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YVONNE CHOQUET BRUHAT

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Global existence theorems for hyperbolic harmonic maps

by

Yvonne CHOQUET BRUHAT
Institut de Mécanique, Université Paris VI

INTRODUCTION

Harmonic maps between properly riemannian manifolds have a long mathematical history, with global results of great interest in geometry and physics (cf. [1] [2] [16]). The harmonic maps with source a riemannian manifold of hyperbolic (lorentzian) signature have been studied more recently, however such maps appear in numerous problems of physics, from the harmonic gauge of General Relativity to the non-linear σ models and many others, as pointed out in [15] [17] and [14].

The natural problem for such « hyperbolic harmonic maps » is the Cauchy problem, that is the determination of the map from its value, and the value of its first derivative, on a space like submanifold of the source. A local existence theorem of a solution of the problem for harmonic maps from M^{n+1} , Minkowski space time of arbitrary dimension, into various compact riemannian manifold has been proved by Ginibre and Velo [10], using I. Segal's theory of non linear semi-group [12]. They also prove for such problems a global existence theorem when $n = 1$, by using energy estimates. A global existence from M^2 into a complete riemannian has been proved, by a different method, for smooth data by Gu Chao Hao [11].

In this article we prove a local existence theorem for harmonic maps from a globally hyperbolic manifold (M, g) into a riemannian manifold (N, h) both arbitrary except for some regularity conditions. The proof uses the embedding of (N, h) in an euclidean space $(\mathbb{R}^q, e)$, like Ginibre and Velo [10]. Another proof, which used only the standard local existence and uniqueness results for hyperbolic equations had been indicated in [3], which treated harmonic gauges in General Relativity.

We prove in § 5 a global existence theorem in the case $n = 1$, using the second order equation satisfied by the differential of an harmonic mapping and, like Ginibre and Velo, the resulting *a priori* estimates.

In § 6 we prove a global existence theorem for $(M, g) = M^{n+1}$, with n odd, arbitrary if the Cauchy data are sufficiently near from those of a constant map. We use the method of conformal transformation as in [6] [7] and [8]. The theorem is valid for $n = 1$, because of conformal invariance, and for $n = 3$ because the operator satisfies an analogue of the condition indicated as sufficient by Christodoulou in [8] (which treats scalar systems).

It results from counter examples constructed by Shatah [9] that this last theorem cannot be true for arbitrary, large, data.

1. DÉFINITIONS

Let (M, g) and (N, h) be two smooth riemannian manifolds of arbitrary signature and dimensions. Let

$$f : M \rightarrow N$$

be a smooth map. The differential of f at $x \in M$ is a linear map

$$\nabla f(x) : T_x M \rightarrow T_{f(x)} N$$

it is therefore an element of $T_x^* M \otimes T_{f(x)} N$. The differential itself, ∇f is a mapping $x \rightarrow \nabla f(x)$, that is a section of the vector bundle with base M and fiber at x the vector space $T_x^* M \otimes T_{f(x)} N$. This vector bundle—one forms on M with values at x in $T_{f(x)} N$ —is denoted $T^* M \otimes f^{-1} N$. The vector bundle with base M and fiber $T_{f(x)} N$ at x is denoted $f^{-1} TN$. If (x^a) and (y^a) are respectively local coordinates in M and N , and f is represented in these coordinates by

$$y^a = f^a(x^a)$$

the derivative f is represented by

$$(x^a) \rightarrow \left(\frac{\partial f^a}{\partial x^b}(x^a) \right).$$

The metrics g on M and h on N endow the fiber at x of the vector bundle $E = T^* M \otimes f^{-1} TN$ with a scalar product $G(x) = g^\#(x) \otimes h(f(x))$, where $g^\#$ is the contravariant tensor canonically associated with g . In coordinates, if u and v are two sections of E :

$$(1) \quad G(x)(u, v) = g^{a\beta}(x^\lambda) h_{ab}(f^c(x^\lambda)) u_\alpha^a(x^\lambda) v_\beta^b(x^\lambda).$$

The vector bundle $E \equiv T^* M \otimes f^{-1} TN$ is endowed with a linear connection ∇ , mapping sections of E into sections of $T^* M \otimes E$, by the usual rules: if s is a section of $f^{-1} TN$ and t a section of $T^* M$ we have:

$$\nabla_v(t \otimes s) = {}^g \nabla_v t \otimes s + t \otimes f^{*h} \nabla_v s$$

with ${}^g\nabla$ and ${}^h\nabla$ the riemannian covariant derivatives in the metrics g and h respectively. In local coordinates if $(x^\alpha) \mapsto (u^a(x^\alpha))$ is a section of E , we have:

$$(2) \quad \nabla_\alpha u_\beta^a(x^\lambda) = \partial_\alpha u_\beta^a(x^\lambda) + \frac{\partial f^b}{\partial x^\alpha} \Gamma_{b\ c}^a(f^d(x^\lambda)) u_\beta^c(x^\lambda) - \Gamma_{\alpha\ \beta}^\mu(x^\lambda) u_\mu^a(x^\lambda)$$

where $\Gamma_{b\ c}^a$ and $\Gamma_{\alpha\ \beta}^\lambda$ denote respectively the riemannian connexions of g and h . The mapping f is called *harmonic* if

$$(3) \quad \text{tr}_g \nabla^2 f = 0$$

that is, in local coordinates

$$g^{\alpha\beta} \nabla_\alpha \partial_\beta f^a \equiv g^{\alpha\beta} (\partial_{\alpha\beta}^2 f^a - \Gamma_{\alpha\ \beta}^\lambda \partial_\lambda f^a + \Gamma_{b\ c}^a \partial_\alpha f^b \partial_\beta f^c) = 0.$$

