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The Eikonal approximation and the asymptotics of the total scattering cross-section for the Schrödinger equation

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ABSTRACT. — One considers the total cross-section for scattering by a potential $gq(x)$, $x \in \mathbb{R}^m$, for large coupling constants g and large wave numbers k. The function q is supposed to behave as a homogeneous function of degree $-\alpha$, $2\alpha > m+1$, at infinity. One shows that the total cross-section is asymptotically equal to $\sigma_1(g/2k)^\varkappa$ if $k \to \infty, g/2k \to \infty, g \leqslant \gamma_0 k^2$ and the constant γ_0 satisfies certain conditions. Here $\varkappa = (m-1)(\alpha-1)^{-1}$ and the coefficient σ_1 is determined only by the asymptotics of $q(x)$ at infinity. Similar asymptotics is obtained for the forward scattering amplitude. The proofs of these results rely on the so-called eikonal approximation for the wave function of the Schrödinger equation.

RÉSUMÉ. — On considère la section efficace totale pour la diffusion par un potentiel $gq(x)$, $x \in \mathbb{R}^m$, pour des grandes constantes de couplage g et des grands nombres d'onde k. La fonction q est supposée se comporter comme une fonction homogène de degré $-\alpha$, $2\alpha > m+1$, à l'infini. On montre que la section efficace totale est asymptotiquement égale à $\sigma_1(g/2k)^\varkappa$ si $k \to \infty, g/2k \to \infty, g \leqslant \gamma_0 k^2$ et la constante γ_0 satisfait certaines conditions. Ici $\varkappa = (m-1)(\alpha-1)^{-1}$ et le coefficient σ_1 est déterminé seulement

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par l'asymptotique de $q(x)$ à l'infini. Un comportement asymptotique similaire est obtenu pour l'amplitude de diffusion vers l'avant. Les démonstrations de ces résultats se basent sur l'approximation eikonale pour la fonction d'onde de l'équation de Schrödinger.

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1. INTRODUCTION

The total scattering cross-section is one of basic physical observables, which describe classical or quantum scattering. A value of the total cross-section, roughly speaking, shows, how strongly a potential perturbs the motion of a free particle. In quantum mechanics the total cross-section is finite if the potential is vanishing sufficiently quickly at infinity, whereas in classical one its finiteness requires that the potential has compact support.

Let $\sigma(\omega; k, gq(x))$ be the total scattering cross-section for the Schrödinger equation with a potential $gq(x)$, $x \in \mathbb{R}^m$, $m \geqslant 2$, at incident momentum $k\omega$, $\omega \in \mathbb{S}^{m-1}$, $k > 0$ (k^2 is an energy of a particle); we set the mass of the particle to be equal to $1/2$, and the Planck constant to 1. Under the assumption ([1])

$$| q(x) | \leqslant C(1 + | x |)^{-\alpha}, \tag{1.1}$$

where $2\alpha > m + 1$, the value of σ is finite. The behaviour of $\sigma(\omega; k, gq)$ for large coupling constants g and large wave numbers k is qualitatively dependent on the relation between g and k. In case $N := g(2k)^{-1} \to 0$, the asymptotics of various scattering objects are described by perturbation theory. Applied to the total cross-section, this gives the formula

$$\sigma(\omega; k, gq) \sim \sigma_0(\omega; q)N^2 . \tag{1.2}$$

The coefficient $\sigma_0(\omega; q)$ (for its explicit expression see, e. g., [*1*], where the background is surveyed in a slightly more detailed way) is a certain integral of q. In case $N \to \infty$ the behaviour of $\sigma(\omega; k, gq)$ is very sensitive to the fall-off of $q(x)$ at infinity. For functions q with compact support $\sigma(\omega; k, gq)$ converges under some assumptions [*2*] [*3*] to twice the classical total cross-section. For general short-range q the classical total cross-section in infinite so that the quantum one grows to infinity [*4*].

The main aim of the present paper is to find the asymptotics of $\sigma(\omega; k, gq)$ as $k \to \infty$, $N \to \infty$ for potentials with a power-like behaviour at infinity. Omitting some technical assumptions, we formulate here the basic result (Theorem 1). Let

$$q(x) = | x |^{-\alpha}\Phi(\hat{x}) + 0(| x |^{-\alpha}), \qquad \hat{x} = x | x |^{-1}, \tag{1.3}$$

([1]) By C and c we denote different positive constants whose precise values are of no importance to us.

as $|x| \to \infty$. In the region $k \to \infty$, $\mathrm{N} \to \infty$, $g \leqslant \gamma_0 k^2$, we show that

$$\sigma(\omega; k, gq) \sim \sigma_1(\omega; \alpha, \Phi)\mathrm{N}^{\varkappa}, \qquad \varkappa = (m-1)(\alpha-1)^{-1}, \tag{1.4}$$

(the explicit expression for σ_1 is given by (3.5)). As far as γ_0 is concerned, in general we require its smallness. Nevertheless, γ_0 may be arbitrary for potentials, satisfying some repulsive condition. Note that the case $g = \gamma k^2 \to \infty$ corresponds to the so-called quasi-classical limit when the Planck constant tends to zero and other parameters of the problem are fixed. In our construction the restriction $g \leqslant \gamma_0 k^2$ is demanded only for the proof of an *a-priori* estimate uniform in g (Theorem 2) for the resolvent of the Schrödinger operator. Once this estimate is proven, the restriction $g \leqslant \gamma_0 k^2$ can be omitted and our considerations go through in an essentially larger region of (k, g) (see Theorem 1′). Together with the total cross-section, we study the forward scattering amplitude. Its asymptotics is given by the formula (3.4), similar to (1.4).

Relations (1.2) and (1.4) are qualitatively different in two respects. Firstly, by (1.2) as $\mathrm{N} \to 0$ the total cross-section is always vanishing as N^2, whereas by (1.4) as $\mathrm{N} \to \infty$ the growth of $\sigma(\omega; k, gq)$ is determined by the fall-off of the potential at infinity. Secondly, as $\mathrm{N} \to 0$ the asymptotics of the total cross-section depends on values of $q(x)$ for all $x \in \mathbb{R}^m$. On the contrary, as $\mathrm{N} \to \infty$ the asymptotics of $\sigma(\omega; k, gq)$ is determined only by the asymptotics of $q(x)$ as $|x| \to \infty$. In the « critical » case $\mathrm{N} = \text{const.}$, $k \to \infty$ the total cross-section converges (see Theorems 4 and 5) to a finite limit. As in the case $\mathrm{N} \to 0$, this limit depends on values of $q(x)$ for all $x \in \mathbb{R}^m$, though formula (1.2) is violated.

The asymptotics (1.4) was derived in the paper [*5*] by manipulation with the asymptotics for large numbers (k and g fixed) of eigenvalues of the scattering matrix. For central potentials $q(x) = q(|x|)$ (1.4) was obtained earlier in the same manner in the book [*6*]. The arguments of [*6*] [*5*] can be regarded only as heuristic. In a precise sense the conditions of the validity of (1.4) were analysed in [*7*] (the survey of these results is given in [*1*]) on the example of central potentials (for $m = 3$). These conditions depend on the fall-off of $q(x)$ at infinity (and are broader for smaller α) but include always the quasi-classical limit $g = \gamma k^2$. For non-central $q(x)$ and large N only sharp-order upper bounds for the total cross-section were known [*8*] [*9*]. Moreover, in order to be estimated, the total cross-section is averaged in [*8*] [*9*] over some small interval of k. This averaged cross-section is estimated by $\mathrm{C}\mathrm{N}^{\varkappa}$ for q obeing (1.1), and by $\mathrm{C}r^{m-1}$ for q having compact support in a ball of radius r, $r \geqslant r_0 > 0$. These bounds are proven in [*8*] [*9*] in a very large region $k \geqslant k_0 > 0$, g arbitrary. However, as shown in [*10*], without averaging these bounds can be violated (for $k = k_0$, $g \to \infty$) even in a central case $q(x) = q(|x|)$.

Our proof of (1.4) relies on the reduction of the problem to the « critical » case N = const. The latter is investigated in section 4 with the help of the so-called eikonal approximation, which, roughly, can be described as follows. One constructs the formal asymptotic expansion for the wave function of the Schrödinger equation $-\Delta\psi=k^2(1-\varepsilon q(x))\psi$, $\varepsilon=gk^{-2}$, as $k \to \infty$. The phase function of this expansion satisfies the eikonal equation $(\nabla\varphi)^2 = 1 - \varepsilon q(x)$ and coefficients of the amplitude satisfy transport equations. Restricting ourselves to the principal term of this expansion, we study only the eikonal equation. As $\varepsilon \to 0$ one can construct the formal solution of the eikonal equation as a series in powers of ε. It suffices to take the first term of this series to obtain the eikonal asymptotics of the wave function. This asymptotics is justified in Theorem 4 for smooth functions q with compact support. The transition to the general case is given in Theorem 5 by perturbation theory. In its turn, the asymptotics of the wave function determines the asymptotics of the scattering amplitude and of the total cross-section.

For the proof of (1.4) we study separately different regions of the configuration space $\mathbb{R}^m$. It appears that the asymptotics of the total cross-section is determined by the region where $|x|$ has an order of N^ν, $\nu=(\alpha-1)^{-1}$. As $N \to \infty$ one can replace here $q(x)$ by its asymptotics for $|x| \to \infty$. Further, by scaling, the problem in this region is reduced to the case N=const. Proofs, that regions $|x|N^{-\nu} \to \infty$ and $|x|N^{-\nu} \to 0$ do not contribute to the asymptotics, require essentially different methods. The first of these regions is treated by perturbation theory (one applies here Theorem 5). The study of the second is somewhat similar to the upper estimate of $\sigma(\omega; k, gq)$ for q, contained in a ball of a radius r. Since in the latter case $\sigma(\omega; k, gq) \leqslant Cr^{m-1}$, it is natural to expect that in our situation the contribution of the ball $|x| \leqslant O(N^\nu)$ is also bounded by $O(N^{\nu(m-1)}) = O(N^\varkappa)$. The demonstration of this fact is a crucial point of our proof. It is exactly at this step, where the restriction $g \leqslant \gamma_0 k^2$ arises. Technically all our considerations rely on a uniform bound for the resolvent of the Schrödinger operator. We note that at all steps the forward scattering amplitude is treated simultaneously with the total cross-section.

This paper is organized as follows. In section 2 we give precise definitions of objects considered in the paper, and collect some necessary information on stationary scattering theory. The main result is formulated in section 3, Theorem 1. In Theorem 2 of the same section the bound for the resolvent is obtained. As a corollary of this bound, in Theorem 3 of section 3 we give the sharp-order upper bounds for the total cross-section and the forward scattering amplitude (without averaging over k). The case N = const. is studied in section 4. Finally, in section 5 the contribution of the domain $|x|N^{-\nu} \to 0$ is estimated, all partial results are put together and the proof of Theorem 1 is concluded.

2. PRELIMINARIES

Here we collect some results, necessary below, of stationary scattering theory for the Schrödinger operator. Essentially, this information is contained in the literature (see, in particular, the book [11]) or can be obtained by the combination of known methods.

Let V be a multiplication by a real function $v(x)$, $H_0 = -\Delta$, $H = H_0 + V$ be self-adjoint operators in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m)$. We always assume that the condition (1.1), where at least $\alpha > 1$, holds. Let

$$R_0(k,\varepsilon) = (H_0 - k^2 - i\varepsilon)^{-1}, \quad R(k,\varepsilon) = (H - k^2 - i\varepsilon)^{-1}, \quad k > 0, \quad \varepsilon > 0,$$

be resolvents of the free Hamiltonian H_0 and of the Schrödinger operator H. By $X_\beta^{(s)}$ we denote a multiplication by the function $(|x| + s)^{-\beta}$, $s > 0$; $X_\beta = X_\beta^{(1)}$. All operator limits are understood in the paper in the sense of norm-convergence. It is well known that for $2\beta > 1$ the operator-function $X_\beta R(k,\varepsilon)X_\beta$ has a limit as $\varepsilon \to 0$ and this limit is continuous in $k > 0$. Not being quite rigorous in notation, we set $R(k) = R(k, +0)$, $R_0(k) = R_0(k, +0)$. Certainly, the operators $R(k)$ and $R_0(k)$ are correctly defined in $\mathcal{H}$ only when multiplied by X_β, $2\beta > 1$, at both sides. For example, the resolvent identity

$$R(k) - R_0(k) = -R_0(k)VR(k) = -R(k)VR_0(k) \tag{2.1}$$

has an operator sense only after such a multiplication.

