D} = \mathcal{D}^\infty(T) = \bigcap_{n>0} D(T^n)$$

(where T is any self-adjoint operator in \mathcal{H}) and \mathcal{D} is equipped with the t_T -topology defined by the family of seminorms $\phi \rightarrow \|T^n \phi\|$, $n \in \mathbb{N}$,

then $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ is a topological quasi $*$ -algebra with distinguished algebra $L^+(\mathbf{V}^\#)$ (This result has been generalized in [14] to the case of arbitrary domains \mathcal{D}).

Recently, in their study of point-like fields, Epifanio and Trapani [15] have exploited systematically the quasi $*$ -algebra structure of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$. This approach is in fact in the spirit of operators on a PIP space \mathbf{V} , since OpV is isomorphic to $\mathcal{L}(\mathbf{V}^\#, \mathbf{V})$ [4].

In Section IV we introduce the concept of OpV -valued fields and Wightman fields. OpV -valued fields may be used in order to give a precise mathematical meaning to relations of the type

$$A(f) = \int d^4x f(x)A(x), \quad f \in \mathcal{S}(M).$$

In this Section, using some results of Ref [15] we compare our approach to that of Fredenhagen and Hertel [8].

2. PIP-SPACES AND OPERATORS ON THEM [4-7]

A PIP-space \mathbf{V} is a complex vector space with the following structure:

i) $\mathcal{I} = \{V_r, r \in I\}$ is a collection of vector subspaces of \mathbf{V} which covers \mathbf{V} and is an involutive lattice with respect to set intersection, vector sum and lattice involution: $V_r \leftrightarrow V_{\bar{r}}$.

Besides elements of \mathcal{I} , we consider the extreme spaces:

$$\mathbf{V}^\# \equiv \bigcap_{r \in I} V_r \quad \text{and} \quad \mathbf{V} \equiv \bigcup_{r \in I} V_r$$

ii) A nondegenerate hermitian form $\langle . | . \rangle$ (the partial inner product) is defined on $\bigcup_{r \in I} V_r \times V_{\bar{r}}$.

iii) There exists a unique element $0 = \bar{0}$ in I such that $V_0 = V_{\bar{0}} \equiv \mathcal{H}$ is a Hilbert space with respect to $\langle . | . \rangle$.

The nondegeneracy assumption $(\mathbf{V}^\#)^\perp = \{0\}$ implies that every pair $\langle V_r, V_{\bar{r}} \rangle$, as well as $\langle \mathbf{V}^\#, \mathbf{V} \rangle$ is a dual pair with respect to the form $\langle . | . \rangle$. We may therefore equip each V_r with its Mackey topology $\tau(V_r, V_{\bar{r}})$ and similarly for $\mathbf{V}^\#, \mathbf{V}$.

An operator A on a PIP space \mathbf{V} is a map $D(A) \rightarrow \mathbf{V}$, where $D(A)$ is the largest union of V_r 's such that the restriction of A to any of them is linear and continuous into \mathbf{V} .

The set of all operators on \mathbf{V} , denoted by OpV is isomorphic to $\mathcal{L}(\mathbf{V}^\#, \mathbf{V}) = \{ \text{linear continuous maps } \mathbf{V}^\# \rightarrow \mathbf{V} \}$. Equivalently OpV is isomorphic to $B(\mathbf{V}^\#, \mathbf{V}^\#) = \{ \text{separately continuous sesquilinear forms on} \}$

$V^\# \times V^\#$ }. Thus, $\text{Op}V$ is a vector space. Moreover, $\text{Op}V$ carries an involution $A \leftrightarrow A^\times$ (adjoint of A), but it is not an algebra since the multiplication is not always defined. Such sets are called partial $*$ -algebras [16].

An operator on a PIP-space V is called regular [5], if $D(A) = D(A^*) = V$. Equivalently, a regular operator is a linear continuous map of $V^\#$ into itself, which maps V into itself continuously. The set of all regular operators on V , denoted by $\text{Reg } V$ is a $*$ -algebra.

We assume that V is quasi complete in its Mackey topology. Then $\text{Reg } V$ is isomorphic to the $*$ -algebra $L^+(V^\#)$ of all closable operators on \mathcal{H} which, together with their (Hilbertian) adjoint leave $V^\#$ invariant. Actually almost all PIP-spaces are quasi complete in the $\tau(V, V^\#)$ -topology, the only known exceptions being quite pathological [17].

We will endow $\text{Op}V$ with the weak topology defined by the family of seminorms

$$A \mapsto |\langle Af, g \rangle|; \quad f, g \in V^\#.$$

On $\text{Reg } V \simeq L^+(V^\#)$ we will consider the weak topology inherited from $\text{Op}V$.

