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Logical Cover Spaces

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ABSTRACT. — We show that a logic with reasonable properties can be constructed from a logical cover space. Conversely, we show that such logics can always be generated in this manner. Finally, we compare the resulting structure with similar structures that have been considered in the literature.

RÉSUMÉ. — On montre qu'on peut construire une logique avec des propriétés raisonnables à partir d'un « espace recouvert logique » (définition dans le texte). Réciproquement, on montre qu'une telle logique peut toujours être engendrée de cette manière. Finalement, on compare la structure qui en résulte avec des structures analogues qui ont été considérées dans la littérature.

1. INTRODUCTION

A cover space H is a nonempty set X together with a collection of nonempty subsets O whose union is X. Such spaces have also been called pre-manuals in the literature [*4*] [*5*] [*6*] [*7*] [*12*] [*13*] [*14*]. If X is finite, then H has been referred to as a hypergraph [*2*] [*3*]. In the operational approach to the foundations of quantum mechanics, X corresponds to

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a set of outcomes and O to a set of operations for a quantum system [5] [6] [12] [13]. Moreover, a subset of an operation corresponds to a quantum event. We say that two quantum events are orthogonal if they are mutually exclusive and can be measured using a single operation. The states of a quantum system correspond to probability measures defined on the set of quantum events. If a set of states Δ satisfies a certain compatibility condition relative to the orthogonality relation, we call (H, Δ) a logical cover space.

In many studies in the foundations of quantum mechanics an important role is played by the « logic » of a quantum system [1] [8] [9] [10] [11] [15]. In the operational approach, methods have been developed for constructing the logic L(H) of a cover space H. However, in order for L(H) to enjoy satisfactory properties, H is usually required to obey a certain coherence condition. Since coherence is fairly strong, it is of interest to investigate what can be done without this condition. In the present work the coherence condition is not assumed, but is replaced by requiring the existence of a set of states Δ such that (H, Δ) is a logical cover space. There is an advantage to such a procedure. In order to justify the coherence condition physically, one can only employ the physical properties of operations. However, to justify the existence of a logical set of states, one can also employ the statistical properties of the system.

There is a basic difference between our work and that of the traditional operational approach [5] [6] [12] [13]. In the traditional approach, L(H) is constructed independently of the state space. The properties of the states are then developed as consequences of the original construction. In our approach, L(H) is constructed using both H and Δ. In this way statistical properties of the system are employed from the beginning.

2. LOGICAL SETS OF STATES

Let X be a nonempty set and let O be a collection of nonempty subsets of X such that $\mathrm{X} = \cup \mathrm{O}$. We then call O a *cover* of X and we call the pair $\mathrm{H} = (\mathrm{X}, \mathrm{O})$ a *cover space*. We call $\mathrm{A} \subseteq \mathrm{X}$ an *event* if $\mathrm{A} \subseteq \mathrm{E}$ for some $\mathrm{E} \in \mathrm{O}$, and denote the set of events by $\tilde{\mathrm{O}}$. If $x \neq y \in \mathrm{X}$ and $\{x, y\} \in \tilde{\mathrm{O}}$ we write $x \perp y$. For $\mathrm{B} \subseteq \mathrm{X}$, we write

$$\mathrm{B}^{\perp} = \{x \in \mathrm{X} : x \perp y \text{ for all } y \in \mathrm{B}\}.$$

If $\mathrm{A}, \mathrm{B} \in \tilde{\mathrm{O}}$ define $\mathrm{A} \perp \mathrm{B}$ if $\mathrm{A} \cap \mathrm{B} \neq 0$ and $\mathrm{A} \cup \mathrm{B} \in \tilde{\mathrm{O}}$. Notice that $\{x\} \perp \{y\}$ for $x, y \in \mathrm{X}$ if and only if $x \perp y$. Also, $\mathrm{A} \perp \mathrm{B}$ implies $\mathrm{A} \subseteq \mathrm{B}^{\perp}$ but the converse need not hold. For A, $\mathrm{B} \in \tilde{\mathrm{O}}$, if $\mathrm{A} \subseteq \mathrm{B}^{\perp}$ implies $\mathrm{A} \perp \mathrm{B}$ we say that H is *coherent;* and if every $\mathrm{E} \in \mathrm{O}$ is maximal in O, we call H *irredundant* [5] [6] [12] [13].

A function $\mu : X \to \mathbb{R}$ is a *state* if $\mu(x) \geq 0$ for all $x \in X$ and $\Sigma_{x \in E}\, \mu(x) = 1$ for any $E \in O$. Denote the set of all states on H by $\Omega(H)$. For $A \in \tilde{O}$, $\mu \in \Omega(H)$, define $\mu(A) = \Sigma_{x \in A}\, \mu(x)$. Then $\mu : \tilde{O} \to \mathbb{R}$ satisfies

(1) $0 \leq \mu(A) \leq 1$ for all $A \in \tilde{O}$;
(2) $\mu(E) = 1$ for all $E \in O$;
(3) *if* $A \perp B$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

We can thus interpret $\mu \in \Omega(H)$ as a probability measure on the set of events $\tilde{O}$. For $\Delta \subseteq \Omega(H)$ and $A \in \tilde{O}$ we define

$$\overline{A} = \overline{A}_\Delta = \{ x \in X : \mu(x) \leq \mu(A) \text{ for all } \mu \in \Delta \}$$
$$\underline{A} = \underline{A}_\Delta = \{ x \in X : \mu(x) \leq 1 - \mu(A) \text{ for all } \mu \in \Delta \}.$$

Notice that if $A \perp B$, then $\mu(A) \leq 1 - \mu(B)$ so $\overline{A} \subseteq \underline{B}$. Conversely, if $\overline{A} \subseteq \underline{B}$ implies $A \perp B$, then we call Δ a *logical* set of states. It is clear that if Δ is logical, then $\overline{A} \subseteq \underline{B}$ if only if $\overline{B} \subseteq \underline{A}$. If Δ is a logical set of states on H we call the pair (H, Δ) a *logical* cover space.

