ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 47, nº 1 (1987), p. 25-37
<http://www.numdam.org/item?id=AIHPA_1987__47_1_25_0>

Ann. Inst. Henri Poincaré,
Vol. 47, n° 1, 1987, p. 25-37. *Physique théorique*

Time-decay of scattering solutions and classical trajectories

by

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ABSTRACT. — We obtain the best time-decay results for scattering solutions of semiclassical Schrödinger equation with various localizations in the phase space. We show also that the non trapping condition in classical mechanics is in fact equivalent with that in quantum mechanics. The proof of these results is based on the construction of temporal global outgoing or incoming h-parametrices.

RÉSUMÉ. — On obtient les meilleurs résultats sur la décroissance en temps des solutions de diffusion pour l'équation de Schrödinger semiclassique avec diverses localisations dans l'espace de phase. On montre aussi que la condition de non capture en mécanique classique est équivalente avec celle en mécanique quantique. La démonstration de ces résultats est basée sur la construction des h-paramétrices sortantes ou entrantes globales en t.

1. INTRODUCTION

Let $H_0(h) = - h^2\Delta$ and $H(h) = H_0(h) + V$ be Schrödinger operators on $\mathbf{R}^n$, where $h \in]0, 1]$ is a small parameter and V a smooth potential on $\mathbf{R}^n$.

* Address after February 1st, 1987.

In [7], we have studied the time-decay of scattering solutions for Schrödinger equation:

$$(1.1) \qquad \begin{cases} ih\partial/\partial t u(t, h) = \mathrm{H}(h)u(t, h), & t \in \mathbf{R} \\ u(0, h) = f(h) & f(h) \in \mathrm{L}^2(\mathbf{R}^n) \end{cases}$$

and proved that for short range potentials satisfying:

$$(1.2)_\rho \qquad |\partial_x^\alpha \mathrm{V}(x)| \leqslant \mathrm{C}_\alpha \langle x \rangle^{-\rho-|\alpha|}$$

for some $\rho > 1$ ($\langle x \rangle = (1 + |x|^2)^{1/2}$), the time-decay uniformly in $h \in]0, 1]$ for the solutions of (1.1) is equivalent to some non-trapping condition on the classical trajectories. The purpose of this paper is to prove that in fact the same results are true for long range potentials satisfying $(1.2)_\rho$ for some $0 < \rho \leqslant 1$.

Let us introduce first the non-trapping condition. Consider the Hamiltonian system:

$$(1.3) \qquad \begin{cases} d/dt x(t) = 2\xi(t) & x(0) = z \\ d/dt \xi(t) = -\nabla \mathrm{V}(x(t)) & \xi(0) = \zeta \end{cases}$$

for $t \in \mathbf{R}$ and $(z, \zeta) \in \mathbf{R}^{2n}$. Let J be an open interval in $]0, +\infty[$. We say that J is an interval of non-trapping energy iff:

(N) for every subinterval $\mathrm{I} \subset\subset \mathrm{J}$ and for every $\mathrm{R} > 0$, there exists $\mathrm{T} = \mathrm{T}(\mathrm{R}, \mathrm{I}) > 0$ such that $|x(t; z, \zeta)| > \mathrm{R}$ for $|t| > \mathrm{T}$ and $(z, \zeta) \in \mathbf{R}^{2n}$ with $|\zeta|^2 + \mathrm{V}(z) \in \mathrm{I}$ and $|z| < \mathrm{R}$.

We want to give a characterization for this non-trapping condition by means of the uniform time-decay of the scattering solutions of (1.1). Our first result is concerned with (micro-) local energy decay.

Introduce a class of microlocalization operators. Let $b_\pm \in \mathrm{C}^\infty(\mathbf{R}^{2n})$ be bounded symbols satisfying:

$$|\partial_x^\alpha \partial_\xi^\beta b_\pm(x, \xi)| \leqslant \mathrm{C}_{\alpha\beta} \langle x \rangle^{-|\alpha|}.$$

Suppose that there exist $d > 0$ and $\sigma_\pm \in]-1, 1[$ such that:

$$(1.4)_\pm \quad b_\pm(x, \xi) = 0, \qquad \text{if } |x| \leqslant d \quad \text{or} \quad |\zeta| \leqslant d \quad \text{or} \quad \pm \hat{x} . \hat{\xi} \leqslant \pm \sigma_\pm$$

where $\hat{x} = x/|x|$ and $\hat{\xi} = \xi/|\xi|$. For a symbol a on $\mathbf{R}^{2n}$, $a(x, h\mathrm{D})$ denotes the associated h-pseudodifferential operator defined by:

$$a(x, h\mathrm{D})f(x) = (2\pi)^{-n} \iint e^{i(x-y).\xi} a(x, h\xi) f(y) dy d\xi .$$

Put: $\mathrm{U}(t, h) = \exp(-ih^{-1}t\mathrm{H}(h))$. Then we have the following results.

