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Generalized atypical supertableaux for superunitary and orthosymplectic groups

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ABSTRACT. — We give a unified presentation of the generalized atypical supertableaux of the superunitary and orthosymplectic groups. The classification and properties of these generalized atypical supertableaux and their interpretation in terms of non-fully reducible representations are discussed.

RÉSUMÉ. — On donne une présentation unifiée des supertableaux généralisés atypiques des groupes superunitaires et orthosymplectiques. On discute la classification et les propriétés de ces supertableaux généralisés atypiques et leur interprétation en termes de représentations non complètement réductibles.

I. INTRODUCTION

Young supertableaux for supergroups offer an elegant graphical description of supertensors as the Young tableaux are for the tensors of the classical simple Lie groups. In their pioneering work Balantekin and Bars [*1*] introduce supertableaux for the superunitary group SU(m/n), the orthosymplectic group OSP($m/2p$) and the strange supergroup P(n) and they give simple and compact formulas for computing characters, dimensions and superdimensions.

The case of superunitary supertableaux has been investigated in more details by Bars, Morel and Ruegg [2] and that of orthosymplectic supertableaux by Farmer and Jarvis [3]. Other interesting results for supertableaux can be found in [4].

The aim of this paper is to give a non-technical account of our own work on the classification and the interpretation of the supertableaux of the superunitary group [5] [6] [7] and of the orthosymplectic group [8] [9] [10] [11]. We shall present here only statements and results whose origin and proof has to be found in the quoted papers.

In order to be readible by non specialists in supergroups and superalgebras we first recall few basic notations and definitions in Section II. The description and the parametrization of the legal supertableaux of $SU(m/n)$ and $OSP(m/2p)$ are given respectively in Section III and in Section IV. The relation between the invariance of a non degenerate form G and the existence of generalized atypical supertableaux and non-fully reducible representations is discussed in Section V. Then we give in Section VI a classification of the legal supertableaux according to their level of atypicity and in Section VII according to their size. Such classifications give rise to three theorems which formalize the results on generalized atypical supertableaux. Finally in Section VIII we briefly discuss some aspects of the non-fully reducible representations described by the generalized atypical supertableaux. All the results given here concern only the superunitary and the orthosymplectic supertableaux. An extension to the supertableaux of $P(n)$ is desirable and probably straightforward.

II. SUPERGROUPS AND SUPERALGEBRAS

1) The general graded linear group $PL(m/n)$ is the set of regular graded linear transformations in a Z_2-graded vector space $V = V_0 \oplus V_1$ of dimension $m + n$. The degree of grade of a vector $x \in V_\alpha$ with $\alpha = 0, 1$ is defined modulo two as α.

Equivalently $PL(m/n)$ can be viewed as the multiplicative group of $(m + n)(m + n)$ graded matrices M for which the inverse matrix exists.

If we impose to the matrix M to be unimodular we get the special graded linear group $SPL(m/n)$

$$SPL(m/n) = \{ M \in PL(m/n) \mid S \det M = 1 \}.$$

2) Let V a graded vector space on the field of complex numbers and G a nondegenerate even sesquilinear form. The superunitary groups $U(m/n)$ and $SU(m/n)$ are defined by

$$U(m/n) = \{ M \in PL(m/n) \mid M^T G M = M \}$$
$$SU(m/n) = \{ M \in U(m/n) \mid S \det M = 1 \}$$

Accordingly to the grading of V we have two classes of representations

$$\text{Class I} \quad \dim V_0 = m \quad \dim V_1 = n$$
$$\text{Class II} \quad \dim V_0 = n \quad \dim V_1 = m\,.$$

Because of the isomorphism of the supergroups SU(m/n) and SU(n/m) the Class I (Class II) representations of SU(m/n) are also Class II (Class I) representations of SU(n/m). In what follows we shall discuss only Class I representations.

3) Let $V = V_S \oplus V_A$ a graded vector space on the field of real numbers and G a non degenerate even bilinear form. The invariance of G is compatible with the grading if and only if the restriction of G to V_S being symmetrical the restriction of G to V_A is skew symmetrical. As a consequence the dimension of V_A must be even $n = 2p$ and we get

$$\dim V_S = m \qquad \dim V_A = 2p\,.$$

The orthosymplectic group OSP($m/2p$) is defined by

$$\mathrm{OSP}(m/2p) = \{\, M \in \mathrm{PL}(m/2p) \mid M^{ST} G M = G \,\}\,.$$

Again we have two classes of representations

Class I	$V_0 = V_S$	$V_1 = V_A$	G is supersymmetric
Class II	$V_0 = V_A$	$V_1 = V_S$	G is superantisymmetric.

Only Class I representations will be considered in what follows.

4) For completeness let us point out that a third possibility is to have a graded vector space V of dimension $2n$ on the field of real numbers and a non-degenerate odd bilinear form G. The corresponding supergroup is P(n)

$$P(n) = \{\, M \in \mathrm{SPL}(n/n) \mid M^{ST} G M = G \,\}\,.$$

This case will not be considered here even if it turns out to be very similar to the two previous ones.

5) A supergroup has a Z_2-graded Lie algebra $L = L_0 \oplus L_1$ we shall call a superalgebra. A systematic study of the superalgebras of interest here can be found in Kac's papers [*12*] and other interesting publications [*13*]. Of course the detailed results will not be reproduced here.