LEMMA 1. — Let $H = (X, O)$ be a cover space which admits a logical set of states Δ. (*a*) For any $x \in X$, there exists a $\mu \in \Delta$ with $\mu(x) > 1/2$. (*b*) H is irredundant.

Proof. — (*a*) Since $\{ x \} \not\perp \{ x \}$ we have $\overline{\{ x \}} \nsubseteq \underline{\{ x \}}$. Hence, there is a $y \in X$, $\mu_0 \in \Delta$ such that $\mu(y) \leq \mu(x)$ for all $\mu \in \Delta$ and $\mu_0(y) > 1 - \mu_0(x)$. We then have

$$\mu_0(x) \geq \mu_0(y) > 1 - \mu_0(x).$$

Hence, $\mu_0(x) > 1/2$. (*b*) Let $E, F \in O$ with $E \subseteq F$ and suppose $x \in F \backslash E$. If $\mu \in \Delta$ we have

$$1 = \Sigma_{y \in E}\, \mu(y) \leq \Sigma_{y \in E}\, \mu(y) + \mu(x) \leq \Sigma_{y \in F}\, \mu(y) = 1.$$

Hence, $\mu(x) = 0$ for all $\mu \in \Delta$. This contradicts (*a*), so $E = F$. It follows that any $E \in O$ is maximal in O. □

We now give an example which shows that a logical cover space can be incoherent.

Example. — Let $X = \{ a, b, c, u, v, w, x, y, z \}$, $E_1 = \{ a, x, y \}$, $E_2 = \{ b, y, z \}$, $E_3 = \{ c, x, z \}$, $E_4 = \{ a, u \}$, $E_5 = \{ b, v \}$, $E_6 = \{ c, w \}$, $O = \{ E_1, \ldots, E_6 \}$, $H = (X, O)$. Then H is an incoherent cover space. Indeed, letting $A = \{ x \}$, $B = \{ y, z \}$ we see that $A, B \in \tilde{O}$, $A \subseteq B^\perp$ but $A \not\perp B$. It can be checked that $(H, \Omega(H))$ is a logical cover space. For example, let μ_0, μ_1 be the unique states satisfying $\mu_0(x) = \mu_0(y) = \mu_0(z) = 1/2$; $\mu_1(y) = \mu_1(z) = 1/4$, $\mu_1(x) = 3/4$. Then $x \in \overline{A}$ but $\mu_0(x) + \mu_0(B) = 3/2 > 1$. Hence, $x \notin \underline{B}$ so $\overline{A} \nsubseteq \underline{B}$. Similarly, $v \in \overline{B}$ but $\mu_1(v) + \mu_1(A) = 5/4 > 1$. Hence, $v \notin \underline{A}$ and $\overline{B} \nsubseteq \underline{A}$.

Let $H = (X, 0)$ be a cover space and let $\Delta \subseteq \Omega(H)$. It is clear that $A \subseteq \overline{A}$ and that $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$ for any $A, B \in \tilde{O}$. However, as we shall later show, $A \subseteq \overline{B}$ need not imply that $\overline{A} \subseteq \overline{B}$ even when Δ is logical.

Let $A \in \tilde{O}$ and suppose $A \subseteq E \in O$. If $A' = E \backslash A$, then $A' \in \tilde{O}$ and we call A' a *complement* of A. We denote the set of complements of A by A^c. The set A^c may contain distinct elements and of course, $A \perp A'$ for all $A' \in A^c$.

LEMMA 2. — Let $H = (X, O)$ be a cover space with $A \in \tilde{O}$. (*a*) If

$$\Delta \subseteq \Gamma \subseteq \Omega(H), \qquad \text{then } \overline{A}_\Gamma \subseteq \overline{A}_\Delta, \underline{A}_\Gamma \subseteq \underline{A}_\Delta.$$

(*b*) If $\{x\} \perp A'$ for some $A' \in A^c$, then $x \in \overline{A}_\Delta$ for any $\Delta \subseteq \Omega(H)$. (*c*) If $\Delta \subseteq \Omega(H)$ is logical and $x \in \overline{A}_\Delta$, then $\{x\} \perp A'$ for any $A' \in A^c$. (*d*) If $\Delta, \Gamma \subseteq \Omega(H)$ and Δ is logical, then $\overline{A}_\Delta \subseteq \overline{A}_\Gamma$.