If f satisfies (3) it is a critical point of the functional

$$f \mapsto E(f) = \int_M G(\nabla f, \nabla f) d\mu(g) = \int_M g^{\alpha\beta}(x^\lambda) h_{ab}(f^c(x^\lambda)) \partial_\alpha f^a \partial_\beta f^b d\mu(g).$$

2. HYPERBOLIC HARMONIC MAPS. ENERGY INTEGRAL

When the metrics g and h are properly riemannian the integral (4) is called the energy of the mapping f . When g is of hyperbolic signature we will define another integral as the energy of f , like for usual wave equations.

We define the stress energy tensor of the map f as the covariant 2-tensor on M given by

$$T = (h \circ f)(\nabla f, \nabla f) - \frac{1}{2} g(g^\# \otimes h \circ f)(\nabla f \otimes \nabla f)$$

that is

$$(2.1) \quad T_{\alpha\beta} = (h_{ab} \circ f) \partial_\alpha f^a \partial_\beta f^b - \frac{1}{2} g_{\alpha\beta} g^{\lambda\mu} (h_{ab} \circ f) \partial_\lambda f^a \partial_\mu f^b$$

We have on M

$$(2.2) \quad \nabla_\alpha T^\alpha_\beta \equiv (h_{ab} \circ f) \partial_\beta f^a \nabla_\alpha \partial_\alpha f^b$$

that is

$$\nabla \cdot T = (h \circ f)(\nabla f, \text{tr}_g \nabla^2 f).$$

Thus $\nabla \cdot T = 0$ if f is a harmonic map.

We suppose that (M, g) is a hyperbolic manifold with $M = S \times \mathbb{R}$, $S_t \equiv S \times \{t\}$ space-like, we denote by n their unit time like normal. Let X be a time like vector field. We define the energy density of f relative to S_t and X by:

$$e(f) = T(X, n) = X^\alpha n^\beta T_{\alpha\beta}$$

we have

$$(2.3) \quad e(f) = \frac{1}{2} \gamma^{\alpha\beta} (h_{ab} \circ f) \partial_\alpha f^a \partial_\beta f^b = \frac{1}{2} \gamma^\# \otimes (h \circ f)(\nabla f, \nabla f)$$

where $\gamma^\#$ is the quadratic form

$$\gamma^{\alpha\beta} = n^\alpha X^\beta + n^\beta X^\alpha - g^{\alpha\beta} X^\lambda n_\lambda.$$

It is well known that this form is positive definite if g of hyperbolic signature $(+ - - \dots)$ with X and n time like.

We deduce from

$$\nabla_\alpha T^{\alpha\beta} = 0$$

when f is a harmonic map that

$$(2.4) \quad \nabla_\alpha (X_\beta T^{\alpha\beta}) = \frac{1}{2} T^{\alpha\beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha).$$

By integration of 2-4 on $S \times [0, t]$ we obtain the following:

PROPOSITION. — Let f be a smooth map such that $\nabla f|_{S_t}$ has a compact support for $0 \leq \tau \leq t$ then, if f is harmonic it satisfies the identity:

$$(2.5) \quad \int_{S_t} N e(f) d\mu_t \equiv \int_{S_0} N e(f) d\mu_0 + \frac{1}{2} \int_0^t \int_{S_\tau} N (T \cdot LX) d\mu_\tau$$

$d\mu_t$ denotes the volume element of the metric $\bar{g}_t$ induced on S_t by g , N is the lapse function, $N = g(X, n)$, that is $N = (g^{00})^{-1/2}$ if X is the tangent vector to the curves $\{x\} \times \mathbb{R}$ and the coordinates are adapted to the product $S \times \mathbb{R}$: the volume element of (M, g) is

$$d\mu(g) = N d\mu_{x^0} dx^0.$$

DEFINITION 1. — The manifold (M, g) is said *regularly hyperbolic* if

1) M and g are smooth and $M = S \times \mathbb{R}$, the metrics $\bar{g}_t$ induced on $S_t = S \times \{t\}$ are (properly) riemannian ⁽¹⁾, and uniformly equivalent to the metric $\bar{g}_0$ which is complete.

2) The tangent vector X to the lines $\{x\} \times \mathbb{R}$ is time like, and there exists strictly positive numbers a and b such that

$$\inf_M g(X, X) \geq a \geq 0 \quad \text{and} \quad \sup_M N \leq b$$

we then have also, since $N = g(X, n) \geq (g(X, X))^{1/2}$

$$\inf_M N \geq a^{1/2} \quad \text{and} \quad \sup_M g(X, X) \leq b^2.$$

If (M, g) is regularly hyperbolic the metric γ on $M = S \times \mathbb{R}$ defined by 2-3 is uniformly equivalent to the metric $\Gamma = (dx^0)^2 - \bar{g}_0$.

We shall then add to the definition of regular hyperbolicity the following.

3) The riemann curvature of g , together with as many of its covariant derivatives as is relevant, is bounded in Γ -norm.