The Bose sector L_0 of the superalgebra is the subset of even generators acting separately on V_0 and V_1. L_0 is an ordinary Lie algebra.

The Fermi sector L_1 of the superalgebra is the subset of odd generators transforming the vectors of $V_0(V_1)$ into vectors of $V_1(V_0)$. L_1 is not a Lie algebra but it has well defined transformation properties under L_0.

The Cartan subalgebra H of L is simply the Cartan subalgebra of L_0. For the superunitary group SU(m/n) the dimension of H is $d = m + n - 1$. For the orthosymplectic group OSP($m/2p$) the dimension of H is $d = \nu + p$ where ν is the integer part of $m/2$.

6) The finite dimensional irreducible representations of the superalgebras are determined by their highest weight Λ. An order relation being defined it is convenient to decompose the superalgebra L in three parts

$$L = H \oplus N^+ \oplus N^- .$$

The Cartan generators of H acting on Λ give the d Kac-Dynkin parameters of the representation. The positive generators of N^+ annihilate Λ and the full representation is built by applying the negative generators of N^- on Λ in all the possible ways. It is usual to classify the weights of the representation R in irreducible representations of L_0. When all the possible L_0-multiplets are coupled to Λ the representation R is typical and, in particular, the dimension of R is simply computed from the Kac-Dynkin parameters. When some L_0-multiplets are decoupled from Λ the representation R is atypical and there exists specific relations satisfied by the d Kac-Dynkin parameters of R. However we dont have compact and simple expressions for determining from the highest weight Λ the L_0 content and the dimension of an irreducible atypical representation and we must proceed in an artisanal way—even with computer—in order to obtain these informations [*14*].

III. SUPERUNITARY SUPERTABLEAUX

1) The purely covariant tensors describe all the finite dimensional irreducible representations of the special unitary group SU(n). Because of the unimodular character of the SU(n) transformations the purely contravariant tensors and the mixed tensors with covariant and contravariant indices are equivalent to purely covariant tensors.

Such an equivalence does not occur for the supertensors of the superunitary group SU(m/n). Purely covariant and purely contravariant supertensors are not related and, in addition, we also need mixed supertensors for the description of the tensor representations of SU(m/n). As a consequence the supertableaux of SU(m/n) may have simultaneously covariant (undotted) and contravariant (dotted) boxes. Let us introduce the usual notation

$$\left| \begin{matrix} b_j \\ \bar{b}_j \end{matrix} \right| \text{ counts the } \left| \begin{matrix} \text{covariant} \\ \text{contravariant} \end{matrix} \right| \text{ boxes of the row } j$$

$$\left| \begin{matrix} c_k \\ \bar{c}_k \end{matrix} \right| \text{ counts the } \left| \begin{matrix} \text{covariant} \\ \text{contravariant} \end{matrix} \right| \text{ boxes of the column } k$$

with the constraints of positivity for the supertableau parameters

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_j \geqslant \ldots \geqslant 0 \quad \bar{b}_1 \geqslant \bar{b}_2 \geqslant \ldots \geqslant \bar{b}_j \geqslant \ldots \geqslant 0$$
$$c_1 \geqslant c_2 \geqslant \ldots \geqslant c_k \geqslant \ldots \geqslant 0 \quad \bar{c}_1 \geqslant \bar{c}_2 \geqslant \ldots \geqslant \bar{c}_k \geqslant \ldots \geqslant 0 \quad (1)$$

2) The Young tableaux of SU(n) have, at most, n rows of arbitrary length, the highest rank of a fully skewsymmetric tensor in a n-dimensional space being precisely n.

The same argument cannot be used for SU(m/n) because of the grading of the space of representation and the subsequent existence of bosonic and fermionic indices. Nevertheless we have precise restrictions of the number of independent parameters necessary for describing a supertableau of SU(m/n). A legal supertableau of SU(m/n) has at most m row and n column parameters of arbitrary length the inequalities (1) being satisfied. The constraint of legality are conveniently expressed by introducing the two following sets of inequalities

$$\bar{b}_{1+\alpha} + b_{m+1-\alpha} \leqslant n \qquad \alpha = 0, 1, \ldots, m \qquad \mathrm{B}_0$$
$$c_{1+\beta} + \bar{c}_{n+1-\beta} \leqslant m \qquad \beta = 0, 1, \ldots, n. \qquad \mathrm{C}_0$$

A Class I supertableau of SU(m/n) is legal if and only if the numbers of covariant and contravariant boxes of its rows satisfy one inequality B_0 or, equivalently, if the number of covariant and contravariant boxes of its columns satisfy one inequality C_0. The two sets B_0 and C_0 are clearly equivalent and a legal supertableau of SU(m/n) satisfy at least one inequality B_0 and one inequality C_0.

