ANNALES DE L'I. H. P., SECTION A

WOLFGANG DREYER

WOLF WEISS

The classical limit of relativistic extended thermodynamics

Annales de l'I. H. P., section A, tome 45, nº 4 (1986), p. 401-418

<http://www.numdam.org/item?id=AIHPA_1986__45_4_401_0>

Ann. Inst. Henri Poincaré,
Vol. 45, n° 4, 1986, p. 401-418. *Physique théorique*

The classical limit of relativistic extended thermodynamics

by

Wolfgang DREYER and Wolf WEISS
Hermann-Föttinger-Institut, Technische Universität Berlin,
Strasse des 17. Juni 135, 1000 Berlin 12

ABSTRACT. — The non-relativistic limit of Relativistic Extended Thermodynamics, a theory that guarantees hyperbolic field equations from the very beginning on, is analysed. The relations between

a) relativistic and non-relativistic variables

b) relativistic and non-relativistic constitutive functions are systematically developed.

Furthermore it is shown that the relativistic dynamic pressure, a quantity that is relativistically small in a non-reacting gas becomes measurable in a reacting gas.

RÉSUMÉ. — On analyse la limite non relativiste de la Thermodynamique relativiste étendue, une théorie qui garantit dès le départ l'hyperbolicité des équations de champ. On développe systématiquement les relations entre

a) les variables relativistes et non relativistes

b) les fonctions constitutives relativistes et non relativistes.

En outre, on montre que la pression dynamique relativiste, qui est petite en théorie relativiste dans un gaz non réactif, devient mesurable dans un gaz réactif.

1. INTRODUCTION

Relativistic extended thermodynamics of ideal Gases is most easily formulated as a field theory which starts with fourteen independant basic fields, namely

N^A-particle flux vector,
T^{AB}-energy momentum tensor, $T^{AB} = T^{BA}$.

These quantities are also called moments of the phase density and within the relativistic kinetic theory of gases one can determine a phase density that depends on these fourteen moments only (e. g. see Marle [*1*], Dreyer [*2*]). Recently Liu, Müller and Ruggeri [*3*] and Dreyer and Müller [*4*] have worked out a phenomenological theory of the fields N^A, T^{AB}.

In the present paper we investigate the classical non-relativistic limit of this theory and compare it with the results of non-relativistic extended thermodynamics of fourteen fields which has been formulated by Kremer [*5*].

In general there is a conceptual difficulty in the transition from the relativistic case to the non-relativistic one: In relativity the fourteen independent basic fields can be understood as particle density, velocity, energy, pressure, pressure deviator and heat flux, whereas classically pressure and energy are not independent.

Indeed, classically it is most natural to consider thirteen basic fields only, namely the first thirteen moments, and this is in fact done in Grad's thirteen moment theory [*6*] and in the more common extended thermodynamics by Liu and Müller [*7*].

Kremer [*5*] has suggested that the proper choice for a 14th basic variable should be the trace of the 4th moment of the phase density. This suggestion is confirmed in this paper, where we show that in the classical limit the trace of the classical fourth moment is related to that particular combination of pressure and energy that vanishes classically.

In addition it will be shown that the non-equilibrium part Π of the pressure is relativistically small. However, this statement has to be changed if a process with a mass defect is considered, because now it can be shown that the magnitude of Π has the same order as the equilibrium part of the pressure.

This paper also establishes the relations between the constitutive coefficients of relativistic and non-relativistic extended thermodynamics.

Since not all readers may be familiar with the tenets of the kinetic theory and extended thermodynamics we have prefaced the limiting procedure, which is the proper subject of this paper, by a reminder of those theories. This reminder also introduces the necessary notations and for maximum clarity it is presented in a synoptic manner.

NOTATION. — Throughout this paper the tensor index notation is used. Capital- and small indices run from 0 to 3 and 1 to 3, respectively. $\psi^{(A_1 \ldots A_N)}$ and $\psi^{\langle A_1 \ldots A_N \rangle}$ denote the symmetric- and traceless symmetric part of a tensor $\underset{\sim}{\psi}$.

2. REMINDER OF KINETIC THEORY AND EXTENDED THERMODYNAMICS

a) Kinetic theory.

Relativistic

We consider a gas from a Lorentz frame, which is spanned by the world axes x^A. The four momentum p^A of an atom with rest mass m obeys the relation

$$(2.1) \qquad p^A p_A = m^2 c^2,$$

or

$$p^0 = p_0 = mc\sqrt{1 + \frac{p^2}{m^2c^2}}.$$

Non-Relativistic

We consider a gas at time t from an inertial frame which is spanned by the position axes x^a. The velocity of an atom is denoted by c^a.

The phase density $f(x^A, p^A)$ and $f_C(x^a, t, c^a)$ obey the Boltzmann equation, which reads

$$(2.2) \qquad p^A \frac{\partial f}{\partial x^A} = S(f)$$

$$\frac{\partial f_C}{\partial t} + c^a \frac{\partial f_C}{\partial x_a} = S_C(f_C), \qquad (II.1)$$

where $S(f)$ is the collision production of phase points.

$fd^3p = fdp^1dp^2dp^3$ is the number density of atoms with momentum p^a in their rest frame

$f_Cd^3c = f_Cdc^1dc^2dc^3$ is the number density of atoms with velocity c^a.

The moments of the phase density are defined as

$$(2.3) \qquad A^{A_1 \dots A_N} := c\int p^{A_1} \dots p^{A_N} f\, dP,$$

where $dP := \frac{1}{p_0} d^3p$ is the invariant element of momentum space. It follows from (2.1) that we have the identity

$$(2.4) \qquad A^{A_1 \dots A_{N-2}A}{}_A = m^2c^2 A^{A_1 \dots A_{N-2}}.$$

There is no need to decompose the $\underset{\sim}{A}$'s into convective and non-convective parts since the $\underset{\sim}{A}$'s are already tensors.

$$F^{i_1 \dots i_n} := m\int c^{i_1} \dots c^{i_n} f_C d^3c. \qquad (II\ 2)$$

Along with these moments the central or non-convective moments are useful quantities of the kinetic theory. These are formed with the peculiar velocity $C^a := c^a - v^a$, where v^a is the velocity of the gas.

