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ERRATUM

The quantum stability problem for time-periodic perturbations of the harmonic oscillator

(Ann. Inst. Henri Poincaré, t. XLVII, n° 1, 1987, 63-83)

by M. Combes

Laboratoire de Physique Théorique et Hautes Énergies,
Bâtiment 211, Université Paris XI, 91405 Orsay, France

In the above mentioned paper there is some error pertaining to the fact that the « forbidden set » I that one has to remove from B (lemma 2.2) should be:

$$I = \{ \Omega \in B_i \mid \inf_j |(b - T_k b)_i| < \gamma |k|^{-\sigma}, \text{ some } k \in \mathbb{Z}^2 \setminus \{0\} \}$$

with Inf instead of Sup, and some instead of $\forall$. But the above I has not a finite Lebesgue measure.

The way out is to modify the paper as follows:

DEFINITION 2.3. — Given a closed Borel set $B \subset \mathbb{R}$ and any non-negative numbers r, σ , let $M_{r,\sigma}(B)$ denote the set of functions A from B to the space of bounded operators in $l^2(\mathbb{Z}^2)$, represented for any $\Omega \in B$ by an infinite matrix $(A_{ij}(\Omega))_{i,j \in \mathbb{Z}^2}$ such that

$$\sup_{\Omega, \Omega' \in B} \sup_{i \in \mathbb{Z}^2} \sum_{j \in \mathbb{Z}^2} e^{r|i-j|} \langle i+j \rangle^\sigma \left(|A_{ij}(\Omega)| + \left| \frac{A_{ij}(\Omega) - A_{ij}(\Omega')}{\Omega - \Omega'} \right| + i \leftrightarrow j \right) \equiv \|A\|_{r,\sigma,B} < \infty.$$

REMARK 2.1. — One sees easily (Schur's lemma), that any $A \in M_{0,0}(B)$ is such that $A(\Omega)$ is a bounded operator in $l^2(\mathbb{Z}^2)$ and this is true *a fortiori* for $M_{r,\sigma}(B)$, $r, \sigma > 0$.

THEOREM 2.1. — Given any $r > 0$, $1 < \sigma < 2$, $B \subset \mathbb{R}$, let $P \in M_{r,4\sigma}(B)$ be such that

$$\|P\|_{r,4\sigma,B} \leq d(\sigma) \gamma^2 \rho^{10\sigma+1}$$

for some $\gamma > 0$, $\rho : 0 < \rho < r$ and $d(\sigma)$ only depending on σ with $\rho \leq \sigma e$.

Let D be the diagonal matrix whose sequence $d_i(\Omega)$ of diagonal elements is

$$d_i(\Omega) = i_1 \Omega + i_2 \quad i = (i_1, i_2) \in \mathbb{Z}^2.$$

Then there exists a closed Borel set $B' \subset B$ s. t.

$$|B \setminus B'| < \gamma \quad (\text{Lebesgue measure})$$

and an invertible $V \in M_{r-\rho, 0}(B')$ satisfying

$$\|V - 1\|_{r-\rho, 0, B} \quad \text{and} \quad \|V^{-1} - 1\|_{r-\rho, 0, B} \leq \frac{\|P\|_{r, 4\sigma, B}}{d(\sigma)\gamma^2 \rho^{10\sigma+1}}$$

such that

$$V^{-1}(D + P)V = \Delta$$

where Δ is a diagonal matrix, whose sequence δ of diagonal elements satisfies

$$\|d - \delta\|_{\mathcal{M}_{B'}} \leq \frac{3}{2} \|P\|_{r, 4\sigma, B}.$$

LEMMA 2.2. — Let $a \in \mathcal{M}_B$ be such that $\|a\|_{\mathcal{M}_B} \leq 1/4$, and let b be the sequence

$$i \in \mathbb{Z}^2 \rightarrow b_i(\Omega) = i_1 \Omega + i_2 + a_i(\Omega).$$

Then for any $\sigma > 1$, there exists a positive constant $C(\sigma)$ such that if:

$$I \equiv \{ \Omega \in B : \exists i \text{ and } j, i \neq j : |b_i(\Omega) - b_j(\Omega)| < \gamma |i - j|^{-\sigma} \langle i + j \rangle^{-2\sigma} / C(\sigma) \}$$

the Lebesgue measure of I satisfies $|I| \leq \gamma$.

Proof. — Given η any positive number $< 1/2$, $i, k \in \mathbb{Z}^2$ we define,

$$I_{ik}(\eta) = \{ \Omega \in B : |b_{\frac{i+k}{2}}(\Omega) - b_{\frac{i-k}{2}}(\Omega)| < \eta \} \quad k \neq 0.$$