THEOREM 1. — Let V be a long range potential satisfying $(1.2)_\rho$ with $\rho > 0$. Let J be an interval of non-trapping energy. Then we have:

i) For every $s > 0$ and $\chi \in C_0^\infty(J)$, there exists $C_{s,\chi} > 0$, such that:

$$\| \langle x \rangle^{-s} \chi(H(h)) U(t, h) \langle x \rangle^{-s} \| \leqslant C_{s,\chi} \langle t \rangle^{-s}, \qquad t \in \mathbf{R}$$

ii) Let $b_\pm$ satisfy $(1.4)_\pm$. Then for every s, $r > 0$ and $\chi \in C_0^\infty(J)$, there exists $C_{r,s,\chi} > 0$ such that:

$$(1.5) \quad \| \langle x \rangle^{b} b_\mp(x, hD) U(t, h) \chi(H(h)) \langle x \rangle^{-r-s} \| \leqslant C_{r,s;\chi} \langle t \rangle^{-s}, \quad \pm t > 0$$

and

$$(1.6) \quad \| \langle x \rangle^{-r-s} \chi(H(h)) U(t, h) b_\pm(x, hD) \langle x \rangle^{r} \| \leqslant C_{r,s;\chi} \langle t \rangle^{-s}, \quad \pm t > 0$$

iii) Assume that $b_\pm$ satisfy $(1.4)_\pm$ with $\sigma_- < \sigma_+$. Then for every $s > 0$, $N > 0$ and $\chi \in C_0^\infty(J)$, there exists $C_{s,N;\chi} > 0$ such that:

$$\| \langle x \rangle^{N} b_\mp(x, hD) \chi(H(h)) U(t, h) b_\pm(x, hD) \langle x \rangle^{N} \| \leqslant C_{s,N;\chi} \langle t \rangle^{-s}, \quad \pm t > 0.$$

Here $\| \, . \, \|$ denotes the norm of bounded operator on L^2 and all the estimates are uniform with respect to $h \in]0, 1]$.

Notice that the results obtained in Theorem 1 are the best possible. In [7] we proved weaker results in the form:

$$(1.7) \qquad \| \langle x \rangle^{-s} \chi(H(h)) U(t, h) \langle x \rangle^{-s} \| \leqslant C_\varepsilon h^{-\varepsilon} \langle t \rangle^{-s}$$

for any $\varepsilon > 0$. The improvement obtained here is important, because it enables us to characterize the condition (N) by the uniform time-decay of the scattering solutions.

THEOREM 2. — Let V satisfy $(1.2)_\rho$ for some $\rho > 0$. Then the following three conditions are equivalent:

i) J is an interval of non-trapping energy: (N);
ii) There exist some $s > 0$, $r > 0$ such that for every $\chi \in C_0^\infty(J)$, we have:

$$\| \langle x \rangle^{-s} \chi(H(h)) U(t, h) \langle x \rangle^{-s} \| \leqslant C_\chi \langle t \rangle^{-r} \qquad t \in \mathbf{R}$$

uniformly in $h \in]0, 1]$;
iii) For every $s > 0$ and $\chi \in C_0^\infty(J)$, we have

$$\| \langle x \rangle^{-s} \chi(H(h)) U(t, h) \langle x \rangle^{-s} \| \leqslant c_{s;\chi} \langle t \rangle^{-s} \qquad t \in \mathrm{R}$$

uniformly in $h \in]0, 1]$.

Theorem 2 is an easy consequence of Theorem 1 and the results on the correspondence between quantum and classical dynamics proved in [5] and [6]. To prove Theorem 1, we construct an outgoing (resp. incoming) h-parametrix and compare the perturbed evolution with the free one. Observe that *ii*) and *iii*) of Theorem 1 are not only an expression of the non trapping condition in various domains of the phase space, but also a useful tool for proving *i*) of the theorem. It should be remarked that in the proof of Theorem 1, we only use the assumption $(1.2)_\rho$ and the estimate on the resolvent, for every $s > 1/2$,

$$\text{(R)} \qquad \| \langle x \rangle^{-s} \mathrm{R}(\lambda \pm i0, h) \langle x \rangle^{-s} \| \leqslant Ch^{-1}$$

locally uniformly in $\lambda \in \mathrm{J}$, which is proved in [4] under the assumptions $(1.2)_\rho$ and (N). Thus in particular we can conclude from Theorem 2 that the condition (N) is necessary to obtain the estimate of the form (R).

The plan of this work is as follows. Admitting Theorem 1, we prove first Theorem 2 in Section 2 by calculating the classical limit of the wave functions. The Sections 3-5 are devoted to the proof of Theorem 1. In Section 3, we apply a commutator method to get an estimate of the form:

$$(1.8) \qquad \| \langle x \rangle^{-1} \chi(\mathrm{H}(h)) \mathrm{U}(t, h) \langle x \rangle^{-1} \| \leqslant c \langle t \rangle^{-\mu}$$

for $0 < \mu < \rho$ and uniformly for $h \in]0, 1]$. In comparing with (1.7), what we gain here is the uniformity in h. Section 4, we sketch the construction of h-parametrices and give results on time-decay in microlocalized forms. Finally we give the proof for Theorem 1 with $0 < s \leqslant 1$ in Section 5. The passage from $s \in]0, 1]$ to arbitrary $s > 0$ is realized by the same inductive argument and partition of unitary as that used in [7]. The interested reader is refered there for more details.

2. CLASSICAL LIMIT OF WAVE FUNCTIONS

In this section we give the proof for Theorem 2, which is easier than that for Theorem 1. Admitting Theorem 1, it is sufficient to show that *ii*) implies *i*) in Theorem 2. This result will be proved by taking the classical limit of wave functions.