(1) These metrics are negative definite: $-\bar{g}_t$ is positive definite.

DEFINITION 2. — A (properly) riemannian manifold is said to be regular if it is smooth, has a non zero injectivity radius (thus is complete), and has a bounded riemannian curvature, as well as its covariant derivatives up to the relevant order.

Tensor products of the metrics Γ and h give scalar products and norms in the fiber at $x \in M$ of vector bundles $(\otimes T_*M)^p (\otimes f^{-1}TN)^q$ or their duals. We denote this norm by $|\cdot|$. We have, for instance,

$$|\nabla f|^2 = \Gamma^{\alpha\beta} (h_{ab} \circ f) \partial_\alpha f^a \partial_\beta f^b.$$

If s and u are two sections of such vector bundles we have, at a point $x \in M$

$$|s \otimes u| = |s| |u|, \quad |s \cdot u| \leq |s| |u|$$

if $s \cdot u$ is some contracted tensor product

Therefore, in particular

$$|LX \cdot T| \leq |LX| |T|.$$

It results from the definition that $|g|$ and $|g^\#|$ are uniformly bounded if (M, g) is regularly hyperbolic (cf. [5]). Thus, due to the expression of T , there exists a constant C such that

$$(2.6) \quad |T| \leq Ce(f)$$

and also, if $|X|$ is uniformly bounded on M , a constant still denoted C such that

$$|LX \cdot T| \leq Ce(f).$$

From the equality (2.5) results then the inequality (C_0 and C positive constants)

$$(2.7) \quad y(t) \leq C_0 y(0) + C \int_0^t y(\tau) d\tau$$

with

$$y(t) = \int_{S_t} |\nabla f|^2 d\mu_0.$$

If y is a continuous function of t we deduce from (2.7), by the Gromwall lemma

$$(2.8) \quad y(t) \leq K(t)y(0)$$

with $K(t)$ the continuous function of t

$$K(t) = C_0 e^{Ct}.$$

3. SECOND ORDER EQUATION FOR f

PROPOSITION. — Every smooth harmonic map $f: (M, g) \rightarrow (N, h)$ satisfies the equation

$$(\nabla \cdot \nabla)(\nabla f) - \text{Ricc}(g)\nabla f + \text{tr}_g(f^* \text{Riem}(h) \cdot \nabla f) = 0$$

that is, in local coordinates

$$(3.1) \quad \nabla^\lambda \nabla_\lambda \partial_\alpha f^a - \mathbf{R}_\alpha^\beta \partial_\beta f^a + \mathbf{R}_{cd}{}^a{}_b \partial_\alpha f^c \partial_\beta f^d \partial^\beta f^b = 0.$$

The proof, independant of signature, is straightforward and given in [1]. We now consider the case g hyperbolic and h properly riemannian. We set (h_{ab} stands always for $h_{ab} \circ f$)

$$(3.2) \quad \mathbf{T}_{\alpha\beta}^{(1)} = e^{\lambda\mu} h_{ab} \left\{ \nabla_\alpha \partial_\lambda f^a \nabla_\beta \partial_\mu f^b - \frac{1}{2} g_{\alpha\beta} g^{\rho\sigma} \nabla_\rho \partial_\lambda f^a \nabla_\sigma \partial_\mu f^b \right\}$$

We have identically, after use of the Ricci identity

$$(3.3) \quad \begin{aligned} \nabla_\alpha \mathbf{T}_{(1)}^{\alpha\beta} &\equiv e^{\lambda\mu} h_{ab} \nabla^\alpha \nabla_\alpha \partial_\lambda f^a \nabla_\beta \partial_\mu f^b \\ &\quad + e^{\lambda\mu} h_{ab} \nabla^\alpha \partial_\lambda f^a (-\mathbf{R}_\alpha^\beta{}_\mu{}^\rho \partial_\rho f^b + \partial_\alpha f^c \partial^\beta f^d \partial_\mu f^e \mathbf{R}_{cd}{}^b{}_e) \\ &\quad + h_{ab} (\nabla^\alpha e^{\lambda\mu}) \left\{ \nabla_\alpha \partial_\lambda f^a \nabla^\beta \partial_\mu f^b - \frac{1}{2} g_\alpha{}^\beta g^{\rho\sigma} \nabla_\rho \partial_\lambda f^a \nabla_\sigma \partial_\mu f^b \right\}. \end{aligned}$$

Using (3.1) we see that, for a harmonic map, $\nabla_\alpha \mathbf{T}_{(1)}^{\alpha\beta}$ is a polynomial $Q(f, \nabla f, \nabla^2 f)$ of degree 2 in $\nabla^2 f$, with coefficients of degree 1 or 3 [respectively 0] in ∇f for the terms of degree 1 [respectively 2] in $\nabla^2 f$. The mapping f itself appears through $h \circ f$ and $\text{Riem}(h) \circ f$.