It is clear that the first component k_1 of k has to be non-zero, in order that $I_{ik}(\eta)$ be non-empty. But if Ω_1 and Ω_2 both belong to $I_{ik}(\eta)$ we have:

$$|\Omega_1 - \Omega_2| < \frac{2\eta}{|k_1| - 2\|a\|_{\mathcal{M}_B}} \leq \frac{2\eta}{|k_1| - 1/2}$$

which implies that the Lebesgue measure of $I_{ik}(\eta)$ is smaller than $2\eta/|k_1| - 1/2$. Now it is clear that

$$I \subset \bigcup_{\substack{i \in \mathbb{Z}^2 \setminus \{0\} \\ k \in \mathbb{Z}^2 \setminus \{0\}}} I_{ik}(\gamma |k|^{-\sigma} \langle i \rangle^{-2\sigma} C(\sigma)^{-1})$$

and therefore

$$|I| \leq \sum_{\substack{i \in \mathbb{Z}^2 \\ k \in \mathbb{Z}^2 \setminus \{0\}}} \frac{2\gamma \langle i \rangle^{-2\sigma} C(\sigma)^{-1}}{|k|^\sigma (|k_1| - 1/2)} \leq \frac{4\gamma h(\sigma)}{C(\sigma)} \sum_1^\infty \frac{\text{Log } 2N}{N^\sigma} \leq \gamma$$

where

$$h(\sigma) \equiv \sum_{i \in \mathbb{Z}^2} \langle i \rangle^{-2\sigma}$$

and

$$C(\sigma) \equiv 4h(\sigma) \left/ \sum_1^{\infty} N^{-\sigma} \log 2N \right.$$

LEMMA 2.3. — Given $\rho, \sigma, r > 0$, $A \in M_{r,\sigma}(B)$ and $C \in M_{r+\rho,0}(B)$ with $\sigma \leq 2$ and $\rho \leq \sigma e$, we have AC and $CA \in M_{r,\sigma}(B)$ with

$$\|AC\|_{r,\sigma,B} \quad \text{and} \quad \|CA\|_{r,\sigma,B} \leq 2^{2+\sigma/2} \left(\frac{\sigma}{e\rho}\right)^{\sigma} \|A\|_{r,\sigma,B} \|C\|_{r+\rho,0,B}.$$

Proof. — Let $D = AC$. Then $D_{ij} = \sum_{k \in \mathbb{Z}^2} A_{ik} C_{kj}$. But

$$\begin{aligned} \sup_i \sum_j e^{r|i-j|} \langle i+j \rangle^{\sigma} |D_{ij}| \\ \leq 2^{\sigma/2} \sup_i \sum_{j,k} e^{r|i-k|+r|k-j|} (\langle i+k \rangle^{\sigma} + \langle j-k \rangle^{\sigma}) |A_{ik}| |C_{kj}| \\ \leq 2^{\sigma/2} \sup_i \sum_k e^{r|i-k|} \langle i+k \rangle^{\sigma} f_{\sigma}(\rho) \sup_k \sum_j e^{(r+\rho)|j-k|} |C_{kj}| \end{aligned}$$

where $f_{\sigma}(\rho) = \sup_x e^{-\rho x} (1+x^2)^{\sigma/2} \leq 1 + \left(\frac{\sigma}{e\rho}\right)^{\sigma} < 2\left(\frac{\sigma}{e\rho}\right)^{\sigma}.$

Proceeding similarly with the other terms of the norm, we get the result.

LEMMA 2.5. — Let D be a diagonal matrix whose diagonal sequence d satisfies for $\Omega \in B$:

$$|d_i(\Omega) - d_j(\Omega)| > \gamma |i-j|^{-\sigma} \langle i+j \rangle^{-2\sigma} C(\sigma)^{-1} \quad i \neq j.$$

Then given any $P \in M_{r,4\sigma}(B)$ with $\text{diag } P = 0$ and given any $\rho : 0 < \rho < r$, there exists a unique $W \in M_{r-\rho,0}(B)$ with $\text{diag } W = 0$ solution of

$$[D, W] + P = 0.$$

Furthermore

$$\|W\|_{r-\rho,0,B} \leq 2C(\sigma)^2 \gamma^{-2} \left(\frac{\sigma}{e\rho}\right)^{2\sigma+1} \|P\|_{r,4\sigma,B}.$$

The proof of lemma 2.5 is immediate, using definition 2.3.

Now the proof of Theorem 2.1 proceeds along exactly the same lines as in the paper, B_n being redefined according to lemma 2.2, and noting

that the power law decay in $\langle i + j \rangle$ propagates from P_n to P_{n+1} , due to lemma 2.3:

$$\begin{aligned} r_n &= r_{n-1} - 2\rho_n \\ \|P_n\|_{r_n, 4\sigma, B_n} &\leq \theta_n^{2^n} \end{aligned}$$

the sequence $(\theta_n)_{n \in \mathbb{N}}$ being defined inductively by

$$\theta_{n+1}^{2^{n+1}} \leq 2^{6(3\sigma+1)} C(\sigma)^2 \left(\frac{\sigma}{e\rho_n} \right)^{10\sigma+1} \gamma_n^{-2} \theta_n^{2^{n+1}}$$

and (2.17) being modified accordingly.

The rest of the paper works without change, except that $r > 9$ in Theorem 3.1, and in assumption (C) of Section 4 has to be replaced by $r > 25$, and (3.2) by

$$|P_{ij}| \leq C\gamma^2 |i - j|^{-r} \langle i + j \rangle^{-4\sigma} \quad (i \neq j).$$