In order to apply the results proved in [5] and [6], we use the quantification of Weyl. For a symbol a, $a^{\mathrm{W}}(h^{1/2}x, h^{1/2}\mathrm{D})$ denotes the operator defined by:

$$a^{\mathrm{W}}(h^{1/2}x, h^{1/2}\mathrm{D})f(x) = (2\pi)^{-n} \iint e^{i(x-y).\xi} a(h^{1/2}(x + y)/2, h^{1/2}\xi) f(y) dx d\xi .$$

Let $\mathrm{P}(h) = -h\Delta + \mathrm{V}(h^{1/2}x)$ and $\mathrm{V}(t, h) = e^{-it\mathrm{P}(h)/h}$. Since $\mathrm{P}(h)$ is unitarily equivalent with $\mathrm{H}(h)$, we conclude that under the assumption *ii*) of Theorem 2, for every $\chi \in \mathrm{C}_0^\infty(\mathrm{J})$,

$$(2.1) \qquad \| \langle h^{1/2}x \rangle^{-s} \chi(\mathrm{P}(h)) \mathrm{V}(t, h) \langle h^{1/2}x \rangle^{-s} \| \leqslant \mathrm{C}_\chi \langle t \rangle^{-r} \qquad t \in \mathbf{R}$$

uniformly in $h \in]0, 1]$. For each $(z, \zeta) \in \mathbf{R}^{2n}$, let $\mathrm{W}(z, \zeta; h)$ denote the operator;

$$\mathrm{W}(z, \zeta; h) = \exp(ih^{-1/2}(\zeta . x - z . \mathrm{D}))$$

Put:

$A(t, h) = W(z, \zeta; h)^* \langle h^{1/2}x \rangle^{-s} \chi(P(h)) V(t, h) \langle h^{1/2}x \rangle^{-s} V(t, h)^* W(z, \zeta; h)$.

Then $A(t, h)$ satisfies the estimate:

$$\| A(t, h) \| \leqslant C \langle t \rangle^{-r} \quad \text{uniformly in } h \in]0, 1]. \tag{2.2}$$

LEMMA 2.1. — Set: $b(x, \xi) = \chi(|\xi|^2 + V(x)) \langle x \rangle^{-s}$. Then we have: $\| A(t, h) - W(z, \zeta; h)^* \langle h^{1/2}x \rangle^{-s} V(t, h) b^W(h^{1/2}x, h^{1/2}D) V(t, h)^* W(z, \zeta; h) \| \leqslant Ch$, for $h \in]0, 1]$ uniformly in $t \in \mathbf{R}$.

Lemma 2.1 follows easily from the result on functional calculus ([1]), which says that $\chi(P(h))$ is a h-pseudodifferential operator with h-principal symbol $\chi(|\xi|^2 + V(x))$.

LEMMA 2.2. — For every $t \in R$, $(z, \zeta) \in \mathbf{R}^{2n}$, we have:

$$s\text{-}\lim_{h \to 0} A(t, h) = \langle z \rangle^{-s} \chi(|\zeta|^2 + V(z)) \langle x(-t; z, \zeta) \rangle^{-s}$$

in $L^2(\mathbf{R}^n)$. Here $(x(t; z, \zeta), \xi(t; z, \zeta))$ is the solution of (1.3).

Proof. — By Theorem 4.2 [6], we have:

$$s\text{-}\lim_{h \to 0} W(z, \zeta; h)^* V(t, h) b^W(h^{1/2}x, h^{1/2}D) V(t, h)^* W(z, \zeta; h) = b(x(-t; z, \zeta), \xi(-t; z, \zeta)) \quad \text{in} \quad L^2.$$

Now the desired result follows from Lemma 2.1 and an easy calculation of the action of $W(z, \zeta; h)$. See [5] [6]. ■

Now Theorem 2 can be easily derived from Lemma 2.3.

Proof of Theorem 2. — We prove that *ii*) implies *i*) in Theorem 3. For every subinterval $I \subset\subset J$, take $\chi \in C_0^\infty(J)$ which is equal to 1 on I. By *ii*), (2.2) is verified. From Lemma 2.2, we obtain:

$$\langle x(t; z, \zeta) \rangle \geqslant C^{-1} \langle t \rangle^{r/s} \langle z \rangle^{-1} \qquad t \in \mathbf{R} \tag{2.3}$$

for $(z, \zeta) \in \mathbf{R}^{2n}$ such that $|\zeta|^2 + V(z) \in I$. Here the constant C is the same as in (2.2). (2.3) means that the condition (N) is satisfied for the interval I. This proves Theorem 2. ■

By the same method, we can also derive from *ii*) and *iii*) the global behavior of classical trajectories with initial data in outgoing or incoming region in the phase space. But we do not go into the details.

3. A COMMUTATOR ESTIMATE

We give in this section a natural generalization of Theorem 4.1 in [7] which is only true for short range potentials.

THEOREM 3.1. — Let $\delta \in]0, \rho[$. Under the assumptions of Theorem 1, for every $\chi \in C_0^\infty(J)$, we have:

$$\| \langle x \rangle^{-1} \chi(H(h))U(t, h) \langle x \rangle^{-1} \| \leqslant C_\chi \langle t \rangle^{-\delta} \tag{3.1}$$

for $t \in \mathbf{R}$ and uniformly in $h \in]0, 1]$.

Notice that for short range potentials, we can take $\delta = 1$. The proof of Theorem 3.1 is based on a commutator relation (see (3.3)). Let us prepare first two lemmas.