We denote these central moments by $p^{i_1 \dots i_n}$, and write

$$p^{i_1 \dots i_n} := m\int C^{i_1} \dots C^{i_n} f\, d^3C \qquad (II.3)$$

The first few moments have an easy interpretation, because we have

Relativistic	Non-Relativistic
$A^A := N^A$-particle number-particle flux vector	$F := \rho$-mass density
$A^{AB} := T^{AB}$-energy-momentum tensor	$F^i := \rho v^i$-momentum density
	$F^{ii} := 2x$ energy density
	$F^{ij} :=$ momentum flux density
	$F^{ijj} := 2x$ energy flux density.

The Boltzmann equation implies the equations of transfer for the moments, namely

(2.5) $$A^{A_1 \dots A_{N-1}A}{}_{,A} = I^{A_1 \dots A_{N-1}}$$

$$\frac{\partial F^{i_1 \dots i_n}}{\partial t} + \frac{\partial F^{i_1 \dots i_n j}}{\partial x^j} = i^{i_1 \dots i_n} \quad \text{(II.4)}$$

$I^{A_1 \dots A_{n-1}}$ and $i^{i_1 \dots i_n}$ are the collision productions of $\underset{\sim}{A}$ and $\underset{\sim}{F}$, respectively. Because of conservation of particle number, momentum and energy we must have

(2.6) $$I = 0, \qquad I^A{}_A = 0$$

and

$$I^A = 0, \qquad I^{AB}{}_B = 0$$

$$i = 0, \qquad i^i = 0, \qquad i^{ii} = 0. \quad \text{(II.5)}$$

b) Extended thermodynamics of 14 fields.

Relativistic || *Non-Relativistic*

Extended thermodynamics is a field theory of the 14 fields

(2.7) $$N^A, \qquad T^{AB}$$

$$F, \quad F^i, \quad F^{ij}, \quad F^{ijj}, \quad F^{iijj}. \quad \text{(II.6)}$$

The field equations for these fields are based upon the conservation laws of particle number, momentum and energy, namely

(2.8) $$N^A{}_{,A} = 0$$

(2.9) $$T^{AB}{}_{,B} = 0$$

$$\frac{\partial F}{\partial t} + \frac{\partial F^i}{\partial x^i} = 0, \quad \text{(II.7)}$$

$$\frac{\partial F^i}{\partial t} + \frac{\partial F^{ij}}{\partial x^j} = 0 \quad \text{(II.8)}$$

$$\frac{\partial F^{ii}}{\partial t} + \frac{\partial F^{iij}}{\partial x^j} = 0 \quad \text{(II.9)}$$

and upon the balance equations of fluxes that are motivated by the equations of transfer of the kinetic theory

(2.10) $A^{ABC}{}_{,C} = I^{AB}$

Note that by (2.4) and (2.6) the set (2.8), (2.9), (2.10) contains only 14 independent equations.

$$\frac{\partial F^{\langle ij\rangle}}{\partial t} + \frac{\partial F^{\langle ij\rangle k}}{\partial x^k} = i^{\langle ij\rangle} \quad \text{(II.10)}$$

$$\frac{\partial F^{ijj}}{\partial t} + \frac{\partial F^{ijjk}}{\partial x^k} = i^{ijj} \quad \text{(II.11)}$$

$$\frac{\partial F^{iijj}}{\partial t} + \frac{\partial F^{iijjk}}{\partial x^k} = i^{iijj} \quad \text{(II.12)}$$

In order to obtain field equations for the fields (2.7)/(II.6) we must supplement the equations (2.8), (2.9), (2.10)/(II.7) through (II.12) by constitutive equations for

(2.11) A^{ABC}, I^{AB}

$F^{\langle ijk\rangle}$, $F^{\langle ij\rangle kk}$, F^{iijjk}, $i^{\langle ij\rangle}$, i^{ijj}, i^{iijj}. (II.13)

These quantities may depend on the variables in a manner that depends on the nature of the gas.

It was mentioned before that there is no need to decompose the relativistic variables and constitutive quantities into convective and non-convective quantities, since N^A, T^{AB}, A^{ABC} and I^{AB} are already tensors with respect to arbitrary transformations.

Nevertheless it is customary, at least for N^A and T^{AB} to make the role of the 4-velocity U^A explicit in those quantities by writing

(2.12) $N^A = nU^A$

(2.13) $T^{AB} = p^{\langle AB\rangle} + Ph^{AB} + \frac{1}{c^2}(q^A U^B + q^B U^A) + \frac{1}{c^2} e U^A U^B,$

where

$$h^{AB} := \frac{1}{c^2} U^A U^B - g^{AB}, \quad U^A U_A = c^2.$$

The following quantities have been introduced in (2.12), (2.13):

(2.14) $n = \frac{1}{c^2} N^A U_A$

It is appropriate to choose the non-convective moments $\underset{\sim}{p}$ as variables and constitutive quantities rather than the moments $\underset{\sim}{F}$, because the p's are objective tensors, i. e. tensors with respect to Euclidean transformations.