For the definition of the scattering matrix consider the operator

$$(Z_0(k)u)(\omega) = 2^{-1/2}k^{\frac{m}{2}-1}(2\pi)^{-m/2}\int e^{-ik\langle\omega,x\rangle}u(x)dx \tag{2.2}$$

on the Schwartz set of u. Integrals in a variable x are always taken over the whole space $\mathbb{R}^m$. Note that $Z_0(k)H_0u = k^2Z_0(k)u$. If $2\beta > 1$, the operator $Z_0(k)X_\beta$ is extended by continuity to a compact operator from $\mathcal{H}$ to $\mathcal{M} = L_2(\mathbb{S}^{m-1})$ and it is continuous in $k > 0$. Now the scattering matrix $S(k)\colon \mathcal{M} \to \mathcal{M}$ for the pair H_0, H and the value k^2 of a spectral parameter (an energy) can be defined by the relation

$$S(k) = I - 2\pi i Z_0(k)(V - VR(k)V)Z_0(k)^*. \tag{2.3}$$

The formula (2.3) needs some explanation. Namely, for $1 < 2\beta \leqslant \alpha$ rewrite (2.3) as

$$S(k) = I - 2\pi i(Z_0(k)X_\beta)[X_{-\beta}VX_{-\beta} - \lim_{\varepsilon\to 0}(X_{-\beta}VR(k,\varepsilon)VX_{-\beta})](Z_0(k)X_\beta)^*. \tag{2.3$_1$}$$

Then the R. H. S. is a combination of bounded operator and hence the last relation has a correct operator sense. It is implied below that all formulae of the type (2.3) should be transformed to the form (2.3_1). By (2.3_1) the operator $S(k) - I$ is compact and continuous in $k > 0$. Moreover, in virtue of (2.1) and of the identity

$$2\pi i Z_0(k)^* Z_0(k) = R_0(k) - R_0(k)^*, \tag{2.4}$$

the operator $S(k)$ is unitary. Note that from a point of view of abstract operator theory a scattering matrix is defined only up to an unitary equivalence. The choice (2.2) of the operator $Z_0(k)$ fix the standard representation of $S(k)$, in terms of which the scattering amplitude and the total cross-section are defined. Together with (2.3), we need also an expression for $S(k)$ obtained in scattering theory with an identification $\mathcal{J}$. Namely, let $\mathcal{J}$ be a multiplication by such a function η that $1-\eta \in C_0^\infty(\mathbb{R}^m)$ and (2)

$$T = H\mathcal{J} - \mathcal{J}H_0 = -2(\nabla\eta)\nabla - \Delta\eta + \eta v. \tag{2.5}$$

Then [12]

$$S(k) = I - 2\pi i Z_0(k)[\mathcal{J}^*T - T^*R(k)T]Z_0(k)^*. \tag{2.6}$$

Clearly, (2.3) is a particular case of (2.6) for $\mathcal{J} = I$. The relation (2.6) can be derived similarly to (2.3) if one inserts $\mathcal{J}$ in a definition of wave operators. Since in our case $\mathcal{J} - I$ is compact with respect to H_0, wave operators and, hence, scattering matrices, corresponding to triples $\{H_0, H, I\}$ and $\{H_0, H, \mathcal{J}\}$ coincide with each other. In Appendix A the coincidence of right-hand sides of (2.3) and (2.6) will be verified directly.

Let now $2\alpha > m + 1$. Then the operator $S(k) - I$ belongs to the Hilbert-Schmidt class and, consequently, is an integral operator. We denote its kernel by $ik^{m-1}(2\pi)^{-m+1}f(\omega, \omega'; k)$, where a function $f(\omega, \omega'; k)$ is defined for almost all $(\omega, \omega') \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1}$ and $f(., .; k) \in L_2(\mathbb{S}^{m-1} \times \mathbb{S}^{m-1})$. In these terms the scattering amplitude $F(\varphi, \omega; k)$, corresponding to an incident direction ω and a scattering direction φ, is introduced by the relation

$$F(\varphi, \omega; k) = (2\pi)^{-\frac{m-1}{2}} k^{\frac{m-1}{2}} e^{-\frac{\pi i}{4}(m-3)} f(\varphi, \omega; k). \tag{2.7}$$

(A strange numerical coefficient appears here because traditionally F is defined as a coefficient at an outgoing spherical wave-see (2.17)). Below the function f is also called the scattering amplitude. In its turn, the scattering amplitude determines the total cross-section

$$\sigma(\omega; k) = \int_{\mathbb{S}^{m-1}} |F(\varphi, \omega; k)|^2 d\varphi = (2\pi)^{-m+1} k^{m-1} \int_{\mathbb{S}^{m-1}} |f(\varphi, \omega; k)|^2 d\varphi \tag{2.8}$$

for an incident direction ω and a wave number k. Note that by unitarity

(2) Often we do not distinguish a notation of some function and of a multiplication by this function.

of $S(k)$ the integrals over φ of functions $|F(\varphi, \omega; k)|^2$ and $|F(\omega, \varphi; k)|^2$ are equal to each other. When averaged over ω, the total cross-section

$$\sigma_{av}(k) = S_{m-1}^{-1} \int_{\mathbb{S}^{m-1}} \sigma(\omega; k) d\omega$$

(S_{m-1} is a surface of $\mathbb{S}^{m-1}$) can be expressed in terms of the Hilbert-Schmidt norm $\| \, . \, \|_2$ of the operator $S(k) - I$:

$$\sigma_{av}(k) = S_{m-1}^{-1}(2\pi)^{m-1}k^{-m+1} \| S(k) - I \|_2^2 .$$

Thus for $2\alpha > m + 1$ the value of $\sigma_{av}(k)$ is finite. We emphasize that in contrast to $\sigma_{av}(k)$ the definitions of $F(\varphi, \omega; k)$ and $\sigma(\omega; k)$ are not unitary invariant.

We use now the definition (2.3) of $S(k)$ to obtain a representation for the scattering amplitude. Let $\psi_0(x; \omega, k) = e^{ik\langle \omega, x \rangle}$ be a plane wave corresponding to incident momentum $k\omega$. Relations (2.2) and (2.3) show that formally

$$f(\omega, \omega'; k) = -(2k)^{-1} [(V\psi_0(\omega', k), \psi_0(\omega, k)) - (VR(k)V\psi_0(\omega', k), \psi_0(\omega, k))], \tag{2.9}$$

where $(.,.)$ stands for a scalar product in $\mathscr{H}$. The precise meaning of (2.9) should be clarified. Note that for $2\beta > m$ a function $X_\beta \psi_0(\omega, k)$ belongs to $\mathscr{H}$ and is continuous in $\mathscr{H}$ for $\omega \in \mathbb{S}^{m-1}$, $k > 0$. If $2\beta < 2\alpha - 1$, an operator $X_{-\beta}VR(k)VX_{-\beta}$ is bounded and continuous in $k > 0$. Thus for $m < 2\beta < 2\alpha - 1$ the second summand in the R. H. S. of (2.9)

$$(VR(k)V\psi_0(\omega', k), \psi_0(\omega, k)) = ([X_{-\beta}VR(k)VX_{-\beta}]X_\beta\psi_0(\omega', k), X_\beta\psi_0(\omega, k))$$

is correctly defined and continuous in $\omega, \omega' \in \mathbb{S}^{m-1}$ and $k > 0$. The first summand

$$(V\psi_0(\omega', k), \psi_0(\omega, k)) = ((X_{-\beta}VX_{-\beta})X_\beta\psi_0(\omega', k), X_\beta\psi_0(\omega, k))$$

has, obviously, these properties only for $m < 2\beta \leqslant \alpha$. Therefore for $\alpha > m$ the scattering amplitude $f(\omega, \omega'; k)$ is continuous in $\omega, \omega' \in \mathbb{S}^{m-1}$, $k > 0$ and is given by the formula (2.9). Quite similarly, (2.2) and (2.6) ensure a more general representation

$$f(\omega, \omega'; k) = -(2k)^{-1}([\mathscr{J}^*T - T^*R(k)T]\psi_0(\omega', k), \psi_0(\omega, k)), \tag{2.10}$$

also used below. Since in (2.5) $\nabla\eta \in C_0^\infty(\mathbb{R}^m)$ and $\Delta\eta \in C_0^\infty(\mathbb{R}^m)$, the second summand in the R. H. S. of (2.10) is again continuous in $\omega, \omega' \in \mathbb{S}^{m-1}$, $k > 0$ for $2\alpha > m+1$, and the first summand, only for $\alpha > m$. So for $\alpha > m$ the forward scattering amplitude $f(\omega, \omega; k)$ is correctly defined and is continuous in $\omega \in \mathbb{S}^{m-1}$, $k > 0$. Moreover, in this case the so-called optical theorem is known. It connects the total cross-section with the forward scattering amplitude:

$$\sigma(\omega; k) = 2 \operatorname{Im} f(\omega, \omega; k). \tag{2.11}$$

The relation (2.11) is, of course, a corollary of unitarity of the operator $S(k)$. Now we give a generalization of (2.11) to a case $2\alpha > m + 1$.

LEMMA 1. — Let the condition (1.1) with $2\alpha > m + 1$ hold. Then the total cross-section $\sigma(\omega; k)$ is continuous in $\omega \in \mathbb{S}^{m-1}$, $k > 0$ and

$$\sigma(\omega; k) = k^{-1} \operatorname{Im} (T^*R(k)T\psi_0(\omega, k), \psi_0(\omega, k)). \tag{2.12}$$

Proof. — Set $\Phi_1 = -2\pi Z_0 \mathcal{J}^* T Z_0^*$, $\Phi_2 = 2\pi Z_0 T^* R T Z_0^*$, $\Phi = \Phi_1 + \Phi_2$. Note that $\Phi_1 - \Phi_1^* = 2\pi Z_0(KH_0 - H_0K)Z_0^*$, where $K = \mathcal{J}^*\mathcal{J} - I$ is a multiplication by a function $|\eta|^2 - 1 \in C_0^\infty(\mathbb{R}^m)$. Therefore

$$Z_0 K H_0 Z_0^* = Z_0 H_0 K Z_0^* = k^2 Z_0 K Z_0^*$$

and hence $\Phi_1 = \Phi_1^*$. Now unitarity of $S(k)$ ensures that

$$\Phi^*\Phi = i(\Phi^* - \Phi) = i(\Phi_2^* - \Phi_2).$$

In terms of kernels of operators Φ and Φ_2 this means that

$$\int_{\mathbb{S}^{m-1}} \overline{\Phi(\omega'', \omega)}\Phi(\omega'', \omega')d\omega'' = i[\overline{\Phi_2(\omega', \omega)} - \Phi_2(\omega, \omega')]$$

for almost all $(\omega, \omega') \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1}$. As was already noted, if $2\alpha > m + 1$, the R. H. S. and hence the L. H. S. of the last equality are, actually, continuous in $\omega, \omega' \in \mathbb{S}^{m-1}$ (and $k > 0$). This permits to set here $\omega = \omega'$, what by definition (2.8) gives (2.12). □ □

We need also a particular case of (2.12), corresponding to $\mathcal{J} = I$:

$$\sigma(\omega; k) = k^{-1} \operatorname{Im} (VR(k)V\psi_0(\omega, k), \psi_0(\omega, k)). \tag{2.13}$$

Below it will be convenient to use a notion of a wave function $\psi(x; \omega, k)$. For $2\alpha > m + 1$ we define a wave function by the relation

$$\psi(\omega, k) = \psi_0(\omega, k) - R(k)V\psi_0(\omega, k), \tag{2.14}$$

where, as usual, k is a wave number and ω is an incident direction. Clearly, $X_\beta(\psi - \psi_0) \in \mathcal{H}$ for $2\beta > 1$. The resolvent identity (2.1) implies the Lippman-Schwinger equation

$$(I + R_0(k)V)\psi(\omega, k) = \psi_0(\omega, k) \tag{2.15}$$

for $\psi(\omega, k)$. In terms of $\psi(\omega, k)$ the representation (2.9) takes a form

$$f(\omega, \omega'; k) = -(2k)^{-1}(V\psi(\omega', k), \psi_0(\omega, k)), \qquad \alpha > m. \tag{2.16}$$

For $\alpha > m$ the wave function and the scattering amplitude can be defined directly in terms of solutions of the Schrödinger equation

$$-\Delta\psi + v(x)\psi = k^2\psi.$$

Namely, for every $\omega \in \mathbb{S}^{m-1}$ this equation has the unique solution $\psi(x;\omega,k)$ with the asymptotics

$$\psi(x;\omega,k)=e^{ik\langle\omega,x\rangle}+F(\hat{x},\omega;k)\,|x|^{-\frac{m-1}{2}}e^{ik|x|}+O(|x|^{-\frac{m-1}{2}}),\quad \hat{x}=x\,|x|^{-1}, \tag{2.17}$$

as $|x| \to \infty$. Thus defined, the solution ψ and the coefficient F coincide respectively with the wave function and the scattering amplitude.