LEMMA 3.2. — For every $\chi \in \mathcal{S}(\mathbf{R})$, $\chi(H(h))$ is uniformly bounded as operator on $L^{2,s}(\mathbf{R}^n)$:

$$\sup_{h \in]0,1]} \| \chi(H(h)) \|_{\mathcal{L}(L^{2,s})} \leqslant C.$$

Here $L^{2,s}(\mathbf{R}^n)$ denotes the weighted L^2-space with the norm: $\| f \|_s = \| \langle x \rangle^s f \|$.

Lemma 3.2 follows easily from the results on functional calculus ([*1*]), which says that $\chi(H(h))$ is a h-pseudodifferential operator with bounded symbol.

LEMMA 3.3. — Under the assumptions of Theorem 3.1, for every $s > 1/2$, we have:

$$\int_{-\infty}^{+\infty} \| \langle x \rangle^{-s} \chi(H(h))U(t, h)f \|^2 dt \leqslant C \| f \|^2, \qquad f \in L^2(\mathbf{R}^n)$$

uniformly in $h \in]0, 1]$.

Proof. — Recall that under the conditions of Theorem 1, it is proved in [*4*] that for every $s > 1/2$,

$$\| \langle x \rangle^{-s} R(\lambda \pm i0, h) \langle x \rangle^{-s} \| \leqslant Ch^{-1} \qquad h \in]0, 1] \tag{3.2}$$

locally uniformly in $\lambda \in J$. Here $R(z, h) = (H(h) - z)^{-1}$. This means that $\langle x \rangle^{-s}$ is locally $H(h)$-smooth on J. By Theorem XIII 25 ([*3*]), we conclude the lemma from (3.2).

Now we pass to the proof of Theorem 3.1.

Proof of Theorem 3.1. — Put: $A(h) = h(x.D_x + D_x.x)/3$. Then we have:

$$H(h)U(t, h) = (2t)^{-1} \left\{ [A(h), U(t, h)] + \int_0^t U(t - s, h)WU(s, h)ds \right\} \tag{3.3}$$

for $t \neq 0$, where $W = -2V + x.\nabla V$. For $\chi \in C_0^\infty(J)$, take ψ, $\varphi \in C_0^\infty(J)$ such that $\varphi(\lambda)\psi(\lambda) = \chi(\lambda)/\lambda$. Let $F(t, h)$ be the operator defined by:

$$F(t, h) = \int_0^t \langle x \rangle^{-1} \varphi(H(h))U(t - s, h)WU(s, h)\psi(H(h)) \langle x \rangle^{-1} ds \tag{3.4}$$

Then (3.3) gives:

$$(3.5)\quad \langle x\rangle^{-1}\chi(\mathrm{H}(h))\mathrm{U}(t,h)\langle x\rangle^{-1} = (2t)^{-1}\{\langle x\rangle^{-1}\varphi(\mathrm{H}(h))[\mathrm{A}(h),\mathrm{U}(t,h)]\times\psi(\mathrm{H}(h))\langle x\rangle^{-1}+\mathrm{F}(t,h)\}.$$

By lemma 3.2, we can easily show that:

$$\sup_{t,h}\|\langle x\rangle^{-1}\varphi(\mathrm{H}(h))[\mathrm{A}(h),\mathrm{U}(t,h)]\psi(\mathrm{H}(h))\langle x\rangle^{-1}\| < +\infty.$$

As a consequence, we get:

$$(3.6)\quad \|\langle x\rangle^{-1}\chi(\mathrm{H}(h))\mathrm{U}(t,h)\langle x\rangle^{-1}\| \leqslant \mathrm{C}\langle t\rangle^{-1}(1+\|\mathrm{F}(t,h)\|).$$

To estimate $\mathrm{F}(t,h)$ we choose a function $\theta\in\mathrm{C}_0^\infty(\mathbf{R}^n)$ which is equal to 1 in a neighborhood of 0. Put:

$$\mathrm{V}_1(x,t)=\theta(x/t)\mathrm{W}(x),\qquad \mathrm{V}_2(x,t)=\mathrm{W}(x)-\mathrm{V}_1(x,t).$$

Let $\mathrm{F}_j(t,h)$ be the operator defined by (3.4) with W replaced by $\mathrm{V}_j(t)$, $j=1,2$. We have, for every $s>1/2$,

$$|\langle f,\mathrm{F}_1(t,h)g\rangle| \leqslant \|\mathrm{V}_1(t)\langle x\rangle^{2s}\|\times$$
$$\times\int_0^t \|\langle x\rangle^{-s}\varphi(\mathrm{H}(h))\mathrm{U}(\tau-t,h)\langle x\rangle^{-1}f\|\ \|\langle x\rangle^{-s}\psi(\mathrm{H}(h))\mathrm{U}(\tau,h)\langle x\rangle^{-s}\|\,d\tau$$

for $f,g\in\mathrm{L}^2(\mathrm{R}^n)$. Applying Lemma 3.3, we get:

$$\|\mathrm{F}_1(t,h)\|\leqslant \mathrm{C}\|\mathrm{V}_1(t)\langle x\rangle^{2s}\|\leqslant \mathrm{C}'\langle t\rangle^{2s-\rho}$$

uniformly in $h\in\,]0,1]$. By the assumption on V, we get also that:

$$\|\mathrm{F}_2(t,h)\|\leqslant \mathrm{C}\langle t\rangle^{1-\rho}$$

uniformly in $h\in\,]0,1]$. These two estimates show that for every $\varepsilon>0$,

$$(3.7)\qquad \|\mathrm{F}(t,h)\|\leqslant \mathrm{C}_\varepsilon\langle t\rangle^{1-\rho+\varepsilon},\qquad t\in\mathbf{R}.$$

Now (3.1) results from (3.6) and (3.7). ■

Notice that Theorem 3.1 shows that *i*) implies *ii*) in Theorem 2. In the remainder of this work, we will show how to derive Theorem 1 from Theorem 3.1.