There is an easy relation between the two sets of moments which follows from a comparison of (II.2) and (II.3). For those moments that are of interest we have the relations

$$F = \rho \quad \text{(II.14)}$$

$$F^i = \rho v^i \quad \text{(II.15)}$$

$$F^{ij} = \rho v^i v^j + p_C^{ij} \quad \text{(II.16)}$$

$$F^{ijk} = \rho v^i v^j v^k + 3p_C^{(ij} v^{k)} + p^{ijk} \quad \text{(II.17)}$$

$$F^{ijjk} = \rho v^2 v^i v^k + 6p_C^{(ij} v^j v^{k)} + 4p^{(ijk} v^{j)} + p^{ijjk} \quad \text{(II.18)}$$

$$F^{iijjk} = \rho v^4 v^k + 10 p_C^{(ij} v^j v^i v^{k)} + 10p^{(ijk} v^i v^{j)} + 5p^{(ijjk} v^{i)} + p^{iijjk} \quad \text{(II.19)}$$

p^{ij} is the pressure tensor which is

(particle number density in rest frame)

(2.15)
$$p^{\langle AB\rangle} = \left(h^A{}_M h^B{}_N - \frac{1}{3} h_{MN} h^{AB}\right) T^{MN}$$

(pressure deviator)

(2.16) $P = \frac{1}{3} h_{MN} T^{MN}$ (pressure)

(2.17) $q^A = - h^A{}_M U_N T^{MN}$

(heat flux)

(2.18) $e = \frac{1}{c^2} U_M U_N T^{MN}$

(energy density)

The names in (2.14) through (2.18) are suggested by the physical meaning of those quantities in the Rest Lorentz Frame « RLF ». Thus for instance $p^{\langle AB\rangle}$ is the pressure deviator in the RLF.

Note that there is no relation between pressure and energy density relativistically.

We do not decompose the constitutive quantities A^{ABC} and I^{AB} in a manner analogous to (2.12), (2.13), since the various parts would not have suggestive interpretations.

usually split into the pressure p_C and the pressure deviator $p_C{}^{\langle ij\rangle}$ according to

$$p_C^{ij} = p_C^{\langle ij\rangle} + p_C \delta^{ij} \quad \text{(II.20)}$$

$\frac{1}{2} p_C^{ij}$ and $\frac{1}{2} p^{iij}$ are the densities of internal energy and the heat flux, and we shall denote these quantities in the sequel as $\rho\varepsilon$ and q_C^i

$$\frac{1}{2} p_C^{ii} = \rho\varepsilon, \qquad \frac{1}{2} p^{ijj} = q_C^i \quad \text{(II.21)}$$

Note that, by (II.20), (II.21), we have the relation

$$p_C = \frac{2}{3} \rho\varepsilon \qquad \text{(II.22)}$$

between pressure and internal energy density.

Just as the moments contain convective and non-convective parts so do the productions i^{ijj} and i^{iijj}. However, the combinations

$$i^i := \frac{1}{2}(i^{ijj} - 2 i^{\langle ij\rangle} v^j)$$

and

$$i := i^{iijj} - 4(i^{ijjj} - i^{\langle ij\rangle} v^j v^i) \quad \text{(II.23)}$$

are objective tensors.

We may now write the constitutive relations in the form

(2.19)
$$A^{ABC} = \hat{A}^{ABC}(n, P, e, U^A, p^{\langle AB\rangle}, q^A)$$

(2.20) $I^{AB} = \hat{I}^{AB}$ (————————)

$$C = \hat{C}(\rho, v^i, \varepsilon, p_C^{\langle ij\rangle}, q_C^i, p^{iijj}), \quad \text{(II.24)}$$

where C is a generic expression which stands for any one of the quantities

$$p^{\langle ijk\rangle}, p^{\langle ij\rangle kk}, p^{iijjk}, i^{\langle ij\rangle}, i^i, i. \quad \text{(II.25)}$$

In equilibrium all collision productions must vanish

(2.21) $I^{AB}|_E = 0$

$$i^{\langle ij\rangle}|_E = 0, \quad i^i|_E = 0, \quad i|_E = 0. \quad \text{(II.25)}$$

This requirement furnishes nine conditions of which eight can be satisfied by setting

(2.22) $p^{\langle AB\rangle}|_E = 0, \quad q^A|_E = 0$ ‖ $p_C^{\langle ij\rangle}|_E = 0, \quad q_C^i|_E = 0$. (II.27)

The remaining 9th condition obviously requires an equilibrium relation between the remaining non-vanishing variables, i. e. between n, p, e in the relativistic case and between ρ, ε nd p^{iijj} in the non-relativistic case.

We may write this condition in the form

(2.23) $P|_E = p(e, n)$ ‖ $p^{iijj}|_E = g(\rho, \varepsilon)$ (II.28)

In order to emphasize that P and p^{iijj} are not independent of the other variables in equilibrium, we write in general

(2.24) $P = p(n, e) + \Pi$ with $\Pi|_E = 0$	$p^{iijj} = g(\rho, \varepsilon) + \Delta$ with $\Delta|_E = 0$ (II.29)

Thus we may rewrite the constitutive relations in the form

(2.25)
$A^{ABC} = \hat{A}^{ABC}(n, e, \Pi, U^A, p^{\langle AC\rangle}, q^A)$ $\quad C = \hat{C}(\rho, \varepsilon, \Delta, v^i, p_C^{\langle ij\rangle}, q_C^i)$ (II.30)

(2.26) $I^{AB} = \hat{I}^{AB}$ (————————) ‖

This new nomenclature has the advantage that equilibrium values of the constitutive functions can be identified by simply setting three variables equal to zero.

The constitutive functions are restricted by invariance requirements, namely

Principle of Relativity	Principle of Material Frame Indifference,

which requires that the constitutive functions are the same ones in

Arbitrary frames This requirement implies that the constitutive functions $\underset{\sim}{\hat{A}}$ and $\underset{\sim}{\hat{I}}$ must be isotropic functions of their variables under arbitrary transformation.	Euclidean frames. This requirement implies that all constitutive quantities (II.25) have to be independent of the velocity and that all constitutive functions must be isotropic functions of their variables under Euclidean transformation.