3) A legal supertableau of SU(m/n) is entirely determined by the lengths b and b of its rows satisfying at least one constraint C_0. In fact for the determination of the highest weight Λ of the supertableau it is more convenient to use a mixed description of the supertableau with both row and column parameters. Of course in order to avoid a double counting we must decide if a given box belongs to a row or to a column. For that purpose we define as J the smallest value of α for which one constraint B_0 is satisfied and by K the smallest value of β for which one constraint C_0 is satisfied

$$0 \leqslant \mathrm{J} \leqslant m \qquad 0 \leqslant \mathrm{K} \leqslant n.$$

The supertableau T of type $\mathrm{T}_{\mathrm{J}}^{\mathrm{K}}$ is described by $m + n$ parameters as follows

J row parameters $\bar{b}$	$m - \mathrm{J}$ row parameters b
K column parameters c	$n - \mathrm{K}$ column parameters $\bar{c}$

The restrictions on these parameters are just the translation of the inequalities of positivity (1)

$$\bar{b}_1 \geqslant \ldots \geqslant \bar{b}_{\mathrm{J}} \geqslant n - \mathrm{K} \qquad b_1 \geqslant \ldots \geqslant b_{m-\mathrm{J}} \geqslant \mathrm{K}$$
$$c_1 \geqslant \ldots \geqslant c_{\mathrm{K}} \geqslant m - \mathrm{J} \qquad \bar{c}_1 \geqslant \ldots \geqslant \bar{c}_{n-\mathrm{K}} \geqslant \mathrm{J}. \qquad (2)$$

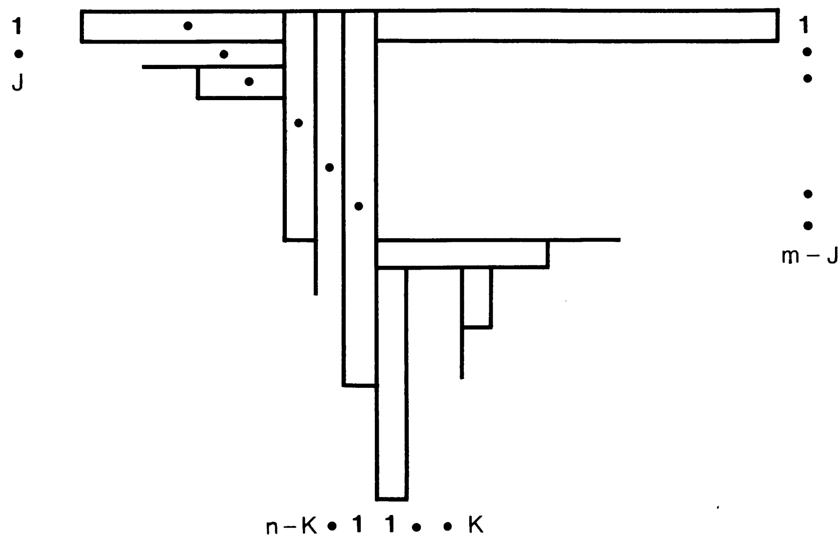

FIG. 1. — Supertableau of $SU(m/n)$ of type T.

A supertableau T of type T_J^K is shown on figure 1.

4) The supertableau T previously defined by $m + n$ parameters is, in fact, a supertableau of the full superunitary group $U(m + n)$ whose rank is precisely $m + n$. The Cartan subalgebra of $U(m/n)$ contains an even generator Q_N commuting with all the generators of $U(m/n)$. Of course the superalgebra of $SU(m/n)$ does not contain the generator Q_N.

The generator Q_N has a well defined eigenvalue for all the weights of a given irreducible representation of $U(m/n)$. The same result extends to a legal supertableau T and the generator Q_N can be normalized in such a way as to measure the difference between the total number of covariant boxes and the total number of contravariant boxes

$$Q_N = \sum_{1}^{m-J} b_j + \sum_{1}^{K} c_k - \sum_{1}^{J} \bar{b}_j - \sum_{1}^{n-K} \bar{c}_k . \tag{3}$$

Such a property is at the origin of the equivalence in $SU(m/n)$ of families of supertableaux and it will be referred as the $U(m/n) \Rightarrow SU(m/n)$ reduction.

IV. ORTHOSYMPLECTIC SUPERTABLEAUX

1) The field on which the orthosymplectic group is defined being that of real numbers the supertableaux of $OSP(m/2p)$ have only undotted

boxes with row parameters b and column parameters c satisfying the constraint of positivity (1).

2) The Young tableaux of the symplectic group $Sp(2p)$ have, at most, p rows [15].

For the Young tableaux of the orthogonal group $O(m)$ the situation is slightly more complicated and the legality constraint is that the sum of the boxes of the two first columns cannot exceed m. Then the permissible tableaux are paired in associate tableaux T and T′ such that the sum of the first column of T and of the first column of T′ is precisely m, the other columns being identical for T and T′. The associate tableaux T and T′ coincide if and only if when m is even, $m = 2\nu$, the tableau T has ν non vanishing rows; in this case T is a self associate tableau. If we restrict now to the special orthogonal group $SO(m)$ the representations corresponding to associate tableaux T and T′ become equivalent and in the case of a self-associate tableau the representation splits into two non equivalent irreducible representations of $SO(m)$ of same dimension [15]. This feature is referred in what follows as the $O(2\nu) \Rightarrow SO(2\nu)$ reduction and it will play an important role for the supertableaux of $OSP(2\nu/2p)$.

3) For an orthosymplectic supertableau T determined by the length b of its rows with the positivity inequalities (1) the legality constraint is simply

$$b_{\nu+1} \leqslant p\,.$$

For an orthosymplectic supertableau T determined by the length c of its columns with the positivity inequalities (1) the legality constraint is simply

$$c_{p+1} \leqslant \nu\,.$$

A more convenient description for the determination of the highest weight Λ of the supertableau uses both row and column parameters and again in order to avoid a double counting we must decide if a given box belongs to a row or to a column. For $m \neq 2$ the usual parametrization is made with ν row and p column parameters

$$\lambda_1 \geqslant \ldots \geqslant \lambda_\nu \geqslant 0 \qquad \mu_1 \geqslant \ldots \geqslant \mu_p \geqslant 0 \tag{4}$$

and we have the supplementary conditions

$$\text{for} \quad \mu_p = b < \nu \qquad \text{then} \quad \lambda_{1+b} = \ldots = \lambda_\nu = 0\,. \tag{5}$$

An orthosymplectic supertableau of $OSP(m/2p)$ is represented on figure 2. For the orthosymplectic supertableaux of $OSP(2/2p)$ it is usual to have a slightly different parametrization as described on figure 3.