4. MICROLOCALIZED ENERGY DECAY

In this section, we will construct an outgoing (resp. incoming) h-parametrix and prove that *i*) of Theorem 1 implies the *ii*) and *iii*).

4.1 Construction of *h*-parametrices.

Let us begin with sketching the construction of h-parametrices for the problem (1.1). See [7] for more details. Introduce first the class of symbols used below.

DEFINITION. — Let $m \in \mathbf{R}$. S^m denotes the class of symbols $a \in C^\infty(\mathbf{R}^{2n})$ satisfying:

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant c_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \quad \text{on} \quad \mathbf{R}^{2n}.$$

If $m = 0$, we put: $S = S^m$. $S_\pm$ denotes the class of symbols $b_\pm \in S$ satisfying $(1.4)_\pm$ for some $d > 0$ and $\sigma_\pm \in]-1, 1[$.

For $\sigma, \lambda > 0$, we define:

$$\Omega_\pm(\sigma, \lambda) = \{ (x, \xi) \in \mathbf{R}^{2n}; \pm \hat{x}.\hat{\xi} > -1 + \sigma, |\xi| > \lambda \}.$$

For $R > 0$, set:

$$\Omega_\pm(\sigma, \lambda, R) = \{ (x, \xi) \in \Omega_\pm(\sigma, \lambda); |x| > R \}.$$

The following result is proved in [2].

PROPOSITION 4.1. — Under the assumption $(1.2)_\rho$, for every $\sigma, \lambda > 0$, there exists $R' > 0$ such that for every $R > R'$, there are two real functions $\phi_\pm \in C^\infty$, which solve the eikonal equation:

$$|\nabla_x \phi_\pm(x, \xi)| + V(x) = |\xi|^2 \quad \text{in} \quad \Omega_\pm(\sigma, \lambda, R)$$

and for every $\alpha, \beta \in \mathbf{N}^n$, one has:

$$|\partial_x^\alpha \partial_\xi^\beta \phi_\pm(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{1-|\alpha|-\tau} R^{-\delta}$$

for any $\tau, \delta \geqslant 0$ such that $\tau + \delta = \rho$.

Let $b_\pm$ be in $S_\pm$. Making use of the phase function ϕ_+ (resp. ϕ_-), we will construct an outgoing (resp. incoming) h-parametrix to approximate $U(t, h)b_+(x, hD)$ (resp. $U(t, h)b_-(x, hD)$). We consider only the outgoing case and the incoming case can be treated in the same way.

Now let $\phi = \phi_+$. We denote $I(a, h)$ the Fourier integral operator defined by:

$$(4.2) \quad I(a, h)f(x) = (2\pi h)^{-1} \iint e^{i(\phi(x,\xi) - y\xi)/h} a(x, \xi) f(y) dy d\xi, \qquad f \in \mathscr{S}(\mathbf{R}^n).$$

Take $a = a(h)$ in the form $a(h) = \Sigma_{j=0}^N h^j a_j$. A simple calculation gives:

$$e^{-i\phi/h}(H(h) - |\xi|^2)(e^{i\phi/h} a(h)) = (|\nabla_x \phi|^2 + V - |\xi|^2) a(h) - ih(2\nabla_x \phi . \nabla_x a_0 + \Delta\phi a_0) - \Sigma_{j=1}^N h^{j+1}(2i\nabla\phi . \nabla a_j + i\Delta\phi a_j + \Delta a_{j-1}) - h^{N+2}\Delta a_N.$$

This relation gives the transport equations:

$$(4.3) \quad \begin{aligned} &2\nabla\phi . \nabla a_0 + \Delta\phi a_0 = 0 \\ &2\nabla\phi . \nabla a_j + \Delta\phi a_j = i\Delta a_{j-1}; \qquad j = 1, 2, \ldots, N \end{aligned}$$

which determine $a_0, a_1, \ldots, a_N$. We can prove that the solutions of (4.3) exist and are C^∞ on $\mathbf{R}^{2n}$:

$$|\partial_x^\alpha \partial_\xi^\beta a_k(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{-k-|\alpha|}, \qquad \text{for} \quad (x, \xi) \in \Omega_+(\sigma, \lambda)$$

$k = 0, 1, \ldots, N$. Let χ be a smooth function with support in $\Omega_+(\sigma, \lambda, R)$ and equal to 1 on $\Omega_+(2\sigma, 2\lambda, 2R)$. Then ϕ solves the eikonal equation on the support of χ. Put:

$$(4.4) \qquad \begin{aligned} a_N(h) &= \chi \Sigma_{j=0}^{N} h^j a_j \\ r_N(h) &= e^{-i\phi/h}(H(h) - |\xi|^2)(e^{i\phi/h} a_N(h)). \end{aligned}$$