On the other hand we have:

$$e_{1,f} \equiv \mathbf{T}_{\alpha\beta}^{(1)} X^\alpha n^\beta = \frac{1}{2} \gamma^{\alpha\beta} \gamma^{\lambda\mu} h_{ab} \nabla_\alpha \partial_\lambda f^a \nabla_\beta \partial_\mu f^b \geq 0.$$

Integrating the identity:

$$\nabla_\alpha (X_\beta \mathbf{T}_{(1)}^{\alpha\beta}) \equiv X_\beta \nabla_\alpha \mathbf{T}_{(1)}^{\alpha\beta} + \frac{1}{2} \mathbf{T}_{(1)}^{\alpha\beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)$$

with $\mathbf{T}_{\alpha\beta}^{(1)}$ and $\nabla_\alpha \mathbf{T}_{(1)}^{\alpha\beta}$ given by (3.2) and (3.3) gives for a harmonic map, with compact support in space, using (3.1), an equality:

$$(3.4) \quad \int_{S_t} \text{Ne}_{1,f} d\mu(\bar{g}_t) = \int_{S_0} \text{Ne}_{1,f} d\mu(\bar{g}_0) + \int_0^t \int_{S_\tau} \text{NQ}_1(f, \nabla f, \nabla^2 f) d\mu(\bar{g}_\tau) d\tau$$

with Q_1 of the type

$$Q_1(f, \nabla f, \nabla^2 f) \equiv \Sigma k \nabla^2 f \cdot \{ \text{Riem}(g) \cdot \nabla f + \text{Riem}(h) \circ f \cdot (\otimes \nabla f)^3 + \nabla^2 f \}$$

with k polynomial in $g^\#, h, X$ and ∇X .

If (M, g) is regularly hyperbolic, and (N, h) regularly riemannian we deduce from (3.4) an inequality, as in § 2, with C_0, C_1, C_2, C_3 positive constants:

$$(3.5) \quad y_1(t) \leq C_0 y_1(0) + C_1 \int_0^t y_1(\tau) d\tau + \int_0^t \int_{S_\tau} (C_2 |\nabla^2 f| |\nabla f| + C_3 |\nabla^2 f| |\nabla f|^3) d\mu_0 d\tau$$

where, by definition

$$y_1(t) = \int_{S_t} |\nabla^2 f|^2 d\mu_0$$

and

$$|\nabla^2 f|^2 = e^{\lambda\mu} e^{\alpha\beta} (h_{ab} \circ f) \nabla_\alpha \partial_\lambda f^a \nabla_\beta \partial_\mu f^b.$$

4. LOCAL EXISTENCE

Let N be a submanifold of the riemannian manifold (Q, q) and h be the metric induced on N by q . We shall suppose that N is defined by p equations

$$N : \Phi^I(z) = 0, \quad z \in Q, \quad I = 1, \dots, p$$

where $\phi = (\Phi^I) : Q \rightarrow \mathbb{R}^p$ is a smooth map of rank p at each point of N . The matrix $m = (m^{IJ})$ given by

$$m = q(\nabla\phi, \nabla\phi), \quad \text{i. e.} \quad m^{IJ} = (q^{AB} \partial_A \phi^I \partial_B \phi^J) \circ f$$

(x^A coordinates in Q) is then positive definite on M when f takes its values in N . We denote by $m^{-1} = m_{IJ}$ the inverse matrix.

LEMMA 1. — A necessary and sufficient condition for the mapping $f : M \rightarrow N \subset Q$ to be a harmonic map from (M, g) into (N, h) is that, as a mapping $M \rightarrow Q$ it satisfies the equations which read in local coordinates x^α in M and x^A in Q :

$$(4.1) \quad \hat{\nabla}^\alpha \nabla_\alpha f^A + \lambda_I (q^{AB} \partial_B \Phi^I) \circ f = 0$$

where $(\hat{\nabla}^\alpha \nabla_\alpha f^A)$ is the tension field of the map $f : (M, g) \rightarrow (Q, q)$ and

$$(4.2) \quad \lambda_I = m_{IJ} g^{\alpha\beta} \partial_\alpha f^A \partial_\beta f^B (\hat{\nabla}_B \partial_A \phi^J) \circ f$$

together with the conditions

$$\phi \circ f = 0.$$

Proof (cf. a particular case in Ginibre and Velo [10]).

A mapping $f : M \rightarrow N$ defines a mapping $F : M \rightarrow Q$ by

$$(4.3) \quad F = i \circ f$$

where i denotes the embedding (identity map) $N \rightarrow Q$.

The integral constructed with $F : M \rightarrow Q$

$$E(F) = \int_M (g^\# \otimes q)(\nabla F, \nabla F) d\mu(g)$$

is equal to the integral (1.4) constructed with f since

$$\nabla F = \nabla i \cdot \nabla f, \quad \text{i. e.} \quad \partial_\alpha F^A = \partial_\alpha i^A \partial_\alpha f^a$$

and h is the metric induced by i on N ; that is $h_{ab} = q_{AB}\partial_a i^A \partial_b i^B$, and

$$(4.4) \quad E(f) = E(F).$$

Thus a critical point of $E(f)$ is a critical point of $E(F)$ with the constraint $\phi(F) = 0$, that is a solution of equations of the form:

$$(4.5) \quad \hat{\nabla}^\alpha \nabla_\alpha F^A + \lambda_1 q^{AB} \partial_B \Phi^I = 0$$

where the λ_1 (Lagrange multipliers) are determined by derivating twice the conditions

$$\Phi^I \circ F = 0$$

with $\Phi^I \circ F$ considered as a mapping $M \rightarrow Q \rightarrow \mathbb{R}^p$, and contracting with g :

$$(4.6) \quad \nabla^\alpha \nabla_\alpha (\Phi^I \circ f) \equiv \partial_A \Phi^I \hat{\nabla}^\alpha \partial_\alpha F^A + g^{\alpha\beta} \partial_\alpha F^A \partial_\beta F^B \nabla_B \partial_A \Phi^I = 0$$

comparing (4.5) and (4.6) gives

$$(4.7) \quad \lambda_1 q^{AB} \partial_B \Phi^I = g^{\alpha\beta} \partial_\alpha F^A \partial_\beta F^B \nabla_B \partial_A \Phi^I$$

(4.7) is equivalent to (4.2) ($F = i \circ f = f$, since i is the identity mapping $N \rightarrow N \subset Q$).