3. $\text{Op}V$ AND THE WEAK SEQUENTIAL COMPLETION OF $L^+(V^\#)$

Following [3] we denote by $S_{V^\#}$ the space of all sesquilinear forms on $V^\# \times V^\#$. It has been proved in [3] that the space $S_{V^\#}$ endowed with the topology of pointwise convergence given by the set of seminorms:

$$F \mapsto |F(f, g)|; \quad f, g \in V^\#$$

is isomorphic to the weak completion of $L^+(V^\#)$, i. e. in notations of [3]

$$S_{V^\#} \simeq \widehat{L^+(V^\#)^w}.$$

On the other hand, it is clear that $S_{V^\#}$ contains the space $\text{Op}V$ which is isomorphic to the space $B(V^\#[\tau], V^\#[\tau])$ of all Mackey separately continuous sesquilinear forms on $V^\# \times V^\#$.

In what follows, we want to answer the following question: given a PIP space V , when is $\text{Op}V$ isomorphic to the weak sequential completion $\widehat{L^+(V^\#)^w}$ of $L^+(V^\#)$? If this isomorphism exists, then the sesquilinear forms which describe quantized fields at points may be considered as elements of $\text{Op}V$ equipped with the weak topology.

In general, for a given PIP space V , whenever $\text{Op}V$ is weakly sequentially complete, we have the following relation between $\text{Op}V$ and $\widehat{L^+(V^\#)^w}$:

$$\widehat{L^+(V^\#)^w} \subseteq \text{Op}V \subseteq \widehat{L^+(V^\#)^w} \simeq S_{V^\#}.$$

We show that this relation holds if in particular $\langle V^\#, V \rangle$ is reflexive dual pair. Indeed we have the following:

PROPOSITION 3.1. — Let V be a PIP space. If $\langle V^\#, V \rangle$ is a reflexive dual pair, then $\text{Op}V$ is weakly sequentially complete.

Proof. — Let $\{T_n\}$ be a weak Cauchy sequence in $\text{Op}V$, i.e. $\forall f \in V^\#, \{T_n f\}$ is a weak Cauchy sequence in V .

Since $\langle V^\#, V \rangle$ is reflexive, it follows that $V^\#$ and V are quasi complete (i.e. closed bounded sets are complete) with respect to the weak topology and therefore $V^\#$ and V are weakly sequentially complete i.e. $w\text{-}\lim_{n \rightarrow \infty} T_n f = T f \in V$. This shows that T is a map from $V^\#$ into V .

In order to show that T is continuous from $V^\#[\tau(V^\#, V)]$ to $V[\tau(V, V^\#)]$, one uses the dual mapping theory [18]. \square

REMARK 3.2. — For $\text{Reg } V \simeq L^+(V^\#)$ one could also try to perform the same proof as in Proposition 3.1, but in general we do not have $D(T) = D(T^*) = V$. So, in general $L^+(V^\#)$ is not weakly sequentially complete. Actually this fits with results of [3] where it is shown that $\overline{L^+(V^\#)}^w$ may contain elements which are not operators.

The condition of reflexivity of the dual pair $\langle V^\#, V \rangle$ is weak enough to cover most spaces of practical interest, in particular, all spaces of distributions.

Typical instances when the dual pair $\langle V^\#, V \rangle$ is reflexive are [7]:

- . $V^\#$ is a Hilbert space; then so is V .
- . $V^\#$ is a reflexive Banach space; then so is V .
- . $V^\#$ is a reflexive Fréchet space; V is the a reflexive complete DF-space [18].
- . $V^\#$ is a Montel space; then so is V .

Now, given a PIP space V , when is $\text{Op}V$ contained in $\overline{L^+(V)}^w$? Let $A \in \text{Op}V, V^\#$ separable, e_ν an orthonormal basis in $V^\#$ and $P_\nu = |e_\nu\rangle\langle e_\nu|$ the orthogonal projection on e_ν . In the terminology of [5], P_ν is a very regular operator.

Consider the operator $P_j A P_j$. Obviously this operator is regular, since the operator itself as well as its adjoint leave $V^\#$ invariant. Let B_{nm} be the

sequence in $L^+(V^\#)$ defined by $B_{nm} = \sum_{j=1}^n \sum_{j'=1}^m P_j A P_{j'}$.

Since $\{e_\nu\}$ is an orthonormal basis, for all $f \in V^\#$ we have $\sum_\nu P_\nu f = f$, and consequently: $\forall f, g \in V^\#$ we get:

$$\begin{aligned} \lim_{m \rightarrow \infty} \langle B_{nm} f, g \rangle &= \lim_{m \rightarrow \infty} \left\langle \sum_{j=1}^n \sum_{j'=1}^m P_j A P_{j'} f, g \right\rangle = \lim_{m \rightarrow \infty} \left\langle \sum_{j'=1}^m A P_{j'} f, \sum_{j=1}^n P_j g \right\rangle = \\ &= \lim_{m \rightarrow \infty} \left\langle \sum_{j'=1}^m A P_{j'} f, g \right\rangle = \lim_{m \rightarrow \infty} \left\langle \sum_{j'=1}^m P_{j'} f, A^* g \right\rangle = \langle f, A^* g \rangle = \langle A f, g \rangle \end{aligned}$$

Thus, the arbitrary element $A \in \text{OpV}$ is the weak limit of a weakly convergent (hence a weak Cauchy) sequence of $L^+(V^\#)$, i. e. $A \in \overline{L^+(V^\#)}^w$.