Proof. — (*a*) This is straightforward. (*b*) If $\{x\} \perp A'$ for $A' \in A^c$, then $\mu(x) \leq 1 - \mu(A') = \mu(A)$ for any $\mu \in \Delta$. Hence, $x \in \overline{A}_\Delta$. (*c*) Suppose $\Delta \subseteq \Omega(H)$ is logical and $x \in \overline{A}_\Delta$. Then for any $y \in \overline{\{x\}}_\Delta$, $\mu \in \Delta$, $A' \in A^c$ we have

$$\mu(y) \leq \mu(x) \leq \mu(A) = 1 - \mu(A').$$

Hence, $\overline{\{x\}}_\Delta \subseteq \underline{A'_\Delta}$ so $\{x\} \perp A'$. (*d*) If $x \in \overline{A}_\Delta$, then by (*c*) $\{x\} \perp A'$ for any $A' \in A^c$. Hence, by (*b*) $x \in \overline{A}_\Gamma$ so $\overline{A}_\Delta \subseteq \overline{A}_\Gamma$. □

The next result shows that $\overline{A}_\Delta$ is independent of the logical Δ and that logicality is hereditary.

THEOREM 3. — Let $H = (X, O)$ be a cover space. (*a*) If $\Delta, \Gamma \subseteq \Omega(H)$ are logical, then $\overline{A}_\Delta = \overline{A}_\Gamma$ for any $A \in \tilde{O}$. (*b*) If $\Delta \subseteq \Gamma \subseteq \Omega(H)$ and Δ is logical, then Γ is logical.

Proof. — (*a*) This follows from Lemma 2(*d*). (*b*) Let $A, B \in \tilde{O}$ and suppose that $\overline{A}_\Gamma \subseteq \underline{B}_\Gamma$. Applying Lemma 2(*a*), (*d*) gives

$$\overline{A}_\Delta \subseteq \overline{A}_\Gamma \subseteq \underline{B}_\Gamma \subseteq \underline{B}_\Delta .$$

Since Δ is logical, we conclude that $A \perp B$. Hence, Γ is logical. □

LEMMA 4. — If (H, Δ) is a logical cover space with $A, B \in \tilde{O}$, then the following statements are equivalent (*a*) $\overline{A} \subseteq \overline{B}$, (*b*) $A \perp B'$ for every $B' \in B^c$, (*c*) $A \perp B'$ for some $B' \in B^c$, (*d*) $\overline{B'} \subseteq \overline{A'}$ for every $B' \in B^c$, $A' \in A^c$, (*e*) $\overline{B'} \subseteq \overline{A'}$ for some $B' \in B^c$, $A' \in A^c$.

Proof. — (*a*) $\Rightarrow$ (*b*) If $B' \in B^c$, then $\overline{A} \subseteq \overline{B} \subseteq \underline{B}'$. Since Δ is logical, $A \perp B'$. (*b*) $\Rightarrow$ (*c*) is trivial. (*c*) $\Rightarrow$ (*d*) Suppose $A \perp B'$ for some $B' \in B^c$ and let $B_1 \in B^c$, $A_1 \in A^c$. If $x \in \overline{B}_1$, $\mu \in \Delta$, then

$$\mu(x) \leq \mu(B_1) = 1 - \mu(B) = \mu(B') \leq 1 - \mu(A) = \mu(A_1).$$

Hence, $x \in \overline{A}_1$ so $\overline{B}_1 \subseteq \overline{A}_1$. (*d*) $\Rightarrow$ (*e*) is trivial. (*e*) $\Rightarrow$ (*a*) Suppose $\overline{B'} \subseteq \overline{A'}$ for some $B' \in B^c$, $A' \subseteq A^c$, and let $x \in \overline{A}$, $\mu \in \Delta$. Since $A \in (A')^c$ we conclude from the above that $B' \perp A$. Hence,

$$\mu(x) \leq \mu(A) \leq 1 - \mu(B') = \mu(B).$$

Thus, $x \in \overline{B}$ and $\overline{A} \subseteq \overline{B}$. □

COROLLARY 5. — If (H, Δ) is a logical cover space with $A, B \in \tilde{O}$, then the following statements are equivalent (a) $\overline{A} = \overline{B}$, (b) $\overline{A'} = \overline{B'}$ for all $A' \in A^c$, $B' \in B^c$, (c) $\overline{A'} = \overline{B'}$ for some $A' \in A^c$, $B' \in B^c$.

If (H, Δ) is a logical cover space we define the *logic* of (H, Δ) to be $L(H) = \{ \overline{A} : A \in \tilde{O} \}$. It follows from Theorem 3 that $L(H)$ does not depend upon which logical set of states is used. For $\overline{A} \in L(H)$ define $(\overline{A})' = \overline{A'}$ for any $A' \in A^c$. It follows from Corollary 5 that $(\overline{A})'$ is well-defined and as above, it is independent of the logical set of states Δ. Notice that if $\overline{A} = \overline{B}$, then $\mu(A) = \mu(B)$ for all $\mu \in \Omega(H)$. Indeed, by Lemma 4 we have $A \perp B'$ for some $B' \in B^c$ and hence

$$1 - \mu(B) + \mu(A) = \mu(B') + \mu(A) = \mu(B' \cup A) \leq 1 .$$

Thus, $\mu(A) \leq \mu(B)$. Similarly, $B \perp A'$ for some $A' \in A^c$ so $\mu(B) \leq \mu(A)$. For $\overline{A} \in L(H)$, $\mu \in \Omega(H)$, we define $\mu(\overline{A}) = \mu(A)$. The above observation shows that $\mu(\overline{A})$ is well-defined.