LEMMA 2. — The equations (4.1) satisfied by a mapping $M \rightarrow Q$, with λ_1 given by (4.2), imply that the mapping $\phi \circ f: M \rightarrow \mathbb{R}^p$ satisfies the homogeneous wave equation on M

$$(4.8) \quad \nabla^\alpha \nabla_\alpha (\phi \circ f) = 0.$$

Proof. — (4.6) implied by (4.1) and (4.2).

DEFINITION 3. — A submanifold N of $\mathbb{R}^q$ given by $N = \{y \in \mathbb{R}^q, \phi(y) = 0\}$ with ϕ a smooth map $\mathbb{R}^q \rightarrow \mathbb{R}^p$ is said to be regularly defined by ϕ if there exists $a > 0$ and $\varepsilon > 0$ such that

$$\inf_{y \in N_\varepsilon} |\det m^{IJ}(y)| \geq a > 0, \quad N_\varepsilon = \{y \in \mathbb{R}^q, d(y, N) < \varepsilon\}$$

d denotes the euclidean distance. The definition means that ϕ is uniformly of rank p in some uniform neighbourhood of N .

DEFINITION 4. — We denote by $H_s(S)$ the Sobolev space of $\mathbb{R}^q$ -valued functions on the regularly riemannian manifold $(S, \bar{g}_0)$, closure of C^∞ , $\mathbb{R}^q$ valued functions with compact support on S in the norm:

$$\|\varphi\|_{H_s}^2 = \int_S \sum_{k=0}^s |D^k \varphi|^2 d\mu_0$$

where D is the covariant derivative in the metric $\bar{g}_0$ for each scalar valued

map $\varphi^A : S \rightarrow \mathbb{R}$, and $|\cdot|_e$ is the $\bar{g}_0$ and e norm of the set $D^k \varphi = (D^k \varphi^A)$, for instance

$$|D^2 \varphi|_e^2 = e_{AB} \bar{g}_0^i \bar{g}_0^{jk} D_{ih}^2 \varphi^A D_{jk}^2 \varphi^B, \quad e_{AB} = \delta_{AB}.$$

THEOREM (local existence). — Let (M, g) , $M = S \times \mathbb{R}$, be a regularly hyperbolic manifold of dimension $n + 1$ (definition 1).

Let (N, h) be a regular riemannian manifold, regularly defined by a mapping $\phi : \mathbb{R}^q \rightarrow \mathbb{R}^p$. Let $\varphi, \dot{\varphi}$ be mappings $S \rightarrow \mathbb{R}^q$, $\varphi \in H_s(S)$, $\dot{\varphi} \in H_{s-1}(S)$, $s > \frac{n}{2} + 1$ such that $\Phi \circ \varphi = 0$, $(\nabla \phi \circ \varphi) \cdot \dot{\varphi} = 0$.

Then there exists $l > 0$ and on $S \times (-l, l)$ a harmonic map $f : (M, g) \rightarrow (N, h)$, with h the metric induced on N by the euclidean metric e of $\mathbb{R}^q$, such that

$$(4.9) \quad f|_{s_0} = \varphi, \quad \partial_0 f|_{s_0} = \dot{\varphi}.$$

Proof. — We apply lemma 1 with $(Q, q) = (\mathbb{R}^q, e)$. Equations (4.1) reads then:

$$(4.10) \quad \nabla^\alpha \nabla_\alpha f^A + (\nabla \phi^T \nabla \phi)_{IJ}^{-1}(f) g^{\alpha\beta} \partial_\alpha f^A \partial_\beta f^B (\partial_{AB}^2 \Phi^J)(f) = 0;$$

they are a system of q , numerical, quasi-linear, quasi-diagonal second order hyperbolic equations on M , with smooth coefficients if $d(f, N) < \varepsilon$. The local existence theorem, on $S \times (-l, l)$ is a standard result, since $d(\varphi, N) = 0$. The solution $f : S \times (-l, l) \rightarrow \mathbb{R}^q$ satisfies $\phi \circ f = 0$ because $\phi \circ f$ satisfies the homogeneous wave equation (4.8) with zero Cauchy data:

$$\phi \circ f|_S = \phi \circ \varphi|_{=0}, \quad \partial_0(\phi \circ f)|_S = (\nabla \phi \circ f) \cdot \partial_0 f|_S = (\nabla \phi \circ \varphi) \cdot \dot{\varphi} = 0.$$

REMARK 1. — The local existence theorem for a numerical hyperbolic system gives that the interval of existence depends continuously on the $H_{s_0} \times H_{s_0-1}$ norm, s_0 smallest integer such that $s_0 > \frac{n}{2} + 1$, of the Cauchy data, and tends to infinity when these norms tend to zero. If N is a submanifold of $\mathbb{R}^q$, we can always, by translation, take the origin of $\mathbb{R}^q$ at some arbitrary given point y_0 of N . The $H_s(S_0)$ norm of a map $\varphi : S_0 \rightarrow \mathbb{R}^q$ is by definition

$$\|\varphi\|_s = \left\{ \sum_{k=0}^s \int_S |\nabla^k \varphi|^2 d\mu_0 \right\}^{1/2}$$

with

$$|\varphi|^2 = \sum_{A=1}^q |\varphi^A|^2.$$

Small H_s norm for φ means then nearness of φ from the constant map $M \rightarrow y_0$.