4) Let us notice that for the orthosymplectic supertableaux the number $\nu + p$ of parameters is equal to the rank of the supergroup. It follows that the correspondance between an orthosymplectic supertableau and the set of Kac-Dynkin parameters of its highest weight Λ is bijective.

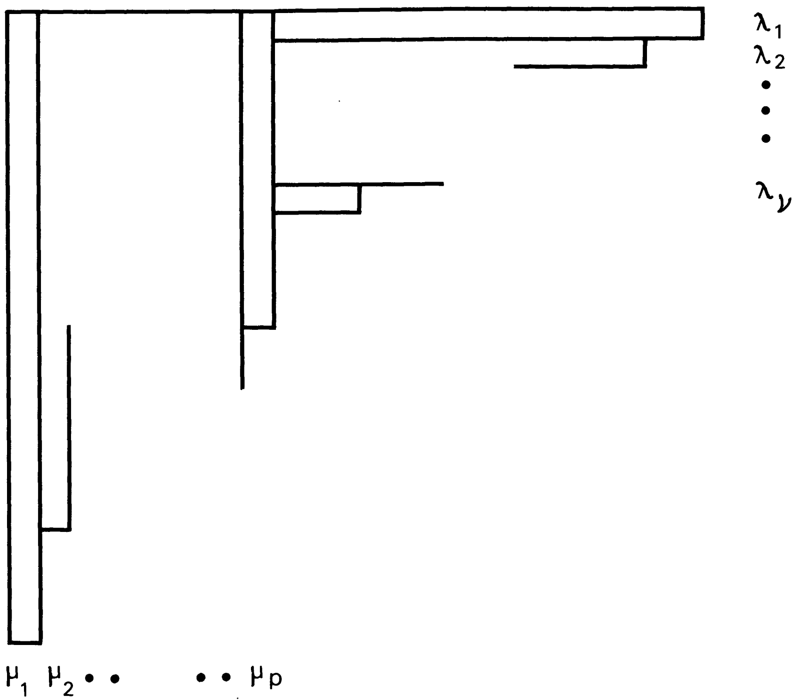

FIG. 2. — Supertableau of OSP($m/2p$), $m = 2$.

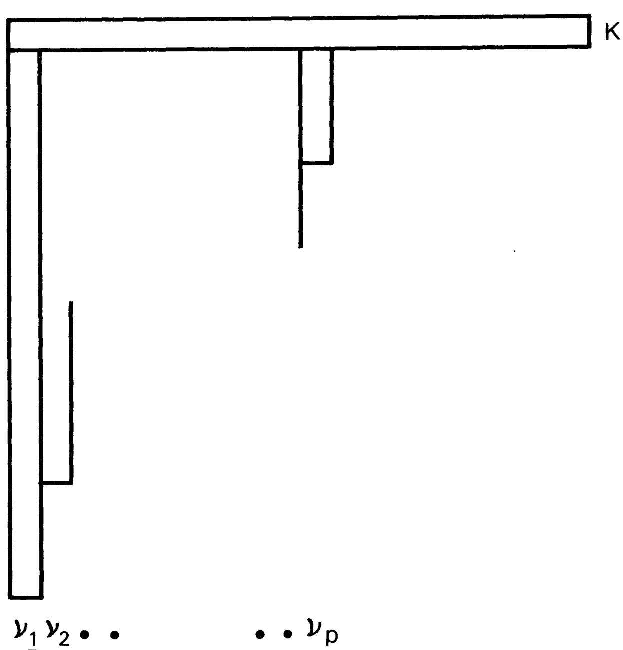

FIG. 3. — Supertableau of OSP($2/2p$).

V. GENERALIZED ATYPICAL SUPERTABLEAUX

For the classical Lie groups the Young tableaux describe irreducible —or a direct sum of two irreducible—representations of the group. For supergroups the same property extends to all the typical supertableaux but it does not do so for all the atypical ones. It is the reason why we must introduce the notion of generalized atypical supertableaux describing non-fully reducible representations of the supergroup. In what follows we call as F the subgroup of PL(m/n) leaving invariant a non degenerate even form G; F is either SU(m/n) or OSP($m/2p$).

To a legal supertableau T of F we associate a supertensor $\mathscr{T}$ of PL(m/n) whose indices have the supersymmetry properties as indicated by the supertableau T. The supertensor $\mathscr{T}$ is an irreducible representation of PL(m/n) but because of the invariance of the form G it is not, in general, an irreducible representation of F. For the supergroup F an irreducible representation is traceless and in order to isolate the irreducible parts of $\mathscr{T}$ under F we must subtract the trace terms in all the possible ways. Let us write symbolically

$$\mathscr{T} = \mathscr{T}_{\mathrm{IRR}} + \sum_{1\,\mathrm{Trace}} \mathrm{G}\mathscr{T}^{(1)}_{\mathrm{IRR}} + \sum_{2\,\mathrm{Traces}} \mathrm{GG}\mathscr{T}^{(2)}_{\mathrm{IRR}} + \dots \tag{6}$$

If r is the rank of the tensor $\mathscr{T}$ then the irreducible tensors $\mathscr{T}_{\mathrm{IRR}}$, $\mathscr{T}^{(1)}_{\mathrm{IRR}}$, $\mathscr{T}^{(2)}_{\mathrm{IRR}}, \dots$ are respectively of ranks r, $r-2$, $r-4$, etc.