If necessary, the dependence of different objects on the potential is specified in the notation, e. g. $R(k) = R(k, v)$, $S(k) = S(k, v)$. On the contrary, if not confusing, the dependence on some parameters, e. g. k, is often dropped out of the notation. Below we need a connection between resolvents and scattering matrices, corresponding to potentials $v(x)$ and $v^{(\rho)}(x) = \rho^2 v(\rho x)$, $\rho > 0$.

LEMMA 2. — For $\alpha > 1$

$$\| X_\beta^{(a)} R(\rho k, v^{(\rho)}) X_\beta^{(a)} \| = \rho^{2\beta-2} \| X_\beta^{(s)} R(k, v) X_\beta^{(s)} \|, \qquad a = \rho^{-1}s, \tag{2.18}$$

$$S(\rho k, v^{(\rho)}) = S(k, v). \tag{2.19}$$

Moreover, for $2\alpha > m + 1$

$$\rho^{m-1} f(\omega,\omega';\rho k, v^{(\rho)}) = f(\omega,\omega';k,v), \quad \text{almost all} \quad (\omega,\omega') \in \mathbb{S}^{m-1}\times\mathbb{S}^{m-1}, \tag{2.20}$$

$$\rho^{m-1}\sigma(\omega;\rho k, v^{(\rho)}) = \sigma(\omega;k,v), \qquad \text{all} \quad \omega\in\mathbb{S}^{m-1}. \tag{2.21}$$

Proof. — Let $\mathcal{U}_\rho$ be an unitary operator of dilations: $(\mathcal{U}_\rho g)(x) = \rho^{m/2} g(\rho x)$. Then $(H_0 + V^{(\rho)})\mathcal{U}_\rho = \rho^2 \mathcal{U}_\rho (H_0 + V)$ and hence

$$R(\rho k, v^{(\rho)})\mathcal{U}_\rho = \rho^{-2}\mathcal{U}_\rho R(k, v).$$

This ensures (2.18). According to the definition (2.3) for the proof of (2.19) one should additionally take into account that, by (2.2), $Z_0(\rho k)\mathcal{U}_\rho = \rho^{-1} Z_0(k)$. Now, (2.20) is a reformulation of (2.19) in terms of kernels of operators $S - I$. Finally, (2.21) is a direct consequence of (2.20) and of the definition (2.8). □

Let us compare scattering amplitudes for two different potentials v and v_1, obeying (1.1) each. To that end we introduce the scattering matrix $\tilde{S}(k) = \tilde{S}(k; v, v_1)$ for the pair $H_1 = H_0 + V_1$, $H = H_0 + V$. The suitable representation for $\tilde{S}(k)$ is defined by the operator

$$Z_1(k) = Z_0(k)[I - V_1 R(k, v_1)^*]. \tag{2.22}$$

If $2\beta > 1$, the operator $Z_1(k)X_\beta : \mathcal{H} \to \mathcal{M}$ is compact and continuous in $k > 0$. The mapping $Z_1(k)$ diagonalizes the operator H_1: $Z_1(k)H_1 u = k^2 Z_1(k) u$. An explicit expression for $\tilde{S}(k)$ will be given immediately in a general form, corresponding to scattering with an identification. Namely, let $\mathcal{J}$ be again a multiplication by η, $1-\eta \in C_0^\infty(\mathbb{R}^m)$, and $\tilde{T}=H\mathcal{J}-\mathcal{J}H_1$. Then, similarly to (2.6),

$$\tilde{S}(k; v, v_1)=I-2\pi i Z_1(k)[\mathcal{J}\tilde{T}-\tilde{T}^*R(k, v)\tilde{T}]Z_1(k)^*. \tag{2.23}$$

The chain rule for wave operators, combined with stationary formulae for wave and scattering operators, shows that

$$S(k, v) = S(k, v_1)\tilde{S}(k; v, v_1). \tag{2.24}$$

Of course, the equality (2.24) may be proven also by direct calculations, using the resolvent identity

$$R(k, v) = R(k, v_1) - R(k, v_1)(V - V_1)R(k, v) \tag{2.25}$$

and (2.4) (see Appendix B).

Let now $2\alpha > m+1$ and $v(x)-v_1(x)=0(|x|^{-m-\varepsilon})$, $\varepsilon > 0$, as $|x| \to \infty$. By (2.22), (2.23) the operator $\tilde{S}(k; v, v_1) - I$ is integral and its kernel $ik^{m-1}(2\pi)^{-m+1}\tilde{f}(\omega, \omega'; k; v, v_1)$ is continuous in ω, $\omega' \in \mathbb{S}^{m-1}$, $k > 0$. Explicitly,

$$\tilde{f}(\omega, \omega'; k; v, v_1)= -(2k)^{-1}([\mathcal{J}\tilde{T}-\tilde{T}^*R(k, v)\tilde{T}]\psi(\omega', k, v_1), \psi(\omega, k, v_1)) \tag{2.26}$$

where $\psi(\omega, k, v_1)= [I-R(k, v_1)V_1]\psi_0(\omega, k)$ is the wave function for the potential v_1. In terms of scattering amplitudes (2.24) reads

$$f(\omega, \omega'; v) - f(\omega, \omega'; v_1) = \tilde{f}(\omega, \omega') + ik^{m-1}(2\pi)^{-m+1}\int_{\mathbb{S}^{m-1}} f(\omega, \varphi; v_1)\tilde{f}(\varphi, \omega')d\varphi \tag{2.27}$$

(almost all $(\omega, \omega') \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1}$). Now we apply the Schwartz inequality to the last integral and take into account that

$$\int_{\mathbb{S}^{m-1}} |\tilde{f}(\varphi, \omega)|^2 d\varphi = 2k^{-m+1}(2\pi)^{m-1}\,\mathrm{Im}\,\tilde{f}(\omega, \omega).$$

Similarly to (2.11), this equality is a corollary of unitarity of $\tilde{S}(k)$. Thus (2.27) ensures that for almost all $(\omega, \omega') \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1}$

$$|f(\omega, \omega'; k, v)-f(\omega, \omega'; k, v_1)| \leqslant |\tilde{f}(\omega, \omega'; k, v, v_1)| + [2\sigma(\omega; k, v_1)\,\mathrm{Im}\,\tilde{f}(\omega', \omega'; k; v, v_1)]^{1/2}. \tag{2.28}$$

In case $\alpha > m$, when both amplitudes $f(\omega, \omega'; v)$ and $f(\omega, \omega'; v_1)$ are continuous in $\omega, \omega' \in \mathbb{S}^{m-1}$, the bound (2.28) is extended to all $\omega, \omega' \in \mathbb{S}^{m-1}$. To perform such an extension for $2\alpha > m + 1$, the L. H. S. of (2.28) should be regularized. To that end note that by (2.9)

$$f(\omega, \omega'; v)-f(\omega, \omega'; v_1)= -(2k)^{-1}\{([V-V_1]\psi_0(\omega'), \psi_0(\omega)) - (VR(v)V\psi_0(\omega'), \psi_0(\omega))+(V_1R(v_1)V_1\psi_0(\omega'), \psi_0(\omega))\}. \tag{2.29}$$

In the R. H. S. of (2.29), which we denote temporarily by $A(\omega, \omega')$, all three summands are continuous functions of $\omega, \omega' \in \mathbb{S}^{m-1}$. Thus (2.28) holds for all $\omega, \omega' \in \mathbb{S}^{m-1}$ if the L. H. S. of (2.28) is replaced by $|A(\omega, \omega')|$. In particular, $|A(\omega, \omega)|$ and, hence $|\operatorname{Im} A(\omega, \omega)|$ are bounded by the R. H. S. of (2.28) at $\omega = \omega'$. Notice now that $\operatorname{Im}([V - V_1]\psi_0(\omega), \psi_0(\omega)) = 0$ and therefore, by (2.13), $2 \operatorname{Im} A(\omega, \omega) = \sigma(\omega; v) - \sigma(\omega; v_1)$. It follows that

$$|\sigma(\omega; k, v) - \sigma(\omega; k, v_1)| \leqslant 2 |\tilde{f}(\omega, \omega; k; v, v_1)| + 2[2\sigma(\omega; k, v_1) \operatorname{Im} \tilde{f}(\omega, \omega; k; v, v_1)]^{1/2}. \quad (2.30)$$

We formulate the obtained results.

LEMMA 3. — Let potentials v_1 and v satisfy the condition (1.1), $v(x) - v_1(x) = O(|x|^{-m-\varepsilon})$, $\varepsilon > 0$, as $|x| \to \infty$, and let the amplitude $\tilde{f}(\omega, \omega'; k; v, v_1)$ be defined by the relation (2.26). Then, if $\alpha > m$, the bound (2.28) holds for all $\omega, \omega' \in \mathbb{S}^{m-1}$. If $2\alpha > m + 1$, the bound (2.30) holds for all $\omega \in \mathbb{S}^{m-1}$.

3. THE MAIN THEOREM. UPPER BOUNDS FOR THE RESOLVENT AND FOR THE TOTAL CROSS-SECTION

The aim of the present paper is to find the asymptotics of the forward scattering amplitude $F(\omega, \omega; k, gq)$ and of the total cross-section $\sigma(\omega; k, gq)$ in case $k \to \infty$, $N := g(2k)^{-1} \to \infty$. Besides the condition (1.3), we suppose that $q(x)$ is twice differentiable with respect to a radial variable $|x|$ and

$$\sup_{x \in \mathbb{R}^m} \left(|q(x)| + |x| \left| \frac{\partial q}{\partial |x|} \right| + |x|^2 \left| \frac{\partial^2 q}{\partial |x|^2} \right| \right) < \infty. \quad (3.1)$$

We assume additionally that $g \leqslant \gamma_0 k^2$ where a constant γ_0 obeys

$$\gamma_0 \inf_{\beta \in (0,2]} \sup_{x \in \mathbb{R}^m} \left(\beta^{-1} |x| \frac{\partial q}{\partial |x|} + q(x) \right) < 1. \quad (3.2)$$

Thus, γ_0 is an arbitrary fixed number if a potential $|x|^\beta q(x)$ is repulsive for some $\beta \in (0, 2]$. In a general case (3.2) requires that γ_0 is small with respect to a size of q.

For a given incident direction $\omega \in \mathbb{S}^{m-1}$ denote by Λ_ω a plane, orthogonal to ω, and let $\mathbb{S}^{m-2}_\omega$ be a unit sphere in the plane Λ_ω. For $\varphi \in \mathbb{S}^{m-2}_\omega$ set

$$\Omega(\omega, \varphi) = \int_0^\pi \Phi(\omega \cos \theta + \varphi \sin \theta) \sin^{\alpha-2} \theta d\theta. \quad (3.3)$$

Now we give a precise formulation of our main result (*).

(*) See also note added in proof.

THEOREM 1. — Assume that $q(x)$ has the asymptotics (1.3) with $\Phi \in C(\mathbb{S}^{m-1})$ and that the condition (3.1) holds. Let $k \to \infty$, $N \to \infty$, $g \leqslant \gamma_0 k^2$, where γ_0 satisfies the estimate (3.2). Then for $\alpha > m$ the forward scattering amplitude has the asymptotics

$$F(\omega, \omega; k, gq) \sim (2\pi)^{-\frac{m-1}{2}} e^{-\frac{\pi i}{4}(m-3)} i(m-1)^{-1}\Gamma(1-\varkappa)$$
$$k^{\frac{m-1}{2}} N^{\varkappa} \int_{\mathbb{S}_\omega^{m-2}} |\Omega(\omega, \varphi)|^{\varkappa} \exp[2^{-1}\pi i \varkappa \operatorname{sgn} \Omega(\omega, \varphi)] d\varphi, \varkappa = (m-1)(\alpha-1)^{-1}. \tag{3.4}$$

For $2\alpha > m + 1$ the total cross-section has the asymptotics

$$\sigma(\omega; k, gq) \sim \pi\left[(m-1)\Gamma(\varkappa)\sin\frac{\pi\varkappa}{2}\right]^{-1} N^{\varkappa} \int_{\mathbb{S}_\omega^{m-2}} |\Omega(\omega, \varphi)|^{\varkappa} d\varphi. \tag{3.5}$$

As was noted in the Introduction, by the proof of Theorem 1 we need uniform in g *a-priori* bound for the resolvent $R(k, gq)$ of the operator $-\Delta + gq$.