By the construction, it is clear that $\{ h^{-1} r_N(h), 0 < h \leqslant 1 \}$ is bounded in S and $\{ h^{-N-1} r_N(h),\ 0 < h \leqslant 1 \}$ is bounded in $S^{-N}(\Omega_+(2\sigma, 2\lambda, 2R))$. Let $b_+ \in S_+$ satisfying $(1.4)_+$ for some $d > 0$ and $\sigma_+ \in]-1, 1[$. Take σ, $\lambda > 0$ to be sufficiently small and $R \gg 1$ to be sufficiently large. Since $a_N(h)$ is elliptic on $\Omega_+(2\sigma, 2\lambda, 2R)$, we can prove that for any $\sigma' > 1 - \sigma_+$ and $d' < d$, there exists a symbol $b_N(h)$ in the form $\Sigma_{j=0}^{N} h^j b_j$, where $b_j \in S^{-j}$ and $\operatorname{supp} b_j \subset \Omega_+(\sigma', d')$ such that:

$$(4.5) \qquad b_+(x, hD) = I(a_N(h), h) I(b_N(h), h)^* + h^{N+1} R_N(h)$$

with $R_N(h)$ uniformly bounded as operator from $L^{2,s}$ to $L^{2,s+N}$. (See [7], § 4). From (4.4) and (4.5), we get, by Duhamel's formula,

$$(4.6) \quad U(t, h) b_+(x, hD) = U_N(t, h) + h^{N+1} U(t, h) R_N(h) + i \int_0^t U(t-s, h) G_N(s, h) ds$$

where

$$\begin{aligned} U_N(t, h) &= I(a_N(h), h) U_0(t, h) I(b_N(h), h)^* \\ G_N(t, h) &= I(r_N(h), h) U_0(t, h) I(b_N(h), h)^* \end{aligned}$$

(4.6) enables us to compare $U(t, h)$ with the free evolution $U_0(t, h)$.

4.2 **Microlocalized Energy Decay.**

In order to apply (4.6), we need to know the behavior of the free unitary group. Let us recall the following result from [7] (Propositions 3.2 and 3.5).

LEMMA 4.2. — Let $b_\pm \in S_\pm$. Then for every $s, r \geqslant 0$, we have:

i) $\| \langle x \rangle^{-s-r} U_0(t, h) I(b_\pm, h)^* \langle x \rangle^r \| \leqslant C_{r,s} \langle t \rangle^{-s} \qquad \pm t > 0.$

ii) If $b_\pm \in S_\pm$ with $\sigma_- < \sigma_+$, then for every $m, s > 0$, we have:

$$\| \langle x \rangle^m I(b_+, h) U_0(t, h) I(b_\pm, h)^* \langle x \rangle^m \| \leqslant C_{m,s} \langle t \rangle^{-s} \ \pm t > 0$$

uniformly in $h \in]0, 1]$.

Making use of Lemma 4.2, we can obtain the following estimates.

LEMMA 4.3. — With the above notations, we have:

i) For every $s, r > 0$,

$$\| \langle x \rangle^{-s-r} U_N(t, h) \langle x \rangle^r \| \leqslant C \langle t \rangle^{-s} \qquad \text{for} \quad t > 0, \quad h \in]0, 1].$$

ii) For every $m, s > 0$ with $2m + s < \mathrm{N}$, we have:

$$\| \langle x \rangle^m \mathrm{G_N}(t, h) \langle x \rangle^m \| \leqslant \mathrm{C} \langle t \rangle^{-s} \qquad \text{for} \quad t > 0, \quad h \in]0, 1].$$

Proof. — *i*) Since $\mathrm{I}(a_\mathrm{N}(h), h)$ is uniformly bounded from $\mathrm{L}^{2,s}$ to $\mathrm{L}^{2,s}$ for every $s \in \mathrm{R}$, it follows from *i*) of Lemma 4.2.

ii) We split suitably $r_\mathrm{N}(h)$ into two terms: $r_\mathrm{N}(h) = r_{\mathrm{N},1}(h) + r_{\mathrm{N},2}(h)$, where $r_{\mathrm{N},1}(h)$ is supported in $\Omega_+(2\sigma, 2\lambda)$, while $r_{\mathrm{N},2}(h)$ is supported in $\Omega_-(1 - 3\sigma, \lambda)$. By the construction of $b_\mathrm{N}(h)$, it is clear that we can apply the *ii*) of Lemma 4.2 to the couple of operators $(\mathrm{I}(r_{\mathrm{N},2}(h), h), \mathrm{I}(b_\mathrm{N}(h), h))$. By the definition of $r_\mathrm{N}(h)$ ((4.4)), $\mathrm{I}(r_{\mathrm{N},1}(h), h)$ is uniformly bounded as operator from $\mathrm{L}^{2,-s-m}$ to $\mathrm{L}^{2,m}$, on condition that $2m + s < \mathrm{N}$. This proves *ii*). ■

Now we can give the main result of this section.