There are representation theorems for isotropic functions but we do not need these in general, because we shall restrict the attention to the simple case of near equilibrium processes. In that case we consider only *the* parts of the constitutive functions that are linear in

$$\Pi,\quad p^{\langle AB\rangle},\quad q^{A} \qquad \Vert \qquad \Delta,\quad p_{C}^{\langle ij\rangle},\quad q_{C}^{i}.$$

and these parts are easily written down without use of formal representation theorems. We obtain

(2.27) $$A^{ABC}=(C_0^1+C_\pi^1\Pi)U^AU^BU^C + \frac{c^2}{6}(nm^2-C_0^1-C_\pi^1\Pi)(g^{AB}U^C + g^{AC}U^B+g^{BC}U^A) + C^3(g^{AB}q^C+g^{AC}q^B+g^{BC}q^A) - \frac{6}{c^2}C^3(U^AU^Bq^C+U^AU^Cq^B + U^BU^Cq^A) + C^5(p^{\langle AB\rangle}U^C+p^{\langle AC\rangle}U^B + p^{\langle BC\rangle}U^A)$$

(2.28) $$I^{AB}=B_\pi^1\Pi g^{AB} - \frac{4}{c^2}B_\pi^1\Pi U^AU^B+B^3p^{\langle AB\rangle} + B^4(q^AU^B+q^BU^A)$$

The coefficients C_0^1 through B^4 have been choosen so that (2.4) and $(2.6)_2$ are automatically satisfied by the representations. All coefficients may depend on n and e.

$$p^{\langle ijk\rangle}=0 \qquad \text{(II.31)}$$

$$p^{\langle ij\rangle kk}=\gamma p_C^{\langle ij\rangle} \qquad \text{(II.32)}$$

$$p^{iijjk}=\nu q_C^k \qquad \text{(II.32)}$$

$$i^{\langle ij\rangle}=\sigma p_C^{\langle ij\rangle} \qquad \text{(II.34)}$$

$$i^i=2\tau q_C^i \qquad \text{(II.35)}$$

$$i=\xi\Delta \qquad \text{(II.36)}$$

The coefficients γ through ξ may depend on ρ and ε. Note that in the relativistic representations there are two more coefficient functions than in the classical ones.

One purpose of this paper is to find the relations between the coefficient functions in the non-relativistic and relativistic representations.

3. THE NON-RELATIVISTIC LIMIT OF RELATIVISTIC EXTENDED THERMODYNAMICS

a) Relativistic and non-relativistic phase densities.

The key argument in the transition from relativistic to the non-relativistic theory is the observation that

$f(x^A, p^a)d^3p$ in relativity is the number density of atomes with momentum p^a in their rest frame, and that

$f_C(x^a, t, c^a)d^3c$ classically is the number density of atoms with velocity c^a.

Both quantities are the same to within $0(c^{-2})$ and therefore it follows that

$$f = \frac{1}{m^3} f_C + 0(c^{-2}) . \tag{3.1}$$

b) **Relativistic and non-relativistic moments.**

The determination of the non-relativistic limit of the relativistic moments

$$\mathrm{N^A} = c\int p^{\mathrm{A}} f d\mathrm{P}, \qquad \mathrm{T^{AB}} = c\int p^{\mathrm{A}}p^{\mathrm{B}} f d\mathrm{P}, \qquad \mathrm{A^{ABC}} = c\int p^{\mathrm{A}}p^{\mathrm{B}}p^{\mathrm{C}} f d\mathrm{P} \tag{3.2}$$

proceeds by replacing p_0 or p^0 in the integrands by the expansion (see (2.1))

$$p_0 = p^0 = mc\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4} + 0(c^{-6})\right). \tag{3.3}$$

In listing the limiting values it is appropriate to first take the purely spatial components of (3.2) and then those with one, two or three temporal indices, because the occurrence of p_0 is similar in those components for all three moments.

Thus we obtain for the purely spatial components

$$\mathrm{N}^a = \frac{1}{m}\int p^a f d^3p - \frac{1}{2m^3c^2}\int p^a p^2 f d^3p + \frac{3}{8m^5c^4}\int p^a p^4 f d^3p + 0(c^{-6}), \tag{3.4}$$

$$\mathrm{T}^{ab} = \frac{1}{m}\int p^a p^b f d^3p - \frac{1}{2m^3c^2}\int p^a p^b p^2 f d^3p + \frac{3}{8m^5c^4}\int p^a p^b p^4 f d^3p + 0(c^{-6}), \tag{3.5}$$

$$\mathrm{A}^{abc} = \frac{1}{m}\int p^a p^b p^c f d^3p - \frac{1}{2m^3c^2}\int p^a p^b p^c p^2 f d^3p + \frac{3}{8m^5c^4}\int p^a p^b p^c p^4 f d^3p + 0(c^{-6}), \tag{3.6}$$

for the moments with one time component

$$\mathrm{N}^0 = c\int f d^3p \tag{3.7}$$

$$\mathrm{T}^{a0} = c\int p^a f d^3p = mc\mathrm{N}^a + \frac{1}{2m^2c}\int p^a p^2 f d^3p - \frac{3}{8m^4c^3}\int p^a p^4 f d^3p + (0c^{-5}), \tag{3.8}$$

$$\mathrm{A}^{ab0} = c\int p^a p^b f d^3p = mc\mathrm{T}^{ab} + \frac{1}{2m^2c}\int p^a p^b p^2 f d^3p - \frac{3}{8m^4c^3}\int p^a p^b p^4 f d^3p + 0(c^{-5}), \tag{3.9}$$

for the moments with two time components

$$(3.10)\quad T^{00}=mc^2\int f d^3p+\frac{1}{2m}\int p^2 f d^3p-\frac{1}{8m^3c^2}\int p^4 f d^3p+0(c^{-4})$$

$$(3.11)\quad =mcN^0+\frac{1}{2m}\int p^2 f d^3p-\frac{1}{8m^3c^2}\int p^4 f d^3p+0(c^{-4})$$

$$(3.12)\quad A^{a00}=mc^2\int p^a f d^3p+\frac{1}{2m}\int p^a p^2 f d^3p-\frac{1}{8m^3c^2}\int p^a p^4 f d^3p+0(c^{-4})$$

$$(3.13)\quad =mcT^{a0}+\frac{1}{2m}\int p^a p^2 f d^3p-\frac{1}{8m^3c^2}\int p^a p^4 f d^3p+0(c^{-4})$$

$$(3.14)\quad =-mc^2N^a+2mcT^{a0}+\frac{1}{4m^3c^2}\int p^a p^4 f d^3p+0(c^{-4}),$$

and for the moment A^{ABC} with three time components

$$(3.15)\quad A^{000}=m^2c^3\int f d^3p+c\int p^2 f d^3p$$

$$(3.16)\quad =mcT^{00}+\frac{c}{2}\int p^2 f d^3p+\frac{1}{8m^2c}\int p^4 f d^3p+0(c^{-3}).$$

$$(3.17)\quad =-m^2c^2N^0+2mcT^{00}+\frac{1}{4m^2c}\int p^4 f d^3p+0(c^{-3}),$$

The forms $(3.8)_2$, $(3.9)_2$ follow by use of (3.4), (3.5) and similarly the alternatives (3.11), (3.13), (3.14), (3.16), (3.17) are obtained by use of the foregoing identification and expansions.