THEOREM 2. — Let conditions (3.1), (3.2) be fulfilled. Then for

$$sk \geqslant a_0 > 0, \; g \leqslant \gamma_0 k^2$$

and every $\beta > 1/2$ the bound

$$\| X_\beta^{(s)} R(k, gq) X_\beta^{(s)} \| \leqslant C k^{-1} s^{1-2\beta} \tag{3.6}$$

holds. A constant C in (3.6) depends only on numbers a_0, γ_0 and on the value of (3.1).

Proof. — By the equality (2.18) with $\rho = k^{-1}$, (3.6) is equivalent to the bound

$$\| X_\beta^{(a)} R(1, \gamma v_k) X_\beta^{(a)} \| \leqslant C a^{1-2\beta}, \tag{3.7}$$

where $v_k(x) = q(k^{-1}x)$, $\gamma = gk^{-2} \leqslant \gamma_0$, $a = sk \geqslant a_0$.

The proof of (3.7) is based on the commutator method of E. Mourre [*13*] (see also the article [*14*]). Set

$$H = H(\gamma, k) = -\Delta + \gamma v_k, \qquad 2iA = x\nabla + \nabla x = 2|x|\frac{\partial}{\partial |x|} + m,$$

$$q' = \frac{\partial q}{\partial |x|}, \; q'' = \frac{\partial^2 q}{\partial |x|^2}, \; Q_\beta = \sup_{x \in \mathbb{R}^m} [\beta^{-1}|x| q'(x) + q(x)].$$

Then $i[H_0, A] = 2H_0$ and

$$i[H, A] = 2H_0 - \gamma k^{-1}|x| q'(k^{-1}x) \geqslant \beta H - \gamma[\beta q(k^{-1}x) + k^{-1}|x| q'(k^{-1}x)] \geqslant \beta(H - \gamma Q_\beta) \tag{3.8}$$

for any $\beta \in (0, 2]$. If $Q_\beta \leqslant 0$ for some $\beta \in (0, 2]$, then the R. H. S. of (3.8) is larger than βH for this β. If $Q_\beta > 0$ for all $\beta \in (0, 2]$, choose β so that

$\gamma_0 Q_\beta < 1$. This is possible in virtue of (3.2). Let now $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(\lambda) = 1$ for $\lambda \in [1 - \sigma, 1 + \sigma]$ and $\varphi(\lambda) = 0$ for $\lambda \notin [1 - 2\sigma, 1 + 2\sigma]$. Take $\sigma < 2^{-1} \min \{ 1, 1 - \gamma_0 Q_\beta \}$. Then (3.8) ensures that

$$i\Phi [H, A] \Phi \geqslant c\Phi^2, \qquad \Phi = \varphi(H), \qquad H = H(\gamma, k), \tag{3.9}$$

with c not dependent on $k > 0$ and $\gamma \leqslant \gamma_0$.

Below we shall prove that the bound (3.7) is a corollary of the inequality (3.9). Compared to the original proof of [*13*] we should keep track of the dependence of (3.7) on the parameter a. This requires some technical modifications. Besides, we should make sure that all estimates are uniform in $k > 0$ and $\gamma \leqslant \gamma_0$, which is essentially standard. Not restricting generality, by the proof of (3.7) we can take $\beta \in (1/2, 1]$. Set $B = i[H, A]$, $M = \Phi B \Phi$, $G_\varepsilon = (H - i\varepsilon M - 1 - i\mu)^{-1}$. All estimates below will be uniform in $\mu > 0$. Let $K_\varepsilon(a)$ be a multiplication by the function

$$(|x| + a)^{-\beta}(\varepsilon |x| + 1)^{\beta - 1}.$$

Then

$$\| K_\varepsilon(a) \| \leqslant C a^{-\beta} \leqslant C_1, \tag{3.10}$$

$$\left\| \frac{d}{d\varepsilon} K_\varepsilon(a) \right\| \leqslant C\varepsilon^{\beta - 1}, \qquad \| K_\varepsilon(a) A (H_0 + I)^{-1} \| \leqslant C\varepsilon^{\beta - 1}. \tag{3.11}$$

We shall establish the bound

$$\| K_\varepsilon(a) G_\varepsilon K_\varepsilon(a) \| \leqslant C a^{1 - 2\beta} \tag{3.12}$$

with a constant C not depending on $\varepsilon > 0$ (and $\mu > 0, k > 0, \gamma \leqslant \gamma_0$ neither). In particular, as $\varepsilon \to 0$, $\mu \to 0$ (3.12) implies the bound (3.7).

We need some facts from papers [*13*] [*14*]. Note firstly that according to (3.8), (3.1) $B = 2H_0 + Y_1$, where $Y_1 = Y_1(k, \gamma)$ is bounded uniformly in $k > 0, \gamma \leqslant \gamma_0$. Similarly, computing the second commutator, we find that

$$i[B, A] = 4H_0 + \gamma [k^{-1} |x| q'(k^{-1}x) + k^{-2} |x|^2 q''(k^{-1}x)] =: 4H_0 + Y_2,$$

where again by (3.1) the operator $Y_2 = Y_2(k, \gamma)$ is bounded uniformly in $k > 0$, $\gamma \leqslant \gamma_0$. It follows that commutators $[\Phi, A]$ and $[M, A]$ are also uniformly bounded. The basic estimate (3.9) is used only to obtain the bound

$$\| \Phi G_\varepsilon f \| \leqslant C\varepsilon^{-1/2} |(f, G_\varepsilon f)|^{1/2}. \tag{3.13}$$

Together with $\| M \| \leqslant C$, this shows that

$$\| (H_0 + I) G_\varepsilon \| \leqslant C\varepsilon^{-1}, \qquad \| (H_0 + I)(I - \Phi) G_\varepsilon \| \leqslant C. \tag{3.14}$$

In its turn, (3.13) and the second estimate (3.14) provide that

$$\| (H_0 + I) G_\varepsilon K_\varepsilon(a) \| + \| K_\varepsilon(a) G_\varepsilon (H_0 + I) \| \leqslant C(\| K_\varepsilon(a) \| + \varepsilon^{-1/2} \| K_\varepsilon(a) G_\varepsilon K_\varepsilon(a) \|^{1/2}). \tag{3.15}$$

Now we can show that the operator $F_\varepsilon(a)=K_\varepsilon(a)G_\varepsilon K_\varepsilon(a)$ satisfies the differential inequality

$$\left\| \frac{dF_\varepsilon(a)}{d\varepsilon} \right\| \leqslant C\varepsilon^{\beta-1}[a^{-\beta} + \varepsilon^{-1/2} \| F_\varepsilon(a) \|^{1/2}]. \tag{3.16}$$

Let us estimate separately the summands in the R. H. S. of the expression

$$\frac{dF_\varepsilon}{d\varepsilon} = \frac{dK_\varepsilon}{d\varepsilon} G_\varepsilon K_\varepsilon + K_\varepsilon G_\varepsilon \frac{dK_\varepsilon}{d\varepsilon} + K_\varepsilon \frac{dG_\varepsilon}{d\varepsilon} K_\varepsilon .$$

The bound (3.16) for norms of first two summands is a direct corollary of (3.11), (3.15). To estimate the last term, we compute

$$\frac{dG_\varepsilon}{d\varepsilon} = iG_\varepsilon M G_\varepsilon$$
$$= -iG_\varepsilon \{ (I-\Phi)B\Phi + B(I-\Phi) - i[H - i\varepsilon M - 1 - i\mu, A] + \varepsilon[M, A] \} G_\varepsilon . \tag{3.17}$$

The contribution to $\left\| K_\varepsilon \frac{dG_\varepsilon}{d\varepsilon} K_\varepsilon \right\|$ of the first summand in the R. H. S. of (3.17)

$$\| K_\varepsilon G_\varepsilon (1-\Phi)B\Phi G_\varepsilon K_\varepsilon \| \leqslant \| K_\varepsilon \| \ \| G_\varepsilon(I-\Phi) \| \ \| B\Phi \| \ \| G_\varepsilon K_\varepsilon \|$$

is estimated by $a^{-\beta} + \varepsilon^{-1/2} \| F_\varepsilon(a) \|^{1/2}$ according to (3.10), (3.14), (3.15). The second summand in the R. H. S. of (3.17) is treated quite similarly. In the next term we expand the commutator. Then

$$\| K_\varepsilon G_\varepsilon [H - i\varepsilon M - 1 - i\mu, A] G_\varepsilon K_\varepsilon \| \leqslant \| K_\varepsilon A(H_0+I)^{-1} \|$$
$$\times (\| (H_0+I)G_\varepsilon K_\varepsilon \| + \| K_\varepsilon G_\varepsilon (H_0+I) \|),$$

which is bounded by the R. H. S. of (3.16) according to (3.10), (3.11), (3.15). Finally, in virtue of bounds $\| [M, A] \| \leqslant C$ and (3.15)

$$\varepsilon \| K_\varepsilon G_\varepsilon [M, A] G_\varepsilon K_\varepsilon \| \leqslant C\varepsilon \| K_\varepsilon G_\varepsilon \| \ \| G_\varepsilon K_\varepsilon \|$$
$$\leqslant C_1(\varepsilon a^{-2\beta} + \| F_\varepsilon \|).$$

Since by (3.10), (3.14)

$$\| F_\varepsilon \| \leqslant Ca^{-2\beta}\varepsilon^{-1}, \tag{3.18}$$

this term is again bounded by the R. H. S. of (3.16). Putting all these inequalities together, we conclude the proof of (3.16).

It remains to prove that the bound (3.12) is a corollary of (3.16), (3.18). The simplest way of doing it is to introduce a new operator-function $\tilde{F}_s(a) = a^{2\beta-1}F_{s/a}(a)$. Then (3.18) reads

$$\| \tilde{F}_s(a) \| \leqslant Cs^{-1}, \tag{3.19}$$

and after substitution $\varepsilon = s/a$ (3.16) provides the inequality

$$\left\| \frac{d\tilde{F}_s(a)}{ds} \right\| \leqslant Cs^{\beta-1}(1+s^{-1/2} \| \tilde{F}_s(a) \|^{1/2}), \qquad a \geqslant a_0 > 0. \tag{3.20}$$

By (3.20) an estimate $\| \tilde{F}_s(a) \| \leqslant Cs^{-\rho}$, $0 < \rho \leqslant 1$, ensures that

$$\left\| \frac{d\tilde{F}_s(a)}{ds} \right\| \leqslant Cs^{\beta-3/2-\rho/2}.$$

After integration this leads to a bound $\| \tilde{F}_s(a) \| \leqslant Cs^{\beta-1/2-\rho/2}$ if $2\beta < \rho+1$, or to a bound $\| \tilde{F}_s(a) \| \leqslant C$ if $2\beta > \rho + 1$. Thus, starting from (3.19), with the help of (3.20) we arrive in a finite number of steps to the bound $\| \tilde{F}_s(a) \| \leqslant C$. This is equivalent to (3.12). □

Comments and remarks.

1) Since it is not assumed that $q(x) \to 0$ as $|x| \to \infty$ in Theorem 2, it can be applied to a multiparticle problem.

2) The bound (3.6) is valid for long-range potentials. In this case it is, perhaps, new even for a fixed g.

3) The bound (3.6) is sharp, even for a fixed g, as far as dependence on k and s is concerned.

4) If (3.7) holds for some fixed a (and arbitrary $\beta > 1/2$), the L. H. S. of (3.7) is automatically bounded by $a^{1-2\beta+\varepsilon}$ for all $a \geqslant a_0 > 0$ and arbitrary $\varepsilon > 0$. This would have given us the bound of the L. H. S. of (3.6) by $k^{-1+\varepsilon}s^{1-2\beta+\varepsilon}$. However, it is important for us to have the precise result.

5) The bound (3.6) for $s = 1$, $g = \gamma k^2$ and functions q with compact supports was established earlier in papers [*15*] [*16*]. Compared to [*15*] [*16*] we have very broad conditions on the behaviour of $q(x)$ at infinity. On the other hand, the so-called non-trapping condition, imposed in [*15*] [*16*], is less restrictive than our assumptions on the coupling constant. Note also that in contrast to [*15*] [*16*] our proof is rather standard from a point of view of scattering theory.