3. QUANTUM LOGICS

Let $(P, \leqslant)$ be a partially ordered set (poset) with first and last elements 0,1, respectively. A map $' : P \to P$ is an *orthocomplementation* if

(1) $a'' = a$ for all $a \in P$,
(2) $a \leqslant b$ implies $b' \leqslant a'$,
(3) $a \vee a' = 1$ for all $a \in P$.

If P admits an orthocomplementation $'$, we call $P = (P, \leqslant, ')$ an *orthocomplemented* poset. For $a, b \in P$ we write $a \perp b$ if $a \leqslant b'$. A map $\mu : P \to [0, 1] \subseteq \mathbb{R}$ is called a *prestate* if $\mu(0) = 0$, and $\mu(a') = 1 - \mu(a)$ for all $a \in P$. We denote the collection of all prestates on P by $\Omega(P)$. A set $\Delta \subseteq \Omega(P)$ is *order-determining* if $\mu(a) \leqslant \mu(b)$ for all $\mu \in \Delta$ implies $a \leqslant b$. A set $\Delta \subseteq \Omega(P)$ is *stately* if $a \perp b$ implies that there is a $c \in P$ such that $a, b \leqslant c$ and $\mu(c) = \mu(a) + \mu(b)$ for all $\mu \in \Delta$. A set $\Delta \subseteq \Omega(P)$ is *orthomodular* if $a \leqslant b$ implies there is a $c \in P$ such that $c \leqslant b$, $c \perp a$, and $\mu(b) = \mu(a) + \mu(c)$ for $\mu \in \Delta$. If $\Delta \subseteq \Omega(P)$ is order-determining and stately, then clearly the element c defined above is unique, and we write $c = a + b$. Also, if $\Delta \subseteq \Omega(P)$ is order-determining and orthomodular, then again c is unique and we write $c = b - a$.

LEMMA 6. — Let P be an orthocomplemented poset and let $\Delta \subseteq \Omega(P)$. Then Δ is stately if and only if Δ is orthomodular.

Proof. — Suppose Δ is stately and $a \leqslant b$. Then $b' \leqslant a'$ so $b' \perp a$. Hence, there is a $d \in P$ such that $a, b' \leqslant d$ and $\mu(d) = \mu(a) + \mu(b')$ for all $\mu \in \Delta$. Letting $c = d'$, it follows that $c \leqslant b$, $c \perp a$ and for any $\mu \in \Delta$ we have

$$\mu(b) = 1 - \mu(b') = \mu(a) + 1 - \mu(d) = \mu(a) + \mu(c) .$$

Hence, Δ is orthomodular. Conversely, suppose Δ is orthomodular, $a \perp b$. Then $a \leqslant b'$ so there is a $d \in \mathrm{P}$ such that $d \leqslant b'$, $d \perp a$ and $\mu(b') = \mu(a) + \mu(d)$ for every $\mu \in \Delta$. Letting $c = d'$, it follows that $a, b \leqslant c$ and for any $\mu \in \Delta$ we have

$$\mu(c) = 1 - \mu(d) = \mu(a) + 1 - u(b') = \mu(a) + \mu(b).$$

Hence, Δ is stately. $\square$

If P is an orthocomplemented poset and $\Delta \subseteq \Omega(\mathrm{P})$ is stately and order-determining, we call (P, Δ) a *quantum logic*. Our next result shows that the logic of a logical cover space forms a quantum logic.

THEOREM 7. — If (H, Δ) is a logical cover space, then $(\mathrm{L(H)}, \subseteq, ', \Delta)$ is a quantum logic.

Proof. — Clearly, $(\mathrm{L(H)}, \subseteq)$ is aposet. Define $0 = \overline{\emptyset}$, $1 = \mathrm{X}$. Notice that $0 = \{ x \in \mathrm{X} : \mu(x) = 0 \text{ for all } \mu \in \Delta \} = \emptyset$ and $\overline{\mathrm{E}} = \mathrm{X}$ for all $\mathrm{E} \in \mathrm{O}$. Hence, $0, 1 \in \mathrm{L(H)}$ and $0 \subseteq \overline{\mathrm{A}} \subseteq 1$ for all $\overline{\mathrm{A}} \in \mathrm{L(H)}$. We now show that $'$ is an orthocomplementation. If $\overline{\mathrm{A}} \subseteq \overline{\mathrm{B}}$, then by Lemma 4 we have

$$(\overline{\mathrm{B}})' = \overline{\mathrm{B}'} \subseteq \overline{\mathrm{A}'} = (\overline{\mathrm{A}})'$$

where $\mathrm{A}' \in \mathrm{A}^c$, $\mathrm{B}' \in \mathrm{B}^c$. Moreover, if $\mathrm{A}' \in \mathrm{A}^c$, then $\mathrm{A} \in (\mathrm{A}')^c$. Hence, $(\overline{\mathrm{A}})'' = (\overline{\mathrm{A}'})' = \overline{\mathrm{A}}$. To show that $\overline{\mathrm{A}} \vee (\overline{\mathrm{A}})' = 1$, suppose $\overline{\mathrm{A}}, (\overline{\mathrm{A}})' \subseteq \overline{\mathrm{B}}$. Then for $\mathrm{B}' \in \mathrm{B}^c$ we have

$$\overline{\mathrm{B}'} = (\overline{\mathrm{B}})' \subseteq (\overline{\mathrm{A}})' \subseteq \overline{\mathrm{B}}.$$