THEOREM 4.4. — Let $s > 0$ and $0 < \mu \leqslant s$ be fixed. Let $\chi \in \mathrm{C}_0^\infty(\mathrm{J})$. Assume that

$$(4.7) \qquad \| \langle x \rangle^{-s} \chi(\mathrm{H}(h)) \mathrm{U}(t, h) \langle x \rangle^{-s} \| \leqslant \mathrm{C} \langle t \rangle^{-\mu} \qquad t \in \mathbf{R}$$

uniformly in $h \in]0, 1]$. Then we have:

i) Let $b_\pm \in \mathrm{S}_\pm$. Then for every $r > 0$:

$$(4.8)_\pm \quad \| \langle x \rangle^{-r-s} \mathrm{U}(t, h) \chi(\mathrm{H}(h)) b_\pm(x, h\mathrm{D}) \langle x \rangle^r \| \leqslant \mathrm{C} \langle t \rangle^{-\mu} \qquad \pm t > 0$$

and

$$(4.9)_\pm \quad \| \langle x \rangle^r b_\mp(x, h\mathrm{D}) \chi(\mathrm{H}(h)) \mathrm{U}(t, h) \langle x \rangle^{-r-s} \| \leqslant \mathrm{C} \langle t \rangle^{-\mu} \qquad \pm t > 0.$$

ii) If $b_\pm \in \mathrm{S}_\pm$ with $\sigma_- < \sigma_+$ (see (1.4)), then for every $m > 0$,

$$(4.10)_\pm \quad \| \langle x \rangle^m b_\mp(x, h\mathrm{D}) \chi(\mathrm{H}(h)) \mathrm{U}(t, h) b_\pm(x, h\mathrm{D}) \langle x \rangle^m \| \leqslant \mathrm{C} \langle t \rangle^{-\mu} \quad \pm t > 0.$$

(4.8)-(4.10) are uniform with respect to $h \in]0, 1]$.

Proof. — *i*) (4.9) can be derived from (4.8) by taking the adjoint. We prove only $(4.8)_+$. $(4.8)_-$ can be proved by constructing an incoming h-parametrix. By Lemma 3.1 and *i*) of Lemma 4.3, we get for every $r > 0$:

$$\| \langle x \rangle^{-s-r} \chi(\mathrm{H}(h)) \mathrm{U_N}(t, h) \langle x \rangle^r \| \leqslant \mathrm{C} \langle t \rangle^{-s} \qquad t > 0, \quad h \in]0, 1].$$

For $r > 0$ and $\mathrm{N} > 2r + 2s + 1$, we get from the assumption of the theorem and (4.5) that:

$$\| \langle x \rangle^{-s-r} \chi(\mathrm{H}(h)) \mathrm{U}(t, h) \mathrm{R_N}(h) \langle x \rangle^r \| \leqslant \mathrm{C} \langle t \rangle^{-\mu} \qquad t \in \mathbf{R}, \quad h \in]0, 1].$$

Applying *ii*) of Lemma 4.3, we have:

$$\left\| \int_0^t \langle x \rangle^{-s-r} \chi(\mathrm{H}(h)) \mathrm{U}(t - \tau, h) \mathrm{G_N}(\tau, h) \langle x \rangle^r d\tau \right\|$$

$$\leqslant c \int_0^t \| \langle x \rangle^{-s} \chi(\mathrm{H}(h)) \mathrm{U}(t - \tau, h) \langle x \rangle^{-s} \| \langle \tau \rangle^{-s-1} d\tau \leqslant \mathrm{C} \langle t \rangle^{-\mu}$$

for $t > 0$ and $h \in]0, 1]$. Here we have used $0 < \mu \leqslant s$. Now (4.7) follows from (4.6) and the above three estimates.

ii) We prove $(4.10)_+$, By (4.6) we have:

$$(4.11)\quad b_-(x, hD)\chi(H(h))U(t, h)b_+(x, hD) = b_-(x, hD)\chi(H(h))U_N(t, h) + h^{N+1}b_-(x, hD)\chi(H(h))U(t, h)R_N(h) + b_-(x, hD)\int_0^t U(t-s, h)G_N(s, h)ds.$$

By the results on the composition of Fourier integral operators with pseudodifferential operators (see for example Appendix in [7]), we can prove that modulo an operator uniformly continuous from $L^{2,-m-1}$ to $L^{2,m}$, $b_-(x, hD)\chi(H(h))I(a_N(h), h)$ is a Fourier integral operator. Since $\sigma_- < \sigma_+$, choosing $R > 0$ large enough (see Proposition 4.1), we can prove that the support of the amplitude of this Fourier integral operator is disjoint from that of $b_N(h)$. Hence applying *ii*) of Lemma 4.2, we get:

$$\| \langle x \rangle^m b_-(x, hD)\chi(H(h))U_N(t, h) \langle x \rangle^m \| \leqslant C \langle t \rangle^{-\mu} \qquad t > 0, \quad h \in]0, 1].$$

By $(4.8)_+$ and the argument used in the proof of *i*), we can show that the last two terms in (4.11) satisfy the similar estimates. We omit the details. This finishes the proof of Theorem 4.4. ■

Notice that by Theorem 3.1, the assumption (4.7) of Theorem 4.4 is satisfied with $s \in [0, 1]$ and $\mu = \rho' s$, $0 < \rho' < \rho$.

5. PROOF OF THEOREM 1

As for short range potentials, the first step in the proof of Theorem 1 is to prove it for $s \in [0, 1]$. Then a suitable partition of unity enables us to pass to the general case by an induction. Now making use of the results established in § § 3, 4, we want to prove Theorem 1 for $s = 1$.