When the term $\mathscr{T}_{\mathrm{IRR}}$ can be isolated then the supertableau T is called an irreducible supertableau of F; it is associated to the irreducible supertensor $\mathscr{T}_{\mathrm{IRR}}$ and it describes an irreducible—or a direct sum of two irreducible—representation of F. Such a situation occurs, in particular, for the typical supertableaux.

When some trace terms cannot be separated from $\mathscr{T}_{\mathrm{IRR}}$ then T is called a non-irreducible supertableau of F; with the other supertableaux corresponding to the trace terms which cannot be separated from $\mathscr{T}_{\mathrm{IRR}}$ the supertableau T generates what we define as a *generalized atypical supertableau* of F (GAST). Such an object describes a *non fully reducible*—or a direct sum of two non fully reducible—*representation* of F (NFRR).

In summary the legal supertableaux T of F are associated to fully reducible and non-fully reducible tensor representations of F.

We now introduce an index ρ for the supertableau T as follows:

1) the supertableau T is irreducible: $\rho = 1$
2) the supertableau T is non irreducible $\rho > 1$.

In the second case T is the leading supertableau of a ρ-GAST containing

ρ atypical supertableaux. The index ρ is then a measure of the number of non-separated terms including $\mathscr{T}_{IRR}$ in the reduction formula (6). It is then straightforward to cheek that ρ is an integer power of 2.

Let us finally remark that for a given supertableau T (supertensor $\mathscr{T}$) the conditions of irreductibility of $T(\mathscr{T}_{IRR})$ are simply functions of the superdimension $(m - n)$ of the vector space V.

VI. CLASSIFICATION OF LEGAL SUPERTABLEAUX

The highest weight Λ_{ST} of a legal supertableau T of the supergroup F is either typical or atypical. When Λ_{ST} is typical the supertableau T is irreducible. When Λ_{ST} is atypical there exists specific relations between the row and the column parameters of T.

1) For a *superunitary supertableau* of type T_J^K as defined in Section III the relations of atypicity are

$$\begin{aligned} \bar{b}_{1+j} + m - j &= c_{1+k} + n - k \quad \text{for} \quad \left| \begin{array}{l} 0 \leqslant j \leqslant J - 1 \\ 0 \leqslant k \leqslant K - 1 \end{array} \right. \\ b_{m-j} + j &= \bar{c}_{n-k} + k \quad \text{for} \quad \left| \begin{array}{l} J \leqslant j \leqslant m - 1 \\ K \leqslant k \leqslant n - 1 . \end{array} \right. \end{aligned} \tag{7}$$

By using the positivity constraints (2) it is straightforward to obtain the maximal number L of independent relations (7) which can be simultaneously satisfied. The result is

$$L = \min [m, n] . \tag{8}$$

2) For an *orthosymplectic supertableau* with parameters λ and μ as defined in Section IV the relations of atypicity are

$$\begin{aligned} \mu_{p-j} &= \alpha - j \\ \mu_{p-j} &= \lambda_{1+\alpha} + m - 2 - \alpha - j \end{aligned} \quad \text{for} \quad \left| \begin{array}{l} 0 \leqslant j \leqslant p - 1 \\ 0 \leqslant \alpha \leqslant v - 1 . \end{array} \right. \tag{9}$$

By using the positivity constraints (4) and (5) it is straightforward to obtain the maximal number L of independent relations (9) which can be simultaneously satisfied. The result is

$$L = \min [m - 1, p] . \tag{10}$$

3) The degeneracy of atypicity δ of the supertableau T is defined as the number of independent atypical relations satisfied by T. We have the obvious upper bound

$$0 \leqslant \delta \leqslant L . \tag{11}$$

To a legal supertableau T of F we have associated two non-negative integers δ and ρ. For a typical supertableau $\delta = 0$ and $\rho = 1$. For an atypical supertableau $\delta \geqslant 1$ and it is straightforward to obtain, at fixed δ, an upper bound for ρ

$$1 \leqslant \rho \leqslant 2^{\delta}. \tag{12}$$

4) We now make a partition of the full set S_0 of legal supertableaux of F in non-empty disjoint classes Δ_l whose union is S_0. The integer l is computed from δ and ρ by the relation

$$2^l = \frac{2^\delta}{\rho}. \tag{13}$$

As a consequence of equations (11), (12) and (13) we have the following bound for l

$$0 \leqslant l \leqslant \delta \leqslant L. \tag{14}$$

The partition of S_0 is then realized with $1 + L$ classes

$$S_0 = \Delta_0 \oplus \Delta_1 \oplus \ldots \oplus \Delta_L. \tag{15}$$

Conversely a supertableau of the class Δ_l with a degeneracy of atypicity δ $l \leqslant \delta \leqslant L$ has an index ρ given by

$$\rho = 2^{(\delta - l)}. \tag{16}$$

5) The previous considerations are conveniently formulated with three theorems:

THEOREM I. — When a legal supertableau T belongs to the class Δ_l, $0 \leqslant l \leqslant L$, then its degeneracy of atypicity δ is lower bounded by l according to equation (14). As two trivial consequences of this theorem we have

i) the typical supertableau all belong to the class Δ_0
ii) the supertableaux of the class Δ_L all have $\delta = L$.