Applying Lemma 4 gives $\mathrm{B}' \perp \mathrm{B}'$. Hence, $\mathrm{B}' \cap \mathrm{B}' = \emptyset$ so $\mathrm{B}' = \emptyset$. Thus, $\mathrm{B} \in \mathrm{O}$ and $\overline{\mathrm{B}} = 1$. We next show that Δ is an order-determining set of prestates on $\mathrm{L(H)}$. If $\mu \in \Delta$, $\overline{\mathrm{A}} \in \mathrm{L(H)}$, it is clear that $0 \leqslant \mu(\overline{\mathrm{A}}) \leq 1$. Moreover, $\mu(0) = 0$. Also, if $\mathrm{A}' \in \mathrm{A}^c$ we have

$$\mu[(\overline{\mathrm{A}})'] = \mu(\overline{\mathrm{A}'}) = \mu(\mathrm{A}') = 1 - \mu(\mathrm{A}) = 1 - \mu(\overline{\mathrm{A}}).$$

Hence, $\Delta \subseteq \Omega[\mathrm{L(H)}]$. Now suppose that $\mu(\overline{\mathrm{A}}) \leqslant \mu(\overline{\mathrm{B}})$ for all $\mu \in \Delta$. If $x \in \overline{\mathrm{A}}$, then for any $\mu \in \Delta$ we have

$$\mu(x) \leqslant \mu(\mathrm{A}) = \mu(\overline{\mathrm{A}}) \leqslant \mu(\overline{\mathrm{B}}) = \mu(\mathrm{B}).$$

Hence, $x \in \overline{\mathrm{B}}$ and $\overline{\mathrm{A}} \subseteq \overline{\mathrm{B}}$. To show that Δ is stately, suppose $\overline{\mathrm{A}} \perp \overline{\mathrm{B}}$. Then $\overline{\mathrm{A}} \subseteq (\overline{\mathrm{B}})' = \overline{\mathrm{B}'}$ where $\mathrm{B}' \in \mathrm{B}^c$. Applying Lemma 4, gives $\mathrm{A} \perp \mathrm{B}$ so $\mathrm{C} = \mathrm{A} \cup \mathrm{B} \in \tilde{\mathrm{O}}$. Now $\overline{\mathrm{A}}, \overline{\mathrm{B}} \subseteq \overline{\mathrm{C}}$ and for any $\mu \in \Delta$ we have

$$\mu(\overline{\mathrm{C}}) = \mu(\mathrm{C}) = \mu(\mathrm{A} \cup \mathrm{B}) = \mu(\mathrm{A}) + \mu(\mathrm{B}) = \mu(\overline{\mathrm{A}}) + \mu(\overline{\mathrm{B}}) \qquad \square$$

We have actually proved the following stronger result.

COROLLARY 8. — If (H, Δ) is a logical cover space, then $(\mathrm{L(H)}, \subseteq, ', \Delta_1)$ is a quantum logic for any logical $\Delta_1 \subseteq \Omega(\mathrm{H})$.

We now prove a converse of Theorem 7. Two quantum logics (P_1, Δ_1)

and (P_2, Δ_2) are *isomorphic* is there exists an isomorphism $J : P_1 \to P_2$ and a bijection $K : \Delta_1 \to \Delta_2$ such that $K\mu(Ja) = \mu(a)$ for all $a \in P_1$, $\mu \in \Delta_1$.

THEOREM 9. — If (P, Δ) is a quantum logic, then there exists a logical cover space (H, Γ) such that $(L(H), \Gamma)$ is isomorphic to (P, Δ). If Δ is convex, then so is Γ and the bijection $K : \Gamma \to \Delta$ is affine.

Proof. — We call a finite set of mutually orthogonal elements $E = \{ a_1, \ldots, a_n \} \subseteq P$ a *partition* if E satisfies

(1) $a_i \neq 0, 1$; $i = 1, \ldots, n$;

(2) $\Sigma_{i=1}^{n} \mu(a_i) = 1$ for every $\mu \in \Delta$;

(3) for any subset $\{ b_1, \ldots, b_n \} \subseteq E$ there exists a $c \in P$ such that $\mu(c) = \Sigma_{i=1}^{m} \mu(b_i)$ for every $\mu \in \Delta$.

Since Δ is order-determining, the element c in (3) is unique and we write $c = \Sigma b_i$. Let O be the set of all partitions of P together with the set $\{ 0, 1 \} \subseteq P$. Every $a \in P$ is contained in some $E \in O$ since $\{ a, a' \} \in O$. Let H be the cover space (P, O). It is clear that $\Delta \subseteq \Omega(H)$. We now show that (H, Δ) is logical. Suppose that $A, B \in \tilde{O}$ and $\overline{A} \subseteq \underline{B}$. Let $c = \Sigma \{ a : a \in A \}$ and $d = \Sigma \{ b : b \in B \}$. Since $c \in \overline{A}$ we have $c \in \underline{B}$ and hence,

$$\mu(c) \leqslant 1 - \mu(B) = 1 - \mu(d) = \mu(d')$$

for every $\mu \in \Delta$. Thus $c \leqslant d'$ so $c \perp d$. Let $e = (c + d)'$. If $a \in A$, $b \in B$, then $a \leqslant c \leqslant d' \leqslant b'$ so $a \perp b$. Moreover, $a \leqslant c \leqslant c + d = e'$ so $a \perp e$ and similarly $b \perp e$. Hence, $A \cup B \cup \{ e \} = E$ is a finite set of mutually orthogonal elements. Moreover, for any $\mu \in \Delta$ we have

$$\Sigma_{f \in E} \mu(f) = \Sigma_{a \in A} \mu(a) + \Sigma_{b \in B} \mu(b) + \mu(e) = \mu(c) + \mu(d) + \mu(e) = \mu(e') + \mu(e) = 1 .$$