LEMMA 5.1. — Let $f \in C_0^\infty(]0, +\infty[)$. Then for every $N \in \mathbf{N}$, $\sigma \in]-1, 1[$ and $\varepsilon > 0$, there exists $b_\pm(h) \in S_\pm$ polynomials in h such that $(1.4)_\pm$ is satisfied with $\sigma_\pm = \sigma + \varepsilon$ and that:

$$f(H(h)) = b_+(x, hD) + b_-(x, hD) + R_N(h)$$

where $R_N(h)$ is uniformly continuous from $L^{2,s}$ to $L^{2,s+N}$, for $s \in \mathbf{R}$.

For the proof of Lemma 5.1, see [7] (Lemma 4.3).

THEOREM 5.2. — Under the assumptions of Theorem 1, for every $\chi \in C_0^\infty(J)$, we have:

$$(5.1)\qquad \| \langle x \rangle^{-1}\chi(H(h))U(t, h) \langle x \rangle^{-1} \| \leqslant C \langle t \rangle^{-1} \qquad t \in \mathbf{R}$$

uniformly in $h \in]0, 1]$.

Proof. — By the expression (3.5), it is sufficient to prove that $F(t, h)$ defined by (3.4) is bounded on $L^2(\mathbf{R}^n)$ uniformly with respect to $t \in \mathbf{R}$ and $h \in]0, 1]$. Take $f \in C_0^\infty(J)$ such that $f\varphi = \varphi$. Then applying Lemma 5.1, we get a corresponding decomposition for $F(t, h)$:

$$F(t, h) = \langle x \rangle^{-1} \int_0^t \psi(H(h))U(t-s, h)W(b_+(x, hD; h) + b_-(x, hD; h) + R_N(h))$$
$$\times\; U(s, h)\psi(H(h)) \langle x \rangle^{-1} ds = F_1(t, h) + F_2(t, h) + F_3(t, h).$$

Fix $\theta \in]0, \rho[$. By Theorem 3.1, we can apply Theorem 4.4 with $s \in]0, 1]$ and $\mu = \theta s$. By a simple calculus of pseudodifferential operators, we have:

$$Wb_+(x, hD) = b_+(x, hD)W + hQ(h)$$

where $Q(h)$ is uniformly continuous from $L^{2,s}$ to $L^{2,s+1+\rho}$. For brevity consider the case $t \geqslant 0$. From *i*) of Theorem 4.4 with $s = \rho$ and $r = 1 - \rho$, we get:

$$\| F_1(t, h) \leqslant C \int_0^t (\| \langle x \rangle^{-1} \varphi(H(h))U(t-s, h)b_+(x, hD) \langle x \rangle^{1-\rho} \|$$
$$+ \langle t - s \rangle^{-\theta\rho}) \langle s \rangle^{-\theta} ds$$
$$\leqslant C' \int_0^t \langle t - s \rangle^{-\theta\rho} \langle s \rangle^{-\theta} ds \leqslant C'' \langle t \rangle^{1-\theta-\mu}$$

with $\mu = \theta\rho$, uniformly in $h \in]0, 1]$. For $F_2(t, h)$ we have a similar estimate:

$$\| F_2(t, h) \| \leqslant C \int_0^t (\| \langle x \rangle^{-1} \varphi(H(h))U(t - s, h) \langle x \rangle^{-1} \|$$
$$\times \| \langle x \rangle^{1-\rho} b_-(x, hD)U(s, h)\psi(H(h)) \langle x \rangle^{-1} \|) ds$$
$$\leqslant C' \langle t \rangle^{1-\mu-\theta}.$$

Applying Theorem 3.1 to $F_3(t, h)$, we obtain:

$$\| F_3(t, h) \| \leqslant C \langle t \rangle^{1-2\theta}.$$

Since $0 < \mu \leqslant \theta$, these three estimates show that:

$$\| F(t, h) \| \leqslant C \langle t \rangle^{1-\mu-\theta} \quad \text{for} \quad t \in \mathrm{R}$$

uniformly in $h \in]0, 1]$. By (3.5) we get an improvement of Theorem 3.1:

(5.3) $\quad \| \langle x \rangle^{-1} \chi(H(h))U(t, h) \langle x \rangle^{-1} \| \leqslant C(\langle t \rangle^{-1} + \langle t \rangle^{-\mu-\theta}).$

If $\mu + \theta < 1$, we can apply (5.3) instead of Theorem 3.1. Repeating the above arguments, we can prove that:

$$\| F(t, h) \| \leqslant C \langle t \rangle^{1-\theta-2\mu}.$$

This gives (5.3) with μ replaced by 2μ. Since $\mu > 0$, repeating these arguments for a sufficient number of times, we obtain (5.1). ■

Now we are able to give the proof for Theorem 1.

Proof of Theorem 1. — Notice that for $s \in [0, 1]$, Theorem 1 is a consequence of Theorems 4.4 and 5.2. In general case, for $\chi \in C_0^\infty(J)$, we take f, $\varphi \in C_0^\infty(J)$ so that $\varphi\chi = \chi$ and $f\varphi = \varphi$. We can write:

$$\chi(H(h))U(t, h) = \chi(H(h))U(t/2, h)f(H(h))U(t/2, h)\varphi(H(h)).$$

Making use of the decomposition for $f(H(h))$ given in Lemma 5.1, we can apply the method for proving Theorem 5.2 to show Theorem 1 with $0 \leqslant s \leqslant 2$. An induction on $s \in \mathbf{N}$ gives the desired results. The reader is refered to [*7*] for more details (see Theorem 5.1, [*7*]). ∎

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(Manuscrit reçu le 3 décembre 1986)