THEOREM II. — When a legal supertableau T belongs to the class Δ_l, $0 \leqslant l \leqslant L$, and has a degeneracy of atypicity $\delta = l$ then T is irreducible. As a consequence the supertableaux of the class Δ_L are all irreducible.

THEOREM III. — When a legal supertableau T belongs to the class Δ_l, $0 \leqslant l \leqslant L - 1$, and has a degeneracy of atypicity δ, $l < \delta \leqslant L$, then T is non irreducible and it is the leading supertableau of a ρ-GAST where ρ is given by the equation (16).

6) When the index ρ for a legal supertableau T has been determined by a direct algebraic computation of the reduction of the supertensor $\mathscr{T}$ as explained in Section V the three previous theorems are just trivial consequences of the equations (14) and (15). However such a computation is complicated and it becomes rapidly untractable for high order supertensors.

Fortunately it is possible to introduce the concept of size for a supertableau and to show that the previous partition of S_0 into $1 + L$ classes is in fact made in relation with the size of the supertableaux. The classes are now characterized by the row and the column parameters of T only and the content of the three theorems become non-trivial. The next section studies this question.

VII. SIZE OF LEGAL SUPERTABLEAUX

We now give the definition of the classes Δ_l of superunitary and orthosymplectic supertableaux by using the row and column parameters.

1) *Superunitary supertableaux.* We define two sets of inequalities B_l and C_l by

$$\bar{b}_{1+\alpha} + b_{m+1-\alpha-l} \leqslant n - l \qquad \alpha = 0, 1, \ldots, n - l \qquad B_l$$

$$c_{1+\beta} + \bar{c}_{n+1-\beta-l} \leqslant m - l \qquad \beta = 0, 1, \ldots, m - l \qquad C_l$$

Because of the positivity of the supertableaux parameters $0 \leqslant l \leqslant L$.

By definition a supertableau T belongs to the subset S_l if and only if one constraint B_l is satisfied or, equivalently, if and only if one constraint C_l is satisfied. As a consequence of the structure of the constraints B_l and C_l we get the following relations of inclusion

$$S_0 \supset S_1 \supset \ldots \supset S_l \supset \ldots \supset S_L . \qquad (17)$$

Of course S_0 is the set of legal supertableaux of $SU(m/n)$ already defined in Section III.

The class Δ_l is now defined as the complement of S_{l+1} in S_l. By extension we put $\Delta_L \equiv S_L$ and we get

$$S_l = \Delta_l \oplus \Delta_{l+1} \oplus \ldots \oplus \Delta_L . \qquad (18)$$

With such a definition of the classes the proof of the Theorems I, II and III has been given in [*5*] and [*6*].

2) In the superunitary case the supertableaux of the class Δ_0 are affected by the equivalence due to the reduction $U(m/n) \Rightarrow SU(m/n)$. Such a feature occurs for both the irreducible supertableaux which are typical and generalized atypical supertableaux the value of Q_N being the same for the ρ atypical supertableaux of a ρ-GAST.

On the other hand the atypical supertableaux of the set S_1 have at most $m + n - 2$ non trivial row and column parameters and it is no more room for the equivalence relation.

3) *Orthosymplectic supertableaux.* We shall use the column parameters of the supertableaux and the legality constraint for supertableaux of OSP($m/2p$) is simply

$$c_{p+1} \leqslant \nu . \qquad (19)$$

The previous inequality defines the full set S_0 of legal supertableaux and we now define the subsets S_l of S_0 with the constraints

$$c_{p+1-l} + c_{p+2-l} \leqslant m - 1 - l \qquad l = 1, 2, \ldots, L . \qquad (20)$$

As previously we have the relations of inclusion (17) and we define the class Δ_l as the complement of S_{l+1} in S_l with $\Delta_L \equiv S_L$. We then obtain a simple characterization of the classes

$$\text{Class } \Delta_l \text{ for } 1 \leqslant l \leqslant L \quad c_{p+1-l} + c_{p+2-l} \leqslant m - 1 - l \leqslant c_{p+l} + c_{p+1-l} \qquad (21)$$

$$\text{Class } \Delta_0 \quad c_{p+1} \leqslant \nu \quad \text{and} \quad c_p + c_{p+1} \geqslant m - 1 \qquad (22)$$

For the orthosymplectic group OSP($1/2p$), $p \geqslant 1$, we have only one class Δ_0 of irreducible typical supertableaux and this case is trivial.

For the orthosymplectic group OSP($2/2p$), $p \geqslant 1$, we have two classes Δ_0 and Δ_1 of irreducible typical or atypical supertableaux and 2-GAST. The proof of the three theorems is simple and it has been given in reference [*9*].

The theorems have also been proven for the three classes of supertableaux of OSP($m/4$), $m \geqslant 3$ [*11*].

In the general case the Theorem I is still simple but for Theorems II and III we have given only plausibility arguments based on the most natural extension of the results obtained in the previous particular cases.