6) J. M. Combes communicated to the author that a bound, similar to (3.6), can be obtained also with the help of R. Lavine's technique instead of that of E. Mourre.

As was mentioned in the Introduction, the restriction $g \leqslant \gamma_0 k^2$ is required in Theorem 1 only for the proof of the bound (3.6). Once this bound is established, the asymptotics (3.4), (3.5) are valid in an essentially broader region of parameters k and g. Namely, the following conditional assertion is true.

THEOREM 1'. — Let $q(x)$ have the asymptotics (1.3) with $\Phi \in C(\mathbb{S}^{m-1})$. Assume that in some region $\mathscr{L}$ of parameters (k, g) the bound (3.6) holds true for all $\beta > 1/2$, $sk \geqslant a_0 > 0$. Then the asymptotics (3.4) (if $\alpha > m$)

and (3.5) (if $2\alpha > m+1$) are valid as $N \to \infty$, $gk^{\alpha-2} \to \infty$, $(k, g) \in \mathscr{L}$.

If at least one of the conditions $N \to \infty$ or $gk^{\alpha-2} \to \infty$ is not fulfilled, the asymptotics (3.4), (3.5) are surely violated. On the other hand for nonnegative central potentials (and $m = 3$) (3.4), (3.5) are valid in the whole region $N \to \infty, gk^{\alpha-2} \to \infty$. This result is obtained in [7] bypassing the bound of the resolvent. Conditions $N \to \infty$, $gk^{\alpha-2} \to \infty$ permit an arbitrary rapid growth of g with respect to k as $k \to \infty$ (or k fixed). Moreover, the case $g \to \infty$, $k \to 0$ is not also totally excluded. Note that for k fixed, $g \to \infty$ the bound, similar to (3.6), was proven in *[10]* for repulsive potentials.

As a corollary of Theorem 2, we shall deduce now sharp upper bounds for the forward scattering amplitude and for the total cross-section. Results of the following theorem are not used by the proof of Theorem 1 but the method of its proof is applied in section 5 in a more complicated situation. Besides, this theorem is, perhaps, of some interest for its own sake. Set $\mathbb{T}_r = \{ x \in \mathbb{R}^m : |x| \leqslant r \}$; by χ_r we denote its characteristic function.

THEOREM 3. — Let q satisfy the condition (3.1) and let $g \leqslant \gamma_0 k^2$ with γ_0 obeying (3.2). If $\operatorname{supp} q \subset \mathbb{T}_r$ and $rk \geqslant a_0 > 0$, then

$$|F(\omega, \omega'; k, gq)| \leqslant Ck^{\frac{m-1}{2}} r^{m-1}, \tag{3.21}$$

$$\sigma(\omega; k, gq) \leqslant Cr^{m-1}. \tag{3.22}$$

If q satisfies the bound (1.1) and $gk^{\alpha-2} \geqslant a_0 > 0$, then

$$|F(\omega, \omega'; k, gq)| \leqslant Ck^{\frac{m-1}{2}} N^{\varkappa}, \qquad \alpha > m, \tag{3.23}$$

$$\sigma(\omega; k, gq) \leqslant CN^{\varkappa}, \qquad 2\alpha > m+1. \tag{3.24}$$

Constants C in (3.21), (3.22) are common for all q, obeying (3.1) uniformly. In (3.23), (3.24) constants C are common for all q, obeying uniformly (3.1) and (1.1).

Proof. — We start from the representation (2.10) for the scattering amplitude. Let $\mathscr{J} = \mathscr{J}_r$ be a multiplication by $\eta(r^{-1}x)$, where $\eta \in C^\infty(\mathbb{R}^m)$, $\eta(x) = 0$ for $|x| \leqslant 1$ and $\eta(x) = 1$ for $|x| \geqslant 2$. Then (cf. (2.5))

$$T = T_r = H\mathscr{J}_r - \mathscr{J}_r H_0$$
$$= -2r^{-1}(\nabla\eta)(r^{-1}x)\nabla - r^{-2}(\Delta\eta)(r^{-1}x) + gq(x)\eta(r^{-1}x). \tag{3.25}$$

Consider firstly potentials with compact supports when $q(x)\eta(r^{-1}x) = 0$. Since $|\psi_0(\omega, k)| = 1$, $|\nabla\psi_0(\omega, k)| = k$, (3.25) ensures that

$$|(\mathscr{J}_r^* T_r \psi_0, \psi_0)| \leqslant Cr^{-1}k \int |(\nabla\eta)(r^{-1}x)|\, dx + Cr^{-2} \int |(\Delta\eta)(r^{-1}x)|\, dx \leqslant C_1 k r^{m-1} \tag{3.26}$$

and, similarly,

$$\| T_r\psi_0 \|^2 \leqslant Ck^2r^{m-2} . \tag{3.27}$$

Further, since $\chi_{2r}T_r = T_r$,

$$| (T_r^*R(k, gq)T_r\psi_0, \psi_0) | \leqslant \| \chi_{2r}R(k, gq)\chi_{2r} \| \ \| T_r\psi_0 \|^2 \leqslant Ckr^{m-1} \tag{3.28}$$

by (3.6) and (3.27). Thus, according to (3.26), (3.28), the modulus of (2.10) does not exceed Cr^{m-1}. Relations (2.7), (2.11) provide now (3.21), (3.22).

Consider potentials satisfying (1.1). In this case the operator T_r contains an additional summand $gq(x)\eta(r^{-1}x)$ and hence

$$| (\mathscr{I}_r^*T_r\psi_0, \psi_0) | \leqslant Ckr^{m-1} + g \int \eta^2(r^{-1}x) | q(x) | dx$$
$$\leqslant C_1kr^{m-1}(1 + Nr^{1-\alpha}), \quad \alpha > m. \tag{3.29}$$

Here and below we suppose that $rk \geqslant a_0 > 0$. Let $1 < 2\beta < 2\alpha - m$. Clearly,

$$\| X^{(r)}_{-\beta}T_r\psi_0 \|^2 \leqslant Ck^2r^{-2} \int | (\nabla\eta)(r^{-1}x) |^2(| x | + r)^{2\beta}dx + Cr^{-4} \int | (\Delta\eta)(r^{-1}x) |^2$$
$$(| x | + r)^{2\beta}dx + Cg^2 \int q^2(x)\eta^2(r^{-1}x)(| x | + r)^{2\beta}dx \leqslant C_1k^2r^{m+2\beta-2}(1 + N^2r^{2-2\alpha}).$$

The bound (3.6) ensures now that

$$| (T_r^*R(k, gq)T_r\psi_0, \psi_0) | \leqslant \| X^{(r)}_\beta R(k, gq)X^{(r)}_\beta \| \ \| X^{(r)}_{-\beta}T_r\psi_0 \|^2$$
$$\leqslant Ckr^{m-1}(1 + N^2r^{2-2\alpha}) . \tag{3.30}$$

Combining (3.29), (3.30) we find that

$$| f(\omega, \omega'; k, gq) | \leqslant Cr^{m-1}(1 + Nr^{1-\alpha} + N^2r^{2-2\alpha}) . \tag{3.31}$$

Set $r = N^{(\alpha-1)^{-1}}$. Then the R. H. S. of (3.31) equals $Cr^{m-1} = CN^{\varkappa}$. As for potentials with compact supports, this gives bounds (3.23), (3.24) for $\alpha > m$. To obtain (3.24) for $2\alpha > m + 1$, one should apply (3.30) to (2.13). □

REMARK 1. — For potentials with supp $q \subset \mathbb{T}_r$ the proof of Theorem 3 can be reduced to a special case $r = 1$. To that end, one should use relations (2.20), (2.21) and the invariance of (3.1) with respect to dilation $x \to r^{-1}x$.

REMARK 2. — The proof of Theorem 3 establishes, actually, bounds of the scattering amplitude by suitable norms of the resolvent. In case supp $q \subset \mathbb{T}_r$ such bound has a form

$$| F(\omega, \omega'; k, gq) | \leqslant Ck^{\frac{m-1}{2}} r^{m-1} [1 + kr^{-1} \| \chi_{2r}R(k, gq)\chi_{2r} \|], \quad rk \geqslant a_0 > 0 .$$

In case of potentials, satisfying (1.1) with $\alpha > m$,

$$| F(\omega, \omega'; k, gq) | \leqslant Ck^{\frac{m-1}{2}} N^{\varkappa}[1 + kr^{2\beta-1} \| X_{\beta}^{(r)} R(k, gq) X_{\beta}^{(r)} \|],$$
$$r = N^{(\alpha-1)^{-1}}, \qquad 1 < 2\beta < 2\alpha - m, \qquad gk^{\alpha-2} \geqslant a_0 > 0.$$

Similar bounds hold, of course, for the total cross-section; there the factor $k^{\frac{m-1}{2}}$ should be omitted and $R(k, gq)$ can be replaced by $R(k, gq) - R(k, gq)^*$. In particular, estimates (3.21)-(3.24) are valid whenever (3.6) is true.

4. THE MODEL PROBLEM

In this section we shall find asymptotics of the wave function, of the scattering amplitude and of the total cross-section in the « critical » case when $k \to \infty$, $N = \text{const}$. We choose coordinates (Z, b) in $\mathbb{R}^m$ so that Z-axes is directed along ω; b is a set of $(m-1)$ variables in a plane Λ_ω, orthogonal to ω. Then $x = b + \omega Z \leftrightarrow (Z, b)$, $b \in \Lambda_\omega$. By ∇_b and Δ_b we denote the gradient and the Laplace operator in variables b. Set

$$\hat{\psi}(x; \omega, k, v) = \exp\left[ikZ - i(2k)^{-1} \int_{-\infty}^{Z} v(Z', b) dZ' \right], \tag{4.1}$$

$$\mathscr{A}(\omega; v) = i \int_{\Lambda_\omega} db \left\{ 1 - \exp\left[-i \int_{-\infty}^{\infty} v(Z, b) dZ \right] \right\}, \tag{4.2}$$

$$\mathscr{A}_0(\omega; v) = 4 \int_{\Lambda_\omega} db \sin^2\left[2^{-1} \int_{-\infty}^{\infty} v(Z, b) dZ \right]. \tag{4.3}$$

Since the incident direction ω is fixed, the notation of dependence of different objects on ω is often omitted. We treat firstly potentials with compact supports.

THEOREM 4. — Let $q \in C_0^{\rho_0}(\mathbb{R}^m)$, where an integer $\rho_0 \geqslant (m+1)/2$, $N \leqslant N_0$, $k \geqslant k_0 > 0$, r is an arbitrary fixed number. Then

$$\| \chi_r(\psi - \hat{\psi})(\omega, k, gq) \| \leqslant Ck^{-1}, \tag{4.4}$$

$$| f(\omega, \omega; k, gq) - \mathscr{A}(\omega; Nq) | \leqslant Ck^{-1}. \tag{4.5}$$

If $2\beta > m$, then

$$\lim_{k \to \infty, N \leqslant N_0} \| X_\beta(\psi - \hat{\psi})(\omega, k, gq) \| = 0. \tag{4.6}$$

Proof. — Our first goal is to derive an expression for $\psi - \hat{\psi}$ in terms of an error

$$\hat{w}(x; k, gq) = - [\Delta + k^2 - gq(x)] \hat{\psi}(x; k, gq), \tag{4.7}$$

arising when $\hat{\psi}$ is inserted into the Schrödinger equation. Note that by a direct differentiation

$$\hat{w}(x; k, gq) = \hat{\psi}(x; k, gq)\left\{ N^2\left[q^2(x) + \left(\int_{-\infty}^{Z} (\nabla_b q)(Z', b)dZ'\right)^2\right] + iN\left[q_Z(x) + \int_{-\infty}^{Z} (\Delta_b q)(Z', b)dZ'\right]\right\}. \quad (4.8)$$