Suppose that $e \neq 0$ (if $e = 0$, we simply delete it). To show that $E \in O$, let $F \subseteq E$. For concreteness, suppose that $F \cap A$, $F \cap B$, $F \cap \{ e \} \neq \emptyset$ (the other cases are even simpler). Let $a_1 = \Sigma \{ a : a \in F \cap A \}, b_1 = \Sigma \{ b : b \in F \cap B \}$. Then for any $\mu \in \Delta$ we have

$$\mu(a_1) + \mu(b_1) \leqslant \mu(c) + \mu(d) = \mu(e') \leqslant 1 .$$

Hence, $a_1 \perp b_1$ and $(a_1 + b_1) \perp e$. Letting $c_1 = (a_1 + b_1) + e$ we have

$$\mu(c_1) = \mu(a_1) + \mu(b_1) + \mu(e) = \Sigma \{ \mu(f), f \in F \} .$$

It follows that $E \in O$ and since $A \cup B \in E$ we have $A \perp B$.

We now describe the logic $L(H)$ of the logical cover space (H, Δ). If $A = \{ a_1, \ldots, a_n \} \in \tilde{O}$, then

$$\overline{A} = \{ a \in P : \mu(a) \leqslant \Sigma\mu(a_i) = \mu(\Sigma a_i) \quad \text{for all } \mu \in \Delta \} .$$

Since Δ is order-determining, $\overline{A}$ is the principle ideal

$$I(\Sigma a_i) = \{ a \in P : a \leqslant \Sigma a_i \} .$$

If $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\} \in \tilde{O}$ and $\overline{A} = \overline{B}$ then $I(\Sigma a_i) = I(\Sigma b_i)$. It follows that $\Sigma a_i = \Sigma b_i$. Hence, for each $A \in \tilde{O}$, there exists a unique $a \in P$ such that $\overline{A} = I(a)$. Define $J : L(H) \to P$ by $J(\overline{A}) = a$ where $\overline{A} = I(a)$. If $\mu \in \Delta$ and $\overline{A} = I(a) \in L(H)$, then clearly $\mu(\overline{A}) = \mu(a)$. If $K : \Delta \to \Delta$ is the identity map, we have $K\mu(J\overline{A}) = \mu(a) = \mu(\overline{A})$. It is straightforward to show that J is an order isomorphism from L(H) onto P. Moreover, let $\overline{A} = I(a)$ where $A = \{a_1, \ldots, a_n\} \in \tilde{O}$ and $\Sigma a_i = a \neq 1$. We can then show, as we did in the previous paragraph, that $\{a_1, \ldots, a_n, a'\} \in O$. Then $A' = \{a'\} \in A^c$ and hence, $(\overline{A})' = \overline{A'} = I(a')$. Therefore, $J[(\overline{A})'] = a' = [J(\overline{A})]'$. If $a = 1$, then $A \in \tilde{O}$ so $\overline{A} = 1$. Then $J[(\overline{A})'] = J(0) = 0 = [J(\overline{A})]'$. It follows that J preserves orthocomplementation. Hence, $(L(H), \Delta)$ and (P, Δ) are isomorphic. $\square$

4. COMPARISONS

In the previous sections we considered a logical set of states Δ on a cover space H. In this section we compare logicality with other conditions on Δ. Moreover, we show that stronger conditions than logicality imply a richer structure for H and its logic L(H).

Let $H = (X, O)$ be a cover space and let $\Delta \subseteq \Omega(H)$. We say that Δ is *full*, *strong*, $\tilde{O}$*-full*, $\tilde{O}$*-strong*, respectively if the following conditions hold, respectively:

(1) For $x, y \in X$, if $\mu(x) + \mu(y) \leqslant 1$ for every $\mu \in \Delta$, then $x \perp y$.
(2) For $x, y \in X$, $\mu \in \Delta$, if $\mu(x) = 1$ implies $\mu(y) = 0$, then $x \perp y$.
(3) For $A, B \in \tilde{O}$, if $\mu(x) + \mu(B) \leqslant 1$ for every $x \in A$, $\mu \in \Delta$, then $A \perp B$.
(4) For $A, B \in \tilde{O}$, $\mu \in \Delta$, if $\mu(B) = 1$ implies $\mu(A) = 0$, then $A \perp B$.

The proof of the following lemma involves a straightforward application of the definitions.

LEMMA 10. — Let $H = (X, O)$ be a cover space and let $\Delta \subseteq \Omega(H)$. (*a*) If Δ is strong, then Δ is full. (*b*) If Δ is logical, then Δ is full. (*c*) If Δ is $\tilde{O}$-full, then Δ is logical and full. (*d*) If Δ is $\tilde{O}$-strong, then Δ is $\tilde{O}$-full, logical, strong, and full.