VIII. NON FULLY REDUCIBLE TENSOR REPRESENTATIONS

1) A ρ-GAST is a collection of ρ atypical supertableaux T_α whose dimension can be computed—at least formally—using the determinant proposed by Balantekin and Bars [*1*]. We then obtain the dimension of the ρ-GAST

$$D_{ST} = \sum_{1}^{\rho} \dim T_\alpha . \qquad (23)$$

The ρ-GAST describes a NFRR of F with N_ρ atypical components R_j.

In principle we know the L_0-content and subsequently the dimension of each R_j. We then obtain the dimension of the NFRR

$$D_R = \sum_{1}^{N_\rho} \dim R_j . \tag{24}$$

In all the particular cases we have studied for orthosymplectic and superunitary supertableaux we have obtained the expected equality

$$D_{ST} = D_R . \tag{25}$$

Let us point out that such an equality is non-trivial because the matching between the supertableaux T_α and the atypical components R_j is meaningless some supertableaux of the ρ-GAST—at least the leading supertableau T_1—having *no individual existence*.

2) The value N_ρ of the number of atypical components of the NFRR described by a ρ-GAST is not only a function of ρ. In our study of superunitary supertableaux by means of the tensor product method we have found [*5*] [*7*]

$$N_2 = 3, \underline{4}, 5 \qquad \text{for} \quad \rho = 2$$
$$N_4 = 16, 17 \qquad \text{for} \quad \rho = 4$$

with the underlined value in the normal case and the other values in particular cases. A possible extrapolation of the normal value of N_ρ for $\rho > 4$ is $N_\rho = \rho^2$. Analogous results have been obtained for the orthosymplectic ρ-GAST with the supplementary complication due to the $O(2\nu) \Rightarrow SO(2\nu)$ reduction for the supertableaux of $OSP(2\nu/2p)$.

3) The leading supertableau T_1 of a generalized atypical supertableau determines completely the ρ_1-GAST and the associated NFRR. Let us call as Λ_1 the highest weight of T_1 with a degeneracy of atypicity δ and Δ_{l_1} the class to which T_1 belongs. Of course $0 \leqslant l_1 \leqslant \delta$ and $\rho_1 = 2^{(\delta - l_1)}$.

i) The $(\rho_1 - 1)$ supertableaux T_α partners of T_1 is the ρ_1-GAST are simply constructed from T_1 by suppressing pairs of boxes associated to non-subtracted trace terms. These atypical supertableaux have the same δ and they belong to a class Δ_{l_α} with $l_1 \leqslant l_\alpha \leqslant \delta$. Some of them might be irreducible supertableaux.

ii) The N_{ρ_1} atypical components of the NFRR are also completely determined by T_1. Among them we have R_1, the atypical component of highest weight Λ_1, which enters just once and the atypical components R_α of highest weights Λ_α those of the previous atypical supertableaux of the ρ_1-GAST and which can enter twice or more in the NFRR.

IX. CONCLUDING REMARKS

1) It is relatively simple to qualitively understand the meaning of the correspondance between the size (class) of the supertableau and the degeneracy of atypicity δ of its highest weight Λ_{ST}. For such a supertableau δ relations exist between the d Kac-Dynkin parameters of Λ_{ST} and therefore δ relations between the row and column parameters of T are satisfied. This implies a minimal size for the supertableau T otherwise more than δ relations would be fullfilled and this minimal size corresponds just to the class Δ_δ. The supertableaux of the class Δ_δ with a degeneracy of atypicity δ are irreducible and they describe—except for the $O(2\nu) \Rightarrow SO(2\nu)$ phenomena—irreducible representations of T of highest weight Λ_{ST}.

At fixed δ we now increase the size of the supertableau T which henceforth belongs to a class Δ_l with $0 \leqslant l < \delta$. The supertableau T becomes too large to accommodate only the previous representation R of highest weight Λ_{ST} and it has more weights. We have a ρ-GAST associated to a NFRR of F and the index ρ increases when the size of the supertableau increases according to the formula (16). For a supertableau T of the class Δ_l with a degeneracy of atypicity δ only l relations of atypicities are natural and $\delta - l$ relations of atypicities are accidental. The difference between a natural and an accidental atypicity is the following. A natural atypicity is coming only from the size of the supertableau because some row and column parameters are frozen whereas an accidental atypicity results of a fortuitous relation between some row and column parameters. In this respect the supertableaux of the class Δ_0 are naturally typical and they may be accidentally atypical.

2) Until now we have described the interpretation of the supertableaux of F in terms of representations of F. Let us now briefly comment on the inverse relation calling as R an irreducible tensor representation of F and Λ its highest weight with a degeneracy of atypicity δ.

When $\delta = 0$ the typical irredubible representation R of F is in general described by a typical irreducible supertableau T as follows:

1° In the superunitary case because of the $U(m/n) \Rightarrow SU(m/n)$ equivalence we have an one parameter family of equivalent typical irreducible supertableaux of the class Δ_0, of same highest weight Λ, same L_0-content and therefore same dimension. All these supertableaux describe the same typical irreducible representation R.

2° In the orthosymplectic case when R is a self-contragradient typical representation there exists one typical irreducible supertableau of the class Δ_0 of highest weight Λ_{ST} uniquely associated to R. When R is not self-contragradient due to the $O(2\nu) \Rightarrow SO(2\nu)$ reduction only the self-contragradient direct sum $R \oplus \overline{R}$ is uniquely described by a typical irreducible supertableau of the class Δ_0.