Consider also an error

$$\tilde{w}(k, gq) = [I + gR_0(k)q]\hat{\psi}(k, gq) - \psi_0(k), \quad (4.9)$$

corresponding to $\hat{\psi}$ in the Lippman-Schwinger equation (2.15). The difference $\psi - \hat{\psi}$ can be easily expressed in terms of $\tilde{w}$. Actually, comparing (2.15) and (4.9) we see that $\tilde{w} = (I + gR_0 q)(\hat{\psi} - \psi)$. Applying the operator $I - gRq$ and taking into account the resolvent identity we arrive at the expression

$$(\psi - \hat{\psi})(k, gq) = -[I - gR(k, gq)q]\tilde{w}(k, gq). \quad (4.10)$$

Now we should find a connection between functions $\tilde{w}$ and $\hat{w}$. Let $\operatorname{supp} q \subset \mathbb{T}_r$ and let $\zeta \in C_0^\infty(\mathbb{R}^m)$, $\zeta(x) = 1$ for $x \in \mathbb{T}_{r_0}$, $r_0 > r$. Since $\zeta q = q$, (4.7) and the free equation $(\Delta + k^2)\psi_0 = 0$ ensure that

$$gR_0(k)q\hat{\psi} = R_0(k)\zeta[(\Delta + k^2)(\hat{\psi} - \psi_0) + \hat{w}]. \quad (4.11)$$

Recall that $R_0(k)$ is an integral operator with a kernel

$$R_0(x, x'; k) = i4^{-1}(2\pi)^{-\left(\frac{m}{2}-1\right)}(k|x - x'|^{-1})^{\frac{m}{2}-1} H^{(1)}_{\frac{m}{2}-1}(k|x - x'|),$$

where $H_s^{(1)}$ is the Hankel function of the first kind and an order s. Denote by $G_0(k, \zeta)$ an integral operator with a kernel

$$G_0(x, x'; k, \zeta) = 2\nabla_{x'} R_0(x, x'; k)\nabla\zeta(x') + R_0(x, x'; k)\Delta\zeta(x'). \quad (4.12)$$

Integrating twice by parts in an integral $R_0(k)\zeta(\Delta + k^2)(\hat{\psi} - \psi_0)$ and taking into account the relation $(\Delta_{x'} + k^2)R_0(x, x'; k) = -\delta(x - x')$, we find that

$$R_0(k)\zeta(\Delta + k^2)(\hat{\psi} - \psi_0) = -\zeta(\hat{\psi} - \psi_0) + G_0(k, \zeta)(\hat{\psi} - \psi_0).$$

Insert now this expression into (4.11). Combined together, (4.9) and (4.11) show that

$$\tilde{w} = (1 - \zeta)(\hat{\psi} - \psi_0) + R_0(k)\zeta\hat{w} + G_0(k, \zeta)(\hat{\psi} - \psi_0). \quad (4.13)$$

Comparing (4.10), (4.13) and using (2.1) we receive, finally, the convenient representation

$$\chi_r(\psi - \hat{\psi})(k, gq) = -\chi_r R(k, gq)\zeta\hat{w}(k, gq) - \chi_r[I - gR(k, gq)q]G_0(k, \zeta)[\hat{\psi}(k, gq) - \psi_0(k)]. \quad (4.14)$$

Now we able to prove the bound (4.4). Choose r_1 so that supp $\zeta \subset \mathbb{T}_{r_1}$. Then (4.14) ensures that

$$\| \chi_r(\psi - \hat{\psi}) \| \leqslant \| \chi_r R \chi_{r_1} \| \; \| \zeta \hat{w} \| + (1 + g \| \chi_r R q \|) . \| \chi_r G_0(\zeta)(\hat{\psi} - \psi_0) \| . \tag{4.15}$$

By (3.6) the factors $\| \chi_r R \chi_{r_1} \|$ and $\| \chi_r R q \|$ are bounded by Ck^{-1}. Moreover, by (4.8) $\| \zeta \hat{w} \| \leqslant C$. Thus the first summand in the R. H. S. of (4.15) does not exceed Ck^{-1}. To obtain the same bound for the second summand, it remains to check that $\| \chi_r G_0(\zeta)(\hat{\psi} - \psi_0) \| \leqslant Ck^{-1}$. We shall prove a somewhat stronger inequality

$$\sup_{x \in \mathbb{T}_r} \left| \int G_0(x, x'; k, \zeta) e^{ik\langle \omega, x' \rangle} \left[e^{-iN \int_{-\infty}^{Z'} q(Z'', b')dZ''} - 1 \right] dx' \right| \leqslant Ck^{-1} . \tag{4.16}$$

Since supp $\nabla \zeta \subset \mathbb{T}_{r_0, r_1} := \mathbb{T}_{r_1} \backslash \mathbb{T}_{r_0}$, where $r < r_0 < r_1$, the integral (4.16) is restricted to $\mathbb{T}_{r_0, r_1}$ which is disjoint by a positive distance from $\mathbb{T}_r$. For $|x - x'| \geqslant c > 0$ the Hankel function in the definition of R_0 can be replaced by its asymptotic expansion at infinity. This implies that the kernel (4.12) admits a representation

$$G_0(x, x'; k, \zeta) = k^{\frac{m-1}{2}} e^{ik|x-x'|} u_0(x, x'; k) + u_1(x, x'; k) , \tag{4.17}$$

where for all multiindices ρ

$$| D_{x'}^{\rho} u_0(x, x'; k) | \leqslant C, \; | u_1(x, x'; k) | \leqslant Ck^{-1}, \qquad x \in \mathbb{T}_r . \tag{4.18}$$

Insert now the representation (4.17) into (4.16). For the summand, corresponding to $u_1(x, x'; k)$, the bound by Ck^{-1} is an immediate consequence of (4.18). Thus (4.16) is reduced to the inequality

$$\sup_{x \in \mathbb{T}_r} \left| \int e^{ik\varphi(x, x')} u(x, x'; k) dx' \right| \leqslant Ck^{-\frac{m+1}{2}} , \tag{4.19}$$

where

$$\varphi(x, x') = | x - x' | + \langle \omega, x' \rangle ,$$

$$u(x, x'; k) = u_0(x, x'; k) \left\{ \exp \left[- iN \int_{-\infty}^{Z'} q(Z'', b')dZ'' \right] - 1 \right\} . \tag{4.20}$$

Since $q \in C_0^{\infty}(\mathbb{R}^m)$ and $N \leqslant N_0$, the function $u(x, x'; k)$ satisfies the first inequality (4.18) for all $0 \leqslant | \rho | \leqslant \rho_0$. Note that the second factor in (4.20) is different from zero only in the region $\Pi_r(\omega)$, consisting of points x' of the form $x' = y + \omega t$ with $y \in \mathbb{T}_r$ and $t \geqslant 0$; thus, $\Pi_r(\omega)$ is a union of the ball $\mathbb{T}_r$ and of its « shadow » for the direction ω of incoming plane wave. So the integration in (4.19) is, actually, restricted to $\mathbb{T}_{r_0, r_1} \cap \Pi_r(\omega)$. It is clear that

$$| \nabla_{x'} \varphi(x, x') | \geqslant \left| \frac{x' - x}{| x' - x |} + \omega \right| \geqslant c > 0, \qquad x \in \mathbb{T}_r, \qquad x' \in \mathbb{T}_{r_0, r_1} \cap \Pi_r(\omega) . \tag{4.21}$$

Now with the help of the identity

$$\int e^{ik\varphi(x,x')}\mathrm{F}(x, x')dx' = -(ik)^{-1}\sum_{j=1}^{m}\int e^{ik\varphi}\frac{\partial}{\partial x'_j}\left(|\nabla_{x'}\varphi|^{-2}\frac{\partial\varphi}{\partial x'_j}\mathrm{F}\right)dx'$$

we integrate ρ_0 times by parts in (4.19). Namely, applying this identity ρ_0 times and using the first estimate (4.18) for u and (4.21) for φ, we find that the integral in (4.19) does not exceed $\mathrm{C}k^{-\rho_0}$, $2\rho_0 \geqslant m+1$. This concludes the proof of (4.19) and, hence, of the bound (4.4).

The asymptotics (4.5) is a corollary of (4.4). Actually, by (2.16)

$$| f(\omega, \omega; k, gq) + \mathrm{N}(q\hat{\psi}(\omega, k, gq), \psi_0(\omega, k)) | \leqslant \mathrm{C}k^{-1}.$$

It remains to notice that according to (4.1), (4.2)

$$\mathrm{N}(q\hat{\psi}(\omega, k, gq), \psi_0(\omega, k)) = -\mathscr{A}(\omega; \mathrm{N}q).$$

To prove the last relation (4.6), we use the bound

$$\| \mathrm{X}_{\beta'}\psi(k, gq) \| \leqslant \mathrm{C},$$

which is valid in virtue of (2.14), (3.6) for $\mathrm{N} \leqslant \mathrm{N}_0$, $k \geqslant k_0 > 0$ and all $\beta' > m/2$. By the definition (4.1) the same bound is valid also for $\hat{\psi}(k, gq)$. Applying these two bounds for $2\beta > 2\beta' > m$ we find that

$$\lim_{r\to\infty} \| (\mathrm{I} - \chi_r)\mathrm{X}_\beta(\psi - \hat{\psi})(k, gq) \| = 0 \qquad (4.22)$$

uniformly in $k \geqslant k_0 > 0$, $\mathrm{N} \leqslant \mathrm{N}_0$. The relation (4.6) is a direct corollary of (4.4), (4.22). □

Now we shall generalize Theorem 4 to potentials with non-compact supports. Simultaneously, we shall consider functions $q^{(\rho)}$, depending on some additional parameter ρ. This is necessary for applications in section 5. As a preliminary, we shall establish that the wave function, the scattering amplitude and the total cross-section are continuous uniformly in $k \geqslant k_0 > 0$, $\mathrm{N} \leqslant \mathrm{N}_0$ as q changes in the metrics corresponding to (1.1).

LEMME 4. — Let q satisfy conditions (1.1) and (3.1). Assume that

$$\lim_{\rho\to\infty} \sup_{x\in\mathbb{R}^m} (1 + |x|)^{\alpha_1} | q^{(\rho)}(x) - q(x) | = 0. \qquad (4.23)$$

Then uniformly in $k \geqslant k_0 > 0$, $\mathrm{N} \leqslant \mathrm{N}_0$,

$$\lim_{\rho\to\infty} \| \mathrm{X}_\beta[\psi(k, gq^{(\rho)}) - \psi(k, gq)] \| = 0,\ 2\beta > 1,\ 2\alpha > m+1,\ 2\alpha_1 > m+1, \qquad (4.24)$$

$$\lim_{\rho\to\infty} | \sigma(\omega; k, gq^{(\rho)}) - \sigma(\omega; k, gq) | = 0,\ 2\alpha > m+1,\ 2\alpha_1 > m+1, \qquad (4.25)$$

$$\lim_{\rho\to\infty} | f(\omega, \omega'; k, gq^{(\rho)}) - f(\omega, \omega'; k, gq) | = 0,\quad \alpha > m,\quad \alpha_1 > m. \qquad (4.26)$$

Proof. — Let us show firstly that

$$\lim_{\rho\to\infty} k \| \mathrm{X}_\beta [\mathrm{R}(k, gq^{(\rho)}) - \mathrm{R}(k, gq)]\mathrm{X}_\beta \| = 0, \qquad \alpha_1 \geqslant 2\beta > 1, \qquad (4.27)$$

uniformly in $k \geqslant k_0 > 0$, $N \leqslant N_0$. We start from the resolvent identity (see (2.25))

$$X_\beta R(k, gq^{(\rho)})X_\beta = X_\beta R(k, gq)X_\beta - g[X_\beta R(k, gq)X_\beta]u_\rho[X_\beta R(k, gq^{(\rho)})X_\beta], \quad (4.28)$$

where $u_\rho(x) = (1 + |x|)^{2\beta}[q^{(\rho)}(x) - q(x)]$. The estimate (3.6) (for $s = 1$), the conditions (4.23) and $N \leqslant N_0$ ensure that

$$\lim_{\rho \to \infty} g \| X_\beta R(k, gq)X_\beta u_\rho \| = 0. \quad (4.29)$$

Thus, (4.28) can be considered as an equation for $X_\beta R(k, gq^{(\rho)})X_\beta$ with a small given operator $gX_\beta R(k, gq)X_\beta u_\rho$ and hence

$$\| X_\beta R(k, gq^{(\rho)})X_\beta \| \leqslant C \| X_\beta R(k, gq)X_\beta \| \leqslant C_1 k^{-1}. \quad (4.30)$$

Using (4.28) once more and applying (4.29), (4.30) we arrive at (4.27). Combined together, relations (4.23), (4.27) show that definitions (2.14) of $\psi(\omega, k, gq^{(\rho)})$, (2.9) of $f(\omega, \omega'; k, gq^{(\rho)})$ and the representation (2.13) for $\sigma(\omega; k, gq^{(\rho)})$ admit taking a limit as $\rho \to \infty$. Moreover, this limit is uniform in $k \geqslant k_0 > 0$, $N \leqslant N_0$. $\square$