With (3.1) and the definitions (II.2) of the non-relativistic moments we thus obtain

$$(3.18)\quad N^a=\frac{1}{m}\rho v^a+0(c^{-2})$$

$$(3.19)\quad T^{ab}=F^{ab}+0(c^{-2})$$

$$(3.20)\quad A^{abc}=mF^{abc}+0(c^{-2})$$

$$(3.21)\quad N^0=c\frac{1}{m}\rho+0(c^{-1})$$

$$(3.22)\quad T^{a0}=c\rho v^a+0(c^{-1})=mcN^a+\frac{1}{2c}F^{abb}+0(c^{-3})$$

$$(3.23)\quad A^{ab0}=cmF^{ab}+0(c^{-1})=mcT^{ab}+\frac{m}{2c}F^{abcc}+0(c^{-3})$$

$$(3.24)\quad T^{00}=c^2\rho+0(1)=mcN^0+\frac{1}{2}F^{aa}+0(c^{-2})$$

$$(3.25)\quad A^{a00}=mc^2\rho v^a+0(1)$$

$$= mc\mathrm{T}^{a0} + \frac{m}{2}\mathrm{F}^{abb} + 0(c^{-2}) \tag{3.26}$$

$$= -\, m^2c^2\mathrm{N}^a + 2mc\mathrm{T}^{a0} + \frac{m}{4c^2}\mathrm{F}^{abbcc} + 0(c^{-4}) \tag{3.27}$$

$$\mathrm{A}^{000} = mc^3\rho + 0(c) \tag{3.28}$$

$$= mc\mathrm{T}^{00} + \frac{m}{2}c\mathrm{F}^{aa} + 0(c^{-1}) \tag{3.29}$$

$$= -\, m^2c^2\mathrm{N}^0 + 2mc\mathrm{T}^{00} + \frac{m}{4c}\mathrm{F}^{aabb} + 0(c^{-3}). \tag{3.30}$$

Thus we see that the relevant classical moments $\underset{\sim}{\mathrm{F}}$ sometimes are found in terms that are of orders of magnitude smaller than the leading terms, where this happens we shall see, however, that those leading terms drop out of the equation of balance by virtue of the exact balance of lower order moments.

c) **Relativistic and classical equations of balance.**

The relativistic equations of balance of extended thermodynamics read

$$\mathrm{N}^{\mathrm{A}}{}_{,\mathrm{A}} = 0, \qquad \mathrm{T}^{\mathrm{AB}}{}_{,\mathrm{B}} = 0, \qquad \mathrm{A}^{\mathrm{ABC}}{}_{,\mathrm{C}} = \mathrm{I}^{\mathrm{AB}}. \tag{3.31}$$

We rewrite these equations by decomposing them into spatial and temporal components and obtain

$$\frac{1}{c}\frac{\partial \mathrm{N}^0}{\partial t} + \frac{\partial \mathrm{N}^a}{\partial x^a} = 0, \tag{3.32}$$

$$\frac{1}{c}\frac{\partial \mathrm{T}^{a0}}{\partial t} + \frac{\partial \mathrm{T}^{ab}}{\partial x^b} = 0, \qquad \frac{1}{c}\frac{\partial \mathrm{T}^{00}}{\partial t} + \frac{\partial \mathrm{T}^{0b}}{\partial x^b} = 0, \tag{3.33}$$

$$\frac{1}{c}\frac{\partial \mathrm{A}^{\langle ab\rangle 0}}{\partial t} + \frac{\partial \mathrm{A}^{\langle ab\rangle c}}{\partial x^c} = \mathrm{I}^{\langle ab\rangle}\,(^1), \qquad \frac{1}{c}\frac{\partial \mathrm{A}^{a00}}{\partial t} + \frac{\partial \mathrm{A}^{a0c}}{\partial x^c} = \mathrm{I}^{a0}, \tag{3.34}$$

$$\frac{1}{c}\frac{\partial \mathrm{A}^{000}}{\partial t} + \frac{\partial \mathrm{A}^{00c}}{\partial x^c} = \mathrm{I}^{00}.$$

Insertion of (3.18) and (3.21) into (3.32) gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v^a}{\partial x^a} + 0(c^{-2}) = 0, \tag{3.35}$$

Which is the classical mass balance to within $0(c^{-2})$.

---

[1] Recall that only nine components of the balance of flux are independent.

Insertion of (3.19) and $(3.22)_1$ into $(3.33)_1$ gives

$$\frac{\partial \rho v^a}{\partial t} + \frac{\partial \mathrm{F}^{ab}}{\partial x^b} + 0(c^{-2}) = 0\,, \tag{3.36}$$

which is the classical momentum balance to within $0(c^{-2})$.

Insertion of $(3.22)_2$ and $(3.24)_2$ into $(3.33)_2$ gives

$$mc\left(\frac{1}{c}\frac{\partial \mathrm{N}^0}{\partial t} + \frac{\partial \mathrm{N}^a}{\partial x^a}\right) + \frac{1}{c}\left[\frac{\partial \frac{1}{2}\mathrm{F}^{aa}}{\partial t} + \frac{\partial \frac{1}{2}\mathrm{F}^{aab}}{\partial x^b}\right] + 0(c^{-3}) = 0\,. \tag{3.37}$$

The first term vanishes by virtue of (3.32), and we obtain

$$\frac{\partial \frac{1}{2}\mathrm{F}^{aa}}{\partial t} + \frac{\partial \frac{1}{2}\mathrm{F}^{aab}}{\partial x^b} + 0(c^{-2}) = 0 \tag{3.38}$$

which is the classical energy balance to within $0(c^{-2})$.