REMARK 2. — Every riemannian manifold (N, h) can be isometrically embedded in a space $(\mathbb{R}^q, e)$ —we have supposed moreover that N is given by equations $\Phi^1 = 0$. We could have proceeded without this hypothesis, either inspired by techniques used by Eells and Sampson in the elliptic case, either by using atlases on M and N , together with local existence and uniqueness theorems (cf. an indication of such a proof in [3]).

5. GLOBAL EXISTENCE WHEN $n = 1$

The global existence of a solution of the Cauchy problem for harmonic maps from 2-dimensional Minkowski space time M^2 into a complete riemannian manifold (N, h) has been proved by Gu Chao Hao [11] for smooth initial data. It has been proved by Ginibre and Velo [10] from the two dimensional Minkowski space into various compact riemannian manifolds for $H_2 \times H_1$ Cauchy data. This result can be generalized:

THEOREM. — Let (M, g) , $M = S \times \mathbb{R}$, be a regularly hyperbolic manifold of dimension 2, and (N, h) be a regular riemannian manifold, regularly defined as a submanifold of $(\mathbb{R}^q, e)$ by mapping $\phi : \mathbb{R}^q \rightarrow \mathbb{R}^p$, $N = \{y \in \mathbb{R}^q, \phi(y) = 0\}$.

Let

$$\varphi \in H_s(S), \quad \dot{\varphi} \in H_{s-1}(S), \quad s \geq 2$$

be given maps $\varphi : S \rightarrow \mathbb{R}^q$, $\dot{\varphi} : S \rightarrow \mathbb{R}^q$, such that $\phi \circ \varphi = 0$, $(\nabla \phi \circ \varphi) \cdot \dot{\varphi} = 0$. Then there exists on M a harmonic map $f : (M, g) \rightarrow (N, h)$ taking on $S_0 = S \times \{0\}$ these Cauchy data.

Proof. — In the case $n = 1$ the local existence theorem is valid with $s \geq 2$. The solution $f : S \times (-l, l) \rightarrow N \subset \mathbb{R}^q$ admits a restriction on each $S_t = S \times \{t\}$, $|t| < l$ which is a mapping $f_t : S_t \rightarrow N \subset \mathbb{R}^q$ which belongs to $H_2(S)$ (definition 4). The derivative $\partial_0 f$ admits a restriction $(\partial_0 f)_t : S_t \rightarrow f^{-1}TN$ by $x \rightarrow T_{f_t(x)}N \subset \mathbb{R}^q$, $(\partial_0 f)_t \in H_1(S)$. The energy inequality (2.9) implies, when $f = i \circ f$ is considered as a mapping into N :

$$(5.1) \quad y(t) \leq K(t)y(0)$$

with

$$y(t) = \int_S |\nabla f|_h^2 d\mu_0$$

with, due to the definition of Γ :

$$(5.2) \quad |\nabla f|_h^2 = (h_{ab} f^a \partial_0 f^b - \bar{g}_0^{ij} \partial_i f^a \partial_j f^b)$$

but we have, since $f = i \circ f$ and $h = i^* e$

$$(5.3) \quad |\nabla f|_h^2 = |\nabla f|_e^2 = e_{AB} (\partial_0 f^A \partial_0 f^B - \bar{g}_0^{ij} \partial_i f^A \partial_j f^B).$$

The inequality (2.9) implies therefore the non-blow up of the $L^2(S, \bar{g}_0)$ norm of Df_t and of $(\partial_0 f)_t$, as mappings $S \rightarrow \mathbb{R}^q$, and thus also of f_t : the norms $\|f\|_{H_1(S)}$ (and thus $\|f_t\|_{C_b^0(S)}$) are bounded by continuous functions of t which extend to $t = +\infty$. To prove the non blow up of the second derivatives we look at the identity (3.4). Due to the regularity hypothesis we have, with C a constant

$$(5.4) \quad \int_{S_t} Q_1 d\mu_t \leq C \int_{S_t} |\nabla^2 f|_h (|\nabla f|_h + |\nabla f|_h^3 + |\nabla^2 f|_h) d\mu_0$$

using the fact that $f = i \circ f$ we find inequalities of the form (C_1 and C_2 positive constants)

$$(5.5) \quad \begin{aligned} |\nabla^2 f|_h^2 &\leq |\hat{\nabla}^2 f|_e^2 + C_1 |\nabla f|^4 \\ |\hat{\nabla}^2 f|_e^2 &\leq |\nabla^2 f|_h^2 + C_2 |\nabla f|^4 \end{aligned}$$

(recall that $\hat{\nabla}$ denotes the covariant derivative of f as a mapping $(M, g) \rightarrow (\mathbb{R}^q, e)$ and ∇ as mapping $M \rightarrow N$).