Note that under the assumption (4.23) all objects (4.1)-(4.3) are also continuous as $\rho \to \infty$, i. e.

$$\lim_{\rho \to \infty} \| X_\beta [\hat\psi(k, gq^{(\rho)}) - \hat\psi(k, gq)] \| = 0, \ 2\beta > 1, 2\alpha > m+1, 2\alpha_1 > m+1, \quad (4.31)$$

$$\lim_{\rho \to \infty} | \mathscr{A}_0(\omega; Nq^{(\rho)}) - \mathscr{A}_0(\omega; Nq) | = 0, \ 2\alpha > m+1, \ 2\alpha_1 > m+1, \quad (4.32)$$

$$\lim_{\rho \to \infty} | \mathscr{A}(\omega; Nq^{(\rho)}) - \mathscr{A}(\omega; Nq) | = 0, \ \alpha > m, \ \alpha_1 > m, \quad (4.33)$$

uniformly in $k \geqslant k_0 > 0$, $N \leqslant N_0$. Let us check, for example, (4.31). Using the inequality $|e^{i\varphi} - 1| \leqslant |\varphi|$, we find that

$$\| X_\beta [\hat\psi(k, gq^{(\rho)}) - \hat\psi(k, gq)] \|^2 \leqslant \int dx (1 + |x|)^{-2\beta} \left[N \int_{-\infty}^{Z} (q^{(\rho)} - q)(Z', b) dZ' \right]^2$$
$$\leqslant C \int_{\Lambda_\omega} db \left[\int_{-\infty}^{\infty} (1 + |Z'| + |b|)^{-\alpha_1} dZ' \right]^2 \sup_{x \in \mathbb{R}^m} [(1 + |x|)^{\alpha_1} | q^{(\rho)}(x) - q(x) |].$$

The integral here is finite, and the second factor tends to zero, as $\rho \to \infty$.

THEOREM 5. — *Let for a family $q^{(\rho)}(x)$ the condition (4.23) hold. Assume that a function q in (4.23) is continuous and satisfies assumptions (1.1) and (3.1). Then for $N \leqslant N_0$, $2\alpha > m + 1$, $2\alpha_1 > m + 1$*

$$\lim_{k \to \infty, \, \rho \to \infty} \| X_\beta [\psi(k, gq^{(\rho)}) - \hat\psi(k, gq)] \| = 0, \qquad 2\beta > m, \quad (4.34)$$

$$\lim_{k \to \infty, \rho \to \infty} | \sigma(\omega; k, gq^{(\rho)}) - \mathscr{A}_0(\omega; Nq) | = 0. \quad (4.35)$$

If, moreover, $\alpha > m$, $\alpha_1 > m$, then

$$\lim_{k \to \infty, \rho \to \infty} | f(\omega, \omega; k, gq^{(\rho)}) - \mathscr{A}(\omega; Nq) | = 0. \quad (4.36)$$

Proof. — Consider a sequence $q_n \in C_0^\infty(\mathbb{R}^m)$ such that the difference $q_n - q$ satisfies (4.23) (for every $\alpha_1 < \alpha$). For fixed n and $k \to \infty$, $N \leqslant N_0$ the asymptotics of $\psi(\omega, k, gq_n)$, $\sigma(\omega; k, gq_n)$ and $f(\omega, \omega; k, gq_n)$ are given by Theorem 4. Apply now Lemma 4 and relations (4.31)-(4.33) to the sequence q_n. This proves formulae (4.34)-(4.36) for the case $q^{(\rho)} = q$. To treat the general case, one should use once more Lemma 4. □

In the following section we shall need also the following auxiliary assertion.

LEMMA 5. — Under the assumptions of Theorem 5 for all $r > 0$

$$\lim_{k\to\infty,\, \rho\to\infty} \| \chi_r \psi(k, gq^{(\rho)}) \| \leqslant C r^{m/2}, \tag{4.37}$$

$$\overline{\lim_{k\to\infty,\, \rho\to\infty}}\; k^{-1} \| \chi_r \nabla\psi(k, gq^{(\rho)}) \| \leqslant C r^{m/2}. \tag{4.38}$$

Proof. — The bound (4.37) is a direct corollary of (4.34) and of the obvious equality $\| \chi_r \hat{\psi}(k, gq) \| = C r^{m/2}$. For the proof of (4.38) we apply an elementary estimate

$$\int | \nabla\psi(x) |^2 \zeta^2(r^{-1}x)dx \leqslant C \left\{ \left[\int | \Delta\psi(x) |^2 \zeta^2(r^{-1}x)dx \,.\, \int | \psi(x) |^2 \zeta^2(r^{-1}x)dx \right]^{1/2} + r^{-2} \int | \psi(x) |^2 \, | (\nabla\zeta)(r^{-1}x) |^2 dx \right\}, \qquad \zeta \in C_0^\infty(\mathbb{R}^m), \tag{4.39}$$

to the function $\psi(x) = \psi(x; k, gq^{(\rho)})$. In virtue of (4.37), when multiplied by k^{-2}, the second summand in the R.H.S. of (4.39) tends to zero as $k \to \infty$, $\rho \to \infty$. On the other hand, it follows from the Schrödinger equation that

$$| \Delta\psi(x; k, gq^{(\rho)}) | \leqslant C k^2 \, | \psi(x; k, gq^{(\rho)}) | .$$

Now (4.39) shows that

$$\overline{\lim_{k\to\infty,\, \rho\to\infty}}\; k^{-2} \int | \nabla\psi(x; k, gq^{(\rho)}) |^2 \zeta^2(r^{-1}x)dx \leqslant C \overline{\lim_{k\to\infty,\, \rho\to\infty}} \int | \psi(x; k, gq^{(\rho)}) |^2 \zeta^2(r^{-1}x)dx .$$

To conclude the proof of (4.38), it remains to use (4.37) once more. □

5. THE PROOF OF THE MAIN THEOREM. THE ASYMPTOTICS OF THE SCATTERING AMPLITUDE AND OF THE TOTAL CROSS-SECTION

In this section we shall complete the proof of Theorem 1. All scattering amplitudes will be considered at the diagonal $\omega = \omega'$. So the dependence of amplitudes on $\omega = \omega'$ and of cross-sections on ω is often dropped out of notation. It is convenient to perform at first a scale transformation $x \to \rho x$, where $\rho^{\alpha-1} = \mathrm{N}$. Then, by formulae (2.20), (2.21)

$$f(k, gq) = \rho^{m-1} f(k_1, g_1 q^{(\rho)}), \qquad \sigma(k, gq) = \rho^{m-1}\sigma(k_1, g_1 q^{(\rho)}), \quad (5.1)$$

with

$$q^{(\rho)}(x) = \rho^\alpha q(\rho x), \qquad k_1 = \rho k, \qquad g_1 = \rho^{2-\alpha} g, \qquad \mathrm{N}_1 := g_1 (2k_1)^{-1} = 1. \quad (5.2)$$

Clearly, $g_1 = 2k_1 \to \infty$ as $k \to \infty$, $\mathrm{N} \to \infty$. Thus our problem is reduced to a similar one for the case $k_1 \to \infty$, $\mathrm{N}_1 = 1$ but for the family of functions $q^{(\rho)}$, depending on the additional parameter ρ. The restriction $\mathrm{N} \to \infty$, $g \leqslant \gamma_0 k^2$ implies that $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$. Since, as $\rho \to \infty$, the family $q^{(\rho)}(x)$ infinitely grows for small $|x|$, results of section 4 can not be applied to the problem (5.1), (5.2) directly.

To use these results, we make a cut-off by zero in a neighbourhood of $x = 0$. Namely, let $\eta_a \in \mathrm{C}^\infty(\mathbb{R}^m)$, $\eta_a(x) = 0$ for $|x| \leqslant a$ and $\eta_a(x) = 1$ for $|x| \geqslant 2a$. Instead of $q^{(\rho)}$ we consider previously an auxiliary potential $\eta_a q^{(\rho)}$. For fixed $a > 0$ this potential satisfies assumptions of Theorem 5. Actually, set $q^{(as)}(x) = |x|^{-\alpha}\Phi(\hat{x})$. The function $\eta_a(x)\, q^{(as)}(x)$ is, obviously, continuous and obeys conditions (1.1) and (3.1). Moreover, by (1.3)

$$\lim_{\rho\to\infty} \sup_{x\in\mathbb{R}^m} (1 + |x|)^\alpha |\eta_a(x)[q^{(\rho)}(x) - q^{(as)}(x)]| = 0.$$

Theorem 5 ensures now that for $\mathrm{N}_1 = 1$ and fixed $a > 0$

$$\lim_{k_1\to\infty,\rho\to\infty} f(k_1, g_1\eta_a q^{(\rho)}) = \mathscr{A}(\eta_a q^{(as)}), \qquad \alpha > m, \quad (5.3)$$

$$\lim_{k_1\to\infty,\rho\to\infty} \sigma(k_1, g_1\eta_a q^{(\rho)}) = \mathscr{A}_0(\eta_a q^{(as)}), \qquad 2\alpha > m+1. \quad (5.4)$$

We shall show, further, that relations (5.3), (5.4) hold true without a cutt-off function $\eta_a(x)$ if ρ is subject to the condition $2\rho^\alpha \leqslant \gamma_0 k_1$, i. e.

$$\lim f(k_1, g_1 q^{(\rho)}) = \mathscr{A}(q^{(as)}), \qquad \alpha > m, \quad (5.5)$$

$$\lim \sigma(k_1, g_1 q^{(\rho)}) = \mathscr{A}_0(q^{(as)}), \qquad 2\alpha > m+1, \quad (5.6)$$

as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$ and $2\rho^\alpha \leqslant \gamma_0 k_1$. To go over from (5.3) to (5.5), we notice first of all that

$$\lim_{a\to 0} \mathscr{A}(\eta_a q^{(as)}) = \mathscr{A}(q^{(as)}), \qquad \alpha > m. \quad (5.7)$$

Actually, according to the definition (4.2)

$$|\mathscr{A}(\eta_a q^{(as)}) - \mathscr{A}(q^{(as)})| \leqslant \int_{\Lambda_\omega} db \left| \exp\left[i \int_{-\infty}^{\infty} ((1-\eta_a)q^{(as)})(Z, b)dZ \right] - 1 \right|. \quad (5.8)$$

Since $\eta_a(Z, b) = 1$ for $|b| \geqslant 2a$ and all $Z \in \mathbb{R}$, the integration over variable b in (5.8) is restricted to a ball $|b| \leqslant 2a$. Consequently the R. H. S. of (5.8) does not exceed Ca^{m-1}, what proves (5.7). To take off η_a in the L. H. S. of (5.3), it suffices to check that as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$

$$\overline{\lim} \, | f(k_1, g_1 \eta_a q^{(\rho)}) - f(k_1, g_1 q^{(\rho)}) | \leqslant \varepsilon_a \tag{5.9}$$

with $\varepsilon_a \to 0$ as $a \to 0$. Actually,

$$\overline{\lim} \, | f(k_1, g_1 q^{(\rho)}) - \mathscr{A}(q^{(as)}) | \leqslant \overline{\lim} \, | f(k_1, g_1 q^{(\rho)}) - f(k_1, g_1 \eta_a q^{(\rho)}) |$$
$$+ \overline{\lim} \, | f(k_1, g_1 \eta_a q^{(\rho)}) - \mathscr{A}(\eta_a q^{(as)}) | + | \mathscr{A}(\eta_a q^{(as)}) - \mathscr{A}(q^{(as)}) | , \tag{5.10}$$

where upper limits are taken for $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$ and fixed a. In the R.H.S. of (5.10) the first and the third summands tend to zero as $a \to 0$ in virtue of (5.9) and (5.7). The second summand is zero by (5.3). Since the L.H.S. of (5.10) does not depend on a, it also equals zero. This proves (5.5). Similarly to (5.7), it can be verified that $\mathscr{A}_0(\eta_a q^{(as)}) \to \mathscr{A}_0(q^{(as)})$ as $a \to 0$, $2\alpha > m + 1$. Thus, by the above arguments to justify the transition from (5.4) to (5.6), it suffices to check that as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$,

$$\overline{\lim} \, | \sigma(k_1, g_1 \eta_a q^{(\rho)}) - \sigma(k_1, g_1 q^{(\rho)}) | \leqslant \varepsilon_a \tag{5.11}$$

with $\varepsilon_a \to 0$ as $a \to 0$.