One can give examples which show that no other implications than those given in Lemma 10 hold in general.

LEMMA 11. — Let $H = (X, O)$ be a cover space with $\Delta \subseteq \Omega(H)$. Then Δ is $\tilde{O}$-full if and only if Δ is logical and for $A, B \in \tilde{O}$, $A \subseteq \overline{B}$ implies $\overline{A} \subseteq \overline{B}$.

Proof. — If Δ is $\tilde{O}$-full, then Δ is logical by Lemma 10(*c*). Suppose $A, B \in \tilde{O}$ and $A \subseteq \overline{B}$. If $B' \in B^c$, $x \in A$, $\mu \in \Delta$, since $\mu(x) \leqslant \mu(B)$ we have

$$\mu(x) + \mu(B') = \mu(x) + 1 - \mu(B) \leqslant 1 .$$

Since Δ is $\tilde{O}$-full, $A \perp B'$. Applying Lemma 4, we conclude that $\overline{A} \subseteq \overline{B}$. Conversely, suppose Δ is logical and $A \subseteq \overline{B}$ implies $\overline{A} \subseteq \overline{B}$. Now suppose $A, B \in \tilde{O}$ and $\mu(x) + \mu(B) \leqslant 1$ for every $x \in A$, $\mu \in \Delta$. If $B' \in B^c$ it follows that $A \subseteq \overline{B}'$. Hence, $\overline{A} \subseteq \overline{B}'$ and again by Lemma 4, $A \perp B$. Thus, Δ is $\tilde{O}$-full. □

LEMMA 12. — Let $H = (X, O)$ be a cover space with $\Delta \subseteq \Omega(H)$. (*a*) If H is full, then H is irreduncant. (*b*) If Δ is $\tilde{O}$-full, then H is coherent. (*c*) If H is coherent and Δ is full, then Δ is $\tilde{O}$-full. (*d*) If H is coherent and Δ is strong, then Δ is $\tilde{O}$-strong.

Proof. — (*a*) The proof is similar to that of Lemma 1. (*b*) Let Δ be $\tilde{O}$-full and suppose $A, B \in \tilde{O}$ satisfy $A \subseteq B^{\perp}$. If $x \in A$, then $\mu(x) + \mu(y) \leqslant 1$ for every $y \in B$. Since Δ is $\tilde{O}$-full, we have $B \perp \{ x \}$. Hence, $\{ x \} \cup B \in \tilde{O}$ so $\mu(x) + \mu(B) \leqslant 1$ for all $x \in A$, $\mu \in \Delta$. Therefore, $A \perp B$ and H is coherent. (*c*) and (*d*) are straightforward. □

Applying Lemma 12 (*b*) and our example in Section 2, we see that Δ logical need not imply Δ is $\tilde{O}$-full. We now compare our operation $A \mapsto \overline{A}$ to other operations that have been used for studying a logic on H. One such operation is $A \mapsto A^{\perp\perp}$ [*5*] [*6*] [*12*] [*13*]. In general, $\overline{A}$ need not equal $A^{\perp\perp}$. For instance, in our example in Section 2,

$$\{ \overline{x, y} \} = \{ x, y, u \} \neq \{ x, y \} = \{ x, y \}^{\perp\perp}.$$

Let $\Delta \subseteq \Omega(H)$ for the cover space $H = (X, O)$. If $A \subseteq X$, $\mu \in \Delta$, we write $\mu(A) = 0$ if $\mu(x) = 0$ for all $x \in A$. For $A \subseteq X$, $\Gamma \subseteq \Delta$, we write

$$A^0 = \{ \mu \in \Delta : \mu(A) = 0 \}$$
$$\Gamma_0 = \{ x \in X : \mu(x) = 0 \quad \text{for all } \mu \in \Gamma \}.$$

It can be shown that $A \mapsto A^0_0$ is a closure operation on the power set $P(X)$ [*8*]. Now suppose that $A \in \tilde{O}$, $x \in \overline{A}$, $\mu \in A^0$. Then $\mu(x) \leqslant \mu(A) = 0$ so $\mu(x) = 0$. Hence, $x \in A^0_0$ and $\overline{A} \subseteq A^0_0$. However, in general, $\overline{A}$ need not equal A^0_0. For instance, in our example in Section 2,

$$\{ \overline{x, a} \} = \{ x, a, z, b \} \neq \{ x, a, z, b, w \} = \{ x, a \}^0_0.$$

An *orthomodular poset* is an orthocomplemented poset P which satisfies:

(1) If $a, b \in P$ with $a \perp b$, then $a \vee b$ exists in P.
(2) If $a, b \in P$ with $a \leqslant b$, then $b = a \vee (b \wedge a')$.