We summarize this analysis in the following:

PROPOSITION 3.3. — If V is a PIP space, then $\text{OpV} \subset \overline{L^+(V^\#)}^w$.

Now, putting together Propositions 3.1 and 3.3 we can state our main result (which shows in particular that a quantized field at a point may be considered as an element of OpV).

PROPOSITION 3.4. — Let V be a PIP space. If $\langle V^\#, V \rangle$ is a reflexive dual pair then OpV is isomorphic to $\overline{L^+(V^\#)}^w$.

4. OpV -VALUED FIELDS AND WIGHTMAN FIELDS

In this section we discuss the concepts of OpV -valued and Wightman fields and in particular using some results of Ref [15] we compare our approach to the work of Fredenhagen and Hertel [8].

DEFINITION 4.1. — We call *OpV-valued field* any mapping A from the Minkowski space-time M into OpV , satisfying the following axioms:

1. *Translation invariance*: There exists in the central Hilbert space \mathcal{H} a strongly continuous unitary representation U of the group of translations of M such that $\forall a \in M$, $U(a)V^\# \subset V^\#$ and

$$U(a)A(x)U(a)^{-1} = A(x + a); \quad x \in M.$$

2. *Existence of a translation invariant vacuum*: There exists a vector $\Omega \in V^\#$ such that $\forall a \in M$,

$$U(a)\Omega = \Omega.$$

3. *Spectral postulate*: The eigenvalues of the energy-momentum operator P^μ do not lie outside the forward light cone.

DEFINITION 4.2. — We call (scalar) *Wightman field* over $V^\#$ a mapping A from $\mathcal{S}(M)$ into $L^+(V^\#)$ such that $\forall \phi, \psi \in V^\#$, the mapping from $\mathcal{S}(M)$ into \mathbb{C} defined by $f \mapsto \langle A(f)\phi, \psi \rangle$ is a tempered distribution i. e. it is continuous.

We assume that the Wightman field satisfies the following axioms:

W1: Translation invariance

W2: Existence of a translation invariant vacuum

W3: *Cyclicity of the vacuum*: Ω is a cyclic vector for the algebra generated by the set of operators $\{A(f) \mid f \in \mathcal{S}(M)\}$.

In [8] a field at a point is defined as being a sesquilinear form on $V^\# \times V^\#$ satisfying a *H-bound condition* i. e. it is assumed that there exists a natural number k such that $R^k A(x) R^k$, with $R = (1 + H)^{-1}$ ($H = P^0$ is the energy operator in \mathcal{H}) is a bounded operator in $V^\#$.

DEFINITION 4. 3. — A point-like field $A(x)$ is said to belong to *the class \mathcal{F}* [8] if for some $k \in \mathbb{N}$, the operator $R^k A(0) R^k$, with $A(0) = U(-x) A(x) U(x)$, is a bounded operator.

In order to compare our approach to the one developed in [8] we will restrict ourselves to a special $V^\#$, namely

$$V^\# = \mathcal{D}^\infty(H) = \bigcap_{n>0} D(H^n).$$

We will consider on $V^\#$ the t_H -topology defined by the seminorms:

$$\phi \rightarrow \|H^n \phi\|, \quad n \in \mathbb{N}.$$

Then, $V^\# [t_H]$ is a reflexive Fréchet space.

PROPOSITION 4. 4. — If $x \rightarrow A(x)$ is an OpV -valued field with $V^\# = \mathcal{D}^\infty(H)$, then $A(x)$ satisfies a *H-bound condition*.

Proof. — See e. g. [15, Proposition 6].

COROLLARY 4. 5. — If $V = \mathcal{D}^\infty(H)$, then every OpV -valued field belongs to the class \mathcal{F} .

PROPOSITION 4. 6. — Let $V^\# = \mathcal{D}^\infty(H)$ and $x \rightarrow A(x)$ be an OpV -valued field.

Then $\forall \phi, \psi \in V^\#$ and $f \in \mathcal{S}(M)$, the integral

$$\langle A(f)\phi, \psi \rangle = \int d^4x f(x) \langle A(x)\phi, \psi \rangle$$

converges and defines a Wightman field i. e. $A(f) \in L^+(V^\#)$.

Proof. — See e. g. [15, Proposition 7].

As a consequence of Proposition 4. 6, our approach may be considered as equivalent to that of Ref [8].

ACKNOWLEDGMENTS

It is a pleasure to thank Prof J.-P. Antoine, Dr F. Mathot and Dr C. Trapani for fruitful discussions. I would also like to thank Prof Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where this work was completed. Finally I thank the referee for constructive remarks and for bringing to my knowledge some references.

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(Manuscrit reçu le 16 décembre 1986)

(Version révisée reçue le 28 octobre 1987)