Insertion of (3.20) and $(3.23)_1$ into $(3.34)_1$ gives

$$\frac{\partial \mathrm{F}^{\langle ab\rangle}}{\partial t} + \frac{\partial \mathrm{F}^{\langle ab\rangle c}}{\partial x^c} + 0(c^{-2}) = \frac{1}{m}\mathrm{I}^{\langle ab\rangle} \tag{3.39}$$

which is the classical balance for the pressure deviator to within $0(c^{-2})$.

Insertion of $(3.23)_2$ and (3.26) into $(3.34)_2$ gives

$$mc\left(\frac{1}{c}\frac{\partial \mathrm{T}^{a0}}{\partial t} + \frac{\partial \mathrm{T}^{ab}}{\partial x^b}\right) + \frac{m}{c}\left[\frac{\partial \frac{1}{2}\mathrm{F}^{abb}}{\partial t} + \frac{\partial \frac{1}{2}\mathrm{F}^{abbc}}{\partial x^c}\right] + 0(c^{-3}) = \mathrm{I}^{a0} \tag{3.40}$$

The first term vanishes by virtue of $(3.33)_1$, and we obtain

$$\frac{\partial \frac{1}{2}\mathrm{F}^{abb}}{\partial t} + \frac{\partial \frac{1}{2}\mathrm{F}^{abbc}}{\partial x^c} + 0(c^{-2}) = \frac{c}{m}\mathrm{I}^{a0} \tag{3.41}$$

which is the classical balance for the energy flux to within $0(c^{-2})$.

Insertion of (3.27) and (3.30) into $(3.34)_3$ gives

$$\begin{aligned} -m^2c^2\left(\frac{1}{c}\frac{\partial \mathrm{N}^0}{\partial t} + \frac{\partial \mathrm{N}^a}{\partial x^a}\right) &+ 2mc\left(\frac{1}{c}\frac{\partial \mathrm{T}^{00}}{\partial t} + \frac{\partial \mathrm{T}^{0a}}{\partial x^a}\right) \\ &+ \frac{m}{4c^2}\left(\frac{\partial \mathrm{F}^{aabb}}{\partial t} + \frac{\partial \mathrm{F}^{aabbc}}{\partial x^c}\right) + 0(c^{-4}) = \mathrm{I}^{00}\,. \end{aligned} \tag{3.42}$$

The first two terms vanish by virtue of (3.32) and $(3.33)_2$, and we obtain

$$\frac{\partial F^{aabb}}{\partial t} + \frac{\partial F^{aabbc}}{\partial x^c} + 0(c^{-2}) = \frac{4c^2}{m} I^{00} \tag{3.43}$$

which is the classical balance for the contracted fourth moment within $0(c^{-2})$.

We remined the reader that F^{aabb} was introduced by Kremer [5] as a basic variable in classical extended thermodynamics in order to have the same number of variables as in the relativistic theory.

Now we have confirmed that relativistic extended thermodynamics based on the equations (3.31) tends to classical extended thermodynamics of 14 fields which is based on the equations (II.7) through (II.12).

At the same time we conclude from (3.39), (3.41) and (3.43) that the collision productions relativistically and classically are related as follows

$$I^{\langle ij\rangle} = mi^{\langle ij\rangle} + 0(c^{-2}), \quad I^{i0} = \frac{m}{2c} i^{ijj} + 0(c^{-3}), \quad I^{00} = \frac{m}{4c^2} i^{iijj} + 0(c^{-4}). \tag{3.44}$$

d) Classical limits of constitutive coefficients, relation between Π and Δ.

What remains to be identify is the relation between the coefficients C and B in the relativistic representations (2.27), (2.28) and the coefficients γ through ξ in the classical representations (II.31) through (II.36). In addition we should like to relate the relativistic pressure, a variable with a suggestive meaning, to the classical contracted fourth moment, a variable without a suggestive meaning.

In order to find those relations it is prospitious to consider a gas at rest. From (2.27) we obtain

$$A^{abc} = - C^3(q^a\delta^{bc} + q^b\delta^{ac} + q^c\delta^{ab}) \tag{3.45}$$

$$A^{ab0} = - \frac{c^3}{6}(nm^2 - C_0^1 - C_\pi^1\Pi)\delta^{ab} + cC^5 p^{\langle ab\rangle} \tag{3.46}$$

$$A^{a00} = - 5C^3 q^a \tag{3.47}$$

$$A^{000} = \frac{c^3}{2}(nm^2 + C_0^1 + C_\pi^1\Pi). \tag{3.48}$$

On the other hand, always in the rest frame we obtain from (3.20), $(3.23)_2$, (3.27) and (3.29) with (2.12) and (2.13)

$$A^{abc} = mp^{\langle abc\rangle} + \frac{2}{5} m(q^a\delta^{bc} + q^b\delta^{ac} + q^c\delta^{ab}) + 0(c^{-2}) \tag{3.49}$$

$$A^{ab0} = cmp^{\langle ab\rangle} + mcP\delta^{ab} + \frac{m}{2c}\left(p^{\langle ab\rangle cc} + \frac{1}{3} p^{ccdd}\delta^{ab}\right) + 0(c^{-3}) \tag{3.50}$$

(3.51) $$A^{a00} = 2mq^a + \frac{m}{4c^2} p^{abbcc} + 0(c^{-4})$$

(3.52) $$A^{000} = -m^2c^3n + 2mce + \frac{m}{4c} p^{aabb} + 0(c^{-3}).$$

We insert $P = p(e, n) + \Pi$ and $p^{aabb} = g(\rho, \varepsilon) + \Delta$ from (2.24) and (II.29) on the right hand sides of (3.50), (3.52).