We deduce from (5.4), by the Cauchy-Schwartz inequality

$$(5.6) \quad \begin{aligned} \int_{S_t} Q_1 d\mu_t &\leq C \int_{S_t} |\nabla^2 f|_h^2 d\mu_0 \\ &+ \left\{ \int_{S_t} |\nabla^2 f|_h^2 d\mu_0 \right\}^{1/2} \left\{ \left(\int_{S_t} |\nabla f|^2 d\mu_0 \right)^{1/2} + \left(\int_{S_t} |\nabla f|^6 d\mu_0 \right)^{1/2} \right\}. \end{aligned}$$

Considering f as mapping $M \rightarrow \mathbb{R}^q$, and setting

$$z(t) = \int_{S_t} |\hat{\nabla}^2 f|_e^2 d\mu_0$$

we obtain, using (3.5) (5.5) and (5.6), with C some constant

$$z(t) \leq C_0 y_1(0) + C \int_0^t \left\{ z(\tau) + \int_{S_\tau} (|\nabla f|^2 + |\nabla f|^4 + |\nabla f|^6) d\mu_0 \right\} d\tau.$$

By the inequality (2.9) we know that

$$\int_{S_\tau} |\nabla f|^2 d\mu_0 = y(t) \leq K(\tau)y(0).$$

We bound the integrals $\int_{S_\tau} |\nabla f|^4 d\mu_0$ and $\int_{S_\tau} |\nabla f|^6 d\mu_0$ by using the following Sobolev inequality, valid in any dimension if $a = \frac{n(p-1)}{p}$ and S admits a uniformly locally finite atlas:

$$\|u\|_{L^p(S)} \leq C \|Du\|_{L^1(S)}^2 \|u\|_{L^1(S)}^{1-a}$$

by taking

$$u = |\nabla f|^2.$$

If $n = 1$, $a = \frac{p-1}{p}$, $p = 1, 2$ or 3 we have therefore:

$$\int_{S_\tau} |\nabla f|^{2p} d\mu_0 = (\|u\|_{L^p})^p \leq C \|Du\|_{L^1(S)}^{(p-1)/p} \|u\|_{L^1(S)}^{1/p}$$

we have

$$\|u\|_{L^1(S)} = y(t)$$

and

$$\|Du\|_{L^1(S)}^2 \leq Cy(t)z(t).$$

We then obtain an integral inequality for $z(t)$, with coefficients continuous functions of t extendable for all t , and at most of degree one in $z(t)$. The non-blow up of $z(t)$ follows.

6. GLOBAL EXISTENCE WHEN $(M, g) = M^{n+1}$, SMALL DATA

Met $M^{n+1} = (\mathbb{R}^{n+1}, \eta)$ be $n + 1$ dimensional Minkowski space time.

Let $\Sigma^{n+1} = (S^n \times \mathbb{R}, g)$ be the Einstein cylinder with its canonical metric. M^{n+1} is known to be conformal to a subset V of Σ^{n+1} , that is:

$$(6.1) \quad g = \Omega^2 \eta \quad \text{on} \quad V = \mathbb{R}^{n+1}$$

the identification of $V \subset \Sigma^{n+1}$ with $\mathbb{R}^{n+1}$ being given in canonical polar coordinates respectively $(t, r, \dots)$ on $\mathbb{R}^{n+1}$ and $(T, \alpha, \dots)$ on Σ^{n+1} by (cf. [7])

$$(6.2) \quad \begin{aligned} T &= \text{Arctg}(t+r) + \text{Arctg}(t-r) \\ \alpha &= \text{Arctg}(t+r) - \text{Arctg}(t-r) \\ V : \alpha - \Pi &< T < \Pi - \alpha \end{aligned}$$

We have

$$(6.3) \quad g = dT^2 - d\alpha^2 - \sin^2 \alpha d\sigma^2 = \Omega^2 \eta = \Omega^2(dt^2 - dr^2 - r^2 g_{S^{n-1}}) \\ \Omega = \cos T + \cos$$

$g_{S^{n-1}}$ metric of the sphere S^{n-1} .

We remark that Ω extends to an analytic function on Σ^{n+1} , which vanishes on ∂V ; the submanifold $t = 0$ (i. e. $\mathbb{R}^n \times \{0\}$) is mapped diffeomorphically onto $T = 0$ (i. e. $S^n \times \{0\}$), minus its north pole $\alpha = \Pi$. On $S^n \times \{0\}$ we have ($\alpha = \Pi$ is $r = +\infty$)

$$\Omega|_{T=0} = 1 + \cos \alpha = 2(1 + r^2)^{-1}.$$

To a mapping $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^q$ corresponds, by the previous diffeomorphism a mapping, still denoted φ , defined almost everywhere on S^n .