When $\delta \geqslant 1$ an atypical irreducible supertableau is not associated to every atypical irreducible representation R of F. Again the discussion is somewhat different for the superunitary and the orthosymplectic supertableaux.

1° In the superunitary case to the atypical highest weight Λ with a degeneracy of atypicity $\delta = 1$ are associated one irreducible supertableau of the class Δ_1 and one parameter family of non-irreducible supertableaux of the class Δ_0 generating an one parameter family of equivalent 2-GAST. For $\delta > 1$ an unique irreducible supertableau of the class Δ_δ exists only when the highest weight Λ is a minimal realization of the degeneracy of atypicity δ [*6*]. In addition for every atypical highest weight Λ we have several non irreducible supertableaux belonging to classes Δ_l with $0 \leqslant l < \delta$. For $l = 0$ due to the $U(m/n) \Rightarrow SU(m/n)$ equivalence we have an one parameter family of ρ-GAST describing the same NFRR of F.

2° In the orthosymplectic case to a self-contragradient highest weight Λ we can associate one atypical supertableau T of highest weight Λ. When Λ is not self contragradient the bijective relation is between $\Lambda \oplus \overline{\Lambda}$ and T using the $O(2\nu) \Rightarrow SO(2\nu)$ reduction. This supertableau T is either irreducible if it belongs to the class Δ_δ or non-irreducible if it belongs to a class Δ_l with $0 \leqslant l < \delta$. In this later case T generates a ρ-GAST and the representation R ($R \oplus \overline{R}$) is part of the associated NFRR. Again a subset of atypical representations is not associated to atypical irreducible supertableaux.

REFERENCES

[*1*] A. B. Balantekin and I. Bars, *J. Math. Phys.*, t. **22**, 1981, p. 1149-1810, t. **23**, 1982, p. 1239.

[*2*] I. Bars, B. Morel and H. Ruegg, *J. Math. Phys.*, t. **24**, 1983, p. 2253.

[*3*] R. J. Farmer and P. D. Jarvis, *J. Phys. A Math. Gen.*, t. **17**, 1984, p. 23655.

[*4*] P. H. Dondi and P. D. Jarvis, *Z. Phys. C*, t. **4**, 1980, p. 201 ; *J. Phys. A Math. Gen.*, t. **14**, 1981, p. 547.

R. C. King, *Lecture notes in Physics*, t. **180**, p. 41 (Springer Verlag, 1983).

S. Hongzhou and H. Qizhi, *Sci. Sin.*, t. **24**, 1981, p. 914 ; *Ph. Energia Fortis*, t. **5**, 1981, p. 646 ; t. **317**, 1982, p. 401.

J. Thierry Mieg and B. Morel, *in Superspace and Supergravity*, edited by S. Hawking and M. Rocek (Cambridge U. P. London, 1981).

J. P. Hurni, *Young Supertableaux of Lie superalgebras*. Preprint Genève, UGVA-DPT, 1984-04 420.

[*5*] F. Delduc and M. Gourdin, *J. Math. Phys.*, t. **25**, 1984, p. 1651.

[*6*] F. Delduc and M. Gourdin, *J. Math. Phys.*, t. **26**, 1985, p. 1865.

[*7*] M. Gourdin, *Naturally atypical supertableaux of the superunitary group SU(3/2)*, Preprint, Paris, LPTHE 84/26 ; *Naturally typical supertableaux of hte superunitary group SU(3/2)*. Preprint, Paris, LPTHE, 84/31 unpublished.

[8] M. GOURDIN, *J. Math. Phys.*, t. **27**, 1986, p. 61.
[9] M. GOURDIN, *J. Math. Phys.*, t. **27**, 1986, p. 2832.
[10] M. GOURDIN, *J. Math. Phys.*, t. **27**, 1986, p. 1980.
[11] M. GOURDIN, Classification and interpretation of the supertableaux of the orthosymplectic groups OSP (m/4). Preprint, Paris, LPTHE, 86/08 to be published in *J. Math. Phys.*, 1987.
[12] V. G. KAC, *Adv. Math.*, t. **26**, 1977, p. 1 ; *Commun. Math. Phys.*, t. **53**, 1977, p. 31, *Lecture Notes in Mathematics*, vol. 676, p. 597 (Springer Verlag, 1978).
[13] L. CORWIN, Y. NE'EMAN and S. RITTENBERG, *Rev. Mod. Phys.*, t. **47**, 1975, p. 575.
A. PAIS and V. RITTENBERG, *J. Math. Phys.*, t. **16**, 1975, p. 2062.
P. G. O. FREUND and I. KAPLANSKY, *J. Math. Phys.*, t. **17**, 1976, p. 228.
W. NAHM and V. RITTENBERG, *J. Math. Phys.*, t. **17**, 1976, p. 868.
M. SCHEUNERT, W. NAHM and V. RITTENBERG, *J. Math. Phys.*, t. **17**, 1976, p. 228 ; t. **17**, 1976, p. 1640 ; t. **18**, 1977, p. 146 ; t. **18**, 1977, p. 155.
M. SCHEUNERT, The theory of Lie superalgebras, *Lecture Notes in Mathematics*, t. **716** (Springer Verlag 1979).
[14] J. THIERRY MIEG, *Table of irreducible representation of the basic classical Lie superalgebras*. Preprint Berkeley UCB-PTH 85/14, LBL-19353.
[15] M. HAMERMESH, *Group Theory and its Applications to Physical Processes* (Addison-Wesley Reading MA 1962).

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