For the proof of (5.9), (5.11) consider the scattering amplitude $\tilde{f}(k_1; v, v_1)$, where $v_1 = g_1 \eta_a q^{(\rho)}$, $v = g_1 q^{(\rho)}$. Recall that $\tilde{f}$ is defined by (2.26), where we choose $\mathscr{J} = \mathscr{J}_a$ to be a multiplication by such a function $\tilde{\eta}(a^{-1}x)$ that $\tilde{\eta} \in C^\infty(\mathbb{R}^m)$, $\tilde{\eta}(x) = 0$ for $|x| \leqslant 2$ and $\tilde{\eta}(x) = 1$ for $|x| \geqslant 3$. Then $\tilde{\eta}(a^{-1}x)\eta_a(x) = \tilde{\eta}(a^{-1}x)$ and

$$\tilde{T} = H\mathscr{J} - \mathscr{J}H_1 = -2a^{-1}(\nabla\tilde{\eta})(a^{-1}x)\nabla - a^{-2}(\Delta\tilde{\eta})(a^{-1}x). \tag{5.12}$$

We shall show that

$$\overline{\lim} \, | \tilde{f}(k_1; v, v_1) | \leqslant Ca^{m-1} \tag{5.13}$$

as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$. Once (5.13) is verified, relations (5.9) and (5.11) follow from inequalities (2.28) and (2.30) correspondingly. Actually, by (5.4), (5.13)

$$\overline{\lim} \{ | \tilde{f}(k_1; v, v_1) | + [2\sigma(k_1, v_1) \operatorname{Im} \tilde{f}(k_1; v, v_1)]^{1/2} \}$$
$$\leqslant C \{ a^{m-1} + [\mathscr{A}_0(\eta_a q^{(as)}) a^{m-1}]^{1/2} \}$$

as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$ and a fixed. Since $\mathscr{A}_0(\eta_a q^{(as)}) \leqslant C$, the R. H. S. here is bounded by $Ca^{\frac{m-1}{2}}$. Inequalities (2.28), (2.30) ensure now (5.9), (5.11) with $\varepsilon_a = Ca^{\frac{m-1}{2}}$. Thus, the demonstration of (5.9), (5.11) is reduced to that of (5.13).

The proof of (5.13) is, essentially, similar to the estimate of the scattering amplitude in Theorem 3 for potentials with compact supports. By (2.26), (5.12)

$$| \tilde{f}(k_1; v, v_1) | \leqslant Ck_1^{-1} \| \chi_{3a}\psi(k_1, v_1) \| [a^{-1} \| \chi_{3a}\nabla\psi(k_1, v_1) \| + a^{-2} \| \chi_{3a}\psi(k_1, v_1) \|] + Ck_1^{-1} \| \chi_{3a}R(k_1, v)\chi_{3a} \| [a^{-2} \| \chi_{3a}\nabla\psi(k_1, v_1) \|^2 + a^{-4} \| \chi_{3a}\psi(k_1, v_1) \|^2],$$
$$v_1 = g_1\eta_a q^{(\rho)}, \qquad v = g_1 q^{(\rho)}. \quad (5.14)$$

The necessary bound for the resolvent $R(k_1, g_1 q^{(\rho)})$ follows from (3.6). Actually, applying (2.18) we find that

$$a^{2\beta} \| X_\beta^{(a)} R(k_1, g_1 q^{(\rho)}) X_\beta^{(a)} \| \leqslant Ck_1^{-1} a, \; g_1 = 2k_1, \; 2\rho^\alpha \geqslant \gamma_0 k_1, \; ak_1 \geqslant c > 0. \quad (5.15)$$

We emphasize that the restriction $2\rho^\alpha \leqslant \gamma_0 k_1$ is required only for a validity of (5.15). It is sufficient for us to have a corollary of (5.15) when $a^\beta(|x|+a)^{-\beta}$ is replaced by a characteristic function $\chi_{3a}(x)$. Taking here an upper limit as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$ and a fixed we find that

$$\overline{\lim}\; k_1 \| \chi_{3a} R(k_1, g_1 q^{(\rho)}) \chi_{3a} \| \leqslant Ca. \quad (5.16)$$

Recall, further, that for fixed a the family $\eta_a q^{(\rho)}$ satisfies assumptions of Theorem 5 and, thus, for functions $\psi(k_1, g_1\eta_a q^{(\rho)})$ the bounds (4.37), (4.38) hold. Now we take in (5.14) an upper limit as $g_1 = 2k_1 \to \infty$, $\rho \to \infty$, $2\rho^\alpha \leqslant \gamma_0 k_1$, a fixed. By (4.37), (4.38), (5.16) two summands in the R. H. S. of (5.14), not containing $\| \chi_{3a}\nabla\psi(k_1, v_1) \|$, vanish in such a limit. The surviving two summands are bounded by a^{m-1}. This concludes the proof of (5.13) and, hence, of (5.5), (5.6).

It remains to make sure that (5.5), (5.6) are equivalent to (3.4), (3.5). By (5.1), (5.2) and (2.7) in terms of initial objects $F(\omega, \omega; k, gq)$, $\sigma(\omega; k, gq)$ relations (5.5), (5.6) can be rewritten as

$$\lim N^{-\varkappa} k^{-\frac{m-1}{2}} F(\omega, \omega; k, gq) = (2\pi)^{-\frac{m-1}{2}} e^{-\frac{\pi i}{4}(m-3)} \mathscr{A}(\omega; q^{(as)}), \quad (5.17)$$

$$\lim N^{-\varkappa} \sigma(\omega; k, gq) = \mathscr{A}_0(\omega; q^{(as)}), \quad (5.18)$$

where $\varkappa = (m-1)(\alpha-1)^{-1}$ and limits are taken for $k \to \infty$, $N \to \infty$, $g \leqslant \gamma_0 k^2$. Let us compute now coefficients $\mathscr{A}(\omega; q^{(as)})$ and $\mathscr{A}_0(\omega; q^{(as)})$. By definition (4.2), choosing spherical coordinates in Λ_ω^{m-1} we find that

$$\mathscr{A}(\omega; q^{(as)}) = i \int_{\mathbb{S}_\omega^{m-2}} d\varphi \int_0^\infty d|b|\, |b|^{m-2} \left[1 - \exp\left(-i \int_{-\infty}^\infty q^{(as)}(|b|\varphi + Z\omega) dZ \right) \right].$$

The substitution $Z = |b| \operatorname{ctg} \theta$ shows that

$$\int_{-\infty}^{\infty} q^{(as)}(|b|\varphi + Z\omega)dZ = |b|^{1-\alpha}\Omega(\omega, \varphi),$$

where the coefficient Ω is defined by (3.3). The integral over $|b|$ can be expressed in terms of the Γ-function:

$$\int_0^{\infty} d|b|\, |b|^{m-2}[1 - \exp(-i|b|^{1-\alpha}\Omega)] = (m-1)^{-1}|\Omega|^{\varkappa} \exp(2^{-1}\pi i\varkappa \operatorname{sgn}\Omega)\Gamma(1-\varkappa).$$

It follows that

$$\mathscr{A}(\omega; q^{(as)}) = i(m-1)^{-1}\Gamma(1-\varkappa)\int_{\mathbb{S}_{\omega}^{m-2}} d\varphi\, |\Omega(\omega, \varphi)|^{\varkappa} \exp[2^{-1}\pi i\varkappa \operatorname{sgn}\Omega(\omega, \varphi)].$$

Now the equivalence between (5.17) and (3.4) becomes apparent. A similar calculation shows that

$$\mathscr{A}_0(\omega; q^{(as)}) = \pi[(m-1)\Gamma(\varkappa)\sin(2^{-1}\pi\varkappa)]^{-1}\int_{\mathbb{S}_{\omega}^{m-2}} d\varphi\, |\Omega(\omega, \varphi)|^{\varkappa}.$$

Thus, (5.18) is equivalent to (3.5). This concludes, finally, the proof of Theorem 1.

The considerations above prove also Theorem 1′. Note in this connection that the condition $N \to \infty$ was used to replace $q(x)$ by its asymptotics $q^{(as)}(x)$ in (5.3), (5.4). The assumption $gk^{\alpha-2} \to \infty$ is required for the parameter k_1, $k_1^{\alpha-1} = 2^{-1}gk^{\alpha-2}$ (see (5.2)), to tend to infinity, which is necessary for the application of Theorem 5.

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APPENDIX

A) Let us give a direct proof of a coincidence of the righthand sides of (2.3) and (2.6). It suffices to prove that

$$\lim_{\varepsilon\to 0} Z_0(k)[\mathcal{J}^* - T^*R(k,\varepsilon)]TZ_0(k)^* = \lim_{\varepsilon\to 0} Z_0(k)[I - VR(k,\varepsilon)]VZ_0(k)^*. \qquad (A.1)$$

Let $K = I - \mathcal{J}$ be a multiplication by a function $1 - \eta \in C_0^\infty(\mathbb{R}^m)$. Then $V - T = HK - KH_0$ and the proof of (A.1) is split up into verifications of two similar equalities

$$\lim_{\varepsilon\to 0} Z_0(k)[K^* - (K^*H - H_0K^*)R(k,\varepsilon)]TZ_0(k)^* = 0, \qquad (A.2)$$

$$\lim_{\varepsilon\to 0} Z_0(k)[I - VR(k,\varepsilon)](HK - KH_0)Z_0(k)^* = 0. \qquad (A.3)$$

We check, for example, (A.2). Since $Z_0(k)H_0u = k^2Z_0(k)u$ and $(H-k^2)R(k,\varepsilon) = I + i\varepsilon R(k,\varepsilon)$, it is sufficient to show that

$$\lim_{\varepsilon\to 0} \varepsilon Z_0(k)KR(k,\varepsilon)TZ_0(k)^* = 0. \qquad (A.4)$$

Recall now that for $2\beta > 1$ the operator $Z_0(k)X_\beta$ is bounded and the operator $X_\beta R(k,\varepsilon)X_\beta$ has a limit as $\varepsilon \to 0$. By the expression (2.5) for T, the operator $Z_0(k)KR(k,\varepsilon)Z_0(k)^*$ also has a limit as $\varepsilon \to 0$. This certainly implies (A.4) and, hence, (A.2). The equality (A.3) can be proven in exactly the same manner. Combining (A.2) and (A.3) together we get (A.1).

B) We shall give here a direct proof of the equality (2.24). For the operator $\tilde{S}(k) = \tilde{S}(k; v, v_1)$ we use the expression (2.23) with $\mathcal{J} = I$ and $\tilde{T} = H - H_1 = V - V_1 =: \tilde{V}$. Similarly to A), one can verify with the help of the relation $Z_1(k)H_1u = k^2H_1u$ that the R. H. S. of (2.23) does not depend on $\mathcal{J}$. By the proof of (2.24) we omit k and set for brevity $R = R(v)$, $R_1 = R(v_1)$, $B = V - VRV$, $B_1 = V_1 - V_1R_1V_1$, $\tilde{B} = \tilde{V} - \tilde{V}R\tilde{V}$. In this notation the equality (2.24) reads

$$Z_0BZ_0^* = Z_0B_1Z_0^* + Z_1\tilde{B}Z_1^* - 2\pi i Z_0B_1Z_0^*Z_1\tilde{B}Z_1^*. \qquad (B.1)$$

We insert here the definition (2.22) of the operator Z_1 and neglect operators Z_0 and Z_0^* on the left and on the right. Thus for the proof of (B.1) it suffices to check that

$$B = B_1 + (I - V_1R_1^*)\tilde{B}(I - R_1V_1) - 2\pi i B_1Z_0^*Z_0(I - V_1R_1^*)\tilde{B}(I - R_1V_1). \qquad (B.2)$$

Note now that by (2.4) and (2.1)

$$2\pi i B_1Z_0^*Z_0(I - V_1R_1^*) = V_1(R_1 - R_1^*)$$

and by the resolvent identity (2.25)

$$\tilde{B}(I - R_1V_1) = \tilde{V}(I - RV).$$

Therefore (B.2) is equivalent to

$$B = B_1 + (I - V_1R_1)\tilde{V}(I - RV).$$

The last equality is again a direct consequence of (2.25). This proves (B.1) and hence (2.24).

Note added in proof. After this work was submitted for publication it was proven by the author that when averaged over some small interval of k the asymptotics (3.4) (for $\alpha > m$) and (3.5) (for $2\alpha > m + 1$) hold true in the whole region $N \to \infty$, $gk^{\alpha-2} \to \infty$.

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