THEOREM. — Let (N, h) be a riemannian submanifold of $(\mathbb{R}^q, e)$ regularly defined by a smooth map $\phi: \mathbb{R}^q \rightarrow \mathbb{R}^p$

$$N = \{y \in \mathbb{R}^q, \phi(y) = 0\}.$$

Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^q$ and $\dot{\varphi}: \mathbb{R}^n \rightarrow \mathbb{R}^q$ be given mappings such that

$$(6.4) \quad \phi \circ \varphi = 0, \quad (\nabla \phi \circ \varphi) \cdot \dot{\varphi} = 0$$

with

$$(1 + \cos \alpha)\varphi \in H_s(S^n), \quad (1 + \cos \alpha)\dot{\varphi} \in H_{s-1}(S^n), \quad s > \frac{n}{2} + 1$$

then there exists a harmonic map $f: M^{n+1} \rightarrow (N, h)$ such that

$$f|_{\mathbb{R}^n} = \varphi, \quad \partial_0 f|_{\mathbb{R}^n} = \dot{\varphi}$$

if the mappings φ and $\dot{\varphi}$ are sufficiently near in the relevant norms respectively from a constant map and zero.

Proof. — It is inspired from the proofs of [7] and [8].

We do not restrict the generality by supposing that N passes through the origin of $\mathbb{R}^q$, that is $\phi(0) = 0$.

To a mapping $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^q$ we associate $\tilde{f}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^q$ defined on V by:

$$(6.5) \quad \tilde{f} = \Omega^{(1-n)/2} f.$$

We deduce then from (6.1), since the scalar curvature of Σ^{n+1} is $n(n-1)$ that

$$(6.6) \quad \square_g \tilde{f} - \frac{(n-1)^2}{4} \tilde{f} = \Omega^{-(3+n/2)} \square_\eta f$$

where $\square_g$ and $\square_\eta$ are the wave operators in the metrics g and η respectively. We see that f satisfies (4.1) (4.2), with $(Q, q) = (\mathbb{R}^q, e)$ that is

$$(6.6 a) \quad \square_\eta f^A + \lambda_1 [(\partial_A \phi^I) \circ f] = 0,$$

$$(6.7 b) \quad \phi(f) = 0$$

with

$$\lambda_1 = m_{IJ}(f) \eta^{\alpha\beta} \partial_\alpha f^A \partial_\beta f^B [(\partial_{AB}^2 \phi^I) \circ f], \quad m^{IJ} = \sum_{A=1}^q (\partial_A \phi^I \partial_B \phi^J) \circ f$$

if and only if

$$(6.8) \quad \square_g \tilde{f} - \frac{(n-1)^2}{4} \tilde{f} + \Omega^{-(3+n/2)} \lambda_1 [(\partial_A \phi^I) \circ \Omega^{(n-1)/2} \tilde{f}] = 0$$

with

$$\lambda_1 = m_{IJ} \Omega^2 g^{\alpha\beta} \partial_\alpha (\Omega^{(n-1)/2} \tilde{f}^A) \partial_\beta (\Omega^{(n-1)/2} \tilde{f}^B) [(\partial_{AB}^2 \phi^I) \circ (\Omega^{(n-1)/2} \tilde{f})]$$

and

$$\phi(\Omega^{(n-1)/2} \tilde{f}) = 0.$$

We have

$$\partial_\alpha (\Omega^{(n-1)/2} \tilde{f}^A) = \Omega^{(n-1)/2} \partial_\alpha \tilde{f}^A + (n-1)/2 \Omega^{(n-3)/2} \partial_\alpha \Omega \tilde{f}^A$$

$\nabla \Omega$ extends to a bounded function on Σ^{n+1} , and so does $\Omega^{-1} g(\nabla \Omega, \nabla \Omega)$ (cf. [8]) since

$$g(\nabla \Omega, \nabla \Omega) = g^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega = \cos^2 \alpha - \cos^2 T = \Omega (\cos \alpha - \cos T).$$

Therefore the equation (6.8) extends to a semi-linear, semi-diagonal second order system with smooth coefficients for a mapping $\tilde{f}$ from an open set U of Σ^{n+1} into $\mathbb{R}^p$ if on the one hand n is odd and if, on the other hand, $\tilde{f}$ is such that

$$d((\Omega^{(n-1)/2} \tilde{f})(X), N) < \eta \quad \forall X \in U$$

this last property will be *a fortiori* satisfied since $0 \in N$ if

$$\sup_{X \in U} |\Omega^{(n-1)/2}(X) \tilde{f}(X)|_e < \eta$$

thus if

$$\sup_{X \in U} |\tilde{f}(X)|_e < \eta.$$

The existence of an open set $U = S^n \times (-l, l)$ where the equation (6.8) has a solution $\tilde{f}$ taking the Cauchy data:

$$\begin{aligned} \tilde{f}|_{S^n \times \{0\}} &= (1 + \cos \alpha)^{(1-n)/2} \varphi \\ \partial_0 \tilde{f}|_{S^n \times \{0\}} &= (1 + \cos \alpha)^{-(1+n)/2} \dot{\varphi} \end{aligned}$$

is then a consequence of the local existence theorem, and Sobolev inequalities, if $s > \frac{n}{2} + 1$. The length depends continuously on the norms of the Cauchy data, and we have $l > \Pi$ if these norms are small enough.

The mapping $f = \Omega^{(n-1)/2} f$ is defined on M^{n+1} , satisfies (4.1), and also (4.2) (lemma 2).

REMARK. — The hypothesis (6.4) on φ implies that φ tends to the constant map $\mathbb{R}^n \rightarrow 0 \in N$, when r tends to infinity.

From the theorem follow decay estimates for f on M^{n+1} (i. e. rate of approximating the constant map $M^{n+1} \rightarrow 0 \in N$) when t or r tend to infinity.

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