Let P be an orthomodular poset. A prestate $\mu \in \Omega(P)$ is called a *state* if $\mu(a \vee b) = \mu(a) + \mu(b)$ whenever $a \perp b$. It is clear that any set of states $\Delta \subseteq \Omega(P)$ is stately and orthomodular. We say that $\Delta \subseteq \Omega(P)$ is *strong* if $a \leqslant b$ whenever

$$\{ \mu \in \Delta : \mu(a) = 1 \} \subseteq \{ \mu \in \Delta : \mu(b) = 1 \}.$$

THEOREM 13. — Let $H = (X, O)$ be a cover space with $\Delta \subseteq \Omega(H)$. (*a*) If Δ is $\tilde{O}$-full, then $\overline{A} = A^{\perp\perp}$, $(\overline{A})' = A^{\perp}$, for every $A \in \tilde{O}$, and $L(H)$ is an orthomodular poset with an order-determining set of states Δ. (*b*) If Δ is $\tilde{O}$-strong, then $\overline{A} = A^{\perp\perp} = A_0^0$ for every $A \in \tilde{O}$, and $L(H)$ is an orthomodular poset with a strong set of states Δ.

Proof. — (*a*) Suppose Δ is $\tilde{O}$-full. It follows from Lemma 12(*b*) that H is coherent. Suppose $x \in \overline{A}$, $y \in A^{\perp}$. Since H is coherent, $\{y\} \cup A \in \tilde{O}$. Hence, $\mu(y) + \mu(x) \leqslant \mu(y) + \mu(A) \leqslant 1$ for all $\mu \in \Delta$. Therefore $x \perp y$ so $x \in A^{\perp\perp}$ and $\overline{A} \subseteq A^{\perp\perp}$. Conversely, let $x \in A^{\perp\perp}$. For $A' \in A^c$, we have $A' \subseteq A^{\perp}$ so $x \in A'^{\perp}$. Since H is coherent, $\{x\} \perp A'$ so by Lemma 2(*b*) we have $x \in \overline{A}$ and $A^{\perp\perp} \subseteq \overline{A}$. For the second statement, let $A' \in A^c$. If $x \in (\overline{A})' = \overline{A'}$, $y \in A$, $\mu \in \Delta$ we have

$$\mu(x) \leqslant \mu(A') = 1 - \mu(A) \leqslant 1 - \mu(y).$$

Hence, $x \perp y$ so $x \in A^{\perp}$ and $(\overline{A})' \subseteq A^{\perp}$, Conversely, if $x \in A^{\perp}$, by coherence we have $\{x\} \perp A$. Applying Lemma 2(*b*) we obtain $x \in \overline{A'} = (\overline{A})'$ so $A^{\perp} \subseteq (\overline{A})'$. Since Δ is $\tilde{O}$-full, by Theorem 7, $L(H)$ is an orthocomplemented poset and Δ is an order-determining set of prestates on $L(H)$. Now suppose $\overline{A}, \overline{B} \in L(H)$ with $\overline{A} \perp \overline{B}$. As in the proof of Theorem 7 we conclude that $A \perp B$. We now show that $\overline{A} \vee \overline{B} = \overline{A \cup B}$. Clearly, $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$. Suppose that $\overline{A}, \overline{B} \subseteq \overline{C}$ for some $\overline{C} \in L(H)$. Then $A \cup B \subseteq \overline{A} \cup \overline{B} \subseteq \overline{C}$. Applying Lemma 11 gives $\overline{A \cup B} \subseteq \overline{C}$. Hence, Condition (1) for an orthomodular poset holds. Moreover, for $\mu \in \Delta$ we have

$$\mu(\overline{A} \vee \overline{B}) = \mu(\overline{A \cup B}) = \mu(A \cup B) = \mu(A) + \mu(B).$$

It follows that Δ is an order-determining set of states on $L(H)$. It is now easy to show that $L(H)$ satisfies Condition (2) for an orthomodular poset. (*b*) Suppose Δ is $\tilde{O}$-strong. By Lemma 10, Δ is $\tilde{O}$-full so by (*a*), $\overline{A} = A^{\perp\perp}$ for every $A \in \tilde{O}$. We have already observed that $\overline{A} \subseteq A_0^0$. Conversely, suppose $x \in A_0^0$. Let $A' \in A^c$ and assume $\mu \in \Delta$ satisfies $\mu(A') = 1$. Then $\mu \in A^0$ so $\mu(x) = 0$. Since Δ is $\tilde{O}$-strong, $\{x\} \perp A'$. Hence, $x \in \overline{A}$ so $A_0^0 \subseteq \overline{A}$. As in (*a*), $L(H)$ is an orthomodular poset and Δ is a set of states on $L(H)$, To show that Δ is strong, suppose $\overline{A}, \overline{B} \in L(H)$ and

$$\{\mu \in \Delta : \mu(\overline{A}) = 1\} \subseteq \{\mu \in \Delta : \mu(\overline{B}) = 1\}.$$

If $B' \in B^c$ we have $\mu(A) = 1$ implies $\mu(B') = 0$. Since Δ is $\tilde{O}$-strong, $A \perp B'$. By Lemma 4, $\overline{A} \subseteq \overline{B}$. □

For a logical cover space (H, Δ), $L(H)$ need not be an orthomodular poset. For instance, in the example in Section 2, we have $\{\bar{x}\} \perp \{\bar{y}\}$, yet $\{\bar{x}\} \vee \{\bar{y}\}$ does not exist. In fact, $\{\bar{x}\}, \{\bar{y}\} \leqslant \{\bar{a}\}', \{\bar{z}\}'$ but $\{\bar{a}\}'$ and $\{\bar{z}\}'$ have no lower bound which is an upper bound for $\{\bar{x}\}$ and $\{\bar{y}\}$.

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