Comparison of (3.48) and (3.52), and of the traces of (3.46) and (3.50) yields

(3.53) $$p = \frac{2}{3}(e - nmc^2) - \frac{g}{12c^2} + 0(c^{-4}),$$

or

(3.54) $$p = \frac{2}{3}\rho\varepsilon + 0(c^{-2})$$

(3.55) $$\Pi = -\frac{1}{12c^2}\Delta + 0(c^{-4})$$

(3.56) $$C_0^1 = nm^2 + \frac{4m}{c^2}(e - nmc^2) + \frac{m}{2c^4} g + 0(c^{-6}),$$

or

(3.57) $$C_0^1 = \rho m + 0(c^{-2})$$

(3.58) $$C_\pi^1 = -6\frac{m}{c^2} + 0(c^{-4}).$$

Furthermore we insert the representations (II.31) through (II.33) on the right hand sides of (3.51) and of the traceless part of (3.50) and compare with (3.47) and the traceless part of (3.46) to get

(3.59) $$C^3 = -\frac{2}{5}m - \frac{m}{20c^2}\nu + 0(c^{-4})$$

(3.60) $$C^5 = m + \frac{m}{2c^2}\gamma + 0(c^{-4}).$$

At least from (2.28) we obtain in the rest frame

(3.61) $$I^{ab} = -B_\pi^1 \Pi \delta^{ab} + B^3 p^{\langle ab \rangle},$$

(3.62) $$I^{a0} = cB^4 q^a,$$

(3.63) $$I^{00} = -3B_\pi^1 \Pi.$$

On the other hand from (3.44) we obtain in the rest frame

(3.64) $$I^{\langle ab \rangle} = mi^{\langle ab \rangle} + 0(c^{-2}), \qquad I^{a0} = \frac{m}{2c} i^a + 0(c^{-3}), \qquad I^{00} = \frac{m}{4c^2} i + 0(c^{-4}).$$

Here we insert the classical representations of $i^{\langle ab \rangle}$, i^a and i, namely (II.34), (II.35) and (II.36), and compare with (3.61), (3.62) and (3.64).

We conclude

$$\mathrm{B}^1_\pi = m\xi + 0(c^{-2}), \tag{3.65}$$

$$\mathrm{B}^3 = m\sigma + 0(c^{-2}), \tag{3.66}$$

$$\mathrm{B}^4 = \frac{m}{c^2}\tau + 0(c^{-4}). \tag{3.67}$$

The relations (3.53) and (3.55) deserve some explanations:

(3.53) relates the difference between the relativistic internal energy density and the equilibrium part p of the relativistic pressure to the equilibrium part g of the classical contracted fourth moment.

(3.55) relates the non-equilibrium part Π of the relativistic pressure to the non-equilibrium part Δ of the classical contracted fourth moment.

The missing interpretation for the variable Δ can now be read off from (3.55). In particular this equation shows why there is no non-equilibrium pressure in classical extended thermodynamics. This is due to the fact that Π is relativistically small.

4. THE MAGNITUDE OF THE DYNAMIC PRESSURE IN A REACTING GAS

As we have seen in the last section the magnitude of Π is $0(c^{-2})$, so that in a non-relativistic theory we shall not be able to discern the effects of the dynamic pressure. However, this statement must be qualified, if a reacting gas is considered in which a mass defect occurs. Under such circumstances m is not constant and it turns out that Π can have the same order of magnitude as p.

This phenomenon is best explained by first looking at the energy equation in a homogeneous body at rest.

This equation reads

$$\frac{\partial \mathrm{T}^{00}}{\partial t} = 0 \tag{4.1}$$

and for an ideal non-degenerate gas we have

$$\mathrm{T}^{00} = nmc^2 + \frac{3}{2}p + \frac{15}{8}\frac{p^2}{nmc^2} - \frac{15}{8}\frac{p^3}{n^2m^2c^4} + 0(c^{-6})\,(^2). \tag{4.2}$$

(2) The calculation that led to (4.3) requires the knowledge of T^{00} up to order c^{-6}. This accurary can not be obtained in the framework of the present paper. (4.2) is taken from [2].

Insertion of (4.2) in (4.1) gives after a little calculation

$$\frac{\partial p}{\partial t} = -\left(\frac{2}{3}nc^2 - \frac{5}{3}\frac{p}{m} + \frac{65}{12}\frac{p^2}{nm^2c^2}\right)\frac{\partial m}{\partial t} + 0(c^{-4}). \tag{4.3}$$

The leading term on the right hand side of (4.3) is the first one which shows that the mass defect $\partial m/\partial t$ multiplied by c^2 increases the equilibrium part of the pressure. This is the well-known dramatic effect that is used e. g. in nuclear explosions.

A very similar argument can be made concerning the temporal component of the balance of flux.

Again in a homogeneous body at rest this equation reads

$$\frac{1}{c}\frac{\partial A^{000}}{\partial t} = I^{00} \tag{4.4}$$

and for an ideal non-degenerate gas we have

$$A^{000} = \frac{c^3}{2}(C_0^1 + nm^2 + C_\pi^1\Pi), \tag{4.5}$$

with [3]

$$C_0^1 = nm^2 + 6\frac{mp}{c^2} + 15\frac{p^2}{nc^4} + 0(c^6) \tag{4.6}$$

$$C_\Pi^1 = -6\frac{m}{c^2} - 63\frac{p}{nc^4} + 0(c^{-6}) \tag{4.7}$$

$$I^{00} = -3B_\pi^1\Pi. \tag{4.8}$$

Insertion of (4.5) and (4.6) in (4.4) gives

$$\frac{\partial \Pi}{\partial t} = \left(-\frac{2}{3}p + 6\Pi\right)\frac{1}{m}\frac{\partial m}{\partial t} + \frac{1}{m}B_\pi^1\Pi + 0(c^{-2}). \tag{4.9}$$

The leading therm on the right hand side of (4.9) is of the same order as p which in turn, by (4.3) is of $0(c^2)$.

Thus we see that in the presence of a reaction with a mass defect $\partial m/\partial t$ the dynamic pressure Π may play a significant role.

---

[3] The calculation that led to (4.9) requires the knowledge of C_0^1 and C_π^1 up to order c^{-4}. This accuracy can not be obtained in the framework of the present paper. Therefore (4.6) and (4.7) are taken from [2].

5. NON-RELATIVISTIC LIMIT OF THE ENTROPY FOUR VECTOR

For completeness we conclude this paper with a brief listing, at least for the non-degenerate case, of the relations between the entropy four vector and its non-relativistic counterparts.

a) **Introduction.**

We start with a juxtaposition of the relativistic and non-relativistic entropy inequality.

Relativistic	*Non-Relativistic*
The entropy four vector is defined as	Entropy density and entropy flux are defined as
(5.1) $\mathrm{S}^{\mathrm{A}} = -ck\int p^{\mathrm{A}} f \ln\left(\frac{1}{y} f\right) d\mathrm{P}$	$h = -k\int f_{\mathrm{C}} \ln\left(\frac{1}{\chi} f_{\mathrm{C}}\right) d^3\mathrm{C}$ (V.1) $\Phi^a = -k\int \mathrm{C}^a f_{\mathrm{C}} \ln\left(\frac{1}{\chi} f_{\mathrm{C}}\right) d^3\mathrm{C}$ (V.2)

k is Boltzmann's constant and y and χ, respectively, are unimportant constants of 0(1).

From Boltzmann's equation one can derive the entropy inequality

(5.2) $\mathrm{S}^{\mathrm{A}}{}_{,\mathrm{A}} \geqslant 0$	$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x^a}(hv^a + \Phi^a) \geqslant 0$ (V.3)

In extended thermodynamics S^{A} and (h, Φ^a) are assumed to be given by constitutive functions.

The second order representations read

(5.3) $\mathrm{S}^{\mathrm{A}} = (\mathrm{A}_0^1 + \mathrm{A}_\pi^1\Pi + \mathrm{A}_{\pi^2}^1\Pi^2 + \mathrm{A}_p^1 p^{\langle\mathrm{BC}\rangle}p_{\langle\mathrm{BC}\rangle} + \mathrm{A}_q^1 q^{\mathrm{B}}q_{\mathrm{B}})\mathrm{U}^{\mathrm{A}} + (\mathrm{A}_0^2 + \mathrm{A}_\pi^2\Pi)q^{\mathrm{A}} + \mathrm{A}^3 p^{\langle\mathrm{AB}\rangle}q_{\mathrm{B}}$	$h = h_0 + h_\Delta\Delta + h_{\Delta^2}\Delta^2 + h_1 p_{\mathrm{C}}^{\langle ab\rangle}p_{\mathrm{C}}^{\langle ab\rangle} + h_2 q_{\mathrm{C}}^a q_{\mathrm{C}}^a$ (V.4) $\Phi^a = (\phi_0^1 + \phi_\Delta^1\Delta)q_{\mathrm{C}}^a + \phi^2 p_{\mathrm{C}}^{\langle ab\rangle}q_{\mathrm{C}}^b$ (V.5)

All coefficients may depend on

n, e,	ρ, ε.

b) The non-relativistic limit of the entropy four vector.

Based on (3.1), namely

$$f = \frac{1}{m^3}{}'f_C + 0(c^{-2}) \tag{5.4}$$

we conclude from (5.1), (V.1), (V.2) that we must have

$$\frac{1}{c}S^0 = h + 0(c^{-2}), \qquad S^a = \Phi^a + 0(c^{-2}). \tag{5.5}$$

In (5.5) we insert S^0, S^a given by (5.3) and h, Φ^a given by (V.4) and (V.5). By virtue of

$$\Delta = -12c^2\Pi + 0(c^{-2}), \qquad p^{\langle ab\rangle} = p_C^{\langle ab\rangle} + 0(c^{-2}), \qquad q^a = q_C^a + 0(c^{-2})$$

we obtain from (5.5)

$$\begin{aligned}
A_0^1 &= h_0 + 0(c^2), & A_\pi^1 &= 12h_\Delta c^2 + 0(1)\,(^4),\\
A_{\pi^2}^1 &= 144h_{\Delta^2}c^4 + 0(c^{-2}), & A_q^1 &= -h_2 + 0(c^{-2}),\\
A_p^1 &= h_1 + 0(c^{-2}), & A_0^2 &= \phi_0^1 + 0(c^{-2}),\\
A_\pi^2 &= -12\phi_\Delta^1 c^2 + 0(1), & A^3 &= -\phi^2 + 0(c^{-2}).
\end{aligned}$$

(4) However, it is shown in [2] [3] that $A_\pi^1 \equiv 0$, $h_\Delta \equiv 0$ must hold, otherwise the entropy inequality could be violated.

REFERENCES

[1] C. MARLE, Sur l'Établissement des Équations de l'Hydrodynamique des Fluides Relativistes Dissipatives. *Ann. Inst. Henri Poincaré*, t. **10**, 1969.

[2] W. DREYER, *Statistical Mechanics of a Relativistic Gas in Non-Equilibrium* (in preparation).

[3] I-Shih LIU, I. MÜLLER, T. RUGGERI, Relativistic Thermodynamics of Gases. *Annals of Physics*, Vol. 169, No. 1, June 1986.

[4] W. DREYER, I. MÜLLER, *Extended Relativistic Thermodynamics of Gases.* Proceeding of the Workshop on Mathematic all Aspects of Fluid and Plasma Dynamics, Trieste, 1984.

[5] G. M. KREMER, Extended Thermodynamics of ideal Gases with 14 Fields. *Ann. I. H. P., Phys. Theor.*, t. **45**, 1986, p. 419-440.

[6] H. GRAD, Principles of the Kinetic Theory of Gases. *Handbuch der Physik XII*, p. 205, Springer, Berlin-Heidelberg-Göttingen, 1958.

[7] I-Shih LIU, I. MÜLLER, Extended Thermodynamics of Classical and Degenerate Ideal Gases. *Arch. Rational Anal.*, t. **83**, 1983.

(Manuscrit reçu le 25